MATHEMATICS

CLASS- 12

BOARD OF SECONDARY EDUCATION, RAJASTHAN
AJMER

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MATHEMATICS

Class - XII

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MATHEMATICS

Class - XII

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PREFACE

This book has been written in accordance with the new syllabus for class XII prescribed by the Board of Secondary Education, Rajasthan Ajmer. In presenting the book the basic object of the syllabus has been fully kept in mind and an attempt has been made to acquaint the students with the contribution of Indian Mathematician towards the development of scientific traditions. The contribution of Indian Mathematician have been mentioned at appropriate places. Every effort has been made to present the subject in simple and lucid manner Important principal have been explained in detail.

In the interest of the students sufficient number the illustrative examples have been given. At the end of each chapter a summary of the chapter is given in the form of important points, which will help the students in revision. In each chapter objective, short and essay type questions have been given in sufficient number in the miscellaneous exercise.

We hope the book will be useful to students. Students, teachers and reviewers are requested to send their comments, suggestions and to point out any shortcoming in the book, so that the desired improvement in the book can be made in the subsequent edition.

Authors

SYLLABUS

MATHEMATICS

Class-XII

Question paper Single	Marks for question paper 80	Sessional Marks 20	Max. Marks 100

Time- 3.15 hours **Max. Marks - 80**

S.N.	Name of Unit	Marks
1.	Composite Function	7
2.	Algebra	10
3.	Calculus	38
4.	Vector and Three-Diamensional Geometry	14
5.	Linear Programming	4
6.	Probability	7

Details of the Syllabus

Unit I. Composite Function 7

1. Function

Introduction and previous learning, properties of composite function, Inverse function, Domain of a inverse function, Range, Properties of inverse function, binary operation, modulo system.

2. Inverse Circular function

Definition, range, domain, principal value, general value, graph of inverse circular functions, relation and properties between inverse circular functions.

Unit II. Algebra 10

1. Matrix

Concept of matrix, notation, order, equality, type of matrices, null matrix, a tranpose of matrix, symmetric and skew-symmetric matrix, Addition of matrices, properties of addition operations, multiplication, properties of multiplication operation and properties of scalar product. Existence of non-zero matrices whose multiplication is a null matrix (the limitation of square matrices upto 2 order). (Here the elements of these matrices are real numbers).

2. Determination

Determinant of a square matrix (upto 3 x 3 square matrices) properties of determinants, minor, co-factor and expansion of determinants, elementary operations, multiplication of determinants.

3. Inverse Matrix and Linear Equations

Introduction, Non-singular matrix, singular matrix, Adjoint of a square matrix, inverse of a matrix, some important theorems, application of determinants - area of triangle, condition of co-linearity of three points, equation of a line passing through two points, solution of system of linear equations - (1) by Cremer's rule (2) help of matrix principal.

Unit III. Calculus 38

1. Continuity and differentiability, Derivative of composite functions, chain rule, inverse trigometrical functions, Derivative implicit functions, concept and derivative of exponential and logarithimic functions, derivative of parametric functions, second order derivatives, Roll's and Lagrange's mean value theorem (without proof) and geometrical meaning of these theorems.

2. Application of derivatives

Application of derivatives : rate of change of quantities, increasing and decreasing functions, Tangents and normals, Approximations from the derivatives, methods to find maximum and minimum values. Simple applications of maxima and minima. (which shows the basic concepts of the subject and related to the real life)

3. Integration

Integration is the inverse process of the differentiation, integration of the different type of functions - by substition, integration by resolving into partial fractions and integration by parts. To evaluate the integrations as follows :-

$$\int \frac{dx}{x^2 \pm a^2}, \int \frac{dx}{\sqrt{x^2 \pm a^2}}, \int \frac{dx}{\sqrt{a^2 - x^2}}, \int \frac{dx}{ax^2 + bx + c}, \int \frac{dx}{\sqrt{ax^2 + bx + c}}$$

$$\int \frac{px+q}{ax^2+bx+c} dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx, \int \sqrt{a^2 \pm x^2} dx \text{ and } \int \sqrt{x^2 - a^2} dx$$

$$\int \sqrt{ax^2 + bx + c}\ dx, \int \underset{\text{II}}{e^{ax}} \underset{\text{I}}{\sin} bx\, dx, \int \underset{\text{II}}{e^{ax}} \underset{\text{I}}{\cos} bx\, dx$$

Definite integral as a limit fo sum, fundamental theorem of calculus (without proof), basic properties of definite integral, to evaluate the definite integrals.

4. Application of integral :

Application : Area under simple curves, specially lines, area of circle / parabolas / ellipses (those in standard forms only), Area between tow curves (those areas which can be reconize easily).

5. Differential equations :

Definition, order and degree, formation of differential equations, general and particular solution of differential equations, solution of first order and first degree differential equations, variable separation form, solution of homogenous equations, linear differential equation solution of differential equations reducable to linear differential equations.

Unit IV. Vector and Three-Dimensional Geometry 14

1. Vector

Vector and scalar, megnitude and direction of a vector, types of vector (equal, megnitude, null, parallel and colinear vector), position vector a point, negative vector, components of a vector. Addition of vectors, multiplication of a vector by a scalar. Vector joining two points, section formula. Position vector of a point divided by section formula. Product of two vectors and its properties, scalar (dot) product and its properties vector, product of three vector, scalar product of three vectors.

2. Three-Dimensional Geometry

Direction ratio and direction cosine of a line passes through two points, equation of a line in cartesian form and vector form. Angle between two lines, intersection of two lines, perpendicular distance of a point from a line, coplanner and skew lines. The shortes distance between two skew lines, distance between two parallel lines cartesian and vector equation of a plane. (1) Angle between two planes, (2) Angle between a plane and a line, distance of a point from a plane.

Unit V. Linear Programming **4**

Introduction, Linear programming problem and its mathematical formulation, the definition of various steps for example : constraints, objective functions, optimal solution, feasible solutions, various types of linear programming problems. Different type of application of linear programming problems.

Unit VI. Probability and probability distribution **7**

Conditional probability, multiplication rule of probability, independent events, total probability, Baye's theorem, random variables and its probability distribution, mean and variance of a random variable. Bernoulli's trials and Binomial distribution.

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01

Composite Functions

1.01 Introduction and Previous Knowledge

We have studied the notion of relations and functions, domain, co-domain and range have been introduced in previous class along with different types of specific real valued functions and their graphs. As the concept of function, we would like to extend our study about function from where we finished earlier. In this section, we would like to study different types of functions.

Function : A function from a non-empty set A to a non-empty set B is defined as a rule in which every element of a set A is *uniquely* associated with the element of set B.

Domain, Co-domain and Range of a Function : If *f* is a function from set A to set B then set A is called as domain of *f* and set B is called as co-domain of *f*. All those elements of set B which are the images of elements of set A are called as range of *f*. It is written as *f*(A).

Constant Function : In this type of function, every element of domain is associated with only one element of co-domain.

Identify Function : A function defined on set A in such a way that every element of A is associated to itself is known as identity function of A. It is written as I_A

Equal Functions : Two functions *f* and *g* are called equal if.

(i) Domain of *f* = Domain of *g* (ii) Co-domain of *f* = Co-domain of *g* (iii) $f(x) = g(x), \ \forall \ x \in$ domain

Type of Functions on the basis of association of elements

(i) **One-One function :** Let $f : A \to B$ is a function, then *f* is one-one if every element of set A has distinct image in set B

(ii) **Many-One function :** Let $f : A \to B$ is a function, then *f* is called many-one if two or more elements of set A has the same image in set B.

(iii) **Onto function :** A function $f : A \to B$ is said to be onto if every element of set B is the image of some element of set A under the function *f* i.e. for every element of B, there exist some pre-image in A i.e. is onto function if *f*(A)=B.

(iv) **Into function :** A function $f : A \to B$ is said to be into if there exist atleast one element in set B which is not the image of element of set A under the function *f* i.e. *f* is into function if $f(A) \neq B$.

(v) **One-One onto function :** A function $f : A \to B$ is said to be One One-Onto if *f* is one-one and onto. This is also called bijective function.

1.02 Composition of Function

Let A, B, C be three non-empty sets and let $f : A \to B$ and $g : B \to C$ are two functions.

Since *f* is a function from A to B therefore every element x of A threre exists a unique $f(x)$ in set B.

Again since g is a function from B to C therefore every $f(x)$ in set B there exists $g[f(x)]$ in set C.

Thus we see that for two functions *f* and *g* we get a new function defined from A to C. This function is said to be a composition of functions and represented by (gof) . It is defined as follows.

Definition : If $f : A \to B$ and $g : B \to C$ be two functions. Then the composition of *f* and *g*, denoted by *gof*, is defined as the function $(gof) : A \to C$,

$$(gof)\,x = g\left[f(x)\right], \quad \forall \; x \in A$$

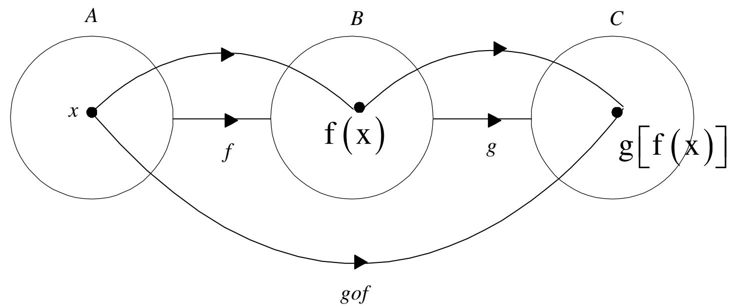


Fig. 1.01

Note : By the definition of *g* o *f*, when every element *x* of set A have *f*(*x*), element of domain of *g*. so that be find image of *g*. Hence *g* o *f* is defind if the range of *f* is the subset of domain of *g* is necessary.

Illustrative Examples

Example 1. If $A = \{1,2,3\}$, $B = \{4,5\}$, $C = \{7,8,9\}$ and $f : A \to B$ and $g : B \to C$ be the functions defined as $f(1) = 4$, $f(2) = 4$, $f(3) = 5$; $g(4) = 8$, $g(5) = 9$ then find $g \circ f$.

Solution : We have $(gof) : A \to C$

$$(gof)(1) = g\left[f(1)\right] = g(4) = 8$$

$$(gof)(2) = g\left[f(2)\right] = g(4) = 8$$

$$(gof)(3) = g\left[f(3)\right] = g(5) = 9$$

$$\therefore \qquad (gof) = \{(1,8),(2,8),(3,9)\}$$

Example 2. If $f : R \to R,\, f(x) = \sin x$ and $g : R \to R,\, g(x) = x^2$ then find $g \circ f$ and $f \circ g$.

Solution : Here the range of *f* is the subset of domain of *g* and range of *g* is the subset of domain of *f*. Therefore (gof) and (fog) both are defined.

$$(gof)(x) = g\left[f(x)\right] = g(\sin x) = (\sin x)^2 = \sin^2 x$$

$$(fog)(x) = f\left[g(x)\right] = f\left(x^2\right) = \sin x^2$$

Here $\qquad (gof) \neq (fog)$

Example 3. If $f : N \to Z,\, f(x) = 2x$

and $\quad g : Z \to Q,\, g(x) = (x+1)/2$ then find $f \circ g$ and $g \circ f$.

Solution : $\quad (gof)(x) = g\left[f(x)\right] = g(2x) = (2x+1)/2, \quad \forall\, x \in N$

here (fog) does not exist.

1.03 Properties of Composite of Functions

(i) The composite of functions is not necessarily commutative

Let $f : A \to B$ and $g : B \to C$ be the two functions, then composite function $(gof) : A \to C$ exists and defined because range of *f* is a subset of domain of *g*. But here (fog) does not exist as range of *g* is not a subset of domain of A of *f*, thus if $C \not\subset A$, (fog) will not exist.

If $C = A$ then $f : A \to B$ and $g : B \to A$

In this case $(gof) : A \to A$ and $(fog) : B \to B$ both will exist and $(gof) \neq (fog)$ as the domain and co-domain are different.

If $A = B = C$ then $(gof) : A \to A$ and $(fog) : A \to A$, but it is not necessary that both will be equal.

Example : If $f : R \to R, f(x) = 2x$ and $g : R \to R, g(x) = x^2$ then $(gof) : R \to R, (fog) : R \to R$ but

$$(gof)(x) = g[f(x)] = g(2x) = (2x)^2 = 4x^2$$

$$(fog)(x) = f[g(x)] = f(x^2) = 2x^2$$

$\therefore$ $$(fog) \neq (fog)$$

Note : (gof) and (fog) are equal only in specific conditions.

Example : If $f : R \to R, f(x) = x^2$

$g : R \to R, g(x) = x^3$ then $(gof) : R \to R, (fog) : R \to R$

and $$(gof)(x) = g[f(x)] = g(x^2) = (x^2)^3 = x^6$$

$$(fog)(x) = f[g(x)] = f(x^3) = (x^3)^2 = x^6$$

$\therefore$ $$(fog) = (gof)$$

This condition does not occur every time.

(ii) Composite of Functions is Associative

Thorem 1.1 If three functions f, g, h are such that the function $f \circ (g \circ h)$ and $(f \circ g) \circ h$ are defined then

$$f \circ (g \circ h) = (f \circ g) \circ h$$

Proof : Let the three functions *f*, *g*, *h* are such that:

$$h : A \to B, \ g : B \to C, \ f : C \to D$$

Now both the functions $fo(goh)$ and $(fog)oh$ are defiend from A to D.

i.e. $$fo(goh) : A \to D \quad \text{and} \quad (fog)oh : A \to D$$

Clearly the domain A and co-domain D of both the functions are same, hence to compare them we have to prove that

$$[fo(goh)](x) = [(fog)oh](x), \ \forall x \in A$$

Let $x \in A, \ y \in B, \ z \in C$ such that

$$h(x) = y \quad \text{and} \quad g(y) = z$$

then
$$[fo(goh)](x) = f[(goh)(x)]$$
$$f[g\{h(x)\}] = f[g(y)] = f(z)$$

$\therefore$
$$[fo(goh)](x) = f(z) \tag{1}$$

again
$$[(fog)oh](x) = (fog)[h(x)] = (fog)(y)$$
$$= f[g(y)] = f(z) \tag{2}$$

from (1) and (2)

$$[fo(goh)](x) = [(fog)oh](x), \ \forall x \in A$$

$\therefore$
$$fo(goh) = (fog)oh$$

This can be shown through the following figure:

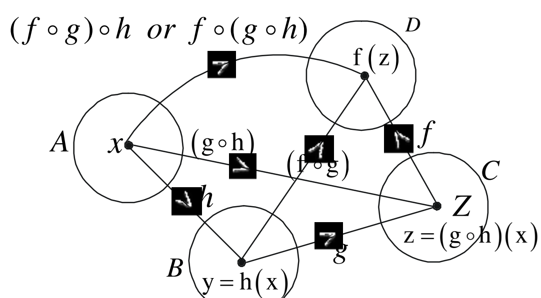


Fig 1.02

(ii) The composite of two bijections is a bijection

Theorem 1.2 If f and g are bijective functions such that (gof) is defined then (gof) is also a bijective function.

Proof : Let $f : A \to B$ and $g : B \to C$ are the two one-one onto functions then composite function (gof) is defined from set A to set C such that,

$$(gof) : A \to C$$

To prove that (gof) is one-one onto function:

One-one : Let $a_1, a_2 \in A$ be such that

$$(gof)(a_1) = (gof)(a_2)$$

$\Rightarrow$
$$g[f(a_1)] = g[f(a_2)]$$

$\Rightarrow \qquad f(a_1) = f(a_2) \qquad [\because g \text{ is one-one}]$

$\Rightarrow \qquad a_1 = a_2 \qquad [\because f \text{ is one-one}]$

$\therefore \qquad (gof)$ is one-one

Onto : If $c \in C$ then

$c \in C \Rightarrow \exists\, b \in B$ is such that $g(b) = c$ $\qquad [\because g \text{ is onto}]$

again $\qquad b \in B \Rightarrow \exists\, a \in A$ is such that $f(a) = b$ $\qquad [\because f \text{ is onto}]$

similarly $\qquad c \in C \Rightarrow \exists\, a \in A$ is such that

$$(gof)(a) = g[f(a)] = g(b) = c$$

i.e. every element of C is the image of some element of A, in other words A has the pre-image of every element of C. Therefore (gof) is onto.

$\therefore \qquad (gof)$ is One-one onto function.

Theorem 1.3 If $f : A \to B$ then $foI_A = I_B of = f$, where I_A and I_B are identity functions defined in set A and B.

i.e. composition of any function with the identity function is function itself.

Proof : $\qquad \because \quad I_A : A \to A$ and $f : A \to B$ $\qquad \therefore \quad (foI_A) : A \to B$

Let $x \in A$ then

$$(foI_A)(x) = f[I_A(x)] = f(x) \qquad [\because I_A(x) = x, \forall\, x \in A]$$

$\therefore \qquad foI_A = f \qquad$ (1)

again $\qquad f : A \to B$ and $I_B : B \to B$ $\qquad \therefore \quad (I_B of) : A \to B$

Let $x \in A$ and $f(x) = y$, where $y \in B$

$$\therefore \qquad (I_B of)(x) = I_B[f(x)] = I_B(y) = y \qquad [\because I_B(y) = y, \ \forall y \in B]$$

$$= f(x) \qquad (2)$$

from (1) and (2) $\qquad (I_B of) = f = (f \circ I_A)$.

Illustrative Examples

Example 4. If $f : R \to R, f(x) = x^3$ and $g : R \to R, g(x) = 3x - 1$ then find $(gof)(x)$ and $(fog)(x)$. Also prove that $fog \neq gof$.

Solution : Clearly $(gof) : R \to R$ and $(fog) : R \to R$

$$(gof)(x) = g[f(x)] = g(x^3) = 3x^3 - 1$$

again $\qquad (fog)(x) = f[g(x)] = f(3x-1) = (3x-1)^3$

$$\because \qquad (3x^3 - 1) \neq (3x-1)^3$$

$$\therefore \qquad (gof) \neq (fog)$$

Example 5. If $f: R \to R,\ f(x) = x^2 + 2$ and $g: R \to R,\ g(x) = \dfrac{x}{x-1}$ then find (gof) and (fog).

Solution : Clearly $(gof): R \to R$ and $(fog): R \to R$ both exist

Let $x \in R$

then $$(g \circ f)(x) = g[f(x)] = g[x^2 + 2] = \frac{x^2 + 2}{x^2 + 2 - 1} = \frac{x^2 + 2}{x^2 + 1}$$

and $$(fog)(x) = f\left[g(x)\right] = f\left(\frac{x}{x-1}\right) = \left(\frac{x}{x-1}\right)^2 + 2 = \frac{x^2 + 2(x-1)^2}{(x-1)^2}$$

Example 6. Verify the associativity of the following functions:

$$f: N \to Z_0,\ f(x) = 2x;\ g:\ Z_0 \to Q,\ g(x) = \frac{1}{x} \text{ and } h: Q \to R,\ h(x) = e^x.$$

Solution : $\because$ $f: N \to Z_0,\ g: Z_0 \to Q,\ h: Q \to R$

$\because$ $(go\,f): N \to Q$ and $(hog): Z_0 \to R$

$\therefore$ $(hog)of: N \to R$

and $h: Q \to R,\ (go\,f): N \to Q \quad \therefore ho(go\,f): N \to R$

Thus both the functions $(hog)of$ and $ho(go\,f)$ are defined on the set from N to R. Now we have to show that

$$\left[(ho\,g)of\right](x) = \left[ho(go\,f)\right](x), \quad \forall\ x \in N$$

Now $$\left[(hog)of\right](x) = \left(hog\left[f(x)\right]\right) = (hog)(2x) = h\left[g(2x)\right] = h\left(\frac{1}{2x}\right) = e^{1/2x} \qquad (1)$$

and $$\left[ho(gof)\right](x) = h\left[(go\,f)(x)\right] = h\left[g\left(f(x)\right)\right]$$

$$= h\left[g(2x)\right] = h\left(\frac{1}{2x}\right) = e^{1/2x} \qquad (2)$$

from (1) and (2)

$$\left[(hog)o\ f\right](x) = \left[ho(go\,f)\right](x).$$

Thus the associativity of the function f, g, h is verified.

Exericse 1.1

1. If $f: R \to R$ and $g: R \to R$ are the two functions defined below then find $(fog)(x)$ and $(gof)(x)$

   (i) $f(x) = 2x + 3,\ g(x) = x^2 + 5$

   (ii) $f(x) = x^2 + 8,\ g(x) = 3x^3 + 1$

   (iii) $f(x) = x,\ g(x) = |x|$

   (iv) $f(x) = x^2 + 2x + 3,\ g(x) = 3x - 4.$

2. If $A=\{a,b,c\},\ B=\{u,v,w\}$

and $f:A\to B$ and $g:B\to A$ are defined as

$f=\{(a,v),(b,u),(c,w)\}$; $g=\{(u,b),(v,a),(w,c)\}$

then find (fog) and (gof) .

3. If $f:R^+\to R^+$ and $g:R^+\to R^+$ are defiend as

$$f(x)=x^2 \text{ and } g(x)=\sqrt{x}$$

then find gof and fog . Are they equal ?

4. If $f:R\to R$ and $g:R\to R$ are two functions such that $f(x)=3x+4$ and $g(x)=\frac{1}{3}(x-4)$ then find $(fog)(x)$ and $(gof)(x)$ also find $(gog)(1)$.

5. If three functions f,g,h defined from R to R in such a way that $f(x)=x^2$, $g(x)=\cos x$ and $h(x)=2x+3$ then find the value of $\{ho\ (gof)\}\sqrt{2\pi}$.

6. If *f* and *g* are defiend as given below then find $(gof)(x)$.

(*i*) $f:R\to R, f(x)=2x+x^{-2}$, $g:R\to R, g(x)=x^4+2x+4.$

7. If $A=\{1,2,3,4\},\ f:R\to R,\ f(x)=x^2+3x+1$

$g:R\to R, g(x)=2x-3$ then find

(*i*) $(fog)(x)$ (*ii*) $(gof)(x)$ (*iii*) $(fof)(x)$ (*iv*) $(gog)(x)$.

1.04 Inverse function

(a) Inverse of an element

Let A and B be two sets and *f* is a function from A to B. i.e. $f:A\to B$ If an element 'a' of set A is associated to an element 'b' of set B under *f* then *b* is the *f* image of *a* under the function *f* is expressed as $b=f(a)$ and element 'a' is called as pre-image or inverse of 'b' under *f* and is denoted by $a=f^{-1}(b)$.

Inverse of an element may be unique, more than one or no one under a function. In fact, this all depend upon the function is one-one, many one, onto or into.

The function *f* is defined as shown in the figure.

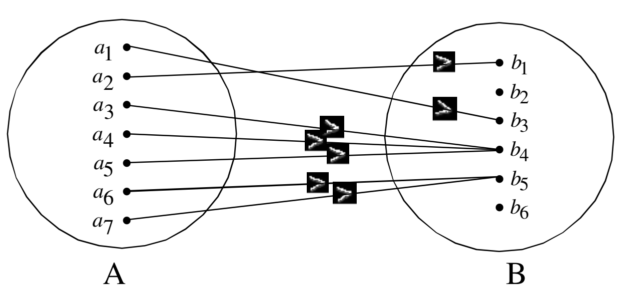


Fig. 1.03

We see that $f^{-1}(b_1) = a_2,$

$f^{-1}(b_2) = \phi,\ f^{-1}(b_3) = a_1,$

$f^{-1}(b_4) = \{a_3, a_4, a_5\},\ f^{-1}(b_5) = \{a_6, a_7\},$

$f^{-1}(b_6) = \phi.$

Example : If $A = \{-1, 1, -2, 2, 3\}$, $B = \{1, 4, 6, 9\}$ and $f : A \to B$, $f(x) = x^2$ are defined as

$f^{-1}(1) = \{-1, 1\},\ f^{-1}(4) = \{-2, 2\},\ f^{-1}(6) = \phi$ and $f^{-1}(9) = \{3\}.$

Example : If $f : C \to C$, $f(x) = x^2 - 1$ then find $f^{-1}(-5)$ and f^{-1}a8f .

Solution : Let $f^{-1}(-5) = x$ then $f(x) = -5$

$\Rightarrow$ $x^2 - 1 = -5 \Rightarrow x^2 = -4 \Rightarrow x = \sqrt{-4}$

$\Rightarrow$ $x = \pm 2i$. both are in C.

again let $f^{-1}(8) = x$ then $f(x) = 8.$

$\Rightarrow$ $x^2 - 1 = 8 \Rightarrow x^2 = 9,\ x = \pm 3$ both are in C

$\therefore$ $f^{-1}(8) = \{-3, 3\}$

i.e. $f^{-1}(-5) = \{2i, -2i\}$ and $f^{-1}(8) = \{-3, 3\}.$

(b) Inverse function

Let A and B be two sets and $f : A \to B$. If we correlate the element of B to their pre-image in A under any rule then we find that there is some element in B which is not associated with any element in A. It happen when it is not onto. therefore, it is necessary that *f* is onto if all element of B would associate any element of A. Just like that if *f* is many-one then some element of B is associated with one or more element of A Therefore, an element of B is associated only one element of A only if *f* is one-one.

Thus we see that if $f : A \to B$ is One-One Onto function then we can define a new function from B to A in which every element y of B is related to its pre-image $f^{-1}(y)$ in A. This function is called as Inverse of *f* and is denoted by f^{-1} .

Definition : If $f : A \to B$ is one-one onto function and inverse of *f* is f^{-1}, then B is a function defined in A in which $b \in B$, is related to $a \in A$ where $f(a) = b.$

$\therefore$ $f^{-1} : B \to A,\ f^{-1}(b) = a \Leftrightarrow f(a) = b$

It is represented as $f^{-1} : \{(b, a) | (a, b) \in f\}$ in terms of ordered pair.

Note: The function f^{-1} is said to be the inverse of *f*, only when it is one-one onto.

1.05 Domain and Range of inverse function

It is clear from the definition that

$$\text{Domain of } f^{-1} = \text{Range of } f$$

and $$\text{Range of } f^{-1} = \text{Domain of } f$$

For Example : If $A = \{1,2,3,4\}$, $B = \{2,5,10,17\}$ and $f(x) = x^2 + 1$ then

$$f(1) = 2,\ f(2) = 5,\ f(3) = 10,\ f(4) = 17$$

$\therefore$ $$f = \{(1,2),(2,5),(3,10),(4,17)\}$$

Clearly f is one-one onto therefore its inverse exists i.e. $f^{-1}: B \to A$ and

$$f^{-1} = \{(2,1),(5,2),(10,3),(17,4)\}.$$

For Example : Let $f : R \to R,\ f(x) = 3x + 4,$ then we can easily prove that f is one-one onto function.

Therefore $f^{-1}: R \to R$ exists

Let $x \in R$ (Domain of f) and $y \in R$ (Co-domain of f)

Let $f(x) = y,$ $\qquad \therefore\ x = f^{-1}(y)$

Now $$f(x) = y \Rightarrow 3x + 4 = y \Rightarrow x = \frac{y-4}{3}$$

$$\Rightarrow\ f^{-1}(y) = \frac{y-4}{3}$$

$\therefore$ $$f^{-1} : R \to R,\ f^{-1}(x) = \frac{x-4}{3} \text{ is defined.}$$

1.06 Properties of Inverse Functions

Theorem 1. The inverse of a bijection is unique.

Proof : Let $f : A \to B$ is one-one, onto function then to prove that inverse of f is unique.

If possible let $g : B \to A$ and $h : B \to A,\ f$ are two inverse function of f. Let y be any element of B.

Let $g(y) = x_1$ and $h(y) = x_2$

Now $g(y) = x_1 \Rightarrow f(x_1) = y$ $\quad$ [$\because$ g is the inverse of f]

and $h(y) = x_2 \Rightarrow f(x_2) = y$ $\quad$ [$\because$ h, is the inverse of f]

$\therefore$ $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ $\quad$ [$\because$ f is one-one]

i.e. $g(y) = h(y),\ \forall\ y \in B$

$\therefore$ $g = h$

Thus inverse of f is unique.

Theorem 1.5 If $f : A \to B$ is bijective function and $f^{-1}: B \to A,\ f$ is the inverse of f then $fof^{-1} = I_B$ and $f^{-1}of = I_A$, where I_A and I_B respectively are the identity function on A and B.

Proof : $f : A \to B$ and $f^{-1} : B \to A$

$\therefore$ $fof^{-1} : B \to B$ and $f^{-1}of : A \to A$

For every $a \in A$ there exist a unique $b \in B$.

Where $f(a) = b$ or $f^{-1}(b) = a$

$\therefore$ $\left(fof^{-1}\right)(b) = f\left[f^{-1}(b)\right] = f(a) = b$

$\therefore$ $\left(fof^{-1}\right)(b) = b, \quad \forall\, b \in B$

$\therefore$ $fof^{-1} = I_B$

similarly $\left(fof^{-1}\right)(a) = f^{-1}\left[f(a)\right] = f^{-1}(b) = a$

$\therefore$ $\left(fof^{-1}\right)(a) = a, \quad \forall\, a \in A$

$\therefore$ $f^{-1}of = I_A$.

Theorem 1.6 The inverse of a bijection is also a bijection.

Proof : Let $f : A \to B$ is one-one, onto function and $g : B \to A$ is the inverse function of *f*. To prove that *g* is also one-one onto function.

Let $a_1, a_2 \in A;\ b_1, b_2 \in B$ are elements such that

$g(b_1) = a_1$ i.e. $f(a_1) = b_1$ [$\because$ g is the inverse of *f*]

and $g(b_2) = a_2$ i.e. $f(a_2) = b_2$ [$\because$ g is the inverse of *f*]

Now $g(b_1) = g(b_2)$ $\Rightarrow$ $a_1 = a_2$

$\Rightarrow$ $f(a_1) = f(a_2)$ $\Rightarrow$ $b_1 = b_2$

$\therefore$ g is one-one

again $a \in A \Rightarrow \exists\, b \in B$ for which $f(a) = b$

Now $f(a) = b \Rightarrow g(b) = a$

$a \in A \Rightarrow \exists\, b \in B$ such that $g(b) = a$

$\therefore$ g is Onto

Thus inverse function *g* is also one-one onto.

Theorem 1.7 If *f* and *g* are two one-one, onto functions such that the composite function gof is defined then there exist an inverse of gof i.e.

$$(gof)^{-1} = f^{-1}og^{-1}$$

Proof : Let $f : A \to B$ and $g : B \to C$ are two one-one, onto functions. Given that $(gof) : A \to C$ is defined therefore from theorem 1.2, gof exist and given by

$$(gof)^{-1} : C \to A$$

To prove that $(gof)^{-1} = f^{-1}og^{-1}$

Now $f : A \to B$ is one-one onto function $\Rightarrow$ $f^{-1}: B \to A$ exists

Again $g : B \to C$ is one-one onto function $\Rightarrow$ $g^{-1} : C \to B$ exists

$\therefore$ $\left(f^{-1}og^{-1}\right): C \to A$ exists

similarly domain and co-domain of $(gof)^{-1}$ and $\left(f^{-1}og^{-1}\right)$ are same.

Let $a \in A, b \in B, c \in C$ are elements such that

$$f(a) = b \quad \text{and} \quad g(b) = c$$

$\therefore$ $(gof)(a) = g[f(a)] = g(b) = c$

$\Rightarrow$ $(gof)^{-1}(c) = a$ (1)

again $f(a) = b \Rightarrow f^{-1}(b) = a$ (2)

$g(b) = c \Rightarrow g^{-1}(c) = b$ (3)

$\therefore$ $\left(f^{-1}og^{-1}\right)(c) = f^{-1}\left[g^{-1}(c)\right] = f^{-1}(b)$ [from (3)]

$= a$ [from (2)] (4)

Therefore from (1) and (4), for any element x of C.

$$(gof)^{-1}(x) = \left(f^{-1}og^{-1}\right)(x)$$

This proves that $(gof)^{-1} = f^{-1} \circ g^{-1}$.

Illustrative Examples

Example 7. If $f : R \to R, f(x) = x^2 + 5x + 9$ then find the value of $f^{-1}(8)$ and $f^{-1}(9)$.

Solution : Let $f^{-1}(8) = x \Rightarrow f(x) = 8$

$\Rightarrow$ $x^2 + 5x + 9 = 8 \Rightarrow x^2 + 5x + 1 = 0$

$\Rightarrow$ $x = \dfrac{-5 \pm \sqrt{25-4}}{2} = \dfrac{-5 \pm \sqrt{21}}{2}$

$\therefore$ $f^{-1}(8) = \left\{\frac{1}{2}\left(-5+\sqrt{21}\right), \frac{1}{2}\left(-5-\sqrt{21}\right)\right\}$

again Let $f^{-1}(9) = x \Rightarrow f(x) = 9$

$\Rightarrow$ $x^2 + 5x + 9 = 9 \Rightarrow x = 0,\ x = -5$

$\therefore$ $f^{-1}(9) = \{0, -5\}$.

Example 8. If $f : R \to R, f(x) = x^2 + 1$ then find the value of $f^{-1}(-5)$ and $f^{-1}(26)$.

Solution : Let $f^{-1}(-5) = x$ then $f(x) = -5$

$\Rightarrow$ $x^2 + 1 = -5 \Rightarrow x^2 = -6 \Rightarrow x = \pm\sqrt{-6}$

then $\sqrt{-6}$ is not a real number

$\therefore \quad \pm\sqrt{-6} \notin R \quad \therefore \quad f^{-1}(-5)=\phi$

again let $\quad f^{-1}(26)=x$ then $f(x)=26$

$\Rightarrow \quad x^2+1=26 \Rightarrow x^2=25 \Rightarrow x=\pm 5$

$\therefore \quad f^{-1}(26)=\{-5,5\}$

Example 9. If $f: R \to R, f(x)=x^3+2$ then prove that *f* is one-one onto function. Also find the inverse of *f*.

Solution : Let $x_1, x_2 \in R$ then $f(x_1)=f(x_2)$

$$\Rightarrow x_1^3+2=x_2^3+2 \quad \Rightarrow x_1^3=x_2^3 \quad \Rightarrow x_1=x_2$$

$\therefore$ f is one-one

again let $y \in R$ then $\exists\ (y-2)^{1/3} \in R$ is such that

$$f\left[(y-2)^{1/3}\right]=(y-2)+2=y$$

Thus function is onto

$\therefore \quad f$ is one-one onto function

since *f* is bijective then $f^{-1}: R \to R$ is defined as

$$f^{-1}(y)=x \quad \Leftrightarrow \quad f(x)=y$$

but $\quad f(x)=x^3+2 \quad \Rightarrow \quad x^3+2=y$

$\Rightarrow \quad x=(y-2)^{1/3}$

$\Rightarrow \quad f^{-1}(y)=(y-2)^{1/3} \quad \Rightarrow \quad f^{-1}(x)=(x-2)^{1/3}$

$\therefore \quad f^{-1}: R \to R, f^{-1}(x)=(x-2)^{1/3}$.

Example 10. If $f: Q \to Q,\ f(x)=2x$ and $g: Q \to Q,\ g(x)=x+2$ then verify the following

$$(gof)^{-1}=f^{-1}og^{-1}$$

Solution : Since *f* and *g* are two linear functions therefore *f* and *g* are one-one onto functions thus their inverse f^{-1} and g^{-1} exist

$$f^{-1}=Q \to Q,\quad f^{-1}(x)=\frac{x}{2}, \quad \forall\ x \in Q \qquad (1)$$

$$g^{-1}=Q \to Q,\quad g^{-1}(x)=x-2 \quad \forall\ x \in Q \qquad (2)$$

We know that composition of two bijective functions is also bijective, therefore $(gof): Q \to Q$ is also bijective and its inverse exists

$\therefore \quad (gof)^{-1}: Q \to Q \quad \because \quad (gof)(x)=g[f(x)]=g(2x)=2x+2$

$$\therefore \quad (gof)^{-1}(x)=(x-2)/2 \qquad (3)$$

again $\quad f^{-1}og^{-1}:Q\to Q$

and $\quad \left(f^{-1}og^{-1}\right)(x)=f^{-1}\left[g^{-1}(x)\right]=f^{-1}(x-2)$ [from (2)]

$=(x-2)/2$ [from (1), (4)]

from (3) and (4) $\quad (gof)^{-1}(x)=\left(f^{-1}og^{-1}\right)(x),\ \forall\, x\in Q$

$\therefore \quad (gof)^{-1}=f^{-1}og^{-1}.$

Exericse 1.2

1. If $A=\{1,2,3,4\}, B=\{a,b,c\}$, then define four one-one onto functions from A to B and also find their inverse function.
2. If $f:R\to R, f(x)=x^3-3$ then prove that f^{-1} exists and find formula of f^{-1} and the values of $f^{-1}(24)$, $f^{-1}(5)$.
3. If $f:R\to R$ is defined as follows

   (i) $f(x)=2x-3$ (ii) $f(x)=x^3+5.$

   then prove that *f* is bijective and also find f^{-1} .
4. If $A=\{1,2,3,4\}, B=\{3,5,7,9\}, C=\{7,23,47,79\}$ and $f:A\to B,\ f(x)=2x+1, g:B\to C, g(x)=x^2-2$ then write $(gof)^{-1}$ and $f^{-1}og^{-1}$ in the form of ordered pair.
5. If $f:R\to R, f(x)=ax+b, a\neq 0$ is defined then prove that *f* is bijective also find the formula of f^{-1} .
6. If $f:R\to R, f(x)=\cos(x+2)$ then does f^{-1} exist?
7. Find f^{-1} (if exist) when $f:A\to B$, where

   (i) $A=\{0,-1,-3,2\}, B=\{-9,-3,0,6\}, f(x)=3x.$

   (ii) $A=\{1,3,5,7,9\}, B=\{0,1,9,25,49,81\}, f(x)=x^2.$

   (iii) $A=B=R,\ f(x)=x^3.$

1.07 Binary operation

Let S be a non-empty set. A function defined from $S\times S$ to S where S is a binary operation i.e. set S is defined in such a way that for every ordered pair (a, b) of set S there exist a unique element in S Generally the binary operation is denoted by symbols $*, o$ or $\oplus$. We denote $*$ by $a*b$ for all $(a,b)\in S\times S$.

Definition : A binary operation $*$ on set S is a function $*:S\times S\to S$ we denote $*(a,b)$ by a * b i.e.

$$a\in S, b\in S \quad\Rightarrow\quad a*b\in S,\quad \forall a,b\in S$$

For Example :

1. Addition $(+)$, substraction $(-)$ and multiplication $(\times)$ of integers are the binary operation on a set of integers Z which relates the elements a,b of Z with $(a+b),(a-b)$ and ab

2. For a power set of any set S, the union of sets $(\bigcup)$ and intersection $(\bigcap)$ are binary operations in P(S) because

$$A \in P(S),\ B \in P(S) \Rightarrow A \bigcup B \in P(S) \text{ and } A \bigcap B \in P(S)$$

3. In a set of rational number Q, *, is defined as

$$a * b = \frac{ab}{2}, \qquad \forall\, a,b \in Q$$

Q is a binary operation as for all $a \in Q, b \in Q \Rightarrow ab/2 \in Q$

4. In a set of real numbers R, $*$, where $*$ is defined as

$$a*b = a + b - ab, \qquad \forall\, a,b \in R$$

R is a binary operation as

$$a \in R,\ b \in R \Rightarrow (a + b - ab) \in R$$

5. In a set of natural numbers N addition and multiplication are binary operations

$$a \in N,\ b \in N \Rightarrow (a+b) \in N,\ \forall\, a,b \in N$$
$$a \in N,\ b \in N \Rightarrow (a \cdot b) \in N,\ \forall\, a,b \in N$$

But difference and divsion are not bianry operations on N.

6. Division is not a binary operation in any of the sets Z, Q, R, C, N but in Q_0, R_0 and C_0 it is a bianry operation.

7. Let S is a set of all defined function in a set A, then composite funcion S is a binary operation as

$$f, g \in S \Rightarrow f : A \to A,\ g : A \to A$$

$$\Rightarrow \qquad (gof) : A \to A$$

1.08 Types of binary operations

(i) Commutativity

Let S be a non-empty set in which a binary operation $*$ is defined $a, b \in S$ then we know that $(a,b) \neq (b,a)$ untill we have $a = b$. Thus it is not necessary that (a,b) and (b,a) defined under $*$ have same image. In other words it is not ncessary that

$$a*b = b*a, \qquad \forall\, a,b, \in S$$

If $a*b = b*a,\ \forall\, a,b, \in S$ then $*$ is commutative operation is S.

Defintion : A bianry operation in set S is said to be commutiatve if $a*b = b*a,\ \forall\, a,b, \in S$.

For Example 1. In a set of Real numbers R addition and multiplication are commutative operations but difference is not.

2. In a power set P(S) of Set S Union of sets $(\bigcup)$ and intersection $(\bigcap)$ are commutative operations but difference of sets is not commutative.

(ii) Associativity

Let S be a non-empty set in which a binary operation $*$ is defined. Let $a,b,c, \in S$. If three elements *a, b, c* are there but binary is defined for two numbers but here are three elements of S.

Therefore we have to foucs on $a*(b*c)$ or $(a*b)*c$ its not always true that

$a*(b*c)=(a*b)*c,\ \forall\ a,b,c \in S$. If $a*(b*c)=(a*b)*c, \forall\ a,b,c \in S$ then operation $*$ is associative.

Definition : A binary operation $*$ defined on set S is said to be associative if $a*(b*c)=(a*b)*c,\ \forall a,b,c \in S.$

For Example

1. Addition and multiplication of set of integers Z are associative but difference is not as

$$a+(b+c)=(a+b)+c,\ \forall\ a,b,c \in Z$$

$$a\cdot(b\cdot c)=(a\cdot b)\cdot c, \qquad \forall\ a,b,c \in Z$$

but $$a-(b-c)\neq(a-b)-c$$

2. For a power set P(S) of set S the union and intersection of sets are associative as for $A, B, C \in P(S)$ we have

$$(A\cup B)\cup C = A\cup(B\cup C)$$

and $$(A\cap B)\cap C = A\cap(B\cap C).$$

3. If A is a non-empty set and S is a set of all functions defined on A then operation defined on set S is a composite function and is associative as

$$(fog)\ oh = fo(goh), \forall\ f, g, h \in S.$$

(iii) Identity element for a binary operation

Let $*$, be a binary operation in set *S*. If there exist an element e in *S* such that

$$a*e=e*a=a,\ \forall\ a \in S,$$

then *e* is called as identity element in *S* under the operation *

For Example 1. In a set of integers *Z*, 0 and 1 are the identity elements of A under addition and multiplication because

for all $a \in Z$ $\qquad 0+a = a+0 = a$

and $\qquad 1\cdot a = a\cdot 1 = a$

2. In a set of natural numbers *N* there is no identity element in addition operation but for mulitplication operation 1 is the identity element.

3. For power set P(S), *S* and ϕ are the identity elements of Union and Intersection because for all $A \in P(S)$

$$A\cap S = S\cap A = A \qquad \text{and} \qquad A\cup\phi = \phi\cup A = A.$$

4. For a set of rational numbers *Q*, $*$ is a binary operation defined

$$a*b=\frac{ab}{2},\ \forall\ a,b \in Q$$

Here $2 \in Q$ is an identity element as for all $a \in Q$

$$2*a=\frac{2\cdot a}{2}=a \qquad \text{and} \qquad a*2=\frac{a\cdot 2}{2}=a.$$

Theorem 1.8 If an identity element of a binary operation in a set exist then it is unique.

Proof : Let *e* and *e*' be the identity element in the binary operation $*$ in a set *S*

$$e*e' = e' = e'*e \qquad [\because\ e \text{ is identity in } S \text{ and } e' \in S] \qquad (1)$$

again $$e'*e = e = e*e' \qquad [\because\ e' \text{ is identity in } S \text{ and } e \in S] \qquad (2)$$

from (1) and (2) $\quad e = e'$

Thus if the identity element of any operation exists then it is unique.

(iv) Inverse Element

Let $*$, be the binary operation in set S and let e be its identity element. Let $a \in S$. Let b be an element in set S such that

$$a*b = b*a = e$$

then b is known as the inverse of a and is denoted by a^{-1}.

The inverse element of a exist in S then a, is known as invertible element, therefore

$$\text{a} \in \text{S is invertible} \iff a^{-1} \in S$$

Note: Let $*$ be the binary operation in set S and let e be its identity element then $e*e = e*e = e$.

For Example 1. In a set of integers Z for every element a, $(-a) \in Z$, is an inverse element

$$a + (-a) = (-a) + a = 0 \text{ (identity)}$$

Thus every element of Z has inverse in addition operation.

2. In a set of rational numbers Q every non-zero number has inverse for multiplication operation and

$$a \in Q \text{ aa} \neq 0\text{f} \Rightarrow a^{-1} = 1/a \text{ because } a \cdot (1/a) = (1/a) \cdot a = 1$$

3. For positive set of rational numbers Q^+ a binary operation is defined as

$$a*b = ab/2, \qquad \forall\, a, b \in Q^+$$

We know that identity element of this operation is 2. The inverse of $a \in Q^+$ is $(4/a) \in Q^+$ as

$$\frac{4}{a} * a = \frac{(4/a) \times a}{2} = 2 \text{ (identity) and } a * \frac{4}{a} = \frac{a \times (4/a)}{2} = 2 \text{ (identity)}$$

Theorem 1.9 : Inverse of any invertible element with respect to a associative operation is uniuqe.

Proof : Let $*$, be a binary operation in Set S, which have identity element e. Let a is an inverse element of S. Let b and c are inverse element of a under S, is possible.

Now, $$b*(a*c) = b*e = b \qquad \left[\because c = a^{-1}\right]$$

and $$\text{a}b*a\text{f}*c = e*c = c \qquad \left[\because b = a^{-1}\right]$$

But by property of Associativity,

$$b*(a*c) = (b*a)*c$$

thus $\quad b = c$

So, inverse of an invertible element is unique.

1.09 Addition and Multiplication operations in modulo system

If a and b are integers and $(a-b)$ is a positive integers divisible by m then $a \equiv b$ (mod m) is denoted by a symbol and read as a is congruent to modulo m.

therefore $a \equiv b \pmod{m} \quad \Leftrightarrow \quad m \mid (a-b)$

For Example : $18 \equiv 6 \pmod{2} \quad \because 18-6=12, 2$ is divisible by 2

$-14 \equiv 6 \pmod{4} \quad \because -14-6=-20, 4$ is divisible by 4

again if m is a positive integer and a, b are two integers then by division algorithm there exist r, q such that

$$a+b=mq+r, \quad 0 \le r < m$$

then r is called as the remainder of addition modulo m of a and b and symbolically $a+b=r\,(\text{mod } m)$ or $a+_m b=r$

therefore $a+_m b=\begin{cases} a+b, & ;\text{if } a+b<m \\ r, & ;\text{if } a+b \ge m \end{cases}$, where r, is the non-negative remainder obtained by dividing $a+b$ by m

For Example $2+_4 3=1 \quad [\because 2+3=5=1\times 4+1]$

$-10+_4 3=1 \quad [\because -10+3=-7=-2\times 4+1]$

similarly m is a positive integer then for two numbers a, b if

$$a\cdot b=mq+r, \quad 0 \le r < m$$

then r is called as the remainder of multiplication modulo m of a and b, symbolically it is written as $a\cdot b=r \pmod{m}$ or is denoted by $a\times_m b=r$

$\therefore \quad a\times_m b=\begin{cases} ab, & ;\text{if } ab<m \\ r, & ;\text{if } ab \ge m \end{cases}$, Where r is the remainder when (a, b) is divided by m

Example $5\times_4 3=3 \quad [\because 15=4\times 3+3]$

$5\times_3 6=0 \quad [\because 5\times 6=30=10\times 3+0]$

1.10 Composition table for a finite set

When a given set A is finite, we can express a binary operation on the finite set A by a table called the *operation table* or *composition table* for the operation. For example:

Example 1. $S=\{(1,\omega,\omega^2);\times\}$ where ω is the cube root of unity

$\times$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

2. $S=\{(0,1,2,3);+_4\}$

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Just like we have following result from composite table:

(i) If table is symmetrical with respect to principle diagonal then defined operation is commutative under the set.

(ii) If row initiated from a_i is superimposed to uppermost row and column initiated from a_j is superimposed to left most column then, identify element of operation is in set S.

(iii) Any element of S is invertible if there is an identity element in corresponding row and column of table.

Illustrative Examples

Example 11. In a set of real number R, $*$ operation is defined as

$$a*b=a+b-ab, \quad \forall\ a,b\in R \text{ and } a\neq 1$$

(i) check the commutativity and associativity of $*$

(ii) find the identity element in $*$ if any

(iii) find the inverse element of $*$ with respect to R

Solution : (i) If $a,b\in R$ then by definition

$$a*b=a+b-ab=b+a-b\cdot a \qquad \text{(commutative)}$$
$$=b*a$$

$\therefore$ $*$ is a commutative

again $(a*b)*c=(a+b-ab)*c$

$$=(a+b-ab)+c-(a+b-ab)\cdot c$$
$$=a+b-ab+c-ac-bc+abc$$
$$=a+b+c-bc-ca-ab+abc \qquad (1)$$

and $a*(b*c)=a*(b+c-bc)$

$$=a+(b+c-bc)-a\cdot(b+c-bc)$$
$$=a+b+c-bc-ca-ab+abc \qquad (2)$$

from (1) and (2) it is clear that $(a*b)*c=a*(b*c)$

$\therefore$ $*$ is associative

(ii) Let e be the identity element of $*$ then for $a\in R$

$$a*e=a \text{ , by definition of identity}$$

$\Rightarrow$ $a+e-ae=a \Rightarrow e(1-a)=0$

$\Rightarrow$ $e=0\in R$ $\qquad [\because a\neq 1]$

0 is the identity element of $*$

(iii) Let $a\in R$ and let x be the inverse element of a then by definition

$$a*x=0 \text{ (identity)}$$

$\Rightarrow$ $a+x-ax=0 \Rightarrow x(a-1)=a$

$\Rightarrow$ $x=\dfrac{a}{a-1}\in R, \quad \because a\neq 1$

$\therefore$ $a\in R\ (a\neq 1)$ is invertible

Example 12. If $S=\{(a,b) | a,b\in R,\ a\neq 0\}$ and an operation $*$ is defined in S in such a way that

$$(a,b)*(c,d)=(ac,bc+d) \quad \text{then}$$

(i) check the commutativity and associativity of $*$

(ii) find the identity element in $*$ if any

(iii) find the inverse element of $*$ with respect to R

Solution : (i) Let $(a,b),(c,d)\in S$

then $(a,b)*(c,d)=(ac,bc+d)$ and $(c,d)*(a,b)=(ca,da+b)$

similarly $(a,b)*(c,d)\neq(c,d)*(a,b)$

$\therefore$ operation $*$ is not commutative

again let $(a,b),(c,d),(e,f)\in S$

now $[(a,b)*(c,d)]*(e,f)=(ac,bc+d)*(e,f)$

$$=(ace,(bc+d)e+f)=(ace,bce+de+f) \quad (1)$$

and $(a,b)*[(c,d)*(e,f)]=(a,b)*(ce,de+f)$
$=(ace,bce+de+f)$

$\therefore$ from (1) and (2) $[(a,b)*(c,d)]*(e,f)=(a,b)*[(c,d)*(e,f)]$ (2)

Thus $*$ is associative operation.

(ii) Let (x,y) be the identity element in S then for all $(a,b)\in S$

$(a,b)*(x,y)=(a,b)$ [by the definition of identity]

$\Rightarrow$ $(ax,bx+y)=(a,b)$

$\Rightarrow$ $ax=a$ and $bx+y=b$

Now $ax=a$ $\Rightarrow$ $x=1$ $[\because a\neq 0]$

and $bx+y=b$ $\Rightarrow$ $b+y=b$ $[\because x=1]$

$\Rightarrow$ $y=0$

$\therefore$ $(x,y)=(1,0)\in S$

$\therefore$ identity element of S is $(1,0)$

because $(a,b)*(1,0)=(a,b)$ and $(1,0)*(a,b)=(a,b)$.

(iii) Let $(a,b)\in S$ and inverse element of (a,b) is (x,y) then by the definition of inverse

$(a,b)*(x,y)=(1,0)$ [identity]

$\Rightarrow$ $(ax,bx+y)=(1,0)$

$\Rightarrow$ $ax=1,\ bx+y=0$

$ax=1,\ \Rightarrow$ $x=(1/a)$ $(a\neq 0)$

and $bx+y=0 \Rightarrow y=(-b/a)$ $(a\neq 0)$

$\therefore$ inverse of (a,b) is $(1/a,-b/a)$

Example 13. If $S=\{A,B,C,D\}$ where $A=\phi, B=\{a,b\}$ $C=\{a,c\}, D=\{a,b,c\}$ prove that the union of set $\cup$ is a binary operation in S but intersection of set $\cap$ is not a binary operation in S.

Solution : We see that

$$A\cup B=\phi\cup\{a,b\}=\{a,b\}=B,\ A\cup C=C, A\cup D=D$$

$$B\cup C=\{a,b\}\cup\{a,c\}=\{a,b,c\}=D$$

$$B\cup D=\{a,b\}\cup\{a,b,c\}=\{a,b,c\}=D,\ C\cup D=D$$

Thus union of set $\cup$ is a binary operation is S but $B\cap C=\{a,b\}\cap\{a,c\}=\{a\}\notin S$ therefore intersection of set $\cap$ is not a binary operation in S

Exercise 1.3

1. Determine whether or not each of the definition of $*$ given below gives a bianry operation. In the event that $*$ is not a binary operation, give justification for this.

   (i) $a*b = a$, on N (ii) $a*b = a + b - 3$, on N

   (iii) $a*b = a + 3b$, on N (iv) $a*b = a/b$, on Q

   (v) $a*b = a-b$, on R

2. For each binary operation * defined below, determine whether it is commutative or associative?

   (i) * on N where $a*b=2^{ab}$ (ii) * on N where $a*b=a+b+a^2b$

   (iii) * on Z where $a*b=a-b$ (iv) * on Q where $a*b=ab+1$

   (v) * on R where $a*b=a+b-7$

3. If in a set of integers Z an operation * is defined as $*, a*b=a+b+1,\ \forall a,b\in Z$ then prove that *, is commutative and associative. Also find its identity element. Find the inverse of any integer.

4. A binary operation defined on a set $R-\{1\}$ is as follows:-

   $$a*b=a+b-ab,\quad \forall\ a,b\in R-\{1\}$$

   Prove that * is commutative and associative also find its identity element and find inverse of any element a.

5. Four functions are defined in set R_0 as follows

   $$f_1(x)=x,\ f_2(x)=-x,\ f_3(x)=1/x,\ f_4(x)=-1/x$$

   Form the composition table for the 'compositive functions f_1, f_2, f_3, f_4 also find the identity element and inverse of every element.

Miscellaneous Exercise – 1

1. If $f:R\to R, f(x)=2x-3;\ g:R\to R, g(x)=x^3+5$ then the value of $(fog)^{-1}(x)$ is

   (a) $\left(\frac{x+7}{2}\right)^{1/3}$ (b) $\left(x-\frac{7}{2}\right)^{1/3}$ (c) $\left(\frac{x-2}{7}\right)^{1/3}$ (d) $\left(\frac{x-7}{2}\right)^{1/3}$.

2. If $f(x)=\frac{x}{1-x}=\frac{1}{y}$, then the vlaue of $f(y)$ is

(a) x (b) $x-1$ (c) $x+1$ (d) $\frac{1-x}{2x-1}$

3. If $f(x)=\frac{x-3}{x+1}$ then the vlaue of $f\left[f\{f(x)\}\right]$ is equal to

(a) x (b) $1/x$ (c) $-x$ (d) $-1/x$.

4. If $f(x)=\cos(\log x)$ then the valeu of $f(x)\cdot f(y)-\frac{1}{2}\left[f(x/y)+f(x\cdot y)\right]$ is

(a) –1 (b) 0 (c) 1/2 (d) –2.

5. If $f:R\to R, f(x)=2x+1$ and $g:R\to R, g(x)=x^3$, then $(gof)^{-1}(27)$ is equal to

(a) 2 (b) 1 (c) –1 (d) 0.

6. If $f:R\to R$ and $g:R\to R$, where $f(x)=2x+3$ and $g(x)=x^2+1$ then the value of $(gof)(2)$ is

(a) 38 (b) 42 (c) 46 (d) 50.

7. If an operation * defiend on Q_0 as $*, a*b=ab/2, \ \forall\, a,b\in Q_0$ then the identity element is

(a) 1 (b) 0 (c) 2 (d) 3.

8. A binary operation defind on R as $a*b=1+ab, \ \forall a,b\in R$ then * is

(a) commutative but no associative
(b) associative but not commutative
(c) neither commutative nor associative
(d) commutative and associative

9. In the set of integers Z the operation subtraction is

(a) commutative and associative
(b) associative but not commutative
(c) neither commutative nor associative
(d) commutative but not associative

10. If an operation $*$ defined on a set of rational numbers Q as $*$, $a*b=a+b-ab, \ \forall\, a,b\in Q$. The inverse of $a(\neq 1)$ with respect to this is

(a) $\frac{a}{a-1}$ (b) $\frac{a}{1-a}$ (c) $\frac{a-1}{a}$ (d) $\frac{1}{a}$

11. Which of the following is commutaitve defiend in a set of R

(a) $a*b=a^2b$ (b) $a*b=a^b$ (c) $a*b=a+b+ab$ (d) $a*b=a+b+a^2b$

12. For the given three functions justify the associativity of composite function operation

$$f:N\to Z_0,\ f(x)=2x \ ; \ g:Z_0\to Q,\ g(x)=1/x \ ; \ h:Q\to R,\ h(x)=e^x$$

13. If $f:R^+\to R^+$ and $g:R^+\to R^+$ are defined as below

$f(x)=x^2, g(x)=\sqrt{x}$ then find gof and fog , are these functions equal ?

14. If $f:R\to R, f(x)=\cos(x+2)$ then justify with reason that whether it is invertible or not.

15. If two functions f and g are defined on $A=\{-1,1\}$ and A where $f(x)=x^2$, $g(x)=\sin\left(\frac{\pi x}{2}\right)$, then prove that g^{-1} exist but f^{-1} does not. Also find g^{-1}.

16. If $f : R \to R$ and $g : R \to R$ are functions such that $f(x) = 3x + 4$ and $g(x) = \frac{(x-4)}{3}$, then find $(fog)(x)$ and $(gof)(x)$. Also fidn the vlaue of $(gog)(1)$.

Important Points

1. If f and g are two functions then gof is defined only when range of f is the subset of domain of g.
2. Composite functions need not satisfy commutative law.
3. Composite function obeys associative law i.e. $(fog)oh = fo(goh)$
4. If two functions are bijective then their compsoite functions are also bijective.
5. Inverse of bijective function is unique.
6. The inverse of one-one onto function is also one-one onto.
7. $(gof)^{-1} = f^{-1}og^{-1}$
8. In set A a binary operation is defined from $A \times A$ to A
9. An element $e \in S$ is the identity element for binary operation *
   If $a*e = e*a = a, \quad \forall a \in S$
10. If $a*b = b*a = e$, then b is inverse of a under $*$ on S.
11. Inverse of a is denoted by a^{-1}.
12. In a set S defined an operation *
    $$a*(b*c) = (a*b)*c, \ \forall a, b, c \in S$$
    then * operation is associative

Answers

Exercise 1.1

1. (i) $(gof)(x) = 4x^2 + 12x + 14, (fog)(x) = 2x^2 + 13$ (ii) $(gof)(x) = 3(x^2+8)^3 + 1, (fog)(x) = 9x^6 + 6x^3 + 9$

(iii) $(gof)(x) = |x|, (fog)(x) = |x|$ (iv) $(gof)(x) = 3x^2 + 6x - 13, (fog)(x) = 9x^2 - 18x + 5$

2. $fog = \{(u,u),(v,v),(w,w)\}; \ gof = \{(a,a),(b,b),(c,c)\}$

3. $(fog)(x) = x, (gof) = x$, Yes, its an identity function

4. $(fog)(x) = x, (gof) = x, (gog)(1) = -5/3$

5. 5

6. (i) $(gof)(x) = (2x + x^{-2})^4 + 2(2x + x^{-2}) + 4$

7. (i) $(fog)(x) = 4x^2 - 6x + 1$ (ii) $(gof)(x) = 2x^2 + 6x - 1$

(iii) $(fog)(x) = (x)^4 + 6x^3 + 14x^2 + 15x + 5$ (iv) $(gog)(x) = 4x - 9$

Exericse 1.2

1. $f_1 = \{(1,a),(2,b),(3,c),(4,d)\};\ f_1^{-1} = \{(a,1),(b,2),(c,3),(d,4)\}$

$f_2 = \{(1,a),(2,c),(3,b),(4,d)\};\ f_2^{-1} = \{(a,1),(c,2),(b,3),(d,4)\}$

$f_3 = \{(1,d),(3,b),(2,a)(4,c)\};\ f_3^{-1} = \{(d,1),(b,3),(a,2),(c,4)\}$

$f_4 = \{(1,a),(3,a),(2,b)(4,c)\};\ f_4^{-1} = \{(a,1),(a,3),(b,2),(c,4)\}$

2. $f^{-1}(x) = (3+x)^{1/3},\ f^{-1}(24) = 3,\ f^{-1}(5) = 2$

3. $f^{-1}(x) = \dfrac{x+3}{2},\ f^{-1}(x) = (x-5)^{1/3}$

4. $(gof)^{-1} = \{(7,1),(23,2),(47,3),(79,4)\} = f^{-1}og^{-1}$

5. $f^{-1}(x) = \dfrac{x-b}{a}$

6. NO

7. $(i)\ f^{-1} = \{(-9,-3),(-3,-1),(0,0),(6,2)\}$ $\quad(ii)\ f^{-1}$ does not exist $\quad(iii)\ f^{-1}(x) = x^{1/3}$

Exercise 1.3

1. (i) yes $\quad(ii)$ no $\quad(iii)$ yes $\quad(iv)$ no $\quad(v)$ yes

2. (i) commutative but not associative
   (ii) neither commutative nor associative
   (iii) neither commutative nor associative
   (iv) commutative but not associative
   (v) commutative and associative

3. $e = -1,\ a^{-1} = -(a+2)$

4. $e = 0,\ a^{-1} = \dfrac{a}{a-1}$

Miscellaneous Exericse 1

1. (d) 2. (d) 3. (a) 4. (b) 5. (b) 6. (d) 7. (c)

8. (a) 9. (c) 10. (a) 11. (c) 13. $(fog)(x) = (gof)(x) = x$ 14. No

15. $g^{-1}(x) = \dfrac{2}{\pi}\sin^{-1}x$

16. $(fog)(x) = (gof)(x) = x;\ (gog)(1) = \dfrac{-5}{3}$

02

Inverse Circular Functions

2.01 Introduction

If $\sin\theta = x$ then we say x is the sin of θ and θ is the sine inverse of x. This statement is written mathematically as $\theta = \sin^{-1} x$ or $\theta = \arcsin x$ is read as sine inverse x.

2.02 Inverse circular functions

We know that $\sin\theta, \cos\theta, \tan\theta$ are trigonometrical circular functions, which for every value of θ gives a fixed value

If $\sin\theta = x$ then $\theta = \sin^{-1} x$

$\sin^{-1} x$ is said to be an inverse circular function. The simialr inverse function are

$\cos^{-1} x, \tan^{-1} x, \cos^{-1} x$ and $\cot^{-1} x$

Note:

1. In functions $\sin^{-1} x, \cos^{-1} x$, -1 is not a exponent but representation of an inverse function as $(\sin x)^{-1} = \dfrac{1}{\sin x}$ therefore $\sin^{-1} x \neq (\sin x)^{-1}$

2. $\sin^{-1} x$ denotes an angle whereas $\sin\theta$ denotes a number where θ is an angle.

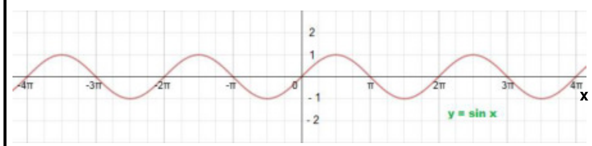
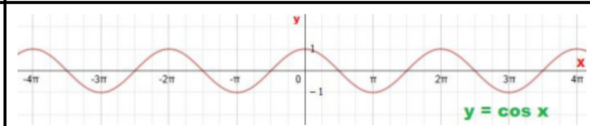
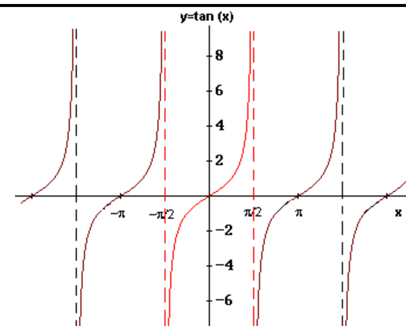
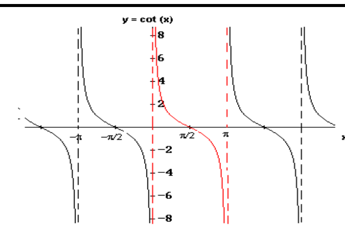
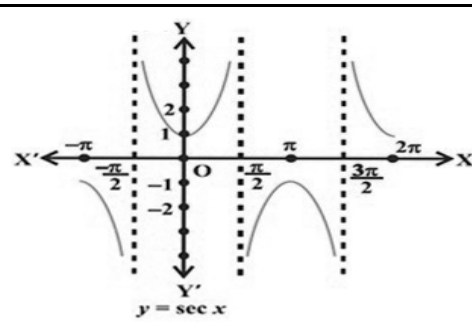
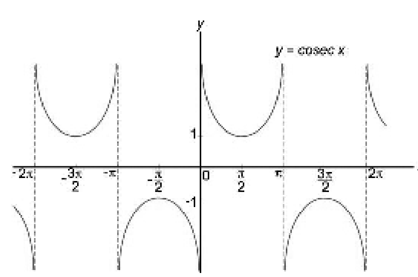
Inverse circular function:

To find the inverse to f i.e. f^{-1} the function f must be one-one onto.

It is clear from the study of trigonometric function that normally they are not bijective. Therefore it is not possible to find their inverse under normal conditions, but on restricting the domain of these function, they become one-one onto and we can easily derive their inverse, under these conditions.

the domain and range of inverse trigonometric function can be understood by the following table-

Table 2.1

Function $y = f(x)$	Domain	Range	Curve
$\sin x$	$x \in R$ or $\ldots\left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right], \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\ldots$	$y \in [-1, 1]$	
$\cos x$	$x \in R$ or $\ldots[-\pi, 0], [0, \pi], [\pi, 2\pi]\ldots$	$y \in [-1, 1]$	
$\tan x$	$x \in R - (2n+1)\frac{\pi}{2}, \;\; \forall \;\; n \in Z$ or $\ldots\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)\ldots$	$y \in R$	
$\cot x$	$x \in R - n\pi \;\; \forall \;\; n \in Z$ or $\ldots(-\pi, 0), (0, \pi), (\pi, 2\pi)\ldots$	$y \in R$	
$\sec x$	$x \in R - (2n+1)\frac{\pi}{2} \;\; \forall \;\; n \in Z$ or $\ldots[-\pi, 0] - \{-\pi/2\},$ $[0, \pi] - \{\pi/2\},$ $[\pi, 2\pi] - \{3\pi/2\}$	$y \in (-\infty, -1] \cup [1, \infty)$ i.e. range does not exist between -1 and 1	
cosec$x$	$x \in R - n\pi \;\; \forall \;\; n \in Z$ $\ldots[-3\pi/2, -\pi/2] - \{-\pi\},$ $[-\pi/2, \pi/2] - \{0\},$ $[\pi/2, 3\pi/2] - \{\pi\}, \ldots$	$y \in (-\infty, -1] \cup [1, \infty)$ i.e. range does not exist between -1 and 1	

Analysing the above table we see that

(i) Circular functiosn are not bijective in their entire domain.

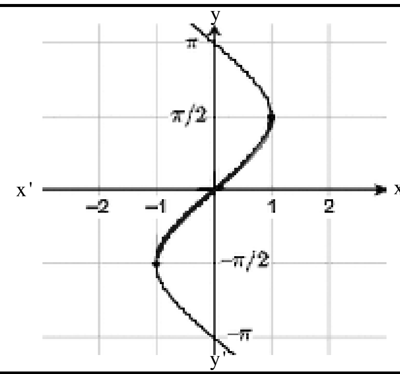
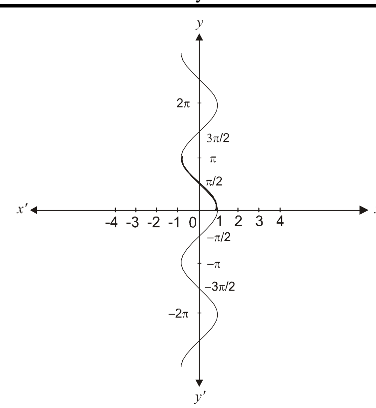
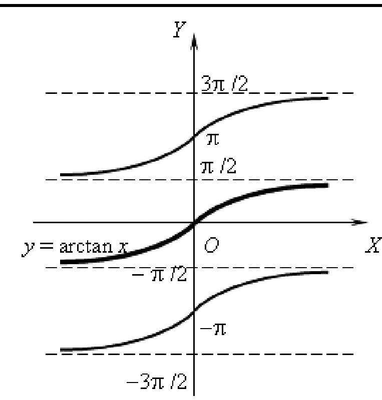
(ii) $\tan, \cot, \sec, \operatorname{cosec}$ functions are not defined on some of its points in their domain.

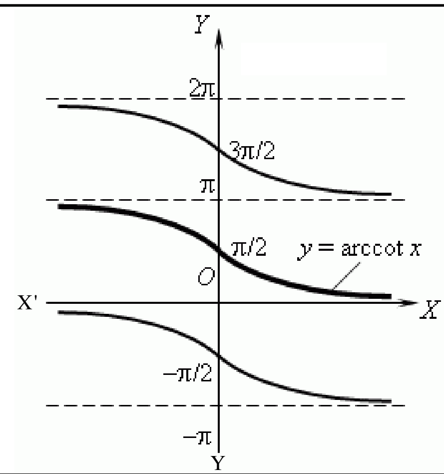
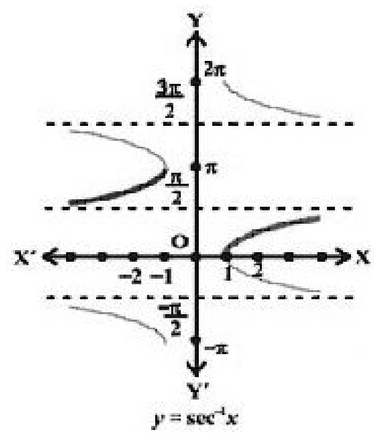
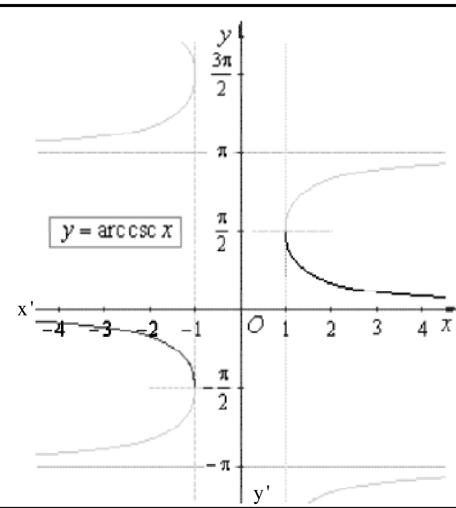
(iii) In sine and cosine functions range is restricted to [–1, 1] whereas in sec and cosec functions range do not lie between the interval (–1, 1)

Now if we have to find the inverse of these functions then we have to restrict their domain and make them one-one onto, for that from the above table the trigonometric fucntions become bijective itself by restricting their domain to any of the given intervals and then their inverse can be found out.

The following table shows the domain and range of inverse trigonometric functions under these bounded conditions. Every interval of range have a branch of inverse function. In these branches there is a principal branch, their range and shape represented by dark black colour.

Table 2.2

<table>
<tr><th>Function
$y =$</th><th>Domain</th><th>Range</th><th>Curve</th></tr>
<tr><td>$\sin^{-1} x$</td><td>$x = [-1, 1]$</td><td>$\ldots\left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right];$
$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right];$
$\left[\frac{\pi}{2}, \frac{3\pi}{2}\right], \ldots$</td><td>
</td></tr>
<tr><td>$\cos^{-1} x$</td><td>$x \in [-1, 1]$</td><td>$\ldots[-\pi, 0];$
$[0, \pi];$
$[\pi, 2\pi], \ldots$</td><td>
</td></tr>
<tr><td>$\tan^{-1} x$</td><td>$x \in R$</td><td>$\ldots\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right);$
$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right); \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ldots$
Note: function is not defined on
$\ldots -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$</td><td>
</td></tr>
</table>

$\cot^{-1} x$	$x \in R$	$...(-\pi, 0)$; $(0, \pi)$; $(\pi, 2\pi),...$ **Note:** function is not defined on $...-\pi, 0, \pi, 2\pi...$	
$\sec^{-1} x$	$x \in (-\infty, -1] \cup [1, \infty)$	$...[-\pi, 0]-\{-\pi/2\}$; $[0, \pi]-\{\pi/2\}$, $[\pi, 2\pi]-\{3\pi/2\},...$ **Note:** function is not defined on $...-\pi/2, \pi/2, 3\pi/2,...$	
$\text{cosec}^{-1} x$	$x \in (-\infty, -1] \cup [1, \infty)$	$...[-3\pi/2, -\pi/2]-\{-\pi\}$; $[-\pi/2, \pi/2]-\{0\}$; $[\pi/2, 3\pi/2]-\{\pi\},...$ **Note:** function is not defined on $...-\pi, 0, \pi,...$	

Note : If $y = f(x)$ then we get $x = f^{-1}(y)$ i.e. in the graph of trigonometric functions if we interchange the X and Y-axis then we get the graph of inverse trigonometric functions.

(i) If the branch of inverse circular function is not defined then we mean the principal branch of the function only.

(ii) If the value of inverse circular functions lies in the principal branch then that value is termed as its principal value, See table 2.3

General values

We know that $\sin\theta = \sin\left\{n\pi + (-1)^n\theta\right\}$, where $n \in Z$ is the set of integers.

Now if $\sin^{-1}x = \theta$ then the general value of $\sin^{-1}x$ is $n\pi + (-1)^n \sin^{-1}x$ and is denoted by $\sin^{-1}x$

Thus $\quad \text{Sin}^{-1}x = n\pi + (-1)^n \sin^{-1}x, \; n \in Z$

Similarly $\quad \text{Cos}^{-1}x = 2n\pi \pm \cos^{-1}x, \; n \in Z$

$\text{Tan}^{-1}x = n\pi + \tan^{-1}x$ etc.

where by $\text{Cos}^{-1}x, \text{Tan}^{-1}x$ we mean the general value of $\cos^{-1}x, \tan^{-1}x$. Similarly $\text{Sec}^{-1}x, \text{Cosec}^{-1}x, \text{Cot}^{-1}x$ we mean the general value of $\sec^{-1}x, \operatorname{cosec}^{-1}x, \cot^{-1}x$

Principal value

The Principal value of inverse circular function is the smallest positive or negative value of θ which satisifies the equation $\sin\theta = x, \cos\theta = x$ For example $\sin^{-1}\left(\frac{1}{2}\right) = 30^\circ, \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$. We denote this by $\sin^{-1}x, \cos^{-1}x, \tan^{-1}x$ etc.

The intervals of inverse circular functions are different:

Table 2.3

Function	Principal Value	Domain
$y = \sin^{-1}x$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$	$-1 \le x \le 1$
$y = \cos^{-1}x$	$0 \le y \le \pi$	$-1 \le x \le 1$
$y = \tan^{-1}x$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$	$-\infty < x < \infty$
$y = \sec^{-1}x$	$0 < y \le \pi, \; y \ne \frac{\pi}{2}$	$(-\infty < x \le -1) \cup (1 \le x < \infty)$
$y = \operatorname{cosec}^{-1}x$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}, \; y \ne 0$	$(-\infty < x \le -1) \cup (1 \le x < \infty)$
$y = \cot^{-1}x$	$0 < y < \pi$	$-\infty < x < \infty$

Note: (i) If $x > 0$ then the principal vlaues of all inverse circular functions lie in the first quadrant $[0, \pi/2]$

(ii) If $x < 0$ then the principal vlaues of $\sin^{-1}x, \tan^{-1}x$ and $\operatorname{cosec}^{-1}x$ lie in the fourth quadrant $[-\pi/2, 0]$ whereas the values of $\cot^{-1}x, \sec^{-1}x$ lie in the second quadrant $[\pi/2, \pi]$

2.03 Relation between Inverse Circular Functions

Let $\theta = \sin^{-1} x$ then $\sin\theta = x$ then $\cos\theta = \sqrt{1-x^2}$ $\left(\because \sin^2\theta + \cos^2\theta = 1\right)$

$$\theta = \cos^{-1}\sqrt{1-x^2}$$

Similarly $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{x}{\sqrt{1-x^2}} \Rightarrow \theta = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{1-x^2}}{x} \Rightarrow \theta = \cot^{-1}\frac{\sqrt{1-x^2}}{x}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1-x^2}} \Rightarrow \theta = \sec^{-1}\frac{1}{\sqrt{1-x^2}}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{x} \Rightarrow \theta = \operatorname{cosec}^{-1}\frac{1}{x}$$

$$\therefore \quad \sin^{-1} x = \cos^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \operatorname{cosec}^{-1}\frac{1}{x}$$

Note: The validity of these formulae is for certain interval.

2.04 Properties of inverse circular functions

(i) $\sin\left(\sin^{-1} x\right) = x, \quad -1 \le x \le 1$ and $\sin^{-1}\left(\sin\theta\right) = x, \quad -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

Proof : $\because \sin^{-1} x = \theta$ then $\sin\theta = x$ [by definition]

putting the vlaue of θ, we have $\sin\left(\sin^{-1} x\right) = x$

again if $\sin\theta = x, -1 \le x \le 1$

then $\theta = \sin^{-1} x, -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}; \theta = \sin^{-1}\left(\sin\theta\right)$

thus from the given table for the values of x and θ we have

$$\cos\left(\cos^{-1} x\right) = x \qquad \cos^{-1}\left(\cos\theta\right) = \theta$$

$$\tan\left(\tan^{-1} x\right) = x \qquad \tan^{-1}\left(\tan\theta\right) = \theta$$

$$\cot\left(\cot^{-1} x\right) = x \qquad \cot^{-1}\left(\cot\theta\right) = \theta$$

$$\sec\left(\sec^{-1} x\right) = x \qquad \sec^{-1}\left(\sec\theta\right) = \theta$$

$$\operatorname{cosec}\left(\operatorname{cosec}^{-1} x\right) = x \qquad \operatorname{cosec}^{-1}\left(\operatorname{cosec}\theta\right) = \theta$$

Note: $\sin^{-1}\left(\sin\dfrac{2\pi}{3}\right) \neq \dfrac{2\pi}{3}$ Since the principal value of $\sin^{-1} x$ is not $\dfrac{2\pi}{3}$

$$\sin^{-1}\left(\sin\frac{2\pi}{3}\right)=\sin^{-1}\left[\sin\left(\pi-\frac{\pi}{3}\right)\right]=\sin^{-1}\left(\sin\frac{\pi}{3}\right)=\frac{\pi}{3}$$

(ii) $\sin^{-1}\frac{1}{x}=\cos ec^{-1}x, \quad R\sim(-1,1)$

Note: $\sin^{-1}\frac{1}{x}=\theta \Rightarrow \sin\theta=\frac{1}{x}\Rightarrow \cos ec\theta=x\Rightarrow \theta=\cos ec^{-1}x\Rightarrow \sin^{-1}\frac{1}{x}=\cos ec^{-1}x$

$$\cos ec^{-1}x=\sin^{-1}\frac{1}{x},\ x\le -1,\ x\ge 1$$

$$\cos^{-1}x=\sec^{-1}\frac{1}{x},\ x\le -1,\ x\ge 1$$

$$\sec^{-1}x=\cos^{-1}\frac{1}{x},\ x\le -1,\ x\ge 1$$

$\tan^{-1}x=\cot^{-1}\frac{1}{x},\ x>0$ and $\cot^{-1}x=\tan^{-1}\frac{1}{x},\ x>0$

(iii) $\sin^{-1}(-x)=-\sin^{-1}x$ and $\cos^{-1}(-x)=\pi-\cos^{-1}x, -1\le x\le 1$

Proof : $\sin^{-1}(-x)=\theta \Rightarrow -x=\sin\theta \Rightarrow x=-\sin\theta=\sin(-\theta)$

or $\sin^{-1}x=-\theta=-\sin^{-1}(-x)$

or $\sin^{-1}(-x)=-\sin^{-1}x$

Similarly if $\cos^{-1}(-x)=\theta$ तो $x=-\cos\theta$

or $x=\cos(\pi-\theta)$

$\therefore$ $\cos^{-1}x=\pi-\theta$

or $\cos^{-1}x=\pi-\cos^{-1}(-x)$

or $\cos^{-1}(-x)=\pi-\cos^{-1}x$

Similarly $\tan^{-1}(-x)=-\tan^{-1}x, \quad \text{cosec}^{-1}(-x)=-\text{cosec}^{-1}x$

$$\sec^{-1}(-x)=\pi-\sec^{-1}x,\ \cot^{-1}(-x)=\pi-\cot^{-1}x$$

2.05 Other important standrad formulae

(i) To Prove that:

(a) $\sin^{-1}x\pm\sin^{-1}y=\sin^{-1}\left\{x\sqrt{1-y^2}\pm y\sqrt{1-x^2}\right\}$

(b) $2\sin^{-1}x=\sin^{-1}\left\{2x\sqrt{1-x^2}\right\}$

(c) $3\sin^{-1}x=\sin^{-1}\left\{3x-4x^3\right\}$

Proof : (a) Let $\sin^{-1} x = \theta_1 \quad \sin\theta_1 = x$ and $\sin^{-1} y = \theta_2$

i.e. $\sin\theta_2 = y$ then $\cos\theta_1 = \sqrt{1-\sin^2\theta_1} = \sqrt{1-x^2}$

similarly $\cos\theta_2 = \sqrt{1-\sin^2\theta_2} = \sqrt{1-y^2}$

we know that

$$\sin(\theta_1 \pm \theta_2) = \sin\theta_1\cos\theta_2 \pm \cos\theta_1\sin\theta_2$$

or $$\theta_1 \pm \theta_2 = \sin^{-1}(\sin\theta_1\cos\theta_2 \pm \cos\theta_1\sin\theta_2)$$

$\therefore$ $$\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right]$$

(b) Let $\sin^{-1} x = \theta$ i.e. $\sin\theta = x$

$\therefore$ $$\sin 2\theta = 2\sin\theta\cos\theta = 2\sin\theta\sqrt{1-\sin^2\theta} = 2x\sqrt{1-x^2}$$

$\Rightarrow$ $$2\theta = \sin^{-1}\left\{2x\sqrt{1-x^2}\right\}$$

$$2\sin^{-1} x = \sin^{-1}\left\{2x\sqrt{1-x^2}\right\}$$

(c) We know that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\therefore$ $$3\theta = \sin^{-1}(3\sin\theta - 4\sin^3\theta)$$

or $$3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$$

(ii) To Prove that

(a) $$\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\left\{xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right\}$$

(b) $$2\cos^{-1} x = \cos^{-1}(2x^2 - 1)$$

(c) $$3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$$

Proof : (a) Let $\cos^{-1} x = \theta_1$ i.e. $\cos\theta_1 = x$

and $\cos^{-1} y = \theta_2$ i.e. $\cos\theta_2 = y$

then $\sin\theta_1 = \sqrt{1-x^2}$ and $\sin\theta_2 = \sqrt{1-y^2}$

Now we know that

$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2$$

or $$\theta_1 \pm \theta_2 = \cos^{-1}(\cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2)$$

$\therefore$ $$\cos^{-1} x \pm \cos^{-1} y = \left[xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right]$$

(b) Let $\cos^{-1} x = \theta$ i.e. $\cos\theta = x$ $\therefore$ $\cos 2\theta = \left(2\cos^2\theta\right) - 1 = 2x^2 - 1$

or $$2\theta = \cos^{-1}\left(2x^2 - 1\right)$$

or $$2\cos^{-1} x = \cos^{-1}\left(2x^2 - 1\right)$$

(c) We know that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ $\therefore 3\theta = \cos^{-1}\left(4\cos^3\theta - 3\cos\theta\right)$

or $$3\cos^{-1} x = \cos^{-1}\left(4x^3 - 3x\right)$$

(iii) To Prove that

(a) $$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

(b) $$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

(c) $$\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$$

(d) $$2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

(e) $$3\tan^{-1} x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

Proof : (a) Let $\tan^{-1} x = \theta_1$ i.e., $\tan\theta_1 = x$ and $\tan^{-1} y = \theta_2$ i.e., $\tan\theta_2 = y$

We know that

$$\tan\left(\theta_1 + \theta_2\right) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2} = \frac{x+y}{1-xy}$$

or $$\theta_1 + \theta_2 = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

or $$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

(b) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$ can be proved in a similar manner as (a)

(c) Now

$$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left(\frac{x+y}{1-xy}\right)+\tan^{-1}z$$

$$=\tan^{-1}\left[\frac{\{(x+y)/(1-xy)\}+z}{1-z\{(x+y)/(1-xy)\}}\right]$$ [(*a*) से]

$$=\tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$$

(d) Let $\tan^{-1}x=\theta$ i.e. $\tan\theta=x$

$\therefore$ $$\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}=\frac{2x}{1-x^2}$$

or $$2\theta=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

or $$2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

(e) we know that $$\tan 3\theta=\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$$

$\therefore$ $$3\theta=\tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$$

or $$3\tan^{-1}x=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

(iv) **To prove that**

(a) $$\cot^{-1}x+\cot^{-1}y=\cot^{-1}\left(\frac{xy-1}{x+y}\right)$$

(b) $$\cot^{-1}x-\cot^{-1}y=\cot^{-1}\left(\frac{xy+1}{y-x}\right).$$

Proof : (a) Let $\cot^{-1}x=\theta_1$ and $\cot^{-1}y=\theta_2$

then $\cot\theta_1=x,$ $\cot\theta_2=y$

we know that $$\cot(\theta_1+\theta_2)=\frac{\cot\theta_1\cot\theta_2-1}{\cot\theta_1+\cot\theta_2}$$

or $$\theta_1+\theta_2=\cot^{-1}\left(\frac{\cot\theta_1\cot\theta_2-1}{\cot\theta_1+\cot\theta_2}\right)$$

or $$\cot^{-1}x+\cot^{-1}y=\cot^{-1}\left(\frac{xy-1}{x+y}\right).$$

(b) $\cot^{-1}x-\cot^{-1}y=\cot^{-1}\left(\frac{xy+1}{y-x}\right)$ can be proved as (a)

(iv) To Prove that

(a) $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$ (b) $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$ (c) $\sec^{-1}x+\operatorname{cosec}^{-1}x=\frac{\pi}{2}$.

Proof : (a) Let $\sin^{-1}x=\theta$ then $\sin^{-1}x=\theta\Rightarrow x=\sin\theta=\cos\left(\frac{\pi}{2}-\theta\right)$

$$\Rightarrow\cos^{-1}x=\frac{\pi}{2}-\theta$$

$$\Rightarrow\cos^{-1}x=\frac{\pi}{2}-\sin^{-1}x$$

$$\Rightarrow\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}.$$

(b) Let $\tan^{-1}x=\theta$ then $\tan^{-1}x=\theta\Rightarrow x=\tan\theta=\cot\left(\frac{\pi}{2}-\theta\right)$

$$\Rightarrow\cot^{-1}x=\frac{\pi}{2}-\theta$$

$$\Rightarrow\cot^{-1}x=\frac{\pi}{2}-\tan^{-1}x$$

$$\Rightarrow\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}.$$

(c) Let $\sec^{-1}x=\theta$ then $\sec^{-1}x=\theta\Rightarrow x=\sec\theta=\operatorname{cosec}\left(\frac{\pi}{2}-\theta\right)$

$$\Rightarrow\operatorname{cosec}^{-1}x=\frac{\pi}{2}-\theta$$

$$\Rightarrow\operatorname{cosec}^{-1}x=\frac{\pi}{2}-\sec^{-1}x$$

$$\Rightarrow\sec^{-1}x+\operatorname{cosec}^{-1}x=\frac{\pi}{2}.$$

Illustrative Examples

Example 1. Find the principal value of

(a) $\sin^{-1}\left(-\frac{1}{2}\right)$ (b) $\tan^{-1}\left(-\sqrt{3}\right)$ (c) $\sec^{-1}\left(\sqrt{2}\right)$.

Solution : (a) Let $\sin^{-1}\left(-\frac{1}{2}\right)=\theta,\quad \sin\theta=-\frac{1}{2}$

since the principal vlaue of $\sin^{-1}x$ lies in the interval $-\frac{\pi}{2}\le\sin^{-1}x\le\frac{\pi}{2}$

$\therefore$ $$-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$$

But $\sin\theta$ is negative

$\therefore$ $$-\frac{\pi}{2}\le\theta\le 0$$

$\Rightarrow$ $$\sin\theta=-\frac{1}{2}=-\sin\frac{\pi}{6}=\sin\left(-\frac{\pi}{6}\right)\Rightarrow\theta=-\frac{\pi}{6}$$

$\therefore$ the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$ is $-\frac{\pi}{6}$

(b) Let $\tan^{-1}\left(-\sqrt{3}\right)=\theta,\quad\Rightarrow\quad \tan\theta=-\sqrt{3}$

since the principal value of $\tan^{-1}x$ lies in the interval $-\frac{\pi}{2}<\tan^{-1}x<\frac{\pi}{2}$

$\therefore$ $$-\frac{\pi}{2}<\theta<\frac{\pi}{2}$$

But $\tan\theta$ is negative

$\therefore$ $$-\frac{\pi}{2}<\theta<0$$

$\Rightarrow$ $$\tan\theta=-\sqrt{3}=-\tan\frac{\pi}{3}=\tan\left(-\frac{\pi}{3}\right)\quad\Rightarrow\quad\theta=-\frac{\pi}{3}$$

$\therefore$ the principal value of $\tan^{-1}(-\sqrt{3})$ is $-\pi/3$

(c) Let $\sec^{-1}\left(\sqrt{2}\right)=\theta,\quad\Rightarrow\quad\sec\theta=\sqrt{2}$

Here since $x\ge 1$ i.e. for $1\le x$ the principal value $\sec^{-1}x$ lies in the interval $0\le\sec^{-1}x<\frac{\pi}{2}$

$\therefore$ $$0<\theta\le\frac{\pi}{2}$$

$\Rightarrow \qquad \sec\theta = \sqrt{2} = \sec\pi/4 \quad \Rightarrow \quad \theta = \pi/4$

$\therefore$ Thus the principal value of $\sec^{-1}(\sqrt{2})$ is $\pi/4$

Example 2. Prove that $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} = \frac{\pi}{4}$

Solution : L.H.S. $= 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$

$$= 2\left(2\tan^{-1}\frac{1}{5}\right) - \left(\tan^{-1}\frac{1}{70} - \tan^{-1}\frac{1}{99}\right)$$

$$= 2\tan^{-1}\frac{2/5}{1-1/25} - \tan^{-1}\frac{1/70 - 1/99}{1+1/70\times 1/99}$$

$$= 2\tan^{-1}\frac{5}{12} - \tan^{-1}\frac{29}{6931}$$

$$= \tan^{-1}\frac{2\times 5/12}{1-25/144} - \tan^{-1}\frac{1}{239}$$

$$= \tan^{-1}\frac{120}{119} - \tan^{-1}\frac{1}{239} = \tan^{-1}\left[\frac{\frac{120}{119} - \frac{1}{239}}{1+\frac{120}{119}\times\frac{1}{239}}\right]$$

$$= \tan^{-1}\frac{28561}{28561} = \tan^{-1}(1) = \frac{\pi}{4} = \text{(RHS)}$$

Example 3. Prove that

$$2\tan^{-1}\left\{\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right\} = \cos^{-1}\left(\frac{b+a\cos x}{a+b\cos x}\right)$$

Solution : Let $\tan^{-1}\left\{\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right\} = \theta$

$\therefore \qquad \tan\theta = \sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}$

$\Rightarrow \qquad \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

$$= \frac{1 - \frac{a-b}{a+b}\tan^2\frac{x}{2}}{1+\frac{a-b}{a+b}\tan^2\frac{x}{2}} = \frac{b\left(1+\tan^2\frac{x}{2}\right) + a\left(1-\tan^2\frac{x}{2}\right)}{a\left(1+\tan^2\frac{x}{2}\right) + b\left(1-\tan^2\frac{x}{2}\right)}$$

$$= \frac{b + a\dfrac{1-\tan^2(x/2)}{1+\tan^2(x/2)}}{a + b\dfrac{1-\tan^2(x/2)}{1-\tan^2(x/2)}}$$ [dividing Nr and Dr by $1+\tan^2 x/2$]

$$= \frac{b + a\cos x}{a + b\cos x}$$

$$\Rightarrow \qquad 2\theta = \cos^{-1}\left(\frac{b + a\cos x}{a + b\cos x}\right)$$

$$\therefore \qquad 2\tan^{-1}\left\{\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right\} = \cos^{-1}\frac{b + a\cos x}{a + b\cos x}.$$

Example 4. Prove that $\tan\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right) + \tan\left(\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right) = \frac{2b}{a}.$

Solution : Let $\frac{1}{2}\cos^{-1}\frac{a}{b} = \theta,$ तब $\cos 2\theta = \frac{a}{b}$

L.H.S. $= \tan\left(\frac{\pi}{4} + \theta\right) + \tan\left(\frac{\pi}{4} - \theta\right)$

$$= \frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4}\tan\theta} + \frac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4}\tan\theta}$$

$$= \frac{1+\tan\theta}{1-\tan\theta} + \frac{1-\tan\theta}{1+\tan\theta}$$

$$= \frac{(1+\tan\theta)^2 + (1-\tan\theta)^2}{(1-\tan\theta)(1+\tan\theta)}$$

$$= 2\left(\frac{1+\tan^2\theta}{1-\tan^2\theta}\right) = \frac{2}{\left(\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right)} = \frac{2}{\cos 2\theta} = \frac{2b}{a} = \text{R.H.S.}$$

Example 5. If $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \alpha$ then Prove that $\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha.$

Solution : Given $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \alpha$

$$\cos^{-1}\left\{\frac{x}{a}.\frac{y}{b}-\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right\}=\alpha$$

$$\Rightarrow \quad \frac{xy}{ab}-\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}=\cos\alpha$$

$$\Rightarrow \quad \left(\frac{xy}{ab}-\cos\alpha\right)^2=\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)$$

$$\Rightarrow \quad \frac{x^2y^2}{a^2b^2}-\frac{2xy}{ab}\cos\alpha+\cos^2\alpha=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2y^2}{a^2b^2}$$

$$\Rightarrow \quad \frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=1-\cos^2\alpha$$

$$\Rightarrow \quad \frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha.$$

Example 6. Solve the following equation :

$$\cos^{-1}\frac{1-a^2}{1+a^2}+\cos^{-1}\frac{1-b^2}{1+b^2}=2\tan^{-1}x.$$

Solution : Let $a=\tan\theta, b=\tan\phi, \quad \Rightarrow \quad \theta=\tan^{-1}a, \phi=\tan^{-1}b$

$$\therefore \quad \frac{1-a^2}{1+a^2}=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\cos 2\theta$$

$$\Rightarrow \quad \frac{1-b^2}{1+b^2}=\frac{1-\tan^2\phi}{1+\tan^2\phi}=\cos 2\phi$$

from the given equation

$$\cos^{-1}(\cos 2\theta)+\cos^{-1}(\cos 2\phi)=2\tan^{-1}x$$

$$\Rightarrow \quad 2\theta+2\phi=2\tan^{-1}x$$

$$\Rightarrow \quad \theta+\phi=\tan^{-1}x$$

$$\Rightarrow \quad \tan^{-1}a+\tan^{-1}b=\tan^{-1}x$$

$$\Rightarrow \quad \tan^{-1}\frac{a+b}{1-ab}=\tan^{-1}x$$

$$\therefore \quad x=\frac{a+b}{1-ab}.$$

Example 7. Prove that

$$\cos\left[\tan^{-1}\left\{\sin\left(\cot^{-1}x\right)\right\}\right]=\sqrt{\frac{x^2+1}{x^2+2}}.$$

Solution : Let $\cot^{-1} x = \theta$, then $\cot\theta = x$

If $\quad \cot\theta = x, \quad \sin\theta = \dfrac{1}{\operatorname{cosec}\theta} = \dfrac{1}{\sqrt{\cot^2\theta + 1}} = \dfrac{1}{\sqrt{x^2+1}}$

$$\theta = \cot^{-1} x = \sin^{-1}\frac{1}{\sqrt{x^2+1}}$$

L.H.S. $\quad = \cos\left[\tan^{-1}\left\{\sin\left(\cot^{-1} x\right)\right\}\right]$

$$= \cos\left[\tan^{-1}\left\{\sin\left(\sin^{-1}\frac{1}{\sqrt{x^2+1}}\right)\right\}\right]$$

$$= \cos\left[\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right]$$

We know that $\tan\phi = \dfrac{1}{\sqrt{1+x^2}}$ then $\cos\phi = \dfrac{\sqrt{1+x^2}}{\sqrt{2+x^2}}$

$\therefore \quad$ L.H.S. $\quad = \cos\left(\cos^{-1}\dfrac{\sqrt{1+x^2}}{\sqrt{2+x^2}}\right)$

$$= \frac{\sqrt{1+x^2}}{\sqrt{2+x^2}} = \sqrt{\frac{x^2+1}{x^2+2}} = \text{R.H.S.}$$

Example 8. Solve the following equation :

$$\tan^{-1}\frac{1}{a-1} = \tan^{-1}\frac{1}{x} + \tan^{-1}\frac{1}{a^2-x+1}.$$

Solution : $\quad \tan^{-1}\dfrac{1}{a-1} - \tan^{-1}\dfrac{1}{x} = \tan^{-1}\dfrac{1}{a^2-x+1}$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{1}{a-1} - \frac{1}{x}}{1+\frac{1}{(a-1)x}}\right) = \tan^{-1}\frac{1}{a^2-x+1}$$

$$\Rightarrow \quad \frac{x-a+1}{ax-x+1} = \frac{1}{a^2-x+1}$$

$$\Rightarrow \quad (x-a+1)\left(a^2-x+1\right) = ax-x+1$$

$$\Rightarrow \quad xa^2 - a^3 - x^2 + a^2 + x - a = 0$$

$$\Rightarrow \quad a^2(x-a) - (x+a)(x-a) + (x-a) = 0$$

$$\Rightarrow \quad (x-a)\left[a^2 - (x+a) + 1\right] = 0$$

$$\Rightarrow \quad (x-a)(a^2 - x - a + 1) = 0$$

$$\Rightarrow \quad x = a \text{ and } x = a^2 - a + 1.$$

Example 9. Solve the following equation :

$$\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3}$$

Solution :

$$\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3}$$

$$\Rightarrow \quad \left(\frac{\pi}{2} - \cos^{-1} x\right) + \left(\frac{\pi}{2} - \cos^{-1} 2x\right) = \frac{\pi}{3}$$

$$\Rightarrow \quad \cos^{-1} x + \cos^{-1} 2x = \frac{2\pi}{3}$$

$$\Rightarrow \quad \cos^{-1}\left[x.2x - \sqrt{1-x^2}\sqrt{1-4x^2}\right] = \frac{2\pi}{3}$$

$$\Rightarrow \quad 2x^2 - \sqrt{1-x^2}\sqrt{1-4x^2} = \cos\frac{2\pi}{3}$$

$$\Rightarrow \quad 2x^2 - \sqrt{1-x^2}\sqrt{1-4x^2} = -\frac{1}{2}$$

$$\Rightarrow \quad 2x^2 + \frac{1}{2} = \sqrt{1-x^2}\sqrt{1-4x^2}$$

$$\Rightarrow \quad 4x^4 + \frac{1}{4} + 2x^2 = (1-x^2)(1-4x^2) \quad \text{[by squaring]}$$

$$\Rightarrow \quad 4x^4 + \frac{1}{4} + 2x^2 = 1 - 5x^2 + 4x^4$$

$$\Rightarrow \quad 7x^2 = \frac{3}{4} \quad \Rightarrow \quad x^2 = \frac{3}{28} \quad \Rightarrow \quad x = \pm\frac{1}{2}\sqrt{\frac{3}{7}}$$

But $x = -\frac{1}{2}\sqrt{\frac{3}{7}}$ does not satisfy the given equation

thus the solution is $x = \frac{1}{2}\sqrt{\frac{3}{7}}.$

Exercise 2.1

1. Find the principal value of the following angles:

(i) $\sin^{-1}(1)$ (ii) $\cos^{-1}\left(-\frac{1}{2}\right)$ (iii) $\sec^{-1}\left(-\sqrt{2}\right)$

(iv) $\operatorname{cosec}^{-1}(-1)$ (v) $\cot^{-1}\left(-\sqrt{\frac{1}{3}}\right)$ (vi) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

Prove that [from 2- 8]

2. $2\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{1}{7}=\frac{\pi}{4}$

3. $\tan^{-1}\frac{17}{19}-\tan^{-1}\frac{2}{3}=\tan^{-1}\frac{1}{7}$

4. $\cos^{-1}\frac{63}{65}+2\tan^{-1}\frac{1}{5}=\sin^{-1}\frac{3}{5}$

5. $\sec^2\left(\tan^{-1}2\right)+\operatorname{cosec}^2\left(\cot^{-1}3\right)=15$

6. $2\tan^{-1}x=\sin^{-1}\frac{2x}{1+x^2}=\cos^{-1}\frac{1-x^2}{1+x^2}$

7. $\tan^{-1}\sqrt{\frac{ax}{bc}}+\tan^{-1}\sqrt{\frac{bx}{ca}}+\tan^{-1}\sqrt{\frac{cx}{ab}}=\pi$, where $a+b+c=x$

8. $\frac{1}{2}\tan^{-1}x=\cos^{-1}\left\{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}\right\}^{\frac{1}{2}}$.

9. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi,$ then prove that $x^2+y^2+z^2+2xyz=1$.

10. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi,$ then prove that $x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2}=2xyz.$

(Hint : If $A+B+C=\pi$ then $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$)

11. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2},$ then prove that $xy+yz+zx=1$.

12. If $\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2}+\frac{1}{2}\cos^{-1}\frac{1-y^2}{1+y^2}+\frac{1}{3}\tan^{-1}\frac{3z-z^3}{1-3z^2}=5\pi,$ then prove that $x+y+z=xyz.$

13. If $\sec^{-1}\left(\sqrt{1+x^2}\right)+\operatorname{cosec}^{-1}\left(\frac{\sqrt{1+y^2}}{y}\right)+\cot^{-1}\left(\frac{1}{2}\right)=3\pi,$ then prove that $x+y+z=xyz.$

14. Prove that $\tan^{-1}x+\cot^{-1}(x+1)=\tan^{-1}(x^2+x+1)$.

15. If $\tan^{-1}x, \tan^{-1}y, \tan^{-1}z$ are in A.P., then prove that $y^2(x+z)+2y(1-xz)-x-z=0$

16. If the roots of $x^3+px^2+qx+p=0$ are α, β, γ, then prove that (except one situation) $\tan^{-1}\alpha+\tan^{-1}\beta+\tan^{-1}\gamma=n\pi$ and also find the situation when it does not happen.

Solve the following equation [Q. 17 to 25]

17. $\sec^{-1}\left(\frac{x}{a}\right)-\sec^{-1}\left(\frac{x}{b}\right)=\sec^{-1}b-\sec^{-1}a$

18. $\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)+\tan^{-1}\left(\frac{2x}{x^2-1}\right)=\frac{2\pi}{3}$

19. $\tan^{-1}\frac{1}{1+2x}+\tan^{-1}\frac{1}{4x+1}=\tan^{-1}\frac{2}{x^2}$

20. $\tan^{-1}\frac{x+7}{x-1}+\tan^{-1}\frac{x-1}{x}=\pi-\tan^{-1}7$

21. $\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)+\cot^{-1}x=\frac{\pi}{4}$

22. $3\tan^{-1}\frac{1}{2+\sqrt{3}}-\tan^{-1}\frac{1}{x}=\tan^{-1}\frac{1}{3}$

23. $\sin 2\left[\cos^{-1}\left\{\cot\left(2\tan^{-1}x\right)\right\}\right]=0$

24. $\tan^{-1}\left(\frac{1}{4}\right)+2\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{6}\right)+\tan^{-1}\left(\frac{1}{x}\right)=\frac{\pi}{4}$

25. $\sin^{-1}x-\sin^{-1}y=\frac{2\pi}{3}$; $\cos^{-1}x-\cos^{-1}y=\frac{\pi}{3}$.

Miscellaneous Exercise - 2

1. The principal value of $\tan^{-1}(-1)$ is

(a) 45° (b) 135° (c) -45° (d) -60°.

2. $2\tan^{-1}(1/2)$ equals

(a) $\cos^{-1}\left(\frac{3}{5}\right)$ (b) $\cos^{-1}\left(\frac{3}{4}\right)$ (c) $\cos^{-1}\left(\frac{5}{3}\right)$ (d) $\cos^{-1}\left(\frac{1}{2}\right)$.

3. If $\tan^{-1}(3/4)=\theta$ then the vlaue of $\sin\theta$ is

(a) $\frac{5}{3}$ (b) $\frac{3}{3}$ (c) $\frac{4}{3}$ (d) $\frac{1}{4}$.

4. The value of $\cot\left[\tan^{-1}\alpha+\cot^{-1}\alpha\right]$ is

(a) 1 (b) ∞ (c) 0 (d) None of these

5. If $\sin^{-1}\left(\frac{1}{2}\right)=x$, then the general value of x is

(a) $2n\pi\pm\frac{\pi}{6}$ (b) $\frac{\pi}{6}$ (c) $n\pi\pm\frac{\pi}{6}$ (d) $n\pi+(-1)^n\frac{\pi}{6}$.

6. The value of $2\tan\left(\tan^{-1}x+\tan^{-1}x^3\right)$ is

(a) $\frac{2x}{1-x^2}$ (b) $1+x^2$ (c) $2x$ (d) None of these

7. If $\tan^{-1}(3x)+\tan^{-1}(2x)=\frac{\pi}{4}$, then the value of x is

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{10}$ (d) $\frac{1}{2}$.

8. The value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)+2\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) π.

9. If $\tan^{-1}(1)+\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=\sin^{-1}x$, then the value of x is

(a) -1 (b) 0 (c) 1 (d) $-\frac{1}{2}$.

10. If $\cot^{-1}x+\tan^{-1}\left(\frac{1}{3}\right)=\frac{\pi}{2}$, then the value of x is

(a) 1 (b) 3 (c) $\frac{1}{3}$ (d) None of these

11. If $4\sin^{-1}x+\cos^{-1}x=\pi$, then find the value of x

12. Find the value of $\cos\left[(\pi/2)+\sin^{-1}(1/3)\right]$

13. If $\sin^{-1}(3/4)+\sec^{-1}(4/3)=x$, then find the value of x

14. Find the value of $\sin^{-1}(4/5)+2\tan^{-1}(1/3)$

15. If $\sin^{-1}(5/x)+\sin^{-1}(12/x)=90^\circ$, then find the value of x

16. Prove that : $\sin^{-1}\frac{3}{5}-\cos^{-1}\frac{12}{13}=\sin^{-1}\frac{16}{65}$.

17. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi,$ then prove that $x+y+z+=xyz.$

18. Prove that : $\tan^{-1}\left(\frac{1}{2}\tan 2A\right)+\tan^{-1}\left(\cot A\right)+\tan^{-1}\left(\cot^2 A\right)=0$.

19. Prove that : $\tan^{-1}x=2\tan^{-1}\left[\operatorname{cosec}\left(\tan^{-1}x\right)-\tan\left(\cot^{-1}x\right)\right].$

20. If $\phi=\tan^{-1}\frac{x\sqrt{3}}{2K-x}$ and $\theta=\tan^{-1}\frac{2x-K}{K\sqrt{3}}$, then prove that the value of $\phi-\theta$ is 30°

21. Prove that : $2\tan^{-1}\left[\tan\left(45^\circ-\alpha\right)\tan\frac{\beta}{2}\right]=\cos^{-1}\left(\frac{\sin 2\alpha+\cos\beta}{1+\sin 2\alpha\cos\beta}\right).$

Important Points

1. If $\sin\theta=x$ then $\theta=\sin^{-1}x$ and $\sin^{-1}x=\theta$ then $\sin\theta=x.$

2. $\sin\left(\sin^{-1}x\right)=x,\ \sin^{-1}\left(\sin x\right)=x;\ \cos\left(\cos^{-1}x\right)=x,\cos^{-1}\left(\cos x\right)=x$ etc.

3. (i) The principal value of $\sin^{-1}x,\tan^{-1}x,\cot^{-1}x,\operatorname{cosec}^{-1}x$ is $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

(ii) The principal value of $\cos^{-1}x$ and $\sec^{-1}x$ lies from 0 to π

4. (i) $\sin^{-1}\left(-x\right)=-\sin^{-1}x,\quad \tan^{-1}\left(-x\right)=-\tan^{-1}x,\quad \operatorname{cosec}^{-1}\left(-x\right)=-\operatorname{cosec}^{-1}x$

(ii) $\cos^{-1}\left(-x\right)=\pi-\cos^{-1}x,\quad \sec^{-1}(-x)=\pi-\sec^{-1}x,\quad \cot^{-1}\left(-x\right)=\pi-\cot^{-1}x$

5. (i) $\sin^{-1}x=\operatorname{cosec}^{-1}\frac{1}{x},\quad \cos^{-1}x=\sec^{-1}\frac{1}{x},\quad \tan^{-1}x=\cot^{-1}\frac{1}{x}$

(ii) $\operatorname{cosec}^{-1}x=\sin^{-1}\frac{1}{x},\quad \sec^{-1}x=\cos^{-1}\frac{1}{x},\quad \cot^{-1}x=\tan^{-1}\frac{1}{x}$

6. $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2},\quad \tan^{-1}x+\cot^{-1}x=\frac{\pi}{2},\quad \sec^{-1}x+\operatorname{cosec}^{-1}x=\frac{\pi}{2}$

7. (i) $\tan^{-1}x\pm\tan^{-1}y=\tan^{-1}\left(\frac{x\pm y}{1\mp xy}\right)$

(ii) $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$

8. $2\tan^{-1}x=\sin^{-1}\frac{2x}{1+x^2}=\cos^{-1}\frac{1-x^2}{1+x^2}=\tan^{-1}\frac{2x}{1-x^2}$

9. $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left(x.\sqrt{1-y^2} \pm y.\sqrt{1-x^2}\right)$

10. $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left(xy \mp \sqrt{1-x^2}\sqrt{1-x^2}\right)$

11. (i) $2\sin^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$ (ii) $2\cos^{-1}x = \cos^{-1}\left(2x^2-1\right)$

12. (i) $3\sin^{-1}x = \sin^{-1}\left(3x-4x^3\right)$ (ii) $3\cos^{-1}x = \cos^{-1}\left(4x^3-3x\right)$

(iii) $3\tan^{-1}x = \tan^{-1}\dfrac{3x-x^3}{1-3x^2}$

Answers

Exercise 2.1

1. (i) $\dfrac{\pi}{2}$ (ii) $\dfrac{2\pi}{3}$ (iii) $\dfrac{3\pi}{4}$ (iv) $-\dfrac{\pi}{2}$ (v) $\dfrac{2\pi}{3}$ (vi) $\dfrac{\pi}{6}$

17. $x = ab$ 18. $x = \tan\left(\dfrac{\pi}{12}\right)$ 19. $x = 0, 3, \dfrac{-2}{3}$ 20. $x = 11 \pm 4\sqrt{6}$

21. $x = 3$ 22. $x = 2$ 23. $x = \pm 1, \pm\left(1 \pm \sqrt{2}\right)$ 24. $x = \dfrac{-461}{9}$

25. $x = \dfrac{1}{2}, y = 1$

Miscellaneous Exericse - 2

1. (c)	2. (a)	3. (b)	4. (c)	5. (d)	6. (a)	7. (a)
8. (c)	9. (c)	10. (c)	11. 1/2	12. −1/3	13. π/2	14. π/2
15. 13						

03

Matrix

3.01 Introduction

In 1857 a mathematician Aurther kelly studied to find the solutions of system of equations and came to know about the concept of Matrix. In this method arrange the quantities or objects in a rectangular arrangement.

3.02 Definition and notation

A *matrix* is an ordered rectangular array of same numbers or quantities. It may be real or complex. The numbers of functions are called the elements or the entries of the matirx. We denote matrices by capital letters A, B, C. . .

The following are some examples of matrices.

$$\begin{pmatrix} 1 & 3 \\ 5 & -2 \\ 0 & 7 \end{pmatrix}, \quad \begin{bmatrix} 1 & 3 \\ 5 & -2 \\ 0 & 7 \end{bmatrix}, \quad \begin{Vmatrix} 1 & 3 \\ 5 & -2 \\ 0 & 7 \end{Vmatrix}$$

Note: Matrix is an arrangement, its value cannot be found out.

3.03 Order of matrix

A matrix having *m* rows and *n* columns is called a matrix of *order* $m \times n$ or simply $m \times n$ matrix (read as a *m* by *n* matrix).

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mj} & & a_{mn} \end{bmatrix}$$

This is a general form of matrix.

Here $a_{11}, a_{12}, ..., a_{mn}$ are the elements of the matrix. a_{ij} represents the *i* th row and *j* th column of the element. In short we represent it as $A = [a_{ij}]_{m \times n}$

Note: In a_{ij}, *i* always represents the row and *j* represents the column.

3.04 Type of matrix

1. Row matrix

A matrix is said to be a *row matrix* if it has only one row. For example

(i) $[2 \;\; 5 \;\; 3]_{1\times 3}$ (ii) $[3 \;\; -4 \;\; 0 \;\; 7 \;\; 1]_{1\times 5}$

2. Column matrix

A matrix is said to be a *column matrix* if it has only one column. For example its order have $m \times 1$ where m number of rows and column is 1.

(i) $\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}_{3\times1}$ (ii) $\begin{bmatrix} -4 \\ 1 \\ 5 \\ 7 \\ 6 \end{bmatrix}_{5\times1}$

3. Zero or Null matrix

A matrix is said to be *zero matrix* or *null matrix* if all its elements are zero. We denote zero matrix by O. **For example**

(i) $\text{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times2}$ (ii) $\text{O} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3\times4}$

4. Square matrix

A matrix in which the number of rows are equal to the number of columns, is said to be a *square matrix*. Thus a $m \times n$ matrix is said to be a square matrix if $m = n$ and is known as a square matrix of order 'n'.

In general, $A = \left[a_{ij}\right]_{m\times n}$ is a square matrix of order m.

(i) $\begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}_{2\times2}$ (ii) $\begin{bmatrix} 0 & 1 & 5 \\ 2 & 3 & 7 \\ 6 & -4 & 8 \end{bmatrix}_{3\times3}$ (iii) $\begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix}_{n\times n}$

Elements $a_{11}, a_{22}, \dots, a_{nn}$ are called the diagonal elements and also termed as Principal diagonal as the subscripts of all the elements are equal.

5. Diagonal matrix

A square matrix $A = \left[a_{ij}\right]_{m\times n}$ is said to be a *diagonal matrix* if all its elements are zero except element of principal diagonal, that is a matrix $A = \left[a_{ij}\right]_{m\times n}$ is said to be a diagonal matrix if $a_{ij} = 0$ when $i \neq j$.

For Example

(i) $[5]_{1\times1}$ (ii) $\begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix}_{2\times2}$ (iii) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{bmatrix}_{3\times3}$

6. Scalar matrix

A diagonal matrix is said to be a *scalar matrix* if its principal diagonal elements are equal, that is, a square matrix

$$A = \left[a_{ij}\right]_{m\times m} \quad \text{if } a_{ij} = \begin{cases} 0 & when \quad i \neq j \\ 1 & when \quad i = j \end{cases}$$

For Example (i) $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}_{2\times 2}$ (ii) $\begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}_{3\times 3}$

7. Unit or Identity matrix

A square matrix in which elements in the principal diagonal are all 1 and rest are all zero is called an *identity matrix.* In other words, the square matirx $A = \left[a_{ij}\right]_{m\times m}$ is an identity matrix, if

$$I_n = \left[a_{ij}\right]_{n\times n}, \quad a_{ij} = \begin{cases} 0 & when \quad i \neq j \\ 1 & when \quad i = j \end{cases}$$

For Example (i) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$ (ii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$

8. Triangular matrix

(*i*) Upper triangular matrix

A square matrix in which all the elements below the principal diagonal elements are zero then it is called as an *Upper Triangular Matrix.*

Therefore, in $A = \left[a_{ij}\right]_{n\times n}$, $a_{ij} = 0$ when $i > j$

For Example (i) $\begin{bmatrix} 9 & 5 \\ 0 & 2 \end{bmatrix}_{2\times 2}$ (ii) $\begin{bmatrix} 1 & -2 & 6 \\ 0 & 4 & 7 \\ 0 & 0 & 3 \end{bmatrix}_{3\times 3}$

(*ii*) Lower triangular matrix

A square matrix in which all the elements above the principal diagonal elements are zero then it is called as an *Lower Triangular Matrix.* Therefore, in $A = \left[a_{ij}\right]_{n\times n}$, $a_{ij} = 0$ when $i < j$

For Example (i) $\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}_{2\times 2}$ (ii) $\begin{bmatrix} 5 & 0 & 0 \\ 6 & 7 & 0 \\ 9 & 2 & -4 \end{bmatrix}_{3\times 3}$

3.05 Properties of matrx

1. Transpose of a matrix

If $A=[a_{ij}]_{m\times n}$ be a $m\times n$ matrix, then the matrix obtained by interchanging the rows and columns of A is called the *transpose* of A. Transpose fo the matrix A is denoted by A^T or A'. In other words, if $A=[a_{ij}]_{m\times n}$ then $A^T=A'=[a_{ji}]_{n\times m}$. For example,

(i) $A=\begin{bmatrix}1&0&2\\5&3&-4\\3&8&6\end{bmatrix}_{3\times3} \Rightarrow A^T=\begin{bmatrix}1&5&3\\0&3&8\\2&-4&6\end{bmatrix}_{3\times3}$ (ii) $A=\begin{bmatrix}1&3\\5&0\\2&-4\end{bmatrix}_{3\times2} \Rightarrow A^T=\begin{bmatrix}1&5&2\\3&0&-4\end{bmatrix}_{2\times3}$

2. Symmetric and skew symmetric matrix

(*i*) Symmetric Matrix

A square matrix $A=[a_{ij}]_{m\times n}$ is said to be *symmetric* if $A=A^T$, for example:

(i) $A=\begin{bmatrix}2&-3\\-3&4\end{bmatrix}_{2\times2}$; $A^T=\begin{bmatrix}2&-3\\-3&4\end{bmatrix}_{2\times2}$

$\therefore$ A is Symmetric matrix

(ii) $A=\begin{bmatrix}a&h&g\\h&b&f\\g&f&c\end{bmatrix}_{3\times3}$; $A^T=\begin{bmatrix}a&h&g\\h&b&f\\g&f&c\end{bmatrix}_{3\times3}$

Note: In Symmetric Matrix, All elements are equal at equidisant with respect to principal diagonal means, $a_{ij}=a_{ji}$.

(*ii*) Skew-Symmetric Matrix

A square matrix $A=[a_{ij}]_{m\times n}$ is said to be skew *symmetric* if $A^T=-A$, for example:

(i) $A=\begin{bmatrix}0&-2\\2&0\end{bmatrix}$; $A^T=\begin{bmatrix}0&2\\-2&0\end{bmatrix}=-A$

(ii) $A=\begin{bmatrix}0&h&g\\-h&0&-f\\-g&f&0\end{bmatrix}$; $A^T=\begin{bmatrix}0&-h&-g\\h&0&f\\g&-f&0\end{bmatrix}=-A$

Note: (a) In Skew-Symmetric Matrix, $a_{ij}=-a_{ji}$ for all possible values of *i* and *j*.

(b) All the diagonal elements of a skew symmetric matrix are zero. If

$$a_{ij}=-a_{ji} \text{ and if } i=1, j=1 \text{ then}$$

$$a_{11}=-a_{11}$$

$$\Rightarrow \quad 2a_{11}=0$$

$$\therefore \quad a_{11}=0=a_{22}=...a_{nn}$$

(c) For any matrices A and B of suitable orders, for addition and multiplication, then

(i) $(A \pm B)^T = A^T \pm B^T$ (ii) $(KA)^T = KA^T$, (where k is any constant) (iii) $(AB)^T = B^T A^T$

(d) If A is a square matrix then

(i) $A + A^T$ is a Symmetric matrix (ii) $A - A^T$ is a Skew-Symmetric matrix

(iii) AA^T and $A^T A$ is a symmetric matrix (iv) $(A^T)^T = A$

(e) Any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix.

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T),$$

where A is a square matrix

$A + A^T$ is a symmetric Matrix

and $A - A^T$ is a skew-symmetric matrix

(f) A matrix is said to be equal if their corresponding elements are equal,

For example: $A = \begin{bmatrix} 2 & -2 & 0 \\ 3 & -4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

$\Rightarrow$ $b_{11} = 2,\ b_{12} = -2,\ b_{13} = 0$

$b_{21} = 3,\ b_{22} = -4,\ b_{23} = 2$

Illustrative Examples

Example 1. The order of A is 3×5 and R is a row matrix of A then write the order of R.

Solution : $\because$ order of matrix A is 3×5

$\therefore$ has 5 elements in each row A

$\therefore$ order of matrix R is 1×5

Example 2. Find a matrix of order 2×3, $A = [a_{ij}]$ whose elements are (i) $a_{ij} = 2i + j$; (ii) $a_{ij} = i^2 - j^2$

Solution : (i) $a_{ij} = 2i + j$ Here $i = 1,\ 2$ and $j = 1,\ 2,\ 3$ as the matrix is of order 2×3

$\therefore$ $a_{11} = 2 + 1 = 3,\ a_{12} = 2 + 2 = 4,\ a_{13} = 2 + 3 = 5$

$a_{21} = 4 + 1 = 5,\ a_{22} = 4 + 2 = 6,\ a_{23} = 4 + 3 = 7$

$\therefore$ Required matrix is $A = \begin{bmatrix} 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix}$

(ii) $a_{ij} = i^2 - j^2$ given matrix is of the order 2×3 thus $i = 1,\ 2$ and $j = 1,\ 2,\ 3$.

$\therefore$ $a_{11} = 1^2 - 1^2 = 0,\ a_{12} = 1^2 - 2^2 = -3,\ a_{13} = 1^2 - 3^2 = -8$

$a_{21} = 2^2 - 1^2 = 3,\ a_{22} = 2^2 - 2^2 = 0,\ a_{23} = 2^2 - 3^2 = -5$

$\therefore$ Required matrix $A = \begin{bmatrix} 0 & -3 & -8 \\ 3 & 0 & -5 \end{bmatrix}$

Example 3. For what vlaues of x, y and z matrices A and B are equal

$$A=\begin{bmatrix} 2 & 0 & x+3 \\ y-4 & 4 & 6 \end{bmatrix}; \quad B=\begin{bmatrix} 2 & 0 & 6 \\ -2 & 4 & 2z \end{bmatrix}$$

Solution : $\because$ A and B are equal matrices, hence their corresponding elemetns are also equal

$\therefore \quad x+3=6,\ y-4=-2,$ and $2z=6$

$\Rightarrow \quad x=3,\ y=2$ and $z=3$

Example 4. If $\begin{bmatrix} 2x+y & 3 & x-2y \\ a-b & 2a+b & -5 \end{bmatrix}=\begin{bmatrix} 3 & 3 & 4 \\ 4 & -1 & -5 \end{bmatrix}$ then find the vlaues of x, y, a and b

Solution : $\because$ Both are equal matrices, hence their corresponding elements are also equal

$\therefore \quad 2x+y=3 \quad (1)$

$x-2y=4 \quad (2)$

Solving (1) and (2) we have

$$x=2,\ y=-1$$

again $\quad a-b=4 \quad (3)$

$2a+b=-1 \quad (4)$

Solving (3) and (4) we have

$$a=1,\ b=-3$$

$\therefore \quad x=2,\ y=-1,\ a=1,\ b=-3$

Exercise 3.1

1. If the matrix $A=[a_{ij}]_{2\times 4}$, then find the number of elements of A
2. Find out the unit matrix of order 4×4
3. If $\begin{bmatrix} k+4 & -1 \\ 3 & k-6 \end{bmatrix}=\begin{bmatrix} a & -1 \\ 3 & -4 \end{bmatrix}$ then find the vlaue of a
4. Find the possible orders of matrix with 6 elements.
5. Find a matrix $A=[a_{ij}]$ of order 2×2 whose elements

   (i) $a_{ij}=\dfrac{2i-j}{3i+j}$ (ii) $a_{ij}=\dfrac{(i+2j)^2}{2i}$ (iii) $a_{ij}=2i-3j$

6. Find a matrix $A=a_{ij}$ of order 2×3 whose elements are $a_{ij}=\frac{1}{2}|2i-3j|$.
7. If $\begin{bmatrix} a+b & 2 \\ 7 & ab \\ -3 & 4 \end{bmatrix}=\begin{bmatrix} 6 & 2 \\ 7 & 8 \\ -3 & 4 \end{bmatrix}$ then find the vlaues of a and b,

8. If $\begin{bmatrix} 2x & 3x+y \\ -x+z & 3y-2p \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -4 & -3 \end{bmatrix}$ then find the vlaues of x, y, z and p.

9. For what values of a, b and c , matrices A and B are equal matrices where.

$$A = \begin{bmatrix} a-2 & 3 & 2c \\ 12c & b+2 & bc \end{bmatrix}; \quad B = \begin{bmatrix} b & c & 6 \\ 6b & a & 3b \end{bmatrix}$$

3.06 Operations on matrix

1. Addition

In general, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are two matrices of the same order, say m x n. then, the sum of the two matrices A and B is defined as a matrix $A + B = [a_{ij} + b_{ij}]_{m\times n}$, for all possible values for i and j.

For example: (i) If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times2}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_{2\times2}$ then

$$A + B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}_{2\times2}$$

(ii) If $A = \begin{bmatrix} 2 & 5 & -3 \\ 4 & 0 & 6 \end{bmatrix}_{2\times3}$ and $B = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 3 & 5 \end{bmatrix}_{2\times3}$ then

$$A + B = \begin{bmatrix} 2+4 & 5+2 & -3-1 \\ 4+1 & 0+3 & 6+5 \end{bmatrix} = \begin{bmatrix} 6 & 7 & -4 \\ 5 & 3 & 11 \end{bmatrix}_{2\times3}$$

2. Subtraction

In general, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are two matrices of the same order, say $m \times n$. Then, the subtraction of the two matrices A and B is defined as a matrix $A - B = [a_{ij} - b_{ij}]_{m\times n}$, for all possible values of i and j.

For example: (i) if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times2}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_{2\times2}$ then

$$A - B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} \\ a_{21}-b_{21} & a_{22}-b_{22} \end{bmatrix}_{2\times2}$$

(ii) If $A = \begin{bmatrix} 5 & 3 & 7 \\ 6 & 2 & 1 \end{bmatrix}_{2\times3}$ and $B = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 4 & 1 \end{bmatrix}_{2\times3}$ then

$$A - B = \begin{bmatrix} 5-2 & 3-4 & 7-6 \\ 6-3 & 2-4 & 1-1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 3 & -2 & 0 \end{bmatrix}_{2\times3}$$

3. Multiplication

For multiplication of two matrices A and B, the number of columnes in A should be equal to the number of rows in B. Further more for getting the elements of the product matrix, we take rows of A and columns of B, multiply them elements-wise and take the sum. The product of two matrices A and B is defined if the number of columns of A is equal to the number of rows of B. Let $A=[a_{ij}]_{m\times p}$ be a $m \times p$ matrix and $B=[b_{ij}]_{p\times n}$ be a $p \times n$ matrix. Then the product of the matrices A and B is the matrix C of order $m \times n$.

Order of matrix AB = No . of rows in A × No. of columns in B

$\therefore$ $A=[a_{ij}]_{m\times p}$ and $B=[b_{ij}]_{p\times n}$ then

order of AB will be $m \times \boxed{p \quad\quad p} \times n = m \times n$

For example : (i) If $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$ and $B=\begin{bmatrix} b_{11} & b_{12} & b_{12} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}_{2\times 3}$ then

order of AB will be $AB \;\; 2\times \boxed{2 \quad\quad 2} \times 3 = 2\times 3$

$$\therefore \quad AB=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$=\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} & a_{11}b_{13}+a_{12}b_{23} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} & a_{21}b_{13}+a_{22}b_{23} \end{bmatrix}_{2\times 3}$$

(ii) If $A=\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}_{2\times 2}$ then $B=\begin{bmatrix} 5 & 4 \\ 6 & 0 \end{bmatrix}_{2\times 2}$

order of AB will be $AB \;\; 2\times \boxed{2 \quad\quad 2} \times 2 = 2\times 2$

$$\therefore \quad AB=\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\times\begin{bmatrix} 5 & 4 \\ 6 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 2\times 5+3\times 6 & 2\times 4+3\times 0 \\ -1\times 5+4\times 6 & -1\times 4+4\times 0 \end{bmatrix}$$

$$=\begin{bmatrix} 10+18 & 8+0 \\ -5+24 & -4+0 \end{bmatrix}=\begin{bmatrix} 28 & 8 \\ 19 & -4 \end{bmatrix}_{2\times 2}$$

4. Scalar Multiplication

In general, we may define multiplication of a matrix by a scalar as follows: if $A=[a_{ij}]_{m\times n}$ is a matrix and n is a scalar, then nA is another matrix which is obtained

$\therefore$ $A=[a_{ij}]_{m\times n}$ then $nA=[na_{ij}]_{m\times n}$

For example (i) if $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3}$ then

$$nA = n\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} na_{11} & na_{12} & na_{13} \\ na_{21} & na_{22} & na_{23} \end{bmatrix}$$

(ii) If $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}_{2\times 2}$ तो $3A = \begin{bmatrix} 6 & 9 \\ 3 & -15 \end{bmatrix}_{2\times 2}$

and $-5A = \begin{bmatrix} -10 & -15 \\ -5 & 25 \end{bmatrix}_{2\times 2}$

3.07 Properties of matrix addition

(i) Commutativity

If A and B are matrices of same order then $A + B = B + A$

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ then clearly $A + B$ and $B + A$ are matrices of same order

$$[A+B]_{m\times n} = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}$$

$$= [a_{ij} + b_{ij}]_{m\times n}$$

$$= [b_{ij} + a_{ij}]_{m\times n} \qquad \text{(Commutative law of addition)}$$

$$= [b_{ij}]_{m\times n} + [a_{ij}]_{m\times n}$$

$$= [B + A]_{m\times n}$$

$\therefore$ $A + B = B + A$

(ii) Associativity

If A, B and C are matrices of same order then $(A+B)+C = A+(B+C)$

Let $A = [a_{ij}]_{m\times n}$; $B = [b_{ij}]_{m\times n}$ and $C = [c_{ij}]_{m\times n}$ then clearly $(A+B)+C$ and $A+(B+C)$ are matrices of same order

$$[(A+B)+C]_{m\times n} = ([a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}) + [c_{ij}]_{m\times n}$$

$$= [a_{ij} + b_{ij}]_{m\times n} + [c_{ij}]_{m\times n}$$

$$= [(a_{ij} + b_{ij}) + c_{ij}]_{m\times n}$$

$$= [a_{ij} + (b_{ij} + c_{ij})]_{m\times n} \qquad \text{(Associative law of addition)}$$

$$= [a_{ij}]_{m\times n} + ([b_{ij}]_{m\times n} + [c_{ij}]_{m\times n})$$

$$= [A + (B + C)]_{m\times n}$$

$\therefore$ $(A+B)+C = A+(B+C)$

(iii) Additive identity

A zero matrix O, $m \times n$ is known as the identity matrix of A as

$$A + O = A = O + A$$

(iv) Additive inverse

For matrix $A=[a_{ij}]_{m\times n}$, if $-A=[-a_{ij}]_{m\times n}$ then $-A$ is the additive inverse of matrix A

as $A+(-A)=O=(-A)+A$, where O is the zero matrix of order $m\times n$

Let $A=[a_{ij}]_{m\times n}$ then $-A=-[a_{ij}]_{m\times n}=[-a_{ij}]_{m\times n}$

$\therefore$ $A+(-A)=[a_{ij}]_{m\times n}+[-a_{ij}]_{m\times n}=0$

and $(-A)+A=A+(-A)$ (Commutative law of addition)

$A+(-A)=O=(-A)+A$

(v) Cancellation law

If A, B and C are three matrices of same order then

$A+B=A+C\Rightarrow B=C$ (Left cancellation law)

and $B+A=C+A\Rightarrow B=C$ (Right cancellation law)

3.08 Properties of Matrix Multiplication

(i) Commutativity

Generally matrix multiplication does not obey Commutative law due to conditions given below:

(a) If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$ then AB and BA can be found out but they are not necessarily equal.

for example let $A=\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ then

$$AB=\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}$$

and $$BA=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$$

$\therefore$ $AB\neq BA$

(b) If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{n\times p}$ then matrix AB can be found but BA cannot be found so no question of proving commutative law.

(c) If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{n\times m}$ then AB and BA can be found out but their order will not be same so $AB\neq BA$

Note: Under certain conditions $AB=BA$ is possible.

(ii) Associativity

If matrix A, B and C are favourable for AB and BC then associative law is verified

i.e. $(AB)C=A(BC)$

(iii) Identity

If I is an unit matrix and A is a matrix of order m × n then

$$I_m A=A=AI_n$$

where I_m, m is the unit matrix of order m and I_n, n is the unit matrix of order n

Note: For square matrix, A, $AI = A = IA$ where I has same order as A.

(iv) Distributivity

If matrices A, B and C are favourable for addition and multiplication then they obey distrbutive law.

(a) $A(B+C) = AB + AC$

(b) $(A+B)C = AC + BC$

3.09 Properties of scalar multiplication of a matirx

If A and B are two matrices of same order and let k and ℓ are two constants then

(i) $(k+\ell)A = kA + \ell A$ (ii) $k(A+B) = kA + kB$

(iii) $k(\ell A) = \ell(kA) = (\ell k)A$ (iv) $1.A = A$

(v) $(-1)A = -A$

3.10 Multiplicative Inverse Matrix

If the product of two square matrices of same order A and B is a Unit matrix then B is known as the multiplicative inverse matrix of A and A is known as the multiplicative inverse matrix of B i.e.

If $AB = I = BA$ then A and B are multiplicative inverse matrix of invertible matrices, for example:

If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 4 \\ 3 & 7 & 5 \end{bmatrix}_{3\times 3}$ and $B = \begin{bmatrix} 3 & -4 & 2 \\ -2 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}_{3\times 3}$ then

$$AB = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 4 \\ 3 & 7 & 5 \end{bmatrix}\begin{bmatrix} 3 & -4 & 2 \\ -2 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-4+2 & -4+2+2 & 2+0-2 \\ 6-10+4 & -8+5+4 & 4+0-4 \\ 9-14+5 & -12+7+5 & 6+0-5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} = I_3$$

and $$BA = \begin{bmatrix} 3 & -4 & 2 \\ -2 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 4 \\ 3 & 7 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3-8+6 & 6-20+14 & 6-16+10 \\ -2+2+0 & -4+5+0 & -4+4+0 \\ 1+2-3 & 2+5-7 & 2+4-5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} = I_3$$

$\therefore$ $AB = I_3 = BA$ thus A amd B are multiplicative inverse matrix of each other.

3.11 Zero Divisors

If the product of two non-zero matrices A and B is a zero matrix then A and B are divisors of zero

$\therefore$ $A = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ are divisors of zero

$$AB = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1+1 & 1-1 \\ -3+3 & 3-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$\therefore$ A and B are divisors of zero.

3.12 Positive Integral Power of a Square Matrix

If a square matrix A is multiplied by itself then we get A^2 , again if A^2 is multiplied with A then we get A^3 similarly when A^{n-1} is multiplied with A then we get A^n i.e.

$$AA = A^2 \qquad A^2A = A^3$$

and $A^{n-1}A = A^n$

If $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ then

$$A^2 = AA = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1+6 & 3+12 \\ 2+8 & 6+16 \end{bmatrix} = \begin{bmatrix} 7 & 15 \\ 10 & 22 \end{bmatrix}$$

and $A^3 = A^2A = \begin{bmatrix} 7 & 15 \\ 10 & 22 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 7+30 & 21+60 \\ 10+44 & 30+88 \end{bmatrix} = \begin{bmatrix} 37 & 81 \\ 54 & 118 \end{bmatrix}$

Illustrative Examples

Example 5. If $A=\begin{bmatrix} 2 & 4 & -1 \\ 3 & 2 & 5 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \end{bmatrix}$ then find $2A-3B$

Solution: $\because \quad A=\begin{bmatrix} 2 & 4 & -1 \\ 3 & 2 & 5 \end{bmatrix}$

$$\therefore \quad 2A = 2\begin{bmatrix} 2 & 4 & -1 \\ 3 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 8 & -2 \\ 6 & 4 & 10 \end{bmatrix} \qquad (1)$$

$$\text{and} \quad 3B = 3\begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 0 \\ -3 & 9 & 12 \end{bmatrix}$$

$$-3B = \begin{bmatrix} -6 & -3 & 0 \\ 3 & -9 & -12 \end{bmatrix}$$

$$\therefore \quad 2A-3B = 2A+(-3B)$$

$$= \begin{bmatrix} 4 & 8 & -2 \\ 6 & 4 & 10 \end{bmatrix} + \begin{bmatrix} -6 & -3 & 0 \\ 3 & -9 & -12 \end{bmatrix}$$

$$= \begin{bmatrix} 4-6 & 8-3 & -2+0 \\ 6+3 & 4-9 & 10-12 \end{bmatrix} = \begin{bmatrix} -2 & 5 & -2 \\ 9 & -5 & -2 \end{bmatrix}$$

Example 6. If $B=\begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix}$ and $C=\begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix}$ then find A where $2A-3B+5C=O$, and O is a zero matrix of order 2×3.

Solution : $\because \quad 2A-3B+5C=O$

$$\therefore \quad 2A = 3B-5C+O$$

$$= 3\begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix} + (-5)\begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 6 & 0 \\ 9 & 3 & 12 \end{bmatrix} + \begin{bmatrix} -10 & 0 & 10 \\ -35 & -5 & -30 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -6-10+0 & 6+0+0 & 0+10+0 \\ 9-35+0 & 3-5+0 & 12-30+0 \end{bmatrix}$$

$$= \begin{bmatrix} -16 & 6 & 10 \\ -26 & -2 & -18 \end{bmatrix}$$

$$\therefore \qquad A = \frac{1}{2}\begin{bmatrix} -16 & 6 & 10 \\ -26 & -2 & -18 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 3 & 5 \\ -13 & -1 & -9 \end{bmatrix}$$

Example 7. If $A = \begin{bmatrix} 4 & 2 & -5 \\ 1 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & -7 & 0 \\ -1 & 2 & 5 \\ 1 & 0 & 3 \end{bmatrix}$ then find AB, and BA if exists.

Solution : $\because$ order of A is 2×3 and order of B is 3×3

$\therefore$ AB exists but BA does not

$$\therefore \qquad AB = \begin{bmatrix} 4 & 2 & -5 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 6 & -7 & 0 \\ -1 & 2 & 5 \\ 1 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 24-2-5 & -28+4+0 & 0+10-15 \\ 6-0+3 & -7+0+0 & 0+0+9 \end{bmatrix} = \begin{bmatrix} 17 & -24 & -5 \\ 9 & -7 & 9 \end{bmatrix}_{2\times 3}$$

Example 8. Find the value of x for which

$$\begin{bmatrix} 1 & x & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = O$$

where O is a zero matrix of order 1×1

Solution :

$$\begin{bmatrix} 1 & x & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = O$$

or $$\begin{bmatrix} 1+2x+15 & 3+5x+3 & 2+x+2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = O$$

or $$\begin{bmatrix} 2x+16 & 5x+6 & x+4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = O$$

or $$\begin{bmatrix} 2x+16+10x+12+x^2+4x \end{bmatrix} = O$$

or $\left[x^2+16x+28\right]=0$

or $x^2+16x+28=0$

or $(x+2)(x+14)=0$

$\Rightarrow$ $x+2=0$ or $x+14=0$

$\Rightarrow$ $x=-2$ or $x=-14$

Example 9. If $A-2I=\begin{bmatrix}-1 & -2 & 3\\ 2 & 1 & -1\\ -3 & 1 & 0\end{bmatrix}$ then find AA^T where I is the identity matrix of order 3×3.

Solution : $\because$ $A-2I=\begin{bmatrix}-1 & -2 & 3\\ 2 & 1 & -1\\ -3 & 1 & 0\end{bmatrix}$

$$\therefore \quad A=\begin{bmatrix}-1 & -2 & 3\\ 2 & 1 & -1\\ -3 & 1 & 0\end{bmatrix}+2\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}-1 & -2 & 3\\ 2 & 1 & -1\\ -3 & 1 & 0\end{bmatrix}+\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}=\begin{bmatrix}1 & -2 & 3\\ 2 & 3 & -1\\ -3 & 1 & 2\end{bmatrix}$$

$$A^T=\begin{bmatrix}1 & 2 & -3\\ -2 & 3 & 1\\ 3 & -1 & 2\end{bmatrix}$$

$$\therefore \quad AA^T=\begin{bmatrix}1 & -2 & 3\\ 2 & 3 & -1\\ -3 & 1 & 2\end{bmatrix}\begin{bmatrix}1 & 2 & -3\\ -2 & 3 & 1\\ 3 & -1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}1+4+9 & 2-6-3 & -3-2+6\\ 2-6-3 & 4+9+1 & -6+3-2\\ -3-2+6 & -6+3-2 & 9+1+4\end{bmatrix}=\begin{bmatrix}14 & -7 & 1\\ -7 & 14 & -5\\ 1 & -5 & 14\end{bmatrix}$$

Example 10. If $A=\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$ then verify the following:

(i) $A^2=2A$ (ii) $A^3=4A$

Solution : (i) L.H.S. $A^2 = AA = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1+1 & -1-1 \\ -1-1 & 1+1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 2A = \text{R.H.S.}$$

(ii) L.H.S. $A^3 = A^2 A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2+2 & -2-2 \\ -2-2 & 2+2 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} = 4\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 4A = \text{R.H.S.}$$

Example 11. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}; B = \begin{bmatrix} 1 & -2 & 2 \\ -3 & 2 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 4 & 1 \end{bmatrix}$ then verify the following.

$$A(B+C) = AB + AC$$

Solution : L.H.S. $= A(B+C)$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \times \left\{ \begin{bmatrix} 1 & -2 & 2 \\ -3 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 6 \\ -1 & 4 & 1 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} 3 & 1 & 8 \\ -4 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 6-12 & 2+18 & 16+15 \\ 3-8 & 1+12 & 8+10 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 20 & 31 \\ -5 & 13 & 18 \end{bmatrix} \qquad (1)$$

R.H.S. $= AB + AC$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \\ -3 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 & 6 \\ -1 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-9 & -4+6 & 4+12 \\ 1-6 & -2+4 & 2+8 \end{bmatrix} + \begin{bmatrix} 4-3 & 6+12 & 12+3 \\ 2-2 & 3+8 & 6+2 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 2 & 16 \\ -5 & 2 & 10 \end{bmatrix} + \begin{bmatrix} 1 & 18 & 15 \\ 0 & 11 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 20 & 31 \\ -5 & 13 & 18 \end{bmatrix} \qquad (2)$$

from (1) and (2) L.H.S. = R.H.S.

Exericse 3.2

1. If $A=\begin{bmatrix}-3 & 2 & 1\\ 1 & -4 & 7\end{bmatrix}$ and $B=\begin{bmatrix}3 & 5 & -2\\ -1 & 4 & -2\end{bmatrix}$ then find $A+B$ and $A-B$.

2. If $A+B=\begin{bmatrix}-7 & 0\\ 2 & -5\end{bmatrix}$ and $A-B=\begin{bmatrix}3 & -2\\ 0 & 3\end{bmatrix}$ then find matrices A and B.

3. If $A=\begin{bmatrix}1 & 3\\ 2 & 1\\ 3 & -1\end{bmatrix}$ and $B=\begin{bmatrix}2 & 1\\ 1 & 2\\ -1 & 0\end{bmatrix}$ then find matrix C where $A+2B+C=O$ and O is a zero matrix.

4. If $A=\begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix}$ and $B=\begin{bmatrix}0 & 4\\ -1 & 7\end{bmatrix}$ then find the value of $3A^2-2B$.

5. If $A=\begin{bmatrix}0 & 1 & 2 & 3\\ 3 & 2 & 1 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & 3\\ 1 & 2\\ 2 & 1\\ 3 & 0\end{bmatrix}$ then show that $AB\neq BA$.

6. If $f(x)=\begin{bmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}$ then show that $f(A)f(B)=f(A+B)$.

7. If $A=\begin{bmatrix}4 & 2 & -5\\ 1 & 0 & 3\end{bmatrix}$ and $B=\begin{bmatrix}6 & -7 & 0\\ -1 & 2 & 5\\ 1 & 0 & 2\end{bmatrix}$ then prove that: $(AB)^T=B^TA^T$

8. Prove that $\begin{bmatrix}x & y & z\end{bmatrix}\begin{bmatrix}a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}ax^2+by^2+cz^2+2hxy+2fyz+2gzx\end{bmatrix}$.

9. If $A=\begin{bmatrix}1 & -2 & 3\\ 2 & 3 & -1\\ -3 & 1 & 2\end{bmatrix}$ and I is the identity matrix of third order then prove that

$$A^2-3A+9I=\begin{bmatrix}-6 & 1 & 2\\ 5 & 4 & 4\\ 2 & 8 & -3\end{bmatrix}$$

10. If $\begin{bmatrix} a & 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 2 \\ 0 & 2 & -4 \end{bmatrix}\begin{bmatrix} a \\ 4 \\ 1 \end{bmatrix} = O$, where O is a zero matrix then find the value of a.

11. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A+B)^2 = A^2 + B^2$ then find the values of a and b

12. If $A = \begin{bmatrix} 0 & -\tan\frac{x}{2} \\ \tan\frac{x}{2} & 0 \end{bmatrix}$ and I is a unit matrix of order 2×2 then prove that

$$I + A = (I - A)\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$$

13. If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then find the value of K where $A^2 = 8A + KI$.

14. If $\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} 1 & -4 & 3 \\ -2 & -10 & 6 \\ 13 & 20 & -9 \end{bmatrix}$ then find the value of A.

15. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ then prove that $A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$, where n is a positive integer

Miscellaneous Exericse - 3

1. If matrix $A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ then find A^2.

2. If $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ then find $(A - 2I).(A - 3I)$.

3. If $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$ then find AB .

4. If $A = \begin{bmatrix} -i & o \\ o & i \end{bmatrix}$ and $B = \begin{bmatrix} o & i \\ i & o \end{bmatrix}$, where $i = \sqrt{-1}$ then find BA .

5. If $A - B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $A + B = \begin{bmatrix} 3 & 5 & -7 \\ -1 & 1 & 4 \\ 11 & 8 & 0 \end{bmatrix}$ then find matrices A and B.

6. If $\begin{bmatrix} -2 & -3 & 1 \\ -y-2 & -1 & 4 \end{bmatrix} = \begin{bmatrix} x+2 & -3 & 1 \\ 5 & -1 & 4 \end{bmatrix}$ then find the values of x and y.

7. The order of matrix A is 3×4 and B is a matrix such that $A^T B$ and AB^T both are defined then find the order of B.

8. If $A = \begin{bmatrix} -2 & -1 & 1 \\ -1 & 7 & 4 \\ 1 & -x & -3 \end{bmatrix}$ is a symmetric matrix then find the value of x.

9. Write a 3×3 matrix $B = [b_{ij}]$ whose elements are $b_{ij} = (i)(j)$.

10. If $A = \begin{bmatrix} 2 & 3 & -4 \\ -1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 3 & 4 \\ -5 & -6 \end{bmatrix}$ then find $A + B^T$.

11. Express the matrix A as the sum of symmetric and skew-symmetric matrix where $A = \begin{bmatrix} 6 & 2 \\ 5 & 4 \end{bmatrix}$.

12. If $A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ then prove that

(i) $(A^T)^T = A$.

(ii) $A + A^T$ is a symmetric matrix.

(iii) $A - A^T$ is a skew-symmetric matrix.

(iv) AA^T and $A^T A$ are symmetric matrices.

13. If $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}; B = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$ and $3A - 2B + C$ is a zero matrix then find matrix C.

14. Write a matrix $B = [b_{ij}]$ of order 2×3 whose elements are $b_{ij} = \dfrac{(i+2j)^2}{2}$.

15. If $A = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 3 & -2 \end{bmatrix}; B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 5 & 7 \end{bmatrix}$ and $C = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ then find the elements of first row of the matrix ABC.

16. If matrix $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ then find AA^T .

17. If $\begin{bmatrix} 1 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = O$ then find the value of x

18. If $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then prove that $B^2 - (a+d)B = (bc - ad)I_2$, where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

19. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ then write $(aA + bB)(aA - bB)$ in the form of matrix A.

20. If $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}$ then prove that $(A - B)^2 \neq A^2 - 2AB + B^2$.

21. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and $A^2 = kA - 2I_2$, then find the value of k.

22. If $A = \begin{bmatrix} i & o \\ o & -i \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} o & i \\ i & o \end{bmatrix}$ where $i = \sqrt{-1}$ then verifty the following expression.

(i) $A^2 = B^2 = C^2 = -I_2$.

(ii) $AB = -BA = -C$.

23. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and $f(A) = A^2 - 5A + 7I$ then find $f(A)$.

24. Prove that

$$\begin{bmatrix} \cos^2 \propto & \cos \propto \sin \propto \\ \cos \propto \sin \propto & \sin^2 \propto \end{bmatrix} \begin{bmatrix} \cos^2 \beta & \cos \beta \sin \beta \\ \cos \beta \sin \beta & \sin^2 \beta \end{bmatrix} = O$$

where $\propto - \beta = (2m - 1)\frac{\pi}{2}; m \in N$.

IMPORTANT POINTS

1. A matrix is an ordered rectangular array of number or functions.
2. **Types of Matrices:** Row Matrix, Column Matrix, Zero Matrix, Square Matrix, Diagonal Matrix, Scalar Matrix. Unit Matrix, Upper Triangular Matrix, Lower Triangular Matrix, Symmetric and Skew-Symmetric Matrices.
3. Addition and substraction of matrices. Addition and substration of two matrices of same order is obtained by addtiion and substraction of their respective elements.
4. **Multiplication of Matrices :** Let two matrices A and B, their multiplication AB is possible when number of column in A is equal to number of row in B and element of AB is obtained by sum of product of element of i^{th} column in A with element of j^{th} row in B.
5. **Scalar Multiplication :** When a non zero scalar is multiplied with matrices A then we have new matrices nA in which all elements is n^{th} time of element of A.
6. Addition of matrices follows commutative and associative law while substraction is not.
7. Multiplication of matrices follows associative law but it doesn't follow commutative law.
8. A matrix having m rows and n columns is called a matrix of order $m \times n$.
9. A $m \times n$ matrix is a square matrix if $m = n$.
10. **Transpose Matrix:** If $A = [a_{ij}]_{m\times n}$ then $A^T = [a_{ji}]_{n\times m}$
11. **Symmetric Matrix:** $A^T = A$
12. **Skew-Symmetric Matrix:** $A^T = -A$
13. If A is a square matrix then
    (i) $A + A^T$ is a symmetric matrix
    (ii) $A - A^T$ is skew symmetric matrix
    (iii) AA^T and $A^T A$ are symmetric matrices
    (iv) $A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right)$
14. If A and B are two matrices then
    (i) $\left(A \pm B\right)^T = A^T \pm B^T$
    (ii) $(A^T)^T = A$
    (iii) $\left(AB\right)^T = B^T.A^T$
    (iv) $\left(kA\right)^T = k.A^T$, where $k \neq 0$

Answers

Exericse 3.1

1. 8

2. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3. $a = 8$

4. $1 \times 6,\ 6 \times 1,\ 2 \times 3,\ 3 \times 2$

5. (i) $\begin{bmatrix} 1/4 & 0 \\ 3/7 & 1/4 \end{bmatrix}$; (ii) $\begin{bmatrix} 9/2 & 25/2 \\ 4 & 9 \end{bmatrix}$; (iii) $\begin{bmatrix} -1 & -4 \\ 1 & -2 \end{bmatrix}$

6. $\begin{bmatrix} 1/2 & 2 & 7/2 \\ 1/2 & 1 & 5/2 \end{bmatrix}$

7. $a = 4, b = 2$ or $a = 2, b = 4$

8. $x = 2,\ y = -1,\ z = -2,\ p = 0$

9. $a = 1, b = 6, c = 3$

Exericse 3.2

1. $A + B = \begin{bmatrix} 0 & 7 & -1 \\ 0 & 0 & 5 \end{bmatrix},\ A - B = \begin{bmatrix} -6 & -3 & 3 \\ 2 & -8 & 9 \end{bmatrix}$

2. $A = \begin{bmatrix} -2 & -1 \\ 1 & -1 \end{bmatrix}, B = \begin{bmatrix} -5 & 1 \\ 1 & -4 \end{bmatrix}$

3. $\begin{bmatrix} -5 & -5 \\ -4 & -5 \\ -1 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 3 & -20 \\ 38 & -11 \end{bmatrix}$

10. $a = -2, -3$

11. $a = 1, b = 4$

13. $k = -7$ 1 4 .

$A = \begin{bmatrix} 1 & -4 & 3 \\ 4 & 2 & 0 \end{bmatrix}$

Miscellaneous Exercise - 3

1. $A^2 = 2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

2. O

3. $\begin{bmatrix} 3 \\ -11 \end{bmatrix}$

4. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

5. $A = \begin{bmatrix} 2 & 3 & -4 \\ 0 & 1 & 2 \\ 6 & 4 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & -3 \\ -1 & 0 & 2 \\ 5 & 4 & 0 \end{bmatrix}$

6. $x = -4,\ y = -7$

7. 3×4

8. -4

9. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 6 & -9 \\ 1 & 6 & -3 \end{bmatrix}$

11. $\begin{bmatrix} 6 & 7/2 \\ 7/2 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -3/2 \\ 3/2 & 0 \end{bmatrix}$

13. $\begin{bmatrix} -16 & -4 \\ 3 & -5 \end{bmatrix}$

14. $\begin{bmatrix} 9/2 & 25/2 & 49/2 \\ 8 & 18 & 32 \end{bmatrix}$

15. 8

16. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or I_2

17. $-9/8$

19. $(a^2 + b^2)A$

21. $k = 1$

23. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

04

Determinants

4.01 Introduction

Consider the following pair of equations

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2,$$

The equations can be solved to find the unique solution if we find $a_1b_2 - b_1a_2$. Therefore number $a_1b_2 - b_1a_2$ is very important and it can be represented as the matrix obtained from the cofficient of x and y

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

The number $a_1b_2 - b_1a_2$ which determines uniqueness of solution is associated with the matrix A = $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ and is called the determinant of A or det A or symbolically we write | A | = $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$. This determinant has two rows and two columns hence it is of order 2.

Note :

1. Only square matrices have determinants
2. A matrix A is said to be Singular matrix if its $|A| = 0$
3. For matrix A, | A | is read as determinmant of A and not modulus of A.

4.02 Definition of determinant

Let $A = [a_{ij}]$ is a square matrix of order n we can associate a unique number $|a_{ij}|$ (real or complex) called determinant of the square matrix A, where $a_{ij} = (i,\ j)$ the element of A. it is denoted by | A |.

4.03 Value of determinant

(i) Determinant of a matrix of order one

Let $A = [a]$ is a square matrix of order one then determinant of $A = |A| = a,$

For Example : If $A = [3]$ then determinant $A = |A| = |3| = 3$

If $A = [3]$ then determinant $A = |A| = |-3| = -3$

(ii) Determinant of a matrix of order two

Let $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is a matrix of order 2, then determinant

$$A = |A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$= a_1 |b_2| - b_1 |a_2|$$

$$= a_1 b_2 - a_2 b_1, \quad \text{value of determinant } A. \qquad (1)$$

$|A|$ = of order 2 = Product of diagonal elements – Product of off-digonal elements.

Example : $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, then

Determinant $A = |A| = \begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} = 2 \cdot (4) - 3\,(-1)$

$$= 8 + 3 = 11.$$

(iii) *Determinant of a matrix of order 3 × 3*

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is a matrix of order 3, then

Determinant $A = |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2) \qquad (2)$$

$$= (a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3) - (a_3 b_2 c_1 + b_3 c_2 a_1 + c_3 a_2 b_1) \qquad (3)$$

Here numbers $a_1, b_1, c_1;\ a_2, b_2, c_2;\ a_3, b_3, c_3$ are called the elements of the determinant. There are a total of $3^2 = 9$ elements in a matrix of order 3. Thus the determinant of a square matrix of order 3 is the sum of the product of elements a_{ij} in first row with $(-1)^{1+j}$ times the determinant of 2 × 2. Sub-matrix obtained by leaving the first row and column passing through the element.

4.04 Rules to expand third order determinant

(i) Write the elements of first row in consecutive positive and negative sign.

(ii) Multiply first element with the second order determinant obtained by deleting the elements of first row (R_1) and first column (C_1). Then multiply 2nd element and the second order determinant obtained by deleting elements of first row (R_1) and 2nd column (C_2). Now multiply third element and the second order determinant obtained by deleting elements of first row (R_1) and third column (C_3) and third column (C_3). To get the value of the determinant add all the three terms.

(iii) The result will be the value of the determinant of order 3.

Example : Evaluate the determinant $\begin{vmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \\ 3 & 0 & 2 \end{vmatrix}$

Solution : $\begin{vmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \\ 3 & 0 & 2 \end{vmatrix} = 1\cdot\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} - 2\cdot\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 0\cdot\begin{vmatrix} 2 & 3 \\ 3 & 0 \end{vmatrix}$

$$= 1(3\times 2 - 1\times 0) - 2(2\times 2 - 3\times 1) + 0(2\times 0 - 3\times 3)$$

$$= 1(6) - 2(1) + 0$$
$$= 6 - 2$$
$$= 4.$$

4.05 Sarrus diagram to determine the value of third order determinant

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix}$$

$$= (a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3) - (a_3b_2c_1 + b_3c_2a_1 + c_3a_2b_1)$$

Note: To evaluate determinant from Sarrus diagram, Like given diagram, we have substract the sum of product of element of leading diagonal to sum of product of element of non-leading diagonal.

Example : Determinant $\begin{vmatrix} 1 & 2 & -1 \\ 3 & 5 & 7 \\ 2 & 4 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 5 & 7 \\ 2 & 4 & 6 \end{vmatrix} \begin{matrix} 1 & 2 \\ 3 & 5 \\ 2 & 4 \end{matrix}$

$$= (30 + 28 - 12) - (-10 + 28 + 36)$$
$$= 46 - 54 = -8.$$

4.06 Difference between matrix and determinant

(i) Matrix is a proper representation of number and does not have a numerical value while determinant has a unique numerical value.

(ii) Matrix can be of any order while determinants are square matrices where number of rows and columns are same.

(iii) If we change the number of rows and columns of the matrix we get a new matrix whereas the value of determinant unchanged.

4.07 Minors and cofactors of a determinant

Minors : Minor of an element a_{ij} of a determinant is the determinant obtained by deleting its ith row and jth column in which element a_{ij} lies. Minor of an element a_{ij} is denoted by A_{ij} .

Example : $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Here element a_2, lies in the second row and first column then leaving the second row and first column in Δ we get the respective determinant.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ or } \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$$ which is the minor of element a_2

similarly the minor of element c_3 of Δ will be

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ or } \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Minor of an element of a determinant of order $n(n \geq 2)$ is a determinatn of order $n - 1$.

Example : The minor of element 1 in the determinant $\begin{vmatrix} -3 & 2 \\ 1 & 5 \end{vmatrix}$ is $|2|$.

The minor of element 3 in the determinant $\begin{vmatrix} 1 & -2 & 3 \\ 7 & 0 & 5 \\ -3 & -1 & 4 \end{vmatrix}$ is $\begin{vmatrix} 7 & 0 \\ -3 & -1 \end{vmatrix}$ and element 7 is $\begin{vmatrix} -2 & 3 \\ -1 & 4 \end{vmatrix}$

Cofactor : Cofactor of an element a_{ij}, denoted by F_{ij} is defined by

$$F_{ij} = (-1)^{i+j} \text{ Minors}$$

$$\Rightarrow \quad F_{ij} = (-1)^{i+j} A_{ij} ,$$

here A_{ij} and F_{ij} denotes the Minors and Cofactors of element a_{ij}

i.e., $$F_{ij} = \begin{cases} A_{ij} & ; \ i+j \text{ is even} \\ -A_{ij} & ; \ i+j \text{ is odd} \end{cases}$$

Example: If $$\Delta = \begin{vmatrix} 7 & 4 & -1 \\ -2 & 3 & 0 \\ 1 & -5 & 2 \end{vmatrix}$$ then

Cofactor of 7 $= (-1)^{1+1} \begin{vmatrix} 3 & 0 \\ -5 & 2 \end{vmatrix} = 6 - 0 = 6$

Cofactor of 5 $= (-1)^{3+2} \begin{vmatrix} 7 & -1 \\ -2 & 0 \end{vmatrix} = -(0-2) = 2$

Cofactor of 4 $= (-1)^{1+2} \begin{vmatrix} -2 & 0 \\ 1 & 2 \end{vmatrix} = -(-4) = 4$

Note: For easy calculation in a matrix of order 2 and 3 the signs of elements to find the cofactor is

$$\begin{vmatrix} + & - \\ - & + \end{vmatrix}, \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

4.08 Expansion of determinants

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$ is a determinant of third order

Expanding along first row we get

$$\Delta = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$= a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13}$, where A_{11}, A_{12} and A_{13} are the minors of corresponding elements

$= a_{11}F_{11} + a_{12}F_{12} + a_{13}F_{13}$, where F_{11}, F_{12} and F_{13} are the cofactors of corresponding elements

Similarly we can see that

$$\Delta = a_{21}F_{21} + a_{22}F_{22} + a_{23}F_{23}$$

$$\Delta = a_{11}F_{11} + a_{21}F_{21} + a_{31}F_{31}$$

$$\Delta = a_{13}F_{13} + a_{23}F_{23} + a_{33}F_{33} \text{ etc}$$

Thus the value of the determinants is the sum of elements with its corresponding cofactors.

Note:

(i) The expansion can be done along any row or column in determinant.

(ii) This rule is valid for any type of determinant.

(iii) Expansion should be done with any row or column with maximum zeroes.

Illustrative Examples

Example 1. Evaluate the determinant $\begin{vmatrix} 2 & 4 \\ 2 & -3 \end{vmatrix}$

Solution : $\begin{vmatrix} 2 & 4 \\ 2 & -3 \end{vmatrix} = (-6)-(8) = -14.$

Example 2. Evaluate the determinant $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

Solution : $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = (\cos^2\theta)-(-\sin^2\theta)$

$= \cos^2\theta + \sin^2\theta = 1.$

Example 3. Evaluate the determinant $\begin{vmatrix} 3 & 11 & -1 \\ 5 & 2 & 0 \\ 10 & 3 & 0 \end{vmatrix}$

Solution : $\begin{vmatrix} 3 & 11 & -1 \\ 5 & 2 & 0 \\ 10 & 3 & 0 \end{vmatrix}$ expanding alogn third column

$$= -1\begin{vmatrix} 5 & 2 \\ 10 & 3 \end{vmatrix} - 0 + 0 = -(15-20) = 5.$$

Example 4. If determinant $\begin{vmatrix} k & 8 \\ 2 & 4 \end{vmatrix} = 4$, then find the value of k.

Solution : Given $\begin{vmatrix} k & 8 \\ 2 & 4 \end{vmatrix} = 4$

$\Rightarrow \quad 4k - 16 = 4$

$\Rightarrow \quad k = 5.$

Example 5. If determinant $\begin{vmatrix} k & 3 \\ -1 & k \end{vmatrix} = 7$ then find the vlaue of k.

Solution : Given $\begin{vmatrix} k & 3 \\ -1 & k \end{vmatrix} = 7$

$\Rightarrow \quad k^2 - (-3) = 7 \quad \Rightarrow \quad k^2 + 3 = 7$

$\Rightarrow \quad k^2 = 4 \quad \Rightarrow \quad k = \pm 2.$

Example 6. Evaluate the determinant $A = \begin{vmatrix} 2 & 4 & 1 \\ 8 & 5 & 2 \\ -1 & 3 & 7 \end{vmatrix}$ and write the cofactors and minors of elements of second row.

Solution: Minors : $A_{21} = \begin{vmatrix} 4 & 1 \\ 3 & 7 \end{vmatrix} = 28 - 3 = 25,\ A_{22} = \begin{vmatrix} 2 & 1 \\ -1 & 7 \end{vmatrix} = 14 - (-1) = 15, A_{23} = \begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} = 6 - (-4) = 10$

$\therefore$ Cofactors $\quad F_{21} = -A_{21} = -25,\ \ F_{22} = A_{22} = 15,\ \ F_{23} = -A_{23} = -10$

Thus the vlaue of determinant A is $= 8 \cdot F_{21} + 5 \cdot F_{22} + 2 \cdot F_{23}$

$$= 8(-25) + 5(15) + 2(-10)$$

$$= -200 + 75 - 20 = -145.$$

Example 7. Evaluate the determinant $\begin{vmatrix} 3 & -7 & 13 \\ 5 & 0 & 0 \\ 0 & 11 & 2 \end{vmatrix}$

Solution : Expanding along second row as it has two zeroes

$$\begin{vmatrix} 3 & -7 & 13 \\ 5 & 0 & 0 \\ 0 & 11 & 2 \end{vmatrix} = 5\times(-1)\begin{vmatrix} -7 & 13 \\ 11 & 2 \end{vmatrix} + 0 - 0$$

$$= -5[-14-143] = 785.$$

Exercise 4.1

1. For what value of k is the value of the determinant $\begin{vmatrix} k & 2 \\ 4 & -3 \end{vmatrix}$ zero?

2. If $\begin{vmatrix} x & y \\ 2 & 4 \end{vmatrix} = 0$ then find the ratio $x : y$.

3. If $\begin{vmatrix} 2 & 3 \\ y & x \end{vmatrix} = 4$ and $\begin{vmatrix} x & y \\ 4 & 2 \end{vmatrix} = 7$ then evaluate x and y.

4. If $\begin{vmatrix} x-1 & x-2 \\ x & x-3 \end{vmatrix} = 0$ then find the value of x.

5. Evaluate the determinant and also find the minors and cofactors of elements of first row

   (i) $\begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$ (ii) $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

6. Evaluate the determinant $\begin{vmatrix} 3 & -11 & 1 \\ 5 & 0 & 0 \\ -10 & 3 & 0 \end{vmatrix}$

7. Prove that $\begin{vmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{vmatrix} = 1 + a^2 + b^2 + c^2$.

4.09 Properties of Determinants

(i) The value of the determinant remains unchanged if its rows and columns are interchanged.

Proof : Let $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

and $$\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

$\therefore$ $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix}$$ (by Sarrus figure)

$$\Delta = (a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3) - (a_3b_2c_1 + b_3c_2a_1 + c_3a_2b_1) \quad (1)$$

and $$\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{matrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{matrix}$$ (by Sarrus figure)

$$\Delta_1 = (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (c_1b_2a_3 + c_2b_3a_1 + c_3b_1a_2) \quad (2)$$

$\therefore$ from (1) and (2) $\Delta = \Delta_1$

$\therefore |A^T| = |A|$, where A^T, is a transpose of square matrix A.

(ii) If any two rows (or columns) of a determinant are interchanged, then sign of determiant changes, but value remains unchanged.

Proof : Let $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and $$\Delta_1 = \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix},$$

(by interchanging the first and second columns of the determinant)

$\therefore$ $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix}$$ (by Sarrus figure)

$$\Delta = (a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3) - (a_3b_2c_1 + b_3c_2a_1 + c_3a_2b_1) \quad (1)$$

and $$\Delta_1 = \left|\begin{array}{ccc|cc} b_1 & a_1 & c_1 & b_1 & a_1 \\ b_2 & a_2 & c_2 & b_2 & a_2 \\ b_3 & a_3 & c_3 & b_3 & a_3 \end{array}\right.$$ (by Sarrus figure)

$$\Delta_1 = \left(b_1a_2c_3 + a_1c_2b_3 + c_1b_2a_3\right) - \left(b_3a_2c_1 + a_3c_2b_1 + c_3b_2a_1\right) \quad (2)$$

$\therefore$ from (1) and (2) $\quad \Delta_1 = -\Delta$

(iii) If any two rows (or columns) of a determinant are identical (all corresponding elements are same), then value of determinant is zero

Proof : $$\begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} = \left|\begin{array}{ccc|cc} a & b & c & a & b \\ a & b & c & a & b \\ x & y & z & x & y \end{array}\right.$$ (by Sarrus figure)

$$= \left(abz + bcx + cay\right) - \left(xbc + yca + zab\right)$$

$$= 0.$$

and $$\begin{vmatrix} x & a & x \\ y & b & y \\ z & c & z \end{vmatrix} = \left|\begin{array}{ccc|cc} x & a & x & x & a \\ y & b & y & y & b \\ z & c & z & z & c \end{array}\right.$$ (by Sarrus figure)

$$= \left(xbz + ayz + xyc\right) - \left(zbx + cyx + zya\right)$$

$$= 0.$$

(iv) If each element of a row (or a column) of a determinant is multiplied by a constant *k*, then its value gets multiplied by *k*.

Proof : Let $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and $$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ ka_3 & kb_3 & kc_3 \end{vmatrix},$$

$\therefore$ By Sarrus figure (multiplying the third row by *k*)

$$\Delta = \left(a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3\right) - \left(a_3b_2c_1 + b_3c_2a_1 + c_3a_2b_1\right) \quad (1)$$

and $$\Delta_1 = \left|\begin{array}{ccc|cc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ ka_3 & kb_3 & kc_3 & ka_3 & kb_3 \end{array}\right.$$ (by Sarrus figure)

$$\Delta_1 = (a_1 b_2 kc_3 + b_1 c_2 ka_3 + c_1 a_2 kb_3) - (ka_3 b_2 c_1 + kb_3 c_2 a_1 + kc_3 a_2 b_1)$$

$$= k\{(a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3) - (a_3 b_2 c_1 + b_3 c_2 a_1 + c_3 a_2 b_1)\}$$

$$= k\Delta$$

$\therefore \qquad \Delta_1 = k\Delta$

Corollory :: Let Δ_1 be the determinant obtained by multiplying the each elements of Δ by k then

$\Delta_1 = k\Delta$, when the order of Δ is one

$\Delta_1 = k^2\Delta$, when the order of Δ is two

$\Delta_1 = k^3\Delta$, when the order of Δ is three

$\Delta_1 = k^4\Delta$, when the order of Δ is four

i.e. $\qquad \Delta_1 = k^n\Delta$ when the order of Δ is n

(v) If each elements of a row or column of a determinant are expressed as sum of two (or more) terms the determinant can be expressed as sum of two (or more) determinants.

Proof : Let $\Delta = \begin{vmatrix} a_1+d_1 & b_1 & c_1 \\ a_2+d_2 & b_2 & c_2 \\ a_3+d_3 & b_3 & c_3 \end{vmatrix}$

Expanding along first row

$$\Delta = (a_1+d_1)\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - (a_2+d_2)\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + (a_3+d_3)\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= \left\{a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\right\} + \left\{d_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - d_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + d_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\right\}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

(vi) If the elements of any row or column of a determinant is added or substracted with any of other row (or column) with a multiple of constant, then the value of the determinant does not changes.

Proof : Let $\qquad \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

and $\qquad \Delta_1 = \begin{vmatrix} a_1+kc_1 & b_1 & c_1 \\ a_2+kc_2 & b_2 & c_2 \\ a_3+kc_3 & b_3 & c_3 \end{vmatrix},$

(by adding first column with k times the third column)

$$\therefore \qquad \Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} kc_1 & b_1 & c_1 \\ kc_2 & b_2 & c_2 \\ kc_3 & b_3 & c_3 \end{vmatrix} \qquad [\text{Property } (v)]$$

$$= \Delta + k \begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix} \qquad [\text{Property (iv)}]$$

$$= \Delta + k \times 0$$

$$= \Delta. \qquad [\text{Property (iii)}]$$

(vii) If elements of one row (or column) are multiplied with cofactors of elements of any other row (or column), then their sum is zero.

Proof : Let

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad (1)$$

$$\Rightarrow \qquad \Delta = a_{11}F_{11} + a_{12}F_{12} + a_{13}F_{13} \quad \text{(Expanding along first rows)} \qquad (2)$$

substituting in (1) of a_{11}, a_{12} and a_{13} by a_{21}, a_{22} and a_{23}

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \qquad [\text{Property (iii)}] \quad (3)$$

thsu from (1) and (3) $\qquad 0 = a_{21}F_{11} + a_{22}F_{12} + a_{23}F_{13}$

similarly $\qquad 0 = a_{31}F_{11} + a_{32}F_{12} + a_{33}F_{13}$ etc.

(viii) If the elements of any row or column of a determinant are zeroes then the vlaue of the determinant is zero.

Proof :

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{expanding along second row}$$

$$= -0 \times \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + 0 \times \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - 0 \times \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$$= 0$$

(ix) In a Triangular matrix the value of the determinant is the product of the elements of the diagonals.

For example: (i) $\begin{vmatrix} a & b \\ 0 & c \end{vmatrix} = ac - 0 = ac$

(ii) $\begin{vmatrix} a & 0 \\ b & c \end{vmatrix} = ac - 0 = ac$

(iii) $\begin{vmatrix} a & b & c \\ 0 & x & y \\ 0 & 0 & \ell \end{vmatrix} = \ell \begin{vmatrix} a & b \\ 0 & x \end{vmatrix} = \ell(ax) = a\ell x$

(iv) $\begin{vmatrix} a & 0 & 0 \\ b & x & 0 \\ c & y & \ell \end{vmatrix} = a \begin{vmatrix} x & 0 \\ y & \ell \end{vmatrix} = a(x\ell - 0) = a\ell x$

Corollory : $|I_n| = 1,$ where I_n, n is the identity matrix of order n

$$\Rightarrow \quad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

(x) If a determinant has polynomial with variable *x* and if by substituting *a* in place of *x* the value of determinant is zero then *x–a* will be a factor of the determinant.

For example : In $\Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & a & a^2 \\ 1 & b & b^2 \end{vmatrix}$ if by substituting $x = a$ and $x = b$ the value of Δ becomes zero

then $(x-a)$ and $(x-b)$ will be the two factors of the determinant.

$\therefore$ To solve for Δ subtracting second row from first and third row from first we have

$$\Delta = \begin{vmatrix} 1 & x & x^2 \\ 0 & a-x & a^2-x^2 \\ 0 & b-x & b^2-x^2 \end{vmatrix} = \begin{vmatrix} a-x & a^2-x^2 \\ b-x & b^2-x^2 \end{vmatrix}$$

$$= (a-x)(b-x)\begin{vmatrix} 1 & a+x \\ 1 & b+x \end{vmatrix}$$

$$= (a-x)(b-x)(b+x-a-x)$$

$$= (a-x)(b-x)(b-a)$$

$$= (x-a)(x-b)(b-a)$$

4.10 Elementary operations

If the order of Δ is $n \geq 2$ then $R_1, R_2, R_3, \ldots$ represents first row, second row, third row. . . and $C_1, C_2, C_3, \ldots$ represents first columns, second column, third column . . . etc.

(i) Operation $R_i \leftrightarrow R_j$ means *i*th and *j*th rows are mutually interchanged and $C_i \leftrightarrow C_j$ means that *i*th and *j*th columns are mutually interchanged.

(ii) Operation $R_i \to kR_i$ means that every element of *i*th row is multiplied by k whereas $C_i \to kC_i$ means that every element of *i*th column is multiplied by *k*.

(iii) Operation $R_i = R_i + kR_j$ refers that every element of *i*th row is added to *k* times the elements in *j*th row similarly $C_i = C_i + kC_j$ refers that every element of *i*th row is added to *k* times the elements in *j*th column

4.11 Product of determinants

I. The product of second order determinant can be done as given below:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1 + b_1\alpha_2 & a_1\beta_1 + b_1\beta_2 \\ a_2\alpha_1 + b_2\alpha_2 & a_2\beta_1 + b_2\beta_2 \end{vmatrix} \quad \text{(Row multiply by column)}$$

and

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix} \quad \text{(Row multiply by Row)}$$

$\because \quad |A^T| = |A|$

II. The product of third order determinant can be done as given below:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1 + b_1\alpha_2 + c_1\alpha_3 & a_1\beta_1 + b_1\beta_2 + c_1\beta_3 & a_1\gamma_1 + b_1\gamma_2 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\alpha_2 + c_2\alpha_3 & a_2\beta_1 + b_2\beta_2 + c_2\beta_3 & a_2\gamma_1 + b_2\gamma_2 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\alpha_2 + c_3\alpha_3 & a_3\beta_1 + b_3\beta_2 + c_3\beta_2 & a_3\gamma_1 + b_3\gamma_2 + c_3\gamma_3 \end{vmatrix}$$

and

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

Note : The product of two different order determinants is also possible.

For example : $\Delta_1 = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix}$

$$\Delta_1 \cdot \Delta_2 = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix}$$

$$\therefore \quad = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 11 \\ 5 & 4 & 10 \end{vmatrix}$$

$$= 1(50-44) - 2(40-55) + 3(16-25)$$

$$= 6 + 30 - 27 = 9. \qquad (1)$$

Now $\qquad \Delta_1 = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 - 4 = -3. \qquad (2)$

and $\qquad \Delta_2 = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix} = 1(4-6) - 2(8-3) + 3(4-1)$

$$= -2 - 10 + 9 = -3. \qquad (3)$$

from (1), (2) and (3)

$$\Delta_1 \cdot \Delta_2 = 9.$$

Illustrative Examples

Example 8. Evaluate the determinant $\begin{vmatrix} 49 & 1 & 6 \\ 39 & 7 & 4 \\ 10 & 2 & 1 \end{vmatrix}$ without cxpansion.

Solution : Using operation $C_1 \to C_1 - 8C_3$

$$\begin{vmatrix} 1 & 1 & 6 \\ 7 & 7 & 4 \\ 2 & 2 & 1 \end{vmatrix} = 0 \qquad [\because C_1 = C_2 \quad \text{Property (iii)}]$$

Example 9. Evaluate the determinant $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ without expansion.

Solution : $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & c+a+b \\ 1 & c & a+b+c \end{vmatrix} \qquad (C_3 \to C_3 + C_2)$

$$= (a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} \qquad [\text{Property (iv)}]$$

$$= (a+b+c)(0)$$

$$= 0 \qquad [\because C_1 = C_3 \quad \text{Property (iii)}]$$

Example 10. Evaluate the determinant $\begin{vmatrix} a-b & m-n & x-y \\ b-c & n-p & y-z \\ c-a & p-m & z-x \end{vmatrix}$ without expansion.

Solution : $\begin{vmatrix} a-b & m-n & x-y \\ b-c & n-p & y-z \\ c-a & p-m & z-x \end{vmatrix}$

Using operation $R_1 \rightarrow R_1 + R_2 + R_3$

$$= \begin{vmatrix} 0 & 0 & 0 \\ b-c & n-p & y-z \\ c-a & p-m & z-x \end{vmatrix} = 0$$ [Using Property (viii)]

Example 11. Prove that

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x) \cdot$$

Solution : $\text{L.H.S.} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

Usign $R_1 \rightarrow R_1 - R_2$ and $R_2 \rightarrow R_2 - R_3$

$$= \begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= (x-y)(y-z) \begin{vmatrix} 0 & 1 & x+y \\ 0 & 1 & y+z \\ 1 & z & z^2 \end{vmatrix}$$ [property (iv)]

Expanding along first column

$$= (x-y)(y-z)\left\{0 - 0 + 1 \begin{vmatrix} 1 & x+y \\ 1 & y+z \end{vmatrix}\right\}$$

$$= (x-y)(y-z)(y+z-x-y)$$

$$= (x-y)(y-z)(z-x).$$

$= \text{R.H.S.}$

Example 12. Without expanding, prove that

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}.$$

Solution :

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$$

$$= \begin{vmatrix} 2c & c+a & a+b \\ 2r & r+p & p+q \\ 2z & z+x & x+y \end{vmatrix}$$ (Property $C_1 \to C_1 + C_2 - C_3$)

$$= 2\begin{vmatrix} c & c+a & a+b \\ r & r+p & p+q \\ z & z+x & x+y \end{vmatrix}$$ [Property (iv)]

$$= 2\begin{vmatrix} c & a & a+b \\ r & p & p+q \\ z & x & x+y \end{vmatrix}$$ (operation $C_2 \to C_2 - C_1$)

$$= 2\begin{vmatrix} c & a & b \\ r & p & q \\ z & x & y \end{vmatrix}$$ (operation $C_3 \to C_3 - C_2$)

$$= -2\begin{vmatrix} a & c & b \\ p & r & q \\ x & z & y \end{vmatrix}$$ (operation $C_1 \leftrightarrow C_2$)

$$= 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$ (operation $C_2 \leftrightarrow C_3$)

Example 13. If x, y, z are different and real,

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$$

then Prove that $xyz = -1$.

Solution :

given $$\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = 0 \qquad \text{[property (v)]}$$

$$\Rightarrow \quad -\begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix} + xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0 \qquad \text{[property (ii) and (iv)]}$$

$$\Rightarrow \quad \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0 \qquad \text{[property (ii)]}$$

$$\Rightarrow \quad (1+xyz)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0 \qquad \text{[from example (11)]}$$

$$\Rightarrow \quad (1+xyz)(x-y)(y-z)(z-x) = 0$$

$\because \quad x \neq y \neq z \Rightarrow x-y \neq 0,\ y-z \neq 0$ तथा $z-x \neq 0$

$\Rightarrow \quad 1+xyz = 0 \Rightarrow xyz = -1\cdot$

Example 14. Evaluate the determinant $\begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ca \\ 1/c & c^2 & ab \end{vmatrix}$

Solution : $$\begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ca \\ 1/c & c^2 & ab \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} 1 & a^3 & abc \\ 1 & b^3 & abc \\ 1 & c^3 & abc \end{vmatrix} \qquad (\text{Operation } R_1 \to aR_1,\ R_2 \to bR_2 \text{ and } R_3 \to cR_3)$$

$$= \frac{abc}{abc}\begin{vmatrix} 1 & a^3 & 1 \\ 1 & b^3 & 1 \\ 1 & c^3 & 1 \end{vmatrix} = 0 \qquad [\because C_1 = C_3, \text{ property (iii)}]$$

Example 15. Prove that

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$

Solution : L.H.S. $= \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix} \qquad \text{(opration } C_1 \to C_1 + C_2 + C_3\text{)}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix} \qquad \text{[property (iv)]}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix} \qquad \text{(operation } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1\text{)}$$

$$= 2(a+b+c)\left\{1 \cdot \begin{vmatrix} b+c+a & 0 \\ 0 & c+a+b \end{vmatrix}\right\}$$

$$= 2(a+b+c)\begin{vmatrix} a+b+c & 0 \\ 0 & a+b+c \end{vmatrix}$$

$$= 2(a+b+c)(a+b+c)^2$$

$$= 2(a+b+c)^3$$

= RHS

Example 16. Prove that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

Solution : $\text{L.H.S.} = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

$$= abc \begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$ (taking a, b and c from first, second and third row)

$$= abc \begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$ (operation $R_1 \to R_1 + R_2 + R_3$)

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$ [property (iv)]

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & \frac{1}{b} \\ 0 & -1 & 1+\frac{1}{c} \end{vmatrix}$$ (Using operation $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$)

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left\{0+0+1\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix}\right\}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(1-0)$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

= R.H.S.

Example 17. Solve the equation $\begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix} = 0$

Solution :

$$\begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix} = 0$$

$$\begin{vmatrix} x+a+b+c & b & c \\ x+a+b+c & x+b & a \\ x+a+b+c & b & x+c \end{vmatrix} = 0 \qquad \text{(operation } C_1 \to C_1 + C_2 + C_3\text{)}$$

or $$(x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+b & a \\ 1 & b & x+c \end{vmatrix} = 0$$

or $$(x+a+b+c)\begin{vmatrix} 0 & -x & c-a \\ 0 & x & a-x-c \\ 1 & b & x+c \end{vmatrix} = 0 \qquad \text{(using operation } R_1 \to R_1 - R_2 \text{ and } R_2 \to R_2 - R_3\text{)}$$

or $$(x+a+b+c)\begin{vmatrix} -x & c-a \\ x & a-x-c \end{vmatrix} = 0 \qquad \text{(expanding } C_1 \text{)}$$

or $$(x+a+b+c)\begin{vmatrix} 0 & -x \\ x & a-x-c \end{vmatrix} = 0 \qquad \text{(operation } R_1 \to R_1 + R_2\text{)}$$

$\Rightarrow$ $$(x+a+b+c)(0+x^2) = 0$$

$\Rightarrow$ $$x^2(x+a+b+c) = 0$$

$\Rightarrow$ $x^2 = 0$ or $x+a+b+c = 0$

$\Rightarrow$ $x = 0$ or $x = -(a+b+c)$

Example 18. Prove that

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = (y-z)(z-x)(x-y)(yz+zx+xy) \cdot$$

Solution : L.H.S. $= \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix}$

$$= \frac{1}{xyz}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ xyz & xyz & xyz \end{vmatrix} \qquad \text{(operation } C_1 \to xC_1, C_2 \to yC_2, C_3 \to zC_3\text{)}$$

$$= \frac{xyz}{xyz}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ 1 & 1 & 1 \end{vmatrix} \qquad \text{(taking out xyz from the operation } R_3\text{)}$$

$$= -\begin{vmatrix} x^2 & y^2 & z^2 \\ 1 & 1 & 1 \\ x^3 & y^3 & z^3 \end{vmatrix} \qquad \text{(operation } R_2 \leftrightarrow R_3\text{)}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} \qquad \text{(operation } R_1 \leftrightarrow R_2\text{)}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ x^2-y^2 & y^2-z^2 & z^2 \\ x^3-y^3 & y^3-z^3 & z^3 \end{vmatrix} \qquad \text{(operation } C_1 \to C_1 - C_2 \text{ and } C_2 \to C_2 - C_3\text{)}$$

$$= \begin{vmatrix} x^2-y^2 & y^2-z^2 \\ x^3-y^3 & y^3-z^3 \end{vmatrix} \qquad \text{(Epanding } R_1\text{)}$$

$$= \begin{vmatrix} (x-y)(x+y) & (y+z)(y-z) \\ (x-y)(x^2+xy+y^2) & (y-z)(y^2+yz+z^2) \end{vmatrix}$$

$$= (x-y)(y-z)\begin{vmatrix} x+y & y+z \\ x^2+xy+y^2 & y^2+yz+z^2 \end{vmatrix}$$

$$=(x-y)(y-z)\begin{vmatrix} x+y & z-x \\ x^2+xy+y^2 & yz+z^2-x^2-xy \end{vmatrix} \qquad \text{(operation } C_2 \to C_2 - C_1 \text{)}$$

$$=(x-y)(y-z)\begin{vmatrix} x+y & z-x \\ x^2+xy+y^2 & (z-x)(z+x)+y(z-x) \end{vmatrix}$$

$$=(x-y)(y-z)\begin{vmatrix} x+y & z-x \\ x^2+xy+y^2 & (z-x)(z+x+y) \end{vmatrix}$$

$$=(x-y)(y-z)(z-x)\begin{vmatrix} x+y & 1 \\ x^2+xy+y^2 & z+x+y \end{vmatrix}$$

$$=(x-y)(y-z)(z-x)\left\{(x+y)(z+x+y)-\left(x^2+xy+y^2\right)\right\}$$

$$=(x-y)(y-z)(z-x)\cdot\left(zx+x^2+xy+yz+xy+y^2-x^2-xy-y^2\right)$$

$$=(x-y)(y-z)(z-x)(xy+yz+zx)$$

= R.H.S.

Example 19. Evaluate the following $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ without expansion.

Solution : We know that $\log_n m = \dfrac{\log m}{\log n}$

$$\therefore \qquad \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix} = \begin{vmatrix} 1 & \dfrac{\log y}{\log x} & \dfrac{\log z}{\log x} \\ \dfrac{\log x}{\log y} & 1 & \dfrac{\log z}{\log y} \\ \dfrac{\log x}{\log z} & \dfrac{\log y}{\log z} & 1 \end{vmatrix}$$

$$= \frac{1}{\log x\cdot\log y\cdot\log z}\begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log z \end{vmatrix}$$

(operation $R_1 \to \log x\cdot R_1; R_2 \to \log y\cdot R_2; R_3 \to \log z\cdot R_3$)

$$= \frac{1}{\log x\cdot\log y\cdot\log z}\times 0 \qquad (\because \quad R_1 = R_2 = R_3\ )$$

= 0

Example 20. Prove that

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3.$$

Solution : L.H.S. $= \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$

$$= \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix} \quad \text{(operation } C_1 \to C_1 - C_3 \text{ and } C_2 \to C_2 - C_3\text{)}$$

$$= \begin{vmatrix} (b+c+a)(b+c-a) & 0 & a^2 \\ 0 & (c+a+b)(c+a-b) & b^2 \\ (c+a+b)(c-a-b) & (c+a+b)(c-a-b) & (a+b)^2 \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix} \quad \text{(Taking out } (a+b+c) \text{ from } C_1 \text{ and } C_2\text{)}$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix} \quad \text{(operation } R_3 \to R_3 - R_1 - R_2\text{)}$$

$$= (a+b+c)^2 \begin{vmatrix} b+c & \frac{a^2}{b} & a^2 \\ \frac{b^2}{a} & c+a & b^2 \\ 0 & 0 & 2ab \end{vmatrix} \quad \text{(operation } C_1 \to C_1 + \frac{C_3}{a} \text{ and } C_2 \to C_2 + \frac{C_3}{b}\text{)}$$

$$= (a+b+c)^2 \left\{ 0+0+2ab \begin{vmatrix} b+c & \dfrac{a^2}{b} \\ \dfrac{b^2}{a} & (c+a) \end{vmatrix} \right\}$$ (Expanding along R_3)

$$= (a+b+c)^2 \cdot 2ab\{(b+c)(c+a)-ab\}$$

$$= (a+b+c)^2 \cdot 2ab(bc+ab+c^2+ca-ab)$$

$$= (a+b+c)^2 \cdot 2ab(bc+c^2+ca)$$

$$= (a+b+c)^2 \cdot 2abc(b+c+a)$$

$$= 2abc(a+b+c)^3 = \text{R.H.S.}$$

Example 21. Prove that

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}.$$

Solution : L.H.S. $= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times (-1) \begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix}$$ $(C_2 \leftrightarrow C_3)$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}$$

$$= \begin{vmatrix} -a^2+bc+bc & -ab+ab+c^2 & -ac+b^2+ac \\ -ab+c^2+ab & -b^2+ac+ac & -bc+bc+a^2 \\ -ac+ac+b^2 & -bc+a^2+bc & -c^2+ab+ab \end{vmatrix}$$ (multiply row by row)

$$= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$$

= R.H.S.

Exercise 4.2

1. If $\begin{vmatrix} \ell & m \\ 2 & 3 \end{vmatrix} = 0$ then find the ratio $\ell : m$

2. Find the minor of the elements of second row of determinant $\begin{vmatrix} 2 & 3 & 4 \\ 3 & 6 & 5 \\ 1 & 8 & 9 \end{vmatrix}$

3. Evaluate the determinant $\begin{vmatrix} 13 & 16 & 19 \\ 14 & 17 & 20 \\ 15 & 18 & 21 \end{vmatrix}$

4. If the first and the third columns of the determinant are interchanged then write the change in the determinant?

5. Prove that

$$\begin{vmatrix} 1 & yz & y+z \\ 1 & zx & z+x \\ 1 & xy & x+y \end{vmatrix} = (x-y)(y-z)(z-x)\cdot$$

6. Evaluate the determinant $\begin{vmatrix} 0 & b^2a & c^2a \\ a^2b & 0 & c^2b \\ a^2c & b^2c & 0 \end{vmatrix}$

7. Solve the following determinant:

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0\cdot$$

8. Without expanding evaluate the determinant

$$\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}\cdot$$

9. Prove that

$$\begin{vmatrix} b+c & a+b & a \\ c+a & b+c & b \\ a+b & c+a & c \end{vmatrix} = a^3+b^3+c^3-3abc\cdot$$

10. Evaluate the determinant $\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix}$

11. If ω is the cube root of unity then find the value of the determinant $\begin{vmatrix} 1 & \omega^3 & \omega^2 \\ \omega^3 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix}$.

12. Prove that :

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2 \cdot$$

13. If in the determinant $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $A_1, B_1, C_1, \ldots$ are the cofactors of elements $a_1, b_1, c_1, \ldots$ then

Prove that

$$\Delta^2 = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}.$$

[HINT : $\Delta \cdot \Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$,

$$= \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3$$

$\therefore$ $\Delta\Delta' = \Delta^3$ or $\Delta' = \Delta^2$

Miscellaneous Exericse – 4

1. The vlaue of the determinant $\begin{vmatrix} \cos 80^\circ & -\cos 10^\circ \\ \sin 80^\circ & \sin 10^\circ \end{vmatrix}$ is

(a) 0 (b) 1 (c) –1 (d) none of these.

2. The cofactors of first column in the determinant $\begin{vmatrix} 5 & 20 \\ 3 & -1 \end{vmatrix}$ are

(a) –1, 3 (b) –1, –3 (c) –1, 20 (d) –1, –20.

3. If $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 4 \end{vmatrix}$ then the value of the determinant $\begin{vmatrix} -2 & -4 & -6 \\ -8 & -10 & -12 \\ -2 & -4 & -8 \end{vmatrix}$ is

(a) -2Δ (b) 8Δ (c) -8Δ (d) -6Δ.

4. Which among the below given determinants is same as determinant $\begin{vmatrix} 1 & 0 & 2 \\ 3 & -2 & -1 \\ 2 & 5 & 4 \end{vmatrix}$?

(a) $\begin{vmatrix} 2 & 5 & 4 \\ 3 & -2 & -1 \\ 1 & 0 & 2 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & 4 \\ 0 & -2 & 5 \end{vmatrix}$ (c) $-\begin{vmatrix} 2 & -1 & 4 \\ 0 & -2 & 5 \\ 1 & 3 & 2 \end{vmatrix}$ (d) $\begin{vmatrix} 2 & 0 & 1 \\ -1 & -2 & 3 \\ 4 & 5 & 2 \end{vmatrix}$.

5. The value of the determinant $\begin{vmatrix} \cos 50° & \sin 10° \\ \sin 50° & \cos 10° \end{vmatrix}$ is

(a) 0 (b) 1 (c) 1 / 2 (d) –1 / 2.

6. The value of the determinant $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$ is

(a) $ab+bc+ca$ (b) 0 (c) 1 (d) abc.

7. If ω is the root of unity then the value of the determinant $\begin{vmatrix} 1 & \omega^4 & \omega^8 \\ \omega^4 & \omega^8 & 1 \\ \omega^8 & 1 & \omega^4 \end{vmatrix}$ is

(a) ω^2 (b) ω (c) 1 (d) 0.

8. If $\begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix}^2 = \begin{vmatrix} 3 & 2 \\ 1 & x \end{vmatrix} - \begin{vmatrix} x & 3 \\ -2 & 1 \end{vmatrix}$ then the value of x is

(a) 6 (b) 7 (c) 8 (d) 0.

9. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and cofactors corresponding to elements $a_{11}, a_{12}, a_{13}, \ldots$ are $F_{11}, F_{12}, F_{13}, \ldots$

then the correct statement is

(a) $a_{12}F_{12} + a_{22}F_{22} + a_{32}F_{32} = 0$ (b) $a_{12}F_{12} + a_{22}F_{22} + a_{32}F_{32} \neq \Delta$

(c) $a_{12}F_{12} + a_{22}F_{22} + a_{32}F_{32} = \Delta$ (d) $a_{12}F_{12} + a_{22}F_{22} + a_{32}F_{32} = -\Delta$.

10. The value of the determinant $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 2 & 2 & 2 \end{vmatrix}$ is

(a) $x+y+z$ (b) $2(x+y+z)$ (c) 1 (d) 0.

11. Solve the following equation $\begin{vmatrix} 1 & 2 & 3 \\ 4 & x & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0.$

12. Evaluate the determinant $\begin{vmatrix} 1 & 3 & 9 \\ 3 & 9 & 1 \\ 9 & 1 & 3 \end{vmatrix}.$

13. Evaluate the determinant $\begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ a & b & 1+c \end{vmatrix}.$

14. Prove that

$$\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2.$$

15. Prove that one root of the equation is $x = 2$ and hence find the remaining roots

$$\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0.$$

Prove that [Q 16 to 20]

16. $\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & c+a+b \end{vmatrix} = 2(a+b)(b+c)(c+a).$

17. $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3.$

18. $\begin{vmatrix} y+z & x & y \\ z+x & z & x \\ x+y & y & z \end{vmatrix} = (x+y+z)(x-z)^2.$

19. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (b-c)(c-a)(a-b)(a+b+c).$

20. $\begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{b^2+c^2}{a} & a \\ b & b & \frac{c^2+a^2}{b} \end{vmatrix} = 4abc$ (Hint: using operation $R_1 \to cR_1, R_2 \to aR_2$ and $R_3 \to bR_3$)

21. If $a+b+c=0$ then solve the following equation

$$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0.$$

22. Prove that

$$\begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} = 9(a+b)b^2$$

23. If $p+q+r=0$ then prove that

$$\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix} = pqr\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

(Hint : L.H.S. $= pqr\left(a^3+b^3+c^3\right)-abc\left(p^3+q^3+r^3\right)$ $\because$ $p+q+r=0 \Rightarrow p^3+q^3+r^3=3pqr$

$\therefore$ L.H.S. $= pqr\left(a^3+b^3+c^3-3abc\right)=$ R.H.S.

24. Prove that

$$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(x-4)^2$$

IMPORTANT POINTS

1. Second order determinant $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$.

2. Third order determinant =

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix}$$

(From Sarrus diagram)

$$= (a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3) - (a_3b_2c_1 + b_3c_2a_1 + c_3a_2b_1)$$

3. Difference between matrix and determinant.
   - (i) There is no value of matrix whereas determinant has a unique value
   - (ii) Matrix can be of any order while determinant is always of order $n \times n$.
   - (iii) In determinant $|A| = |A^T|$ whereas in matrix $[A] \neq [A^T]$.

4. Minor of an element a_{ij} of the determinant of matrix A is the determinant obtained by deleting ith row and jth column and denoted by Aij.

5. Cofactor of element $a_{ij} = (-1)^{i+j}$ Minor

   $\Rightarrow$ Cofactor of $a_{ij} = a_{ij}$, when $i + j$ is even

   $= -(a_{ij}$ Minor of), when $i + j$ is odd

6. Expansion of determinant $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$
   - (i) in terms of minors $\Delta = a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13}$
   - (ii) in terms of co-factors $\Delta = a_{11}F_{11} + a_{12}F_{12} + a_{13}F_{13}$

7. For any square matrix A, the |A| satisfies following properties.
   - (i) If we interchange any two rows (or columns), then sign of determinant changes, but value remains unchanged.
   - (ii) If any two rows or any two columns are identical or propertional, then value of determinant is zero.
   - (iii) If we multiply each element of a row or a column of a determinant by constant k, then vlaue of determinant is multiplied by k.

(iv) Multiplying a determinant by k means multiply elements of only one row (or one column) by k.

(v) If elements of a row or a column in a determinant can be expressed as sum of two or more elements, then the given determinant can be expressed as sum of two or more determinants.

(vi) If each element of a row or a column of a determinant the equimultiples of corresponding elements of other rows or columns are added or subtracted, then value of determinant remains same.

(vii) If all rows are converted into columns or all columns converted in rows in any determinant the value of determinant remains same.

(viii) If any row or column contains all its element as zero then the value of determinant will be zero.

(ix) Value of Determinant of triangular matrices is equal to product of element of principal diagonal.

(x) Multiplication of determinant is done by row to column and row to row law.

Answers

Exericse 4.1

1. $\frac{-8}{3}$ 2. 1 : 2 3. $x=\frac{-5}{2},\ y=-3$ 4. $\frac{3}{2}$

5. (i) $A_{11}=-12,\ A_{21}=-16,\ A_{31}=-4$

$F_{11}=-12,\ F_{21}=16,\ F_{31}=-4,\quad 40$

(ii) $A_{11}=bc-f^2,\ A_{21}=hc-fg,\ A_{31}=hf-bg$

$F_{11}=bc-f^2,\ F_{21}=fg-hc,\ F_{31}=hf-bg;$

$abc+2fgh-af^2-bg^2-ch^2$

6. 15

Exercise 4.2

1. 2 : 3 2. Minor of $3=\begin{vmatrix}3 & 4\\ 8 & 9\end{vmatrix}$, Minor of $6=\begin{vmatrix}2 & 4\\ 1 & 9\end{vmatrix}$ and Minor of $5=\begin{vmatrix}2 & 3\\ 1 & 8\end{vmatrix}$.

3. 0 4. The sign of the determinant changes 6. $2a^3b^3c^3$

7. x = 4 10. = –8 11. 3

Miscellaneous Exercise . 4

1. (b) 2. (d) 3. (c) 4. (c) 5. (c) 6. (b) 7. (d)

8. (a) 9. (c) 10. (d) 11. 5 12. –676 13. $1+a+b+c$

15. 1, –3 21. $0, \pm\sqrt{\frac{3}{2}\left(a^2+b^2+c^2\right)}$

❑

05

Inverse of a Matrix and Linear Equations

5.01 Non-singular matrix

If the determinant of any square matrix A is non-zero i.e. $|A| \neq 0$ then matrix A is termed as non-singular matrix.

For Example : $A = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$ is a non-singular matrix

$\because$ $|A| = \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} = 10 - 12 = -2 \neq 0$

5.02 Singular matrix

If the determinant of any square matrix A is zero i.e. $|A| = 0$ then matrix A is termed as singular matrix.

For Example : $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is a singular matrix as $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 6 - 6 = 0$

5.03 Adjoint of a square matrix

The adjoint of a square matrix $A = [a_{ij}]_{m \times n}$ is defined as the transpose of the matrix $[F_{ij}]$ where F_{ij} is the cofactor of the element a_{ij} Adjoint of the matrix A is denoted by $adjA$.

i.e. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Cofactors of elements of $|A|$

$$[F_{ij}] = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix}$$

$$\therefore \quad [F_{ij}]^T = \begin{bmatrix} F_{11} & F_{21} & F_{31} \\ F_{12} & F_{22} & F_{32} \\ F_{13} & F_{23} & F_{33} \end{bmatrix} = AdjA$$

For Example : (i) Matrix $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}_{2 \times 2} \Rightarrow |A| = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}_{2 \times 2}$

$\therefore$ Elements of $|A|$ = cofactor of $a_{11}(=2), = |5| = 5$

cofactor of $a_{12}(=3), = -|4| = -4$

cofactor of $a_{21}(=4), = -|3| = -3$

cofactor of $a_{22}(=5), = |2| = 2$

$\therefore$ Matrix of cofactors of determinant $|A|$ is $B = \begin{bmatrix} 5 & -4 \\ -3 & 2 \end{bmatrix}_{2\times 2}$

$\therefore$ Adjoint matrix of matrix A is $adjA = B^T = \begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix}$

Note: The adjoint can be found directly of a 2×2 matrix by interchanging the diagonal elements and changing the sign of the off-digonal elements.

(ii) Matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 4 & 6 & 4 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 4 & 6 & 4 \end{vmatrix}$

$\therefore$ Cofactors of $a_{11}(=1)$ is $= \begin{vmatrix} -1 & 1 \\ 6 & 4 \end{vmatrix} = -10$

Cofactors of $a_{12}(=2)$ is $= -\begin{vmatrix} 3 & 1 \\ 4 & 4 \end{vmatrix} = -8$

Cofactors of $a_{13}(=0)$ is $= \begin{vmatrix} 3 & -1 \\ 4 & 6 \end{vmatrix} = 22$

Cofactors of $a_{21}(=3)$ is $= -\begin{vmatrix} 2 & 0 \\ 6 & 4 \end{vmatrix} = -8$

Cofactors of $a_{22}(=-1)$ is $= \begin{vmatrix} 1 & 0 \\ 4 & 4 \end{vmatrix} = 4$

Cofactors of $a_{23}(=1)$ is $= -\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} = 2$

Cofactors of $a_{31}(=4)$ is $= \begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} = 2$

Cofactors of $a_{32}(=6)$ is $= -\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = -1$

Cofactors of $a_{33}(=4)$ is $=\begin{vmatrix}1 & 2\\3 & -1\end{vmatrix}=-7$

$\therefore$ Matrix of cofactors $B=\begin{bmatrix}-10 & -8 & 22\\-8 & 4 & 2\\2 & -1 & -7\end{bmatrix}$

Adjoint of a matrix $adjA=B^T=\begin{bmatrix}-10 & -8 & 2\\-8 & 4 & -1\\22 & 2 & -7\end{bmatrix}$

5.04 Inverse of a matrix of invertible matrix

If A is a square matrix of order *m*, and if there exists another square matrix B of the same order *m*, such that $AB = I = BA$, then B is called the inverse matrix of A and it is denoted by A^{-1}. In that case A is said to be invertible.

Thus, $B = A^{-1} \Rightarrow AA^{-1} = I = A^{-1}A$, from the relation $AB = BA$ it is clear that A is the inverse of B i.e. if two matrices A and B are such that $AB = I = BA$ then matrix A and B are inverse matrices of each other.

5.05 Some Important Theorems

Thorem 1. A square matrix A is invertible if and only if A is non singular matrix i.e. $|A| \neq 0$

Proof : Let A be invertible matrix of order *n* and I be the identity matrix of order *n*. Then, there exists a square matrix B of order *n* such that $AB = BA = I$

$\Rightarrow \quad |AB| = |I|$

$\Rightarrow \quad |A|.|B| = 1 \qquad [\because |I| = 1]$

$\Rightarrow \quad |A| \neq 0$

let A be non singualr. Then $|A| \neq 0$,

$$A\cdot(adjA) = |A|I = (adjA).A$$

diving by $|A|$

$$A.\frac{adjA}{|A|} = I = \frac{(adjA)}{|A|}.A \qquad [\because |A| \neq O]$$

which is of the form $A.B = I = B.A$

Hence $\quad A^{-1} = B = \dfrac{adjA}{|A|}$

$\Rightarrow \quad A^{-1} = \dfrac{adjA}{|A|}$

Thus A is an invertible matrix.

Theorem 2. If A is a square matrix of order 3 then

$$A\cdot(adjA)=|A|I_3=(adjA)\cdot A, \qquad \text{where } I_3 \text{ is an identity matrix of order 3}$$

Proof : Let $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a third order matrix

$$\therefore \quad adjA=\begin{bmatrix} F_{11} & F_{21} & F_{31} \\ F_{12} & F_{22} & F_{32} \\ F_{13} & F_{23} & F_{33} \end{bmatrix}$$

$$\therefore \quad A.(adjA)=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} F_{11} & F_{21} & F_{31} \\ F_{12} & F_{22} & F_{32} \\ F_{13} & F_{23} & F_{33} \end{bmatrix}=\begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

$$=|A|\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=|A|I_3 \qquad (1)$$

similarly, we can prove that

$$(adjA)\cdot A=|A|I_3 \qquad (2)$$

Hence from (1) and (2), we have

$$A\cdot(adjA)=|A|I_3=(adjA)\cdot A$$

Note: If A and B are square matrices of order n then

(i) $A.(adjA)=|A|I_n=(adjA).A$

(ii) $adj(adjA)=|A|^{n-2}A$

(iii) $adjA^T=(adjA)^T$

(iv) $adj(AB)=adjB.adjA$

Theorem 3. Inverse matrix of non-singular matrix is unique.

Proof : Let $A=[aij]$ be a non-singular matrix of order *m*. If possible, let B and C be two inverse matrices of A. We shall show that B = C. We know that Since B is the inverse of A.

$AB=BA=I$ (1)

and $AC=CA=I$ (2)

then $AB=I \Rightarrow C(AB)=CI \Rightarrow (CA)B=CI$

$\Rightarrow$ $I\,B=CI$ [using (2)]

$\Rightarrow$ $B=C$

Thus Inverse of a non-singular matrix, is unique

Theorem 4. **If A and B are non-singular matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$.**

Proof : $\because$ A are B are non-signualr matrices

$\therefore$ multiplication AB is possible

$\because$ A are B are non-singular matrices

$\therefore$ $|A| \neq 0$ and $|B| \neq 0$

$\Rightarrow$ $|AB| = |A||B| \neq 0$

$\Rightarrow$ AB is non-singular square matrix.

let a matrix C be such that $C = B^{-1}A^{-1}$

$$\therefore \quad (AB)C = (AB)(B^{-1}A^{-1})$$

$$= A.(BB^{-1})A^{-1}$$

$$= A.I.A^{-1} \qquad [\because BB^{-1} = I]$$

$$= AA^{-1} = I$$

similarly $$C(AB) = (B^{-1}A^{-1})(AB)$$

$$= B^{-1}(A^{-1}A)B = B^{-1}IB \qquad [\because A^{-1}A = I]$$

$$= B^{-1}B = I$$

$$\therefore \quad (AB)C = C(AB)$$

$$\therefore \quad (AB)^{-1} = B^{-1}A^{-1}$$

Generalisation : $(ABC...XYZ)^{-1} = Z^{-1}Y^{-1}X^{-1}...B^{-1}A^{-1}$

Theorem 5. **If A is a non-singualr matrix then matrix A^T will also be non singualr matrix and $(A^T)^{-1} = (A^{-1})^T$**

Proof : $\because$ $|A| = |A^T|$ $\quad |A| \neq 0$ $\qquad (\because A$ is non-singular)

$\therefore$ $|A^T| \neq 0$

Thus matrix A^T is also non-singular

$\because$ A is non-singular $\Rightarrow A^{-1}$ exists such that

$AA^{-1} = I = A^{-1}A$

$\Rightarrow$ $(AA^{-1})^T = I^T = (A^{-1}A)^T$

$\Rightarrow$ $(A^{-1})^T A^T = I = A^T (A^{-1})^T \qquad [\because (AB)^T = B^T A^T]$

$\Rightarrow$ The inverse of A^T is $(A^{-1})^T$

$\Rightarrow$ $(A^T)^{-1} = (A^{-1})^T$

Illustrative Examples

Example 1. If matrix $A=\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}$ then

(i) Find the adjoint of A $(adjA)$

(ii) Prove that $A.(adjA)=|A|I_2=(adjA).A$

(iii) Find A^{-1}

(iv) Prove that $(A^{-1})^T=(A^T)^{-1}$

Solution : (i) $\because$ Given matrix $A=\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}$

$\therefore$ Cofactor of $a_{11}(=1)$ is $=4$

Cofactor of $a_{12}(=3)$ is $=-2$

Cofactor of $a_{21}(=2)$ is $=-3$

Cofactor of $a_{22}(=4)$ is $=1$

$$\therefore \quad adjA=\begin{bmatrix}4 & -2\\ -3 & 1\end{bmatrix}^T=\begin{bmatrix}4 & -3\\ -2 & 1\end{bmatrix} \qquad (1)$$

(ii) $|A|=\begin{vmatrix}1 & 3\\ 2 & 4\end{vmatrix}=4-6=-2.$

$$\therefore \quad A\cdot(adjA)=\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}\cdot\begin{bmatrix}4 & -3\\ -2 & 1\end{bmatrix}=\begin{bmatrix}4-6 & -3+3\\ 8-8 & -6+4\end{bmatrix}$$

$$=\begin{bmatrix}-2 & 0\\ 0 & -2\end{bmatrix}=-2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=|A|I_2 \qquad (2)$$

$$\therefore \quad (adjA).A=\begin{bmatrix}4 & -3\\ -2 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}=\begin{bmatrix}4-6 & 12-12\\ -2+2 & -6+4\end{bmatrix}$$

$$=\begin{bmatrix}-2 & 0\\ 0 & -2\end{bmatrix}=-2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=|A|I_2. \qquad (3)$$

from (2) and (3) $\quad A\cdot(adjA)=|A|I_2=(adjA)\cdot A$ $\quad$ Hence Proved.

$$\text{(iii)} \quad A^{-1}=\frac{adjA}{|A|}=\frac{-1}{2}\begin{bmatrix}4 & -3\\ -2 & 1\end{bmatrix}=\begin{bmatrix}-2 & 3/2\\ 1 & -1/2\end{bmatrix} \qquad (4)$$

(iv) $\because \quad A^{-1} = \begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix}$

$\therefore \quad (A^{-1})^T = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$ (5)

and $A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow |A^T| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$

$\therefore \quad (A^T)^{-1}$ Exists.

$$adj\left(A^T\right) = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\because \quad \left(A^T\right)^{-1} = \frac{adj\left(A^T\right)}{|A^T|} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \qquad (6)$$

from (5) and (6) $\quad (A^{-1})^T = (A^T)^{-1}$. Hence Proved.

Example 2. If matrix $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ then find A^{-1}.

Solution : $\because \quad A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$\therefore \quad |A| = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$.

$\therefore \quad |A| \neq 0$ i.e. A^{-1} exists

$$adjA = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{adjA}{|A|} = \frac{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}{1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Example 3. If matrix $A=\begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\\3 & 1 & 2\end{bmatrix}$ then find A^{-1} and prove that $A^{-1}A=I_3$.

Solution : Given matrix $A=\begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\\3 & 1 & 2\end{bmatrix}$

$$\therefore \quad |A|=\begin{vmatrix}1 & 2 & 3\\2 & 3 & 1\\3 & 1 & 2\end{vmatrix}=1(6-1)-2(4-3)+3(2-9)=5-2-21=-18\neq 0.$$

$\therefore \quad A^{-1}$ exists

$$\text{Now} \quad adjA=\begin{bmatrix}5 & -1 & -7\\-1 & -7 & 5\\-7 & 5 & -1\end{bmatrix}^T=\begin{bmatrix}5 & -1 & -7\\-1 & -7 & 5\\-7 & 5 & -1\end{bmatrix}$$

$$\therefore \quad A^{-1}=\frac{adjA}{|A|}=-\frac{1}{18}\begin{bmatrix}5 & -1 & -7\\-1 & -7 & 5\\-7 & 5 & -1\end{bmatrix}$$

$$\therefore \quad A^{-1}A=-\frac{1}{18}\begin{bmatrix}5 & -1 & -7\\-1 & -7 & 5\\-7 & 5 & -1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\\3 & 1 & 2\end{bmatrix}$$

$$=-\frac{1}{18}\begin{bmatrix}5-2-21 & 10-3-7 & 15-1-14\\-1-14+15 & -2-21+5 & -3-7+10\\-7+10-3 & -14+15-1 & -21+5-2\end{bmatrix}$$

$$=-\frac{1}{18}\begin{bmatrix}-18 & 0 & 0\\0 & -18 & 0\\0 & 0 & -18\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}=I_3.$$

Example 4. If matrix $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$ then prove that $(AB)^{-1} = B^{-1}A^{-1}$.

Solution : Here $|A| = \begin{vmatrix} 3 & 7 \\ 2 & 5 \end{vmatrix} = 1 \neq 0$ (1)

$\therefore$ A^{-1} exists

and $|B| = \begin{vmatrix} 6 & 8 \\ 7 & 9 \end{vmatrix} = -2 \neq 0$ (2)

$\therefore$ B^{-1} exists

$\therefore$ $AB = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 18+49 & 24+63 \\ 12+35 & 16+45 \end{bmatrix}$

$= \begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix}$ (3)

$\therefore$ $(AB)^{-1} = -\frac{1}{2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$ (4)

$A^{-1} = \frac{1}{1}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$ (5)

and $B^{-1} = -\frac{1}{2}\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$ (6)

$\because$ $B^{-1}A^{-1} = -\frac{1}{2}\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

$= -\frac{1}{2}\begin{bmatrix} 45+16 & -63-24 \\ -35-12 & 49+18 \end{bmatrix}$

$= -\frac{1}{2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$ (7)

$\therefore$ from (4) and (7), $(AB)^{-1} = B^{-1}A^{-1}$. Hence Proved.

Example 5. If matrix $A=\begin{bmatrix}2 & 3\\1 & 2\end{bmatrix}$ then prove that $A^2-4A+I=0$, where $O=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ and find A^{-1}.

Solution : $\because \quad A=\begin{bmatrix}2 & 3\\1 & 2\end{bmatrix}$

$$\therefore \quad A^2=\begin{bmatrix}2 & 3\\1 & 2\end{bmatrix}\begin{bmatrix}2 & 3\\1 & 2\end{bmatrix}=\begin{bmatrix}4+3 & 6+6\\2+2 & 3+4\end{bmatrix}=\begin{bmatrix}7 & 12\\4 & 7\end{bmatrix}$$

$$\therefore \quad A^2-4A+I=\begin{bmatrix}7 & 12\\4 & 7\end{bmatrix}-4\begin{bmatrix}2 & 3\\1 & 2\end{bmatrix}+\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}7 & 12\\4 & 7\end{bmatrix}+\begin{bmatrix}-8 & -12\\-4 & -8\end{bmatrix}+\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}7-8+1 & 12-12+0\\4-4+0 & 7-8+1\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}=0. \text{ Here } |A|=\begin{vmatrix}2 & 3\\1 & 2\end{vmatrix}=4-3=1\neq 0.$$

$\therefore \quad A^{-1}$ Exists

Now $A^2-4A+I=0 \qquad \Rightarrow A^2-4A=-I \qquad \Rightarrow A(A-4I)=-I$

$\Rightarrow A^{-1}A(A-4I)=-A^{-1}I \qquad \Rightarrow (A^{-1}A)(A-4I)=-A^{-1} \qquad \Rightarrow I(A-4I)=-A^{-1}$

$\Rightarrow A-4I=-A^{-1} \qquad \Rightarrow A^{-1}=4I-A$

$$\Rightarrow A^{-1}=4\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}-\begin{bmatrix}2 & 3\\1 & 2\end{bmatrix}=\begin{bmatrix}4 & 0\\0 & 4\end{bmatrix}-\begin{bmatrix}2 & 3\\1 & 2\end{bmatrix}=\begin{bmatrix}2 & -3\\-1 & 2\end{bmatrix}.$$

Exercise 5.1

1. For what value of x is the matrix $\begin{bmatrix}1 & -2 & 3\\1 & 2 & 1\\x & 2 & -3\end{bmatrix}$ singular?

2. If matrix A is $\begin{bmatrix}1 & -1 & 2\\3 & 0 & -2\\1 & 0 & 3\end{bmatrix}$ then find $adjA$ and prove that $A\cdot(adjA)=|A|I_3=(adjA)\cdot A$.

3. Find the non-singualr matrix of the following:

(i) $\begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$

4. If matrix $A = F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then find A^{-1} and prove that

(i) $A^{-1}A = I_3$ (ii) $A^{-1} = F(-\alpha)$ (iii) $A \cdot (adjA) = |A| I = (adjA) \cdot A$

5. If $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$ then prove that $A^{-1} = A^T$

6. If matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ the prove that $A^{-1} = A^3$

7. If $A = \begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ then find $(AB)^{-1}$.

8. If $A = \begin{bmatrix} 1 & \tan\alpha \\ -\tan\alpha & 1 \end{bmatrix}$ then prove that $A^T A^{-1} = \begin{bmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{bmatrix}$.

9. Prove that the matrix $A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$ satisfies the equation $A^2 - 6A + 7 = 0$ and find A^{-1} .

10. If matrix $A = \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix}$ then prove that $A^2 + 4A - 42I = 0$ then find A^{-1} .

5.06 Applications of Determinants

1. Area of a triangle

If the coordinates of vertices of a triangle are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) then we know that

area of triangle $\Delta = \frac{1}{2}\left[x_1(y_2 - y_1) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]$ (1)

and $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = x_1\begin{vmatrix} y_2 & 1 \\ y_3 & 1 \end{vmatrix} - x_2\begin{vmatrix} y_1 & 1 \\ y_3 & 1 \end{vmatrix} + x_3\begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix}$ (Expanding, along first column)

$$= x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \quad (2)$$

from (1) and (2) $$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Thus area of triangle is $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

Note: Since area is always positive hence the value of the determinant is always taken positive.

For Example : Find the area of the triangle if the vertices are $A(-3, 3)$, $B(2, 3)$ and $C(2, -2)$.

Solution : $$\Delta = \frac{1}{2}\begin{vmatrix} -3 & 3 & 1 \\ 2 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left\{-3\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 2 & -2 \end{vmatrix}\right\}$$

$$= \frac{1}{2}\{-3(3+2) - 3(2-2) + 1(-4-6)\}$$

$$= \frac{1}{2}(-15 + 0 - 10)$$

$$= \frac{-25}{2} = -12.5 \text{ sq. Units}$$

$\because$ Area is positive therefore $\Delta = 12.5$ sq. units

2. Condition of collinearity of three points

If the points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear then the area of triangle ABC is zero

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

For Example : Points $A(3,-2)$, $B(5,2)$ and $C(8,8)$ are collinear hence

$$\Delta = \frac{1}{2}\begin{vmatrix} 3 & -2 & 1 \\ 5 & 2 & 1 \\ 8 & 8 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\{3(2-8)+2(5-8)+1(40-16)\}$$

$$= \frac{1}{2}(-18-6+24) = 0$$

3. Equation of a line passing through two points

Let there be two points $A(x_1, y_1)$ and $B(x_2, y_2)$ and let $P(x, y)$, AB lies on a line passing through AB then P, A and B are collinear, if

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

which is the required equation.

For Example : Equation of line passing through $A(3,1)$ and $B(9,3)$ is $\begin{vmatrix} x & y & 1 \\ 3 & 1 & 1 \\ 9 & 3 & 1 \end{vmatrix} = 0$

$\Rightarrow$ $x(1-3)-y(3-9)+1(9-9)=0$

$\Rightarrow$ $-2x+6y=0$

$\Rightarrow$ $x-3y=0$

5.07 Solution of system of linear equations

If a given system of equations

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

$b_1 = b_2 = b_3 = 0$ then it is said to be homogeneous otherwise it is called non-homogeneous

Here we shall find the solution of non-homogenous system of linear equations.

1. Cramer's Rule:

(i) Solution of system of linear equations of two variables

System of linear equation with two variables

$$a_1x + b_1y = c_1 \qquad (1)$$

$$a_2x + b_2y = c_2 \qquad (2)$$

solving through Cramer's rule

$$x = \frac{\Delta_1}{\Delta}, \qquad y = \frac{\Delta_2}{\Delta}$$

or $$\frac{x}{\Delta_1} = \frac{y}{\Delta_2} = \frac{1}{\Delta}, \ \Delta \neq 0 \qquad \text{(Symmetric form)}$$

where $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

Proof : $\because$ $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

$\therefore$ $x\Delta = x\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1x & b_1 \\ a_2x & b_2 \end{vmatrix}$

$\Rightarrow$ $x\Delta = \begin{vmatrix} a_1x + b_1y & b_1 \\ a_2x + b_2y & b_2 \end{vmatrix} = \Delta_1$ (operation $C_1 \to C_1 + yC_2$)

$\Rightarrow$ $x\Delta = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = \Delta_1$ (using equation (1) and (2))

similarly $y\Delta = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = \Delta_2$

$$x = \frac{\Delta_1}{\Delta} \text{ and } y = \frac{\Delta_2}{\Delta}, \text{ where } \Delta \neq 0$$

Special case : This equation represents two equations of straight line

(A) If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ then solution of the equation is unique and the equation is consistent and independent

(B) If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ then there is no solution and the equation is inconsistent.

(C) If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ then there are infinite solutions and the equation is consistent but not independent

(ii) Solution of system of linear equation for three variables

System of equations with three variables

$$a_1x + b_1y + c_1z = d_1 \quad (1)$$

$$a_2x + b_2y + c_2z = d_2 \quad (2)$$

$$a_3x + b_3y + c_3z = d_3 \quad (3)$$

Solving by Cramer's rule $\quad x = \frac{\Delta_1}{\Delta}, \ y = \frac{\Delta_2}{\Delta}, \ z = \frac{\Delta_3}{\Delta}$

or $\quad \frac{x}{\Delta_1} = \frac{y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{1}{\Delta} \quad ; \Delta \neq 0$ [symmetric form]

where $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ and $\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Proof : $\because \quad \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$\therefore \quad x\Delta = x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

or $\quad x\Delta = \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix}$ $\quad (C_1 \to C_1 + yC_2 + zC_3)$

or $\quad x\Delta = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = \Delta_1$ $\quad$ [using equation (1), (2) and (3)]

Similarly $\quad y\Delta = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} = \Delta_2$ and $z\Delta = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = \Delta_3$

$\therefore \quad x = \frac{\Delta_1}{\Delta}, \ y = \frac{\Delta_2}{\Delta}$ and $z = \frac{\Delta_3}{\Delta}$ if $\Delta \neq 0$

Special case :

(i) If $\Delta \neq 0$ then equation is consistent and the solution is unique.

(ii) If $\Delta = 0$ and $\Delta_1 = \Delta_2 = \Delta_3 = 0$ then system of equations can be consistent or inconsistent, if it is consistent then the solution are infinite.

(iii) If $\Delta = 0$ and amongst $\Delta_1, \Delta_2, \Delta_3$ any one is non-zero then equations are inconsistent with no solution.

2. Solution of system of linear equations using matrix method:

Consider the system of equations

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{aligned}\right\} \quad (1)$$

The above equations can be written in a matrix form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad (2)$$

or $$AX = B \quad (3)$$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

If $|A| \neq 0$ then from equation (3)

$$AX = B$$

$\Rightarrow$ $A^{-1}(AX) = A^{-1}B$

$\Rightarrow$ $(A^{-1}A)X = A^{-1}B$

$\Rightarrow$ $I\,X = A^{-1}B$

$\Rightarrow$ $X = A^{-1}B$

Note: (i) $|A| \neq 0$, then A^{-1} exists

(ii) $|A| = 0$, then A^{-1} does not exist, that does not mean the equation cannot be solved.

Example : $x + 3y = 5$

$2x + 6y = 10,$

Here $|A| = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 0$ but it will have infinite solutions.

Illustrative Examples

Example 6. Find the area of the triangle whose vertices are $A(2,3), B(-5,4)$ and $C(4,3)$.

Solution : Area of triangle ABC

$$\Delta = \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ -5 & 4 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\{2(4-3)+5(3-3)+4(3-4)\}$$

$$= \frac{1}{2}(2+0-4)$$

$$= -1$$

= 1 (numerical value) square units

Example 7. If points $(x,-2), (5,2), (8,8)$ are collinear then find the value of x.

Solution : $\because$ Given points $(x,-2), (5,2)$ and $(8,8)$ are collinear

$$\therefore \quad \begin{vmatrix} x & -2 & 1 \\ 5 & 2 & 1 \\ 8 & 8 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad x(2-8)+2(5-8)+1(40-16)=0$$

$$\Rightarrow \quad -6x-6+24=0$$

$$\Rightarrow \quad -6x+18=0$$

$$\Rightarrow \quad x=3.$$

Example 8. Prove that $[bc, a(b+c)]$, $[ca, b(c+a)]$ and $[ab, c(a+b)]$ are collinear.

Solution : Three points are collinear

$$\therefore \quad \begin{vmatrix} bc & a(b+c) & 1 \\ ca & b(c+a) & 1 \\ ab & c(a+b) & 1 \end{vmatrix} = \begin{vmatrix} bc+ab+ca & a(b+c) & 1 \\ ca+bc+ab & b(c+a) & 1 \\ ab+ca+bc & c(a+b) & 1 \end{vmatrix} \qquad (C_1 \to C_1 + C_2)$$

$$= (ab+bc+ca)\begin{vmatrix} 1 & a(b+c) & 1 \\ 1 & b(c+a) & 1 \\ 1 & c(a+b) & 1 \end{vmatrix}$$

$$= (ab+bc+ca).0 \qquad (\because \text{two equal columns})$$

$$= 0$$

Thus given points are collinear

Example 9. Find the equation of line joining the points $A(4, 3)$ and $B(-5, 2)$ also find the value of k if the area fo the triangle ABC is 2 Sq. units where, $C(k, 0)$.

Solution : Let $P(x, y)$ be any point on AB then area of triangle ABC = 0

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} 4 & 3 & 1 \\ -5 & 2 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \frac{1}{2}\left[4(2-y)-3(-5-x)+1(-5y-2x)\right]=0$$

$$\Rightarrow \quad 8-4y+15+3x-5y-2x=0$$

$$\Rightarrow \quad x-9y+23=0.$$

which is the required equation of AB

Now area of triangle ABC = 2 Sq. units

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} 4 & 3 & 1 \\ -5 & 2 & 1 \\ k & 0 & 1 \end{vmatrix} = \pm 2$$

$$\Rightarrow \quad \frac{1}{2}\left[4(2-0)-3(-5-k)+1(0-2k)\right]=\pm 2$$

$$\Rightarrow \quad \frac{1}{2}\left[8+15+3k-2k\right]=\pm 2$$

$$\Rightarrow \quad 23+k=\pm 4$$

$$\Rightarrow \quad k=\pm 4-23$$

$$\Rightarrow \quad k=-19, -27$$

Example 10. If the solution of two below given equation is possible then solve using the Cramer's rule.

(i) $2x-3y=3$
$2x+3y=9$

(ii) $x+2y=5$
$2x+4y=10$

Solution : (i)

$$2x-3y=3$$
$$2x+3y=9$$

Here $\Delta = \begin{vmatrix} 2 & -3 \\ 2 & 3 \end{vmatrix} = 6+6=12 \neq 0$, $\Delta_1 = \begin{vmatrix} 3 & -3 \\ 9 & 3 \end{vmatrix} = 9+27=36 \neq 0$ and $\Delta_2 = \begin{vmatrix} 2 & 3 \\ 2 & 9 \end{vmatrix} = 18-6=12 \neq 0$

$\because$ $\Delta \neq 0,\ \Delta_1 \neq 0,\ \Delta_2 \neq 0$

$\therefore$ Equation is consistent and independent so its solution is finite.

Now using Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{36}{12} = 3, \qquad y = \frac{\Delta_2}{\Delta} = \frac{12}{12} = 1$$

$\Rightarrow$ $x = 3,\ y = 1$.

(ii) $x + 2y = 5$
$2x + 4y = 10$

$\Rightarrow$ $\Delta = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0$, $\Delta_1 = \begin{vmatrix} 5 & 2 \\ 10 & 4 \end{vmatrix} = 20 - 20 = 0$ and $\Delta_2 = \begin{vmatrix} 1 & 5 \\ 2 & 10 \end{vmatrix} = 10 - 10 = 0$

$\because$ $\Delta = 0,\ \Delta_1 = 0,\ \Delta_2 = 0$

$\therefore$ Equation is inconsistent so its solution is infinite.

Let $y = k$ then $x + 2k = 5 \Rightarrow x = 5 - 2k$ therefore $x = 5 - 2k,\ y = k$ are the solutions where k is a real number

Example 11. Prove that the system of equations is inconsistent with no solution.

$$x + y + z = 2$$
$$x + 2y + 3z = 5$$
$$2x + 3y + 4z = 11.$$

Solution : Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix} = 1(8-9) - 1(4-6) + 1(3-4) = -1 + 2 - 1 = 0$.

$$\Delta_1 = \begin{vmatrix} 2 & 1 & 1 \\ 5 & 2 & 3 \\ 11 & 3 & 4 \end{vmatrix} = 2(8-9) - 1(20-33) + 1(15-22) = -2 + 13 - 7 = 4 \neq 0.$$

$$\Delta_2 = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 5 & 3 \\ 2 & 11 & 4 \end{vmatrix} = 1(20-23) - 2(4-6) + 1(11-10) = -13 + 4 - 1 = -8 \neq 0.$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 3 & 11 \end{vmatrix} = 1(22-15) - 1(11-10) + 2(3-4) = 7 - 1 - 2 = 4 \neq 0.$$

$\because$ $\Delta = 0$ and $\Delta_1 \neq 0, \Delta_2 \neq 0, \Delta_3 \neq 0$.

$\therefore$ system of equations is inconsistent with no solution.

Example 12. Solve the following system of equations using Cramer's rule

$$x+y+z=9$$
$$2x+5y+7z=52$$
$$2x+y-z=0$$

Solution : Here $\Delta=\begin{vmatrix}1 & 1 & 1\\ 2 & 5 & 7\\ 2 & 1 & -1\end{vmatrix}=1(-5-7)-1(-2-14)+1(2-10)=-12+16-8=-4\neq 0$.

$$\Delta_1=\begin{vmatrix}9 & 1 & 1\\ 52 & 5 & 7\\ 0 & 1 & -1\end{vmatrix}=9(-5-7)-1(-52-0)+1(52-0)=-108+52+52=-4\neq 0.$$

$$\Delta_2=\begin{vmatrix}1 & 9 & 1\\ 2 & 52 & 7\\ 2 & 0 & -1\end{vmatrix}=1(-52-0)-9(-2-14)+(0-104)=-52+144-104=-12\neq 0.$$

$$\Delta_3=\begin{vmatrix}1 & 1 & 9\\ 2 & 5 & 52\\ 2 & 1 & 0\end{vmatrix}=1(0-52)-1(0-104)+9(2-10)=-52+104-72=-20\neq 0.$$

sign Cramer Rule

$$x=\frac{\Delta_1}{\Delta}=\frac{-4}{-4}=1,\qquad y=\frac{\Delta_2}{\Delta}=\frac{-12}{-4}=3,\qquad z=\frac{\Delta_3}{\Delta}=\frac{-20}{-4}=5$$

$\therefore$ $x=1,\ y=3,\ z=5$.

Example 13. Solve the system of equation using matrix inverse method.

$$5x-3y=2$$
$$x+2y=3.$$

Solution : Matrix form of the equation

$$\begin{bmatrix}5 & -3\\ 1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}2\\ 3\end{bmatrix}$$

i.e. $AX=B$

where $A=\begin{bmatrix}5 & -3\\ 1 & 2\end{bmatrix},\ x=\begin{bmatrix}x\\ y\end{bmatrix}$ तथा $B=\begin{bmatrix}2\\ 3\end{bmatrix}$

$\because$ $|A|=\begin{vmatrix}5 & -3\\ 1 & 2\end{vmatrix}=10+3=13\neq 0$

$\therefore$ A^{-1} exists

$$A^{-1} = \frac{adjA}{|A|} = \frac{1}{13}\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$$

$\Rightarrow$ $$X = A^{-1}B = \frac{1}{13}\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} 4+9 \\ -2+15 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 13 \\ 13 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\Rightarrow$ $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\Rightarrow$ $x = 1,\ y = 1$.

Example 14. Write the following system of equations in matrix form

$$2x - y + 3z = 9$$
$$x + y + z = 6$$
$$x - y + z = 2.$$

If $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ then find A^{-1} and solve the equations.

Solution : $\because AX = B$, where $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$

Matrix form of the equation is

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$$

here $$|A| = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2(1+1) + 1(1-1) + 3(-1-1) = 4 + 0 - 6 = -2 \neq 0$$

$\therefore$ A^{-1} exists

$\therefore$ $$A^{-1} = \frac{adjA}{|A|} = -\frac{1}{2}\begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1/2 & -1/2 \\ 1 & 1/2 & -3/2 \end{bmatrix}$$

$\because \quad X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 18-12-8 \\ 0-6+2 \\ -18+6+6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow \quad x = 1,\ y = 2,\ z = 3.$

Example 15. If $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$ then find AB and solve the following equations

$x - y = 3;\quad 2x + 3y + 4z = 17,\quad y + 2z = 7$.

Solution :

$$AB = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 6I_3$$

$$\Rightarrow \quad A \cdot \left(\frac{1}{6}B\right) = I_3$$

$$\Rightarrow \quad A^{-1} = \frac{1}{6}B = \frac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix} \qquad (1)$$

Now matrix form of the given equation

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$\Rightarrow \quad AX = C$

$\Rightarrow \quad X = A^{-1}C$

$\Rightarrow \qquad X = \frac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}.\begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$

$\Rightarrow \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6+34-28 \\ -12+34-28 \\ 6-17+35 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 12 \\ -6 \\ 24 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

$\Rightarrow \qquad x = 2, \quad y = -1, \quad z = 4.$

Example 16. Solve the following system of equations

$$\begin{bmatrix} 3 & 0 & 3 \\ 2 & 1 & 0 \\ 4 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 2y \\ z \\ 3y \end{bmatrix}$$

Solution : Given system of equation is

$$\begin{bmatrix} 3 & 0 & 3 \\ 2 & 1 & 0 \\ 4 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 2y \\ z \\ 3y \end{bmatrix}$$

$\Rightarrow \qquad \begin{bmatrix} 3x+3z \\ 2x+y \\ 4x+2z \end{bmatrix} = \begin{bmatrix} 8+2y \\ 1+z \\ 4+3y \end{bmatrix}$

$\therefore \qquad \left.\begin{array}{rcr} 3x+3z=8+2y & \Rightarrow & 3x-2y+3z=8 \\ 2x+y=1+z & \Rightarrow & 2x+y-z=1 \\ 4x+2z=4+3y & \Rightarrow & 4x-3y+2z=4 \end{array}\right\} \qquad (1)$

Matrix form of the given equations is (1)

$$\begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

i.e. $\qquad AX = B$

$\Rightarrow \qquad X = A^{-1}B$

$$= -\frac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix} \qquad \left[\because A^{-1} = \frac{adjA}{|A|}\right]$$

$$= -\frac{1}{17}\begin{bmatrix} -8-5-4 \\ -64-6+36 \\ -80+1+28 \end{bmatrix} = -\frac{1}{17}\begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \qquad x = 1,\ y = 2,\ z = 3\,.$$

Exercise 5.2

1. Find the area of triangle using the determinats whose vertices are:
   (i) (2, 5), (–2, –3) and (6, 0)
   (ii) (3, 8), (2, 7) and (5, –1)
   (iii) (0, 0), (5, 0) and (3, 4)
2. Usign determinants find the area of the triangle with vertices (1, 4), (2, 3) and (–5, –3), are the given points collinear?
3. Find the value of *k* if the area of triangle is 35 Sq. units and the vertices are (*k*, 4) (2, –6) and (5, 4).
4. Usign determinants find the value of *k* if the points (*k*, 2 –2 *k*), (–*k* + 1, 2*k*) and (–4 – *k*, 6 –2*k*) are collinear.
5. If points (3, –2), (*x*, 2) and (8, 8) are collinear then find the vlaue of *x* using determinant.
6. Using determinants, find the equation of line passing through the points (3, 1) and (9, 3) and also find the area of the triangle if the third point is (–2, –4).
7. Solve the following system of equations using Cramer's rule.

   (i) $2x+3y=9$, $3x-2y=7$

   (ii) $2x-7y-13=0$, $5x+6y-9=0$
8. Prove that the following system of equations are incosnstent:

   (i) $3x+y+2z=3$, $2x+y+3z=5$, $x-2y-z=1$

   (ii) $x+6y+11=0$, $3x+20y-6z+3=0$, $6y-18z+1=0$

9 Solve the equations using Cramer's rule:

   (i) $x+2y+4z=16$, $4x+3y-2z=5$, $3x-5y+z=4$

   (ii) $2x+y-z=0$, $x-y+z=6$, $x+2y+z=3$

10. Solve the equations using determinants :

(i) $6x+y-3z=5$
$x+3y-2z=5$
$2x+y+4z=8$

(ii) $\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4$
$\frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1$
$\frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2$

11. Solve the equations using matrix method:

(i) $2x-y=-2$
$3x+4y=3$

(ii) $5x+7y+2=0$
$4x+6y+3=0$

(iii) $x+y-z=1$
$3x+y-2z=3$
$x-y-z=-1$

(iv) $6x-12y+25z=4$
$4x+15y-20z=3$
$2x+18y+15z=10$

12. If $A=\begin{bmatrix}1 & -2 & 0\\ 2 & 1 & 3\\ 0 & -2 & 1\end{bmatrix}$ then find A^{-1} and solve the system of equations:

$x-2y=10, \quad 2x+y+3z=8, \quad -2y+z=7.$

13. Find the product of matrices $\begin{bmatrix}-4 & 4 & 4\\ -7 & 1 & 3\\ 5 & -3 & -1\end{bmatrix}$ and $\begin{bmatrix}1 & -1 & 1\\ 1 & -2 & -2\\ 2 & 1 & 3\end{bmatrix}$ and solve the system of equations using the above product

$$x-y+z=4$$
$$x-2y-2z=9$$
$$2x+y+3z=1\cdot$$

14. Find the inverse of the matrix $\begin{bmatrix}1 & -1 & 1\\ 2 & 1 & -3\\ 1 & 1 & 1\end{bmatrix}$ and with the help of this solve the system of equations

$$\begin{bmatrix}1 & 0 & 1\\ 2 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{2}\begin{bmatrix}2y\\ 6z\\ -2x\end{bmatrix}+2\begin{bmatrix}2\\ 0\\ 1\end{bmatrix}$$

15. If the side of an equilateral triangle is a and vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) then prove that $\begin{vmatrix}x_1 & y_1 & 2\\ x_2 & y_2 & 2\\ x_3 & y_3 & 2\end{vmatrix}^2=3a^4$

Miscellaneous Exercise - 5

1. If $A=\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}$ then find A^{-1}.

2. If $A=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}$ then find A^{-1}.

3. If Matrix $A=\begin{bmatrix}1 & -2 & 3\\ 1 & 2 & 1\\ x & 2 & -3\end{bmatrix}$ is a singualr matrix then find the vlaue of x

4. Solve the equations using Cramer's rule

(i) $2x-y=17$
$3x+5y=6.$

(ii) $3x+ay=4$
$2x+ay=2, \quad a\neq 0$

(iii) $x+2y+3z=6$
$2x+4y+z=7$
$3x+2y+9z=14.$

5. Solve the equations using Cramer's rule and show that the equation are inconsistent:

(i) $2x-y=5$
$4x-2y=7$

(ii) $x+y+z=1$
$x+2y+3z=2$
$3x+4y+5z=3$

6. Find the matrix A of order 2 if

$$A\begin{bmatrix}1 & -2\\ 1 & 4\end{bmatrix}=\begin{bmatrix}6 & 0\\ 0 & 6\end{bmatrix}$$

7. If $A=\begin{bmatrix}-8 & 5\\ 2 & 4\end{bmatrix}$ then prove that that $A^2+4A-42I=0$ and using this find A^{-1}.

8. If $A=\begin{bmatrix}2 & 3\\ 5 & -2\end{bmatrix}$ then prove that $A^{-1}=\frac{1}{19}A$.

9. If $A=\begin{bmatrix}1 & 3 & 3\\ 1 & 4 & 3\\ 1 & 3 & 4\end{bmatrix}$ then find A^{-1} and show that $A^{-1}A=I_3$.

10. If $A=\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}$ then prove that $A^2-4A-5I=0$ and using this find A^{-1}.

11. Solve the following system of equations using the matrix method.

(i) $5x-7y=2$
$7x-5y=3$

(ii) $3x+y+z=3$
$2x-y-z=2$
$-x-y+z=1$

(iii) $x+2y-2z+5=0$
$-x+3y+4=0$
$-2y+z-4=0$

12. Find the area triangle ABC for the vertices given below:

(i) $A(-3,5), B(3,-6), C(7,2)$

(ii) $A(2,7)\ B(2,2)\ C(10,8)$

13. If the points $(2,-3),(\lambda,-2)$ and $(0,5)$ are collinear then find the vlaue of λ.

14. Find the matrix A where

$$\begin{bmatrix}1&2\\2&3\end{bmatrix}A\begin{bmatrix}4&7\\3&5\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

15. If $A=\begin{bmatrix}1&1&1\\1&2&-3\\2&-1&3\end{bmatrix}$ then find A^{-1} and using this solve the equations:

$$x+y+2z=0,\quad x+2y-z=9,\quad x-3y+3z=-14$$

16. If $A=\begin{bmatrix}a&b\\c&\frac{1+bc}{a}\end{bmatrix}$ then find A^{-1} and solve that $aA^{-1}=\left(a^2+bc+1\right)I-aA$.

17. Solve the system of equations using determinants

$$x+y+z=1$$
$$ax+by+z=k$$
$$a^2x+b^2y+c^2z=k^2.$$

18. If $A=\begin{bmatrix}1&2&-3\\2&3&2\\3&-3&-4\end{bmatrix}$ then find A^{-1} then using this solve the following system of equations

$$x+2y-3z=-4,\quad 2x+3y+2z=2,\quad 3x-3y-4z=11.$$

19. If $\begin{bmatrix}1&-4\\3&-2\end{bmatrix}X=\begin{bmatrix}-16&-6\\7&2\end{bmatrix}$ then find the value of X.

20. If the system of equations have infinite solutions then find the vlaues of a and b

$$2x+y+az=4$$
$$bx-2y+z=-2$$
$$5x-5y+z=-2.$$

Important Points

1. **Singular matrix:** A Square matrix A, whose $|A|=0$
2. **Non-Singular matrix:** A Square matrix A, whose $|A|\neq 0$
3. **Adjoint of a matrix:** Adjoint of a matrix is a transpose of a matrix, obtained by co-factors of elements of |A| adjoint of the matrix A is written as *adjA*
4. **Inverse of a matrix:** If a square matrix is non-singualr i.e. $|A|\neq 0$ then $A^{-1}=\frac{adjA}{|A|}$
5. **Important theorems:**

   (i) For a matrix A to be non-singualr $|A|\neq 0$

   (ii) If A is a matrix of order n then $A.(adjA)=|A|\,I_n=(adjA).A$

   (iii) $(AB)^{-1}=B^{-1}A^{-1}$

   (iv) $(A^T)^{-1}=(A^{-1})^T$
6. For variables x, y, z the system of equations are

$$\left.\begin{aligned} a_{11}x+a_{12}y+a_{13}z=b_1\\ a_{21}x+a_{22}y+a_{23}z=b_2\\ a_{31}x+a_{32}y+a_{33}z=b_3 \end{aligned}\right\} \qquad (1)$$

the solutions can be found out using the determinants or matrix method

(i) Cramer's rule usign determinants

For the above equation (1)

$$\Delta=\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}, \Delta_1=\begin{vmatrix} b_1 & a_{12} & a_{13}\\ b_2 & a_{22} & a_{23}\\ b_3 & a_{32} & a_{33}\end{vmatrix}, \quad \Delta_2=\begin{vmatrix} a_{11} & b_1 & a_{13}\\ a_{21} & b_2 & a_{23}\\ a_{31} & b_3 & a_{33}\end{vmatrix} \text{ and } \Delta_3=\begin{vmatrix} a_{11} & a_{12} & b_1\\ a_{21} & a_{22} & b_2\\ a_{31} & a_{32} & b_3\end{vmatrix} \text{ then}$$

Case-I: when $\Delta\neq 0$ then solution is unique $\frac{x}{\Delta_1}=\frac{y}{\Delta_2}=\frac{z}{\Delta_3}=\frac{1}{\Delta}$

Case-II: when $\Delta=0$ and $\Delta_1\neq 0$ or $\Delta_2\neq 0$ or $\Delta_3\neq 0$ then there will be infinite solutions

Case-III: when $\Delta=0$ and $\Delta_1=\Delta_2=\Delta_3=0$ then there will be infinite solutions

(ii) **Matrix method:** $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

i.e. $AX = B$

$\Rightarrow \quad X = A^{-1}B$, where $A^{-1} = \dfrac{adjA}{|A|}$.

Answers

Exercise 5.1

1. $x = -1$ 2. $\begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$ 3. (i) $\dfrac{1}{27}\begin{bmatrix} 4 & 17 & 3 \\ -1 & -11 & 6 \\ 5 & 1 & -3 \end{bmatrix}$; (ii) $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$; (iii) $\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$

4. $\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 7. $\begin{bmatrix} -2 & 19 & -27 \\ -2 & 18 & -25 \\ -3 & 29 & -42 \end{bmatrix}$ 10. $\dfrac{1}{42}\begin{bmatrix} -4 & 5 \\ 2 & 8 \end{bmatrix}$

Exercise 5.2

1. (i) 26 Sq. Units; (ii) 11 / 2 Sq. Units; (iii) 10 Sq. Units 2. 13 / 2 Sq. Units, No

3. $x = -2, 12$ 4. $k = -1, 1/2$ 5. $x = 5$ 6. $x - 3y = 0$, 10 Sq. Units

7. (i) $x = 3, y = 1$ (ii) $x = 3, y = -1$ 9. (i) $x = 2, y = 1, x = 3$; (ii) $x = 2, y = -1, z = 3$

10. (i) $x = 1, y = 2, z = 1$; (ii) $x = 2, y = 3, z = 5$

11. (i) $x = \dfrac{-5}{11}, y = \dfrac{12}{11}$; (ii) $x = \dfrac{9}{2}, y = \dfrac{-7}{2}$; (iii) $x = 2, y = 1, z = 2$; (iv) $x = \dfrac{1}{2}, y = \dfrac{1}{3}, z = \dfrac{1}{5}$

12. $A^{-1} = \dfrac{1}{11}\begin{bmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{bmatrix}$; $x = 4, y = -3, z = 1$ 13. $\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}, x = 3, y = -2, z = -1$

14. $\dfrac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}, x = 2, y = -1, z = 1$

Miscellaneous Exercise - 5

1. $\frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$ 2. $\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ 3. $x = -1$

4. (i) $x = 7,\ y = -3$; (ii) $x = 2,\ y = \frac{-2}{a}$; (iii) $x = y = z = 1$

6. $\begin{bmatrix} 4 & 1 \\ -1 & 1 \end{bmatrix}$ 7. $\frac{1}{42}\begin{bmatrix} -4 & 5 \\ 2 & 8 \end{bmatrix}$ 9. $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ 10. $A^{-1} = \frac{1}{5}2\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$

11. (i) $x = \frac{11}{24}, y = \frac{1}{24}$; (ii) $x = 1,\ y = -1,\ z = 1$; (iii) $x = 1,\ y = -1,\ z = 2$

12. (i) 46 Sq. Units; (ii) 20 Sq. Units 13. $\lambda = \frac{7}{4}$ 14. $A = \begin{bmatrix} 21 & -29 \\ -13 & 18 \end{bmatrix}$

15. $A^{-1} = -\frac{1}{11}\begin{bmatrix} 3 & 4 & 5 \\ 9 & -1 & -1 \\ 5 & -3 & -1 \end{bmatrix}$, $x = 1,\ y = 3,\ z = -2$ 16. $A^{-1} = \begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$

17. $x = \frac{(c-k)(k-b)}{(c-a)(a-b)},\ y = \frac{(k-c)(a-k)}{(b-c)(a-b)},\ z = \frac{(b-k)(k-a)}{(b-c)(c-a)}$

18. $A^{-1} = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$, $x = 3,\ y = -2,\ z = 1$ 19. $X = \begin{bmatrix} 6 & 2 \\ 11/2 & 2 \end{bmatrix}$ 20. $a = -2,\ b = 1$

06

Continuity and Differentiability

6.01 Introduction

Graphically, a function is continuous in the given interval, if its graph can be drawn at this point without raising the pencil (or pen), otherwise it is discontinuous in that interval. But, only graphical understanding of the concept of continuity is not sufficient. So we must have an analytical approach to analyse the continuity of a function. We shall understand this approach with the help of limits.

6.02 Cauchy's definition of continuity

Let $f(x)$ be a function, then it is continuous at a point a in its domain, if for a small positive number $\in$, there exists a positive number δ such that

$$|f(x)-f(a)|<\in \text{ when } |x-a|<\delta$$

In other words, function $f(x)$ is called a continuous function at a point a in its domain if for every $\in>0$, for every point in interval $(a-\delta, a+\delta)$ the numerical difference of $f(x)$ and $f(a)$ may be lesser than $\in$.

6.03 Alternate definition of continuity

Let $f(x)$ is a real function on a subset of the real numbers and let a be a point in the domain of f, then f is continuous if and only if $\lim_{x\to a} f(x)$ exists and is equal to $f(a)$, i.e.

$$\lim_{x\to a} f(x)=f(a)$$

$$\Leftrightarrow \qquad \lim_{x\to a^+} f(x)=\lim_{x\to a^-} f(x)=f(a)$$

$$\text{or} \qquad f(a+0)=f(a-0)=f(a)$$

i.e., Right hand limit of $f(x)$ at a = Left hand limit of $f(x)$ at a = Value of function at a

6.04 Continuity at a point from left and right

Any function $f(x)$ at a point a of its domain.

(i) is continuous from left, if

$$\lim_{x\to a^-} f(x)=f(a)$$

$$\text{or} \qquad f(a-0)=f(a)$$

(ii) is continuous from right, if

$$\lim_{x\to a^+} f(x)=f(a)$$

$$\text{or} \qquad f(a+0)=f(a)$$

6.05 Continuous function in an open interval

A function $f(x)$ is called continuous in open interval (a, b) if it is continuous at every point in the interval.

6.06 Continuous function in a closed interval

Function $f(x)$ is called continuous in closed interval $[a, b]$ if it is

(i) Continuous from right at point a

(ii) Continuous from left at point b

(iii) Continuous in open interval (a, b)

6.07 Continuous function

If a function is continuous at every point of its domain then it is called a continuous function. Some examples of continuous function are

(i)	Identity function	$f(x) = x,$
(ii)	Constant function	$f(x) = c,$ where c is a constant
(iii)	Polynomial function	$f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n,$
(iv)	Trigonometric function	$f(x) = \sin x, \cos x$
(v)	Exponential function	$f(x) = a^x, a > 0$
(vi)	Logarithmic function	$f(x) = \log_e x$
(vii)	Absolute valued function	$f(x) = \|x\|, x + \|x\|, x - \|x\|$

6.08 Discontinuous function

A function is discontinuous in its domain D if it is not continuous at atleast one point in the domain. Some examples of discontinuous function are

(i) $f(x) = [x]$, greatest integer function

(ii) $f(x) = x - [x]$, discontinuous at every integer.

(iii) $f(x) = \tan x, \sec x$, discontinuous at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, ...$

(iv) $f(x) = \cot x, \operatorname{cosec} x$, discontinuous at $x = 0, \pm\pi, \pm 2\pi, ...$

(v) $f(x) = \sin\frac{1}{x}, \cos\frac{1}{x}$ discontinuous at $x = 0$

(vi) $f(x) = e^{1/x}$ discontinuous at $x = 0$

(vii) $f(x) = \frac{1}{x}$, discontinuous at $x = 0$

6.09 Properties of continuous function

(i) If $f(x)$ and $g(x)$ are two continuous functions in domain D then $f(x) \pm g(x), f(x) \cdot g(x), cf(x)$ will be continuous in D However $\frac{f(x)}{g(x)}$ will be continuous for all points in D where $g(x) \neq 0, \quad \forall \ x \in D$.

(ii) If $f(x)$ and $g(x)$ are two continuous functions in their respective domains then their composite function $(g \circ f)(x)$ will be continuous.

Illustrative Examples

Example 1. Examine the continuity of function $f(x) = \begin{cases} \dfrac{x-|x|}{x}, & x \neq 0 \\ 1 & , x = 0 \end{cases}$ at $x = 0$.

Solution : We know that $|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$

then the given function may be defined as $f(x) = \begin{cases} 0 \ , & x > 0 \\ 2 \ , & x < 0 \\ 1 \ , & x = 0 \end{cases}$

continuity at $x = 0$

From definition of function

$$f(0) = 1$$

$$\therefore \quad f(0-0) = \lim_{h \to 0} f(0-h) = 2$$

$$f(0+0) = \lim_{h \to 0} f(0+h) = 0$$

$$\therefore \quad f(0) \neq f(0-0) \neq f(0+0)$$

hence $f(x)$ is not continuous at $x = 0$

Example 2. Examine the continuity of $f(x) = |x| + |x-1|$ at $x = 0$ and $x = 1$

Solution : $f(x)$ may be written as $f(x) = \begin{cases} 1-2x, & \text{if } x \leq 0 \\ 1, & \text{if } 0 < x < 1 \\ 2x-1, & \text{if } x \geq 1 \end{cases}$

Continuity at $x = 0$

Here $f(0) = 1 - 2(0) = 1$

$$f(0-0) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (1-2x)$$

$$= \lim_{h \to 0} \{1 - 2(0-h)\} = 1$$

$$f(0+0) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 = 1$$

so $f(0-0) = f(0+0) = f(0)$

hence function $f(x)$ is continuous at $x = 0$

Continuity at $x = 1$

From deinition of function

$$f(1) = 2(1) - 1 = 1$$

$$f(1-0) = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 1 = 1$$

$$f(1+0) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x-1)$$

$$= \lim_{x \to 1^+} [2(1+h)-1] = 1$$

so $$f(1-0) = f(1+0) = f(1)$$

$f(x)$ is continuous at $x = 1$.

Example 3. Show that the following function $f(x)$ is not continuous at $x = 0$.

$$f(x) = \begin{cases} \dfrac{e^{1/x}}{1+e^{1/x}} & ; \ x \neq 0 \\ 0 & ; \ x = 0 \end{cases}$$

Solution : From definition of function $f(0) = 0$

Right hand limit at $x = 0$ $$f(0+0) = \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} \frac{e^{1/(0+h)}}{1+e^{1/(0+h)}}$$

$$= \lim_{h \to 0} \frac{1}{e^{-1/h}+1} = \frac{1}{0+1} = 1$$

Left hand limit at $x = 0$ $$f(0-0) = \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} \frac{e^{1/(0-h)}}{1+e^{1/(0-h)}}$$

$$= \lim_{h \to 0} \frac{e^{-1/h}}{1+e^{-1/h}} = \frac{0}{1+0} = 0$$

so $$f(0-0) \neq f(0+0)$$

hence $f(x)$ is not continuous at $x = 0$

Example 4. Examine the continuity of function $f(x)$ at $x = 2$.

$$f(x) = \begin{cases} x^2 & ; \quad x < 1 \\ x & ; \ 1 \leq x < 2 \\ \dfrac{x^3}{4} & ; \quad x \geq 2 \end{cases}$$

Solution : From definition of function $f(2) = \dfrac{2^3}{4} = 2$

Right hand limit at $x = 2$ $$f(2+0) = \lim_{h \to 0} f(2+h)$$

$$= \lim_{h\to 0} \frac{(2+h)^3}{4}$$

$$= \frac{(2+0)^3}{4} = 2$$

Left hand limit at $x = 2$ $\qquad f(2-0) = \lim_{h\to 0} f(2-h)$

$$= \lim_{h\to 0}(2-h) = 2$$

so $f(2-0) = f(2+0) = f(2) = 2$

Hence $f(x)$ is continuous at $x = 2$.

Example 5. If the following function is continuous at $x = 0$, find the vlaue of c.

$$f(x) = \begin{cases} \dfrac{1-\cos(cx)}{x\sin x} & ; \quad x \neq 0 \\ \dfrac{1}{2} & ; \quad x = 0 \end{cases}$$

Solution : From definition of function $f(0) = \dfrac{1}{2}$ (1)

at $x = 0$ finding limit of $f(x)$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1-\cos(cx)}{x\sin x}$$

$$= \lim_{x\to 0} \frac{2\sin^2(cx/2)}{x\sin x}$$

$$= \lim_{x\to 0} \frac{(c^2/2)\left(\dfrac{\sin(cx/2)}{cx/2}\right)^2}{(\sin x/x)}$$

$$= \frac{(c^2/2)\cdot 1^2}{1} = \frac{c^2}{2} \qquad (2)$$

$\because$ $\quad f(x)$ is continuous at $x = 0$, so

$$\lim_{x\to 0} f(x) = f(0)$$

From (1) and (2)

$\Rightarrow \quad \dfrac{c^2}{2} = \dfrac{1}{2} \quad \Rightarrow \quad c^2 = 1 \quad \Rightarrow \quad c = \pm 1$

Example 6. Find the values of a and b if the given function is continuous in [4, 6]

$$f(x) = \begin{cases} 3 & ; \quad x \le 4 \\ ax+b & ; \quad 4 < x < 6 \\ 7 & ; \quad x \ge 6 \end{cases}$$

Solution : Given that function is continuous is [4, 6]

Right hand limit of $f(x)$ at $x = 4$

$$f(4+0) = \lim_{h \to 0} f(4+h)$$

$$= \lim_{h \to 0} \{a(4+h)+b\}$$

$$= 4a+b \qquad (1)$$

and $$f(4) = 3 \qquad (2)$$

Left hand limit of $f(x)$ at $x = 6$

$$f(6-0) = \lim_{h \to 0} f(6-h)$$

$$= \lim_{h \to 0} \{a(6-h)+b\}$$

$$= 6a+b \qquad (3)$$

and $$f(6) = 7 \qquad (4)$$

$\because$ function $f(x)$ is continuous at left extreme point at $x = 4$ of [4, 6], so $f(4+0) = f(4)$

$\Rightarrow$ $$4a+b = 3 \qquad (5)$$

Similarly $f(x)$, is continuous at right extreme point at $x = 6$ of [4, 6], so $f(6-0) = f(6)$

$\Rightarrow$ $$6a+b = 7 \qquad (6)$$

solving equations (5) and (6)

$$a = 2, \; b = -5$$

which are required values of a and b.

Example 7. Find the condition for m, for which the funcion $f(x)$ is continous at $x = 0$.

$$f(x) = \begin{cases} x^m \sin(1/x) & ; \quad x \ne 0 \\ 0 & ; \quad x = 0 \end{cases}$$

Solution : From definition of function $f(0) = 0$

$$f(0-0) = \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} (0-h)^m \sin\left(1/(0-h)\right)$$

$$= (-1)^{m+1} \lim_{h \to 0} h^m \sin\left(1/h\right)$$

$$= (-1)^{m+1} (0)^m \times (\text{a finite number between } -1 \text{ and } 1)$$

$$= 0, \text{ if } m > 0$$

Similarly $f(0+0)=0,$ if $m>0$

So $f(0-0)=f(0+0)=f(0)=0,$ if $m>0$

have $f(x)$ is continuous at $x = 0$, only when $m > 0$

Example 8. Examine the continuity of function $f(x)$ at $x = 0$.

$$f(x)=\begin{cases}(\sin x)/x+\cos x & ; \quad x\neq 0\\ 2 & ; \quad x=0\end{cases}$$

Solution : From definition of function

$$f(0)=2$$

$$f(0-0)=\lim_{h\to 0} f(0-h)$$

$$=\lim_{h\to 0}\left\{\frac{\sin(-h)}{(-h)}+\cos(-h)\right\}$$

$$=\lim_{h\to 0}\left\{\frac{\sin h}{h}+\cos h\right\}=1+1=2$$

and $$f(0+0)=\lim_{h\to 0} f(0+h)$$

$$=\lim_{h\to 0}\left\{\frac{\sin h}{h}+\cos h\right\}=\{1+1\}=2$$

so $$f(0-0)=f(0+0)=f(0)=2$$

Hence $f(x)$ is continous at $x = 0$.

Exercise 6.1

1. Examine the continuity of following functions

(a) $$f(x)=\begin{cases}x\{1+(1/3)\sin(\log x^2)\} & ; \quad x\neq 0\\ 0 & ; \quad x=0\end{cases}$$

at $x = 0$.

(b) $$f(x)=\begin{cases}\dfrac{e^{1/x}}{x} & ; \quad x\neq 0\\ 0 & ; \quad x=0\end{cases}$$

at $x = 0$

(c) $$f(x)=\begin{cases}1+x & ; \quad x\leq 3\\ 7-x & ; \quad x>3\end{cases}$$

at $x = 3$

(d) $$f(x)=\begin{cases}\sin x & ; \quad \dfrac{-\pi}{2}<x\leq 0\\ \tan x & ; \quad 0<x<\dfrac{\pi}{2}\end{cases}$$

at $x = 0$

(e) $f(x) = \begin{cases} \cos(1/x) & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$

at $x = 0$

(f) $f(x) = \begin{cases} \dfrac{1}{(x-a)}.\cos ec(x-a) & ; \quad x \neq a \\ 0 & ; \quad x = a \end{cases}$

at $x = a$

(g) $f(x) = \begin{cases} \dfrac{x^2}{a} - a, x < a & ; \quad x < a \\ 0 & ; \quad x = a \\ a - \dfrac{a^3}{x^2} & ; \quad x > a \end{cases}$

at $x = a$

2. Examine the continuity of $f(x) = x - [x]$ at $x = 3$.

3. Find the vlaue of k is the following function is continuous at $x = 2$

$$f(x) = \begin{cases} \dfrac{x^3 + x^2 - 16x + 20}{(x-2)^2} & ; \quad x \neq 2 \\ k & ; \quad x = 2 \end{cases}$$

4. Examine the continuity of following function in [–1, 2]

$$f(x) = \begin{cases} -x^2 & ; \quad -1 \leq x < 0 \\ 4x - 3 & ; \quad 0 < x \leq 1 \\ 5x^2 - 4x & ; \quad 1 < x \leq 2 \end{cases}$$

6.10 Differentiability

In previous class we had defined the derivative of a real value function and first principle of derivatives. Here we shall study a method of find derivative with special limit method. if equation of curved is $y = f(x)$ then this function is differentiable at $x = a$ if a tangent to the curve can be drawn through this point. If curve has a break or changes its direction then $f(x)$ is not differentiable at $x = a$. Mathematically we will study differentiability as follows:

1. A function $f:(a,b)\to R$ is differentiable at $c\in(a,b)$ if $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ exists. This limit of $f(x)$ at point c is called derivative of f and is expressed as $f'(c)$.
2. Function f is differentiable at c is for every $\in>0,\ \exists\ \delta>0$ so that

$$\left|\frac{f(x)-f(c)}{x-c}-f'(c)\right|<\in \text{ where } |x-c|<\delta$$

i.e.
$$\Rightarrow f'(c)-\in\ <\ \frac{f(x)-f(c)}{x-c}\ <\ f'(c)+\in$$

6.11 Left hand derivative of a function

A function $f(x)$ is said to be differentiable from left hand side at a point c in its domain if $\lim_{h\to 0}\frac{f(c-h)-f(c)}{-h}, h>0$ exists and is finite.

The value of this limit is represented by $LDf(c)$ or $Lf'(c)$ or $f'(c-0)$ and it is called the left hand derivative of f at c.

6.12 Right hand derivative of a function

A function $f(x)$ is said to be differentiable from right hand side at a point c in its domain if $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}, h>0$ exists and is finite.

The value of this limit is represented by $RDf(c)$ or $Rf'(c)$ or $f'(c+0)$ and it is called the right hand derivative of f at c.

6.13 Differentiable function

A function f is differentiable at a point c in its domain if both left hand derivative and right hand derivative are finite and equal.

i.e. $$f'(c-0)=f'(c+0)$$

$$\lim_{h\to 0}\frac{f(c-h)-f(c)}{-h}=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

Note: In the following cases $f(x)$ is not differentiable at a point c if

(i) $f'(c-0)\neq f'(c+0)$

(ii) $f'(c-0)$ and $f'(c+0)$ either or both infinite

(iii) $f'(c-0)$ and $f'(c+0)$ either or both do not exists.

6.14 Differentiability in an interval

1. Function f is differentiable in open interval (a, b) if $f(x)$ is differentiable at every point of interval.
2. Function f is differentiable is closed interval $[a, b]$ if
   (i) $f'(c)$ exists when $c\in(a,b)$
   (ii) Right hand derivative of $f(x)$ exists at poitn a
   (iii) Left hand derivative of $f(x)$ exists at point b

6.15 Some important results

(i) If a function f is diferentiable at a point c, then it is also continuous at that point but the converse of above statement needs not to be true. It is clear that if a function is not cotinuous then surely it will not be differentiable.

Note:

(i) While examining differentiability of any function, Firstly its continuity should be examined.

(ii) Every polynomial, exponential and constant functions are alway differentiable in R

(iii) Logarithmic and trigonometric functions are differentiable in their domains.

(iv) Composite functions, sum, difference, product and quotient (when denominator is not zero) of two differentiable functions are always differntiable.

Illustrative Examples

Example 9. If the following function is continuous at $x = 0$ then examine its differentiablity at $x = 0$

$$f(x) = \begin{cases} x^2\left(\dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}\right) & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$$

Solution : Left hand derivative of $f(x)$ at $x = 0$

$$f'(0-0) = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{(-h)^2\left(\dfrac{e^{-1/h} - e^{-(-1/h)}}{e^{-1/h} + e^{-(-1/h)}}\right) - 0}{-h}$$

$$= \lim_{h\to 0} -h\left(\frac{e^{-2/h} - 1}{e^{-2/h} + 1}\right)$$

$$= 0 \times \left(\frac{0-1}{0+1}\right) = 0$$

and Right hand derivative of $f(x)$ at $x = 0$

$$f'(0+0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{(h)^2\left(\dfrac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}}\right) - 0}{h}$$

$$= \lim_{h\to 0} h\left(\frac{1 - e^{-2/h}}{1 + e^{-2/h}}\right)$$

$$= 0 \times \left(\frac{1-0}{1+0}\right) = 0$$

so $f'(0-0) = f'(0+0)$

hence function $f(x)$ is differentiable at $x = 0$

Example 10. If the following function is continuous everywhere then examine its differentiability at $x = 0$

$$f(x) = \begin{cases} x\left(1 + \frac{1}{3}\sin(\log x^2)\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

Solution : Right hand derivative of $f(x)$ at $x = 0$.

$$f'(0+0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h\left(1 + 1/3.\sin(\log h^2)\right) - 0}{h}$$

$$= \lim_{h \to 0} \{1 + 1/3.\sin(\log h^2)\}$$

This limit does not exist because $\lim_{h \to 0} \sin(\log h^2), -1$, is between -1 and 1 hence $\lim_{h \to 0} \{1 + 1/3.\sin(\log h^2)\}$, $2/3$ and 4 / 3. Hence $f(x)$ is not differentiable at $x = 0$.

Example 11. For what values of m is the following function differentiable at $x = 0$ and $f'(x)$ is continuous

$$f(x) = \begin{cases} x^m \sin\frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

Solution : differentiability at $x = 0$ Let hand derivative of $f(x)$ at $x = 0$

$$f'(0-0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{(-h)^m \sin\frac{1}{(-h)} - 0}{-h}$$

$$= \lim_{h \to 0} (-1)^m h^{m-1} \sin\frac{1}{h} \qquad (1)$$

right hand derivative of $f(x)$ at $x = 0$

$$f'(0+0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^m \sin \frac{1}{h} - 0}{h}$$

$$= \lim_{h \to 0} h^{m-1} \sin \frac{1}{h} \qquad (2)$$

If $f(x)$ is differentiable at $x = 0$ then $f'(0-0) = f'(0+0)$, which is possible only when $m-1>0$ or $m>1$ hence the given functionis differentiable at $x = 0$ if $m > 1$.

Test of continuity at $x = 0$

$$f'(x) = mx^{m-1} \sin(1/x) - x^{m-2} \cos(1/x) \neq 0$$

$$f'(0) = 0$$

$f'(x)$ is continuous at $x = 0$ if $m > 2$

Hence required condition is $m > 2$.

Example 12. If the function $f(x) = |x-1| + 2|x-2| + 3|x-3|, \ \forall \ x \in R$ is continuous at points $x = 1, \ 2, \ 3$ then examine the differentiability of function at these points.

Solution : We can write the function as

$$f(x) = \begin{cases} 14-6x, & \text{if} \quad x \le 1 \\ 12-4x, & \text{if} \quad 1 < x \le 2 \\ 4, & \text{if} \quad 2 < x \le 3 \\ 6x-14, & \text{if} \quad x > 3 \end{cases}$$

differentiability at $x = 1$

Let hand derivative of $f(x)$ at $x = 1$

$$f'(1-0) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{\{14 - 6(1-h)\} - \{14 - 6(1)\}}{-h}$$

$$= \lim_{h \to 0} \frac{(6h)}{-h} = -6 \qquad (1)$$

Right hand derivative of $f(x)$ at $x = 1$

$$f'(1+0) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{\{12 - 4(1+h)\} - \{14 - 6(1)\}}{h}$$

$$= \lim_{h \to 0} \frac{-4h}{h} = -4 \qquad (2)$$

From (1) and (2)

$$f'(1-0) \neq f'(1+0)$$

Hence function $f(x)$ is not differentiable at $x = 1$, similarly we can prove that $f(x)$ is not differentiable at $x = 2$ and $x = 3$ also.

Example 13. Test the differentiability of following function at $x = 0$.

$$f(x) = \begin{cases} e^{-1/x^2} \cdot \sin(1/x), & \text{if} \quad x \neq 0 \\ 0, & \text{if} \quad x = 0 \end{cases}$$

Solution : Left hand derivative of $f(x)$ at $x = 0$

$$f'(0-0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{e^{-1/(-h)^2} \cdot \sin(1/(-h)) - 0}{-h}$$

$$= \lim_{h \to 0} \frac{\sin(1/h)}{he^{1/h^2}} \qquad (1)$$

Now,

$$= \lim_{h \to 0} \frac{\sin(1/h)}{h\left[1 + \frac{1}{h^2} + \frac{1}{\lfloor 2\, h^4} + \ldots\right]}$$

$$= \text{(a finite number between } -1 \text{ and } 1\text{)} \Big/ \lim_{h \to 0} \left\{h + \frac{1}{h} + \frac{1}{\lfloor 2} \cdot \frac{1}{h^3} + \ldots\right\} = 0 \qquad (2)$$

Right hand limit of $f(x)$ at $x = 0$

$$f'(0+0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{e^{1/h^2} \cdot \sin(1/h) - 0}{h}$$

$$= \lim_{h \to 0} \frac{\sin 1/h}{he^{-1/h^2}}$$

$$= 0 \text{ (as above)}$$

so $\quad f'(0-0) = f'(0+0) = 0$

Hence $f(x)$ is differentiable at $x = 0$.

Example 14. Is function $f(x) = |x-2|$, differentiable at $x = 2$?

Solution : Left hand derivative of $f(x)$ at $x = 2$

$$f'(2-0)=\lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$$

$$=\lim_{h\to 0}\frac{|2-h-2|-0}{-h}=\lim_{h\to 0}\frac{|-h|}{-h}$$

$$=\lim_{h\to 0}\frac{h}{-h}=\lim_{h\to 0}(-1)=-1 \qquad (1)$$

Right hand derivative of $f(x)$ at $x = 2$

$$f'(2+0)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$=\lim_{h\to 0}\frac{|2+h-2|-0}{h}=\lim_{h\to 0}\frac{|h|}{h}$$

$$=\lim_{h\to 0}\frac{h}{h}=\lim_{h\to 0}(1)=1 \qquad (2)$$

From (1) and (2)

Hence $f(x)$ is not differentiable at $x = 2$.

Exercise 6.2

1. Prove that following functions are differentiable for every value of x.
   (i) Identity function $f(x)=x$
   (ii) Constant function $f(x)=c$, where c is constant
   (iii) $f(x)=e^x$
   (iv) $f(x)=\sin x$.
2. Prove that function $f(x)=|x|$ is not differentiable at $x = 0$.
3. Examine the differentiability of the function $f(x)=|x-1|+|x|$, at $x = 0$ and 1.
4. Examine the differentiability of the function $f(x)=|x-1|+|x-2|$, in [0, 2].
5. Examine the differentiability of $f(x)=\begin{cases} x\tan^{-1}x & ; \quad x\neq 0 \\ 0 & ; \quad x=0 \end{cases}$
6. Examine the differentiability of $f(x)=\begin{cases} \dfrac{1-\cos x}{2} & ; \quad x\leq 0 \\ \dfrac{x-2x^2}{2} & ; \quad x>0 \end{cases}$
7. Prove that the following function $f(x)$.

$$f(x)=\begin{cases} x^m\cos(1/x) & ; \quad x\neq 0 \\ 0 & ; \quad x=0 \end{cases}$$

(i) is continuous at $x = 0$, if $m > 0$

(ii) is differentiable at $x = 0$ if $m > 1$

8. Examine the differentiability of following function at $x = 0$

$$f(x) = \begin{cases} \dfrac{1}{1+e^{1/x^2}} & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$$

9. Examine the differentiability of following function at $x = 0$.

$$f(x) = \begin{cases} \dfrac{|x|}{x} & ; \quad x \neq 0 \\ 1 & ; \quad x = 0 \end{cases}$$

10. Examine the Differentiability of following function at $x = \pi/2$

$$f(x) = \begin{cases} 1+\sin x & ; \quad 0 < x < \pi/2 \\ 2+\left(x-\pi/2\right)^2 & ; \quad x \geq \pi/2 \end{cases}$$

11. Find the values of m and n if

$$f(x) = \begin{cases} x^2+3x+m, & \text{when} \quad x \leq 1 \\ nx+2, & \text{when} \quad x > 1 \end{cases}$$

if differentiable at every point

Miscellaneous Exercise 6

1. If $f(x) = \dfrac{x^2 - 9}{x - 3}$ is continuous at $x = 3$ then the value of $f(3)$ will be

(a) 6 (b) 3 (c) 1 (d) 0.

2. If $f(x) = \begin{cases} \dfrac{\sin 3x}{x} & ; \quad x \neq 0 \\ m & ; \quad x = 0 \end{cases}$, $x = 0$ is continuous at $x = 0$ then the value of m

(a) 3 (b) 1 / 3 (c) 1 (d) 0.

3. If $f(x) = \begin{cases} \dfrac{\log(1+mx)-\log(1-nx)}{x} & ; \quad x \neq 0 \\ k & ; \quad x = 0 \end{cases}$, is continuous at $x = 0$, the value of k will be

(a) o (b) $m+n$ (c) $m-n$ (d) $m \cdot n$.

4. If $f(x) = \begin{cases} x+\lambda & ; \quad x < 3 \\ 4 & ; \quad x = 3 \\ 3x-5 & ; \quad x > 3 \end{cases}$, is continuous at $x = 3$, then the vlaue of λ is

(a) 4 (b) 3 (c) 2 (d) 1.

5. If $f(x) = \cot x$, is not continuous at $x = \frac{n\pi}{2}$ when

(a) $n \in Z$ (b) $n \in N$ (c) $n/2 \in Z$ (d) only $n = 0$.

6. The set of those points on $f(x) = x|x|$, where the function is differentiable

(a) $(0, \infty)$ (b) $(-\infty, \infty)$ (c) $(-\infty, 0)$ (d) $(-\infty, 0) \cup (0, \infty)$

7. Which of the following function is not differentiable at $x = 0$

(a) $x|x|$ (b) $\tan x$ (c) e^{-x} (d) $x + |x|$

8. The value of left hand derivative of $f(x)$ at $x = 2$ is; $f(x) = \begin{cases} 1+x, & \text{when} \quad x \le 2 \\ 5-x, & \text{when} \quad x > 2 \end{cases}$

(a) -1 (b) 1 (c) -2 (d) 2.

9. Function $f(x) = [x]$ is not differentiable at

(a) every integer (b) every rational number

(c) origin (d) everywhere

10. The vlaue of right hand derivative of $f(x)$ at $x = 0$ is; $f(x) = \begin{cases} \frac{\sin x^2}{x}, & \text{when } x \ne 0 \\ 0, & \text{when } x = 0 \end{cases}$,

(a) -1 (b) 1 (c) 0 (d) Infinite

11. Examine the continuity of following function $f(x) = |\sin x| + |\cos x| + |x|, \ \forall \ x \in R$

12. Find the value of m, when the following function is continous at $x = 0$

$$f(x) = \begin{cases} \frac{\sin(m+1)x + \sin x}{x} & ; \quad x < 0 \\ 1/2 & ; \quad x = 0 \\ \frac{x^{3/2}+1}{2} & ; \quad x > 0 \end{cases}$$

13. Find the values of m and n when the following function is continuous

$$f(x) = \begin{cases} x^2 + mx + n & ; \quad 0 \le x < 2 \\ 4x - 1 & ; \quad 2 \le x \le 4 \\ mx^2 + 17n & ; \quad 4 < x \le 6 \end{cases}$$

14. Examine the continuity of the function $f(x) = \begin{cases} \frac{\tan x}{\sin x} & ; \quad x \ne 0 \\ 1 & ; \quad x = 0 \end{cases}$, at $x = 0$

15. Examine the continuity of following function at $x = 1$ and 3. $f(x) = \begin{cases} |x-3| & ; \quad x \geq 1 \\ \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4} & ; \quad x < 1 \end{cases}$,

16. Find the values of a, b and c when the following is continuous at $x = 0$

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x}, & \text{if} \quad x < 0 \\ c, & \text{if} \quad x = 0 \\ \dfrac{\sqrt{x+bx^2} - \sqrt{x}}{bx\sqrt{x}}, & \text{if} \quad x > 0 \end{cases}$$

17. Examine the continuity of $f(x) = \dfrac{|3x-4|}{3x-4}$ at $x = \dfrac{4}{3}$

18. Examine the continuity of $f(x) = |x| + |x-1|$ in the interval $[-1, 2]$

19. Find the value of $f(0)$ if the following function is continuous at $x = 0$; $f(x) = \dfrac{\sqrt{1+x} - \sqrt[3]{1+x}}{x}$

20. Examine the continuity of $f(x)$ at $x = 0$ when $f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{-1/x}+1}, & \text{when} \quad x \neq 0 \\ 1 & \text{when} \quad x = 0 \end{cases}$

21. For what vlaues of x, $f(x) = \sin x, x$ is not differentiable.

22. Examine the differentiability of $f(x) = \begin{cases} x^2 \sin x & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$, when $x \in R$ also find the value of $f'(0)$.

23. Examine the differetiability of following function at $x = a$

$$f(x) = \begin{cases} (x-a)^2 \sin\left(\dfrac{1}{x-a}\right) & ; \quad x \neq a \\ 0 & ; \quad x = a \end{cases}$$

24. Prove that the following function uis not differentiable at $x = 1$

$$f(x) = \begin{cases} x^2 - 1 & ; \quad x \geq 1 \\ 1 - x & ; \quad x < 1 \end{cases},$$

25. Examine the diferentiablity of following function at $x = 0$

$$f(x) = \begin{cases} -x & ; \quad x \leq 0 \\ x & ; \quad x > 0 \end{cases},$$

26. Prove that the following function is differentiable at $x = 0$

$$f(x) = \begin{cases} \dfrac{x\log_e \cos x}{\log_e(1+x^2)} & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$$

27. Examine the differentiability of $f(x) = |x-2| + 2|x-3|$ in the interval [1, 3].

28. If the function $f(x) = x^3, x = 2$, is differentiable at $x = 2$ then find the vlaue of $f'(2)$

29. Prove that the greatest integer function $f(x) = [x]$ is not differentiable at $x = 2$

30. If $f(x) = \begin{cases} x-1 & ; \quad x < 2 \\ 2x-3 & ; \quad x \geq 2 \end{cases}$ then find $f'(2-0)$.

IMPORTANT POINTS

1. **Cauchy's definition of continuity**

   Let $f(x)$ be a function, then it is continuous at a point a in its domain if for a small positive number $\in$ there exists a positive δ such that $|f(x) - f(a)| < \in$ when $|x - a| < \delta$.

2. Alternate definition of continuity:

   A function $f(x)$ is continuous at a point a in its domain if $\lim_{x \to a} f(x) = f(a)$

   i.e. $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$

   or $f(a-0) = f(a+0) = f(a)$

3. Continuous function in domain.

   Any function $f(x)$ is called continuous in its domain if $f(x)$ is continuous at every point of domain D.

4. Now continuous function

   (i) A function $f(x)$ is called non continuous at a point a if $f(x)$ is not continuous at this point.

   (ii) Function $f(x)$ is called non continuous in its domain D if it is not continuous at at least one point of D.

5. **Properties of continuity**

   (i) If $f(x)$ and $g(x)$ are two continuous functions in domain D then $f(x) \pm g(x)$, $c \cdot f(x)$, will be continuous in D. How ever $\dfrac{f(x)}{g(x)}, D$ will be continuous for all points in D where $g(x) \neq 0$.

   (ii) If $f(x)$ and $g(x)$ are two continuous functions in their respective domains then their composit function $(f \circ g)(x)$ will be continuous.

6. **Differentiability.**

   A function $f(x)$ is derivable at $x = a$, if

   or, $f'(c-0) = f'(c+0)$

   or, $\lim_{h \to 0} \dfrac{f(c-h) - f(c)}{-h} = \lim_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$

7. **Non-differentiablity at a point**

$f(x)$ is not differentiable at a point c if

(i) $f'(c-0) \neq f'(c+0)$

or

(ii) $f'(c-0)$ and $f'(c+0)$ either or both infinite

or

(iii) $f'(c-0)$ or $f'(c+0)$ either or both do not exist.

Answers

Exercise 6.1

1. (a) continuous ; (b) not continuous ; (c) continuous ; (d) not continuous ; (e) not continuous ; (f) not continuous ; (g) continuous

2. Not continuous

3. $k = 7$

4. Not continuous

Exercise 6.2

3. Not differentiable
4. Not differentiable
5. Not differentiable
6. Not differentiable
7. Not differentiable
8. Not differentiable
9. Not differentiable
10. Not differentiable
11. $m = 3, n = 5$

Miscellaneous Exercise - 6

1. (a) 2. (a) 3. (b) 4. (d) 5. (c) 6. (b) 7. (d)
8. (b) 9. (a) 10. (b)

11. Everywhere continuous in R

12. $m = \frac{-3}{2}$

13. $m = 2, n = -1$

14. continuous

15. continuous

16. $a = -3/2, c = 1/2$ and $b \in R$

17. not continuous

18. continuous in $[-1, 2]$

19. $1/6$

20. Not continuous

21. R

22. differentiable for every $x \in R$ and $f'(0) = 0$

23. Not differentiable

25. Not differentiable

27. Not differentiable at $x = 2$

28. 12

30. 1

07

Differentiation

7.01 Introduction

In the previous class, we have learnt the differentiation by using first principle and derived some formulae given below:-

Standard Results

(i) $\frac{d}{dx}(x^n) = nx^{n-1}$

(ii) $\frac{d}{dx}(e^x) = e^x$

(iii) $\frac{d}{dx}(a^x) = a^x \log_e a$

(iv) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(v) $\frac{d}{dx}(\sin x) = \cos x$

(vi) $\frac{d}{dx}(\cos x) = -\sin x$

(vii) $\frac{d}{dx}(\tan x) = \sec^2 x$

(viii) $\frac{d}{dx}(\cot x) = -\cos ec^2 x$

(ix) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(x) $\frac{d}{dx}(\cos ecx) = -\cos ecx \cot x$

By using the above formulae the derivative of various other functions can be found.

7.02 Derivative of composite functions

Theorem : If the functions f and g are differentiable at any point c in the interval then $f \pm g$, fg and f/g are also differentiable at point c and

(i) $D(f \pm g)(c) = f'(c) \pm g'(c)$

(ii) $D(fg)(c) = f'(c)g(c) + f(c)g'(c)$

(iii) $D\{f/g\}(c) = \frac{g(c)f'(c) - g'(c)f(c)}{[g(c)]^2}$, when $g(c) \neq 0$

Proof : Since functions f and g are differentiable at point $c \in [a, b]$ and $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$

and $\lim_{x \to c} \frac{g(x) - g(c)}{x - c} = g'(c)$

(i) $$D(f \pm g)(c) = \lim_{x \to c} \frac{(f \pm g)(x) - (f \pm g)(c)}{x - c}$$

$$= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \pm \lim_{x \to c} \frac{g(x) - g(c)}{x - c} = f'(c) \pm g'(c).$$

(ii) $$D(fg)(c) = \lim_{x \to c} \frac{(fg)(x) - (fg)(c)}{x - c}$$

$$= \lim_{x \to c} \frac{f(x)g(x) - f(c)g(c)}{x - c}$$

$$= \lim_{x \to c} \frac{f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)}{x - c}$$

$$= \lim_{x \to c} \frac{g(x)\{f(x) - f(c)\} + f(c)\{g(x) - g(c)\}}{x - c}$$

$$= \lim_{x \to c} g(x). \lim_{x \to c} \frac{f(x) - f(c)}{x - c} + f(c) \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

$$= g(c)f'(c) + f(c)g'(c).$$

(iii) $$D(f/g) = \lim_{x \to c} \frac{(f/g)(x) - (f/g)(c)}{x - c}$$

$$= \lim_{x \to c} \frac{f(x)/g(x) - f(c)/g(c)}{x - c}$$

$$= \lim_{x \to c} \frac{f(x)g(c) - g(x)f(c)}{g(x)g(c)(x - c)}$$

$$= \lim_{x \to c} \frac{f(x)g(c) - f(c)g(c) + f(c)g(c) - g(x)f(c)}{g(x)g(c)(x - c)}$$

$$= \lim_{x \to c} \frac{g(c)\{f(x) - f(c)\} - f(c)\{g(x) - g(c)\}}{g(x)g(c)(x - c)}$$

$$= \lim_{x \to c} \frac{1}{g(x)g(c)} \cdot \left[g(c) \lim_{x \to c} \frac{f(x) - f(c)}{x - c} - f(c) \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \right]$$

$$= \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}, \quad g(c) \neq 0.$$

7.03 Derivative of a function of funcitons or chain rule of derivative

Let $y = f(u)$ i.e. y is a function of u and $u = \phi(x)$ i.e. u itself is a function of x. Let there be small increment δx, $\delta y, \delta u$ corresponding to x, y and u then

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \cdot \frac{\delta u}{\delta x}$$

Now if $\delta x \to 0$ then $\delta u \to 0$ therefore

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta u \to 0} \frac{\delta y}{\delta u} \cdot \lim_{\delta x \to 0} \frac{\delta u}{\delta x}$$

or $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Illustrative Examples

Example 1. Differentiate the following functions with respect to x

(i) $\log_e \log_e x^2$ (ii) $e^{\sin x^2}$ (iii) $\tan\left(\log_e \sqrt{1+x^2}\right)$

Solution : (i) Let $y = \log_e \log_e x^2$

Let $\log_e x^2 = u,\ x^2 = v$

then $y = \log_e u,\ u = \log_e v,\ v = x^2$

$$\therefore \quad \frac{dy}{du} = \frac{1}{u},\quad \frac{du}{dv} = \frac{1}{v},\quad \frac{dv}{dx} = 2x$$

Now $$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx} = \frac{1}{u}.\frac{1}{v}.2x = \frac{1}{\log_e x^2}.\frac{1}{x^2}.2x = \frac{2}{x\log_e x^2}$$

Alternate Method: Let $y = \log_e \log_e x^2$

$$\frac{dy}{dx} = \frac{d}{dx}\log_e \log_e x^2 = \frac{1}{\log x^2}\cdot\frac{d}{dx}\log_e x^2$$

$$= \frac{1}{\log x^2}\cdot\frac{1}{x^2}\cdot\frac{d}{dx}\left(x^2\right) = \frac{2x}{x^2\cdot\log x^2}\cdot\frac{2}{x\log x^2}$$

(ii) Let $y = e^{\sin x^2}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sin x^2}\right)$$

$$= e^{\sin x^2}\frac{d}{dx}\left(\sin x^2\right) = e^{\sin x^2}\left(\cos x^2\right)\frac{d}{dx}\left(x^2\right)$$

$$= e^{\sin x^2}(\cos x^2)(2x) = 2x\cos x^2.e^{\sin x^2}$$

(iii) Let $y = \tan\left(\log_e \sqrt{1+x^2}\right)$

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\tan(\log_e \sqrt{1+x^2})\right\}$$

$$= \sec^2(\log_e \sqrt{1+x^2})\frac{d}{dx}(\log_e \sqrt{1+x^2})$$

$$= \sec^2(\log_e \sqrt{1+x^2})\frac{1}{\sqrt{1+x^2}}\frac{d}{dx}(\sqrt{1+x^2})$$

$$= \frac{1}{\sqrt{1+x^2}}.\sec^2(\log_e \sqrt{1+x^2}).\frac{1}{2\sqrt{1+x^2}}.\frac{d}{dx}(1+x^2)$$

$$= \frac{1}{\sqrt{1+x^2}}.\sec^2(\log_e \sqrt{1+x^2}).\frac{1}{2\sqrt{1+x^2}}(0+2x)$$

$$= \frac{x}{(1+x^2)}.\sec^2(\log_e \sqrt{1+x^2}).$$

Example 2. Differentiate the following functions with respect to x

(i) $\dfrac{\sin(ax+b)}{\cos(cx+d)}$ (ii) $\cos x^3.\sin^2(x^5)$ (iii) $\sec(\tan\sqrt{x})$

Solution : (i) Let $y = \dfrac{\sin(ax+b)}{\cos(cx+d)}$

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\frac{\sin(ax+b)}{\cos(cx+d)}\right\}$$

$$= \frac{\cos(cx+d)\frac{d}{dx}\sin(ax+b) - \sin(ax+b)\frac{d}{dx}\cos(cx+d)}{\cos^2(cx+d)}$$

$$= \frac{\cos(cx+d).\cos(ax+b)\frac{d}{dx}(ax+b) - \sin(ax+b)\{-\sin(cx+d)\}\frac{d}{dx}(cx+d)}{\cos^2(cx+d)}$$

$$= \frac{\cos(cx+d)\cos(ax+b)(a) + \sin(ax+b)\sin(cx+d)(c)}{\cos^2(cx+d)}.$$

(iii) Let $y = \cos x^3 \cdot \sin^2(x^5)$

$$\frac{dy}{dx} = \frac{d}{dx}\{\cos x^3 \cdot \sin^2(x^5)\}$$

$$= \cos x^3 \frac{d}{dx}\sin^2(x^5) + \sin^2(x^5)\frac{d}{dx}\cos x^3$$

$$= \cos x^3.2\sin(x^5)\frac{d}{dx}\sin(x^5) + \sin^2(x^5)(-\sin x^3)\frac{d}{dx}(x^3)$$

$$= \cos x^3.2\sin(x^5)\cos(x^5).\frac{d}{dx}(x^5) - \sin^2(x^5)\sin x^3(3x^2)$$

$$= \cos x^3.2\sin(x^5)\cos(x^5).5x^4 - \sin^2(x^5)\sin x^3(3x^2)$$

$$= 10x^4\cos x^3.\sin(x^5)\cos(x^5) - 3x^2\sin^2(x^5)\sin x^3.$$

(iii) Let $y = \sec(\tan\sqrt{x})$

$$\frac{dy}{dx} = \frac{d}{dx}\sec(\tan\sqrt{x})$$

$$= \sec(\tan\sqrt{x}).\tan(\tan\sqrt{x}).\frac{d}{dx}(\tan\sqrt{x})$$

$$= \sec(\tan\sqrt{x}).\tan(\tan\sqrt{x})\sec^2\sqrt{x}\,\frac{d}{dx}(\sqrt{x})$$

$$= \sec(\tan\sqrt{x})\tan(\tan\sqrt{x})\sec^2\sqrt{x}.\frac{1}{2}x^{1/2-1}$$

$$= \frac{1}{2}\sec(\tan\sqrt{x})\tan(\tan\sqrt{x})\sec^2\sqrt{x}.\frac{1}{\sqrt{x}}.$$

Example 3. Diferentiate the following functions with respect to x

(i) $2\sqrt{\cot\left(x^2\right)}$ (ii) $\cos\left(\sqrt{x}\right)$

Solution : (i) Let $y = 2\sqrt{\cot(x^2)}$

$$\frac{dy}{dx} = 2\frac{d}{dx}(\sqrt{\cot x^2})$$

$$= 2.\frac{1}{2\sqrt{\cot x^2}}.\frac{d}{dx}(\cot x^2)$$

$$= \frac{1}{\sqrt{\cot x^2}}.\left\{-\cos ec^2(x^2)\right\}\frac{d}{dx}(x^2)$$

$$= -\frac{\operatorname{cosec}^2(x^2)}{\sqrt{\cot x^2}}.(2x) = -\frac{2x\sqrt{\tan x^2}}{\sin^2(x^2)}$$

$$= \frac{-2x\sqrt{\sin x^2}}{\sin^2(x^2)\sqrt{\cos x^2}} = \frac{-2x}{\sin(x^2)\sqrt{\sin x^2\cos x^2}}$$

$$= \frac{-2\sqrt{2}x}{\sin(x^2)\sqrt{2\sin x^2\cos x^2}} = \frac{-2\sqrt{2}x}{\sin(x^2)\sqrt{\sin(2x^2)}}.$$

(ii) Let $y = \cos\left(\sqrt{x}\right)$

$$\frac{dy}{dx} = \frac{d}{dx}(\cos\sqrt{x}) = -\sin\sqrt{x}\,\frac{d}{dx}(\sqrt{x}) = \frac{-\sin\sqrt{x}}{2\sqrt{x}}.$$

Exercise 7.1

Differentiate the following functions with respect to x

1. $\sin x^2$
2. $\tan(2x+3)$
3. $\sin\left\{\cos\left(x^2\right)\right\}$
4. $\dfrac{\sec x-1}{\sec x+1}$
5. $\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$
6. $\sin x^{o}$
7. $\log_e\sqrt{\dfrac{1-\cos x}{1+\cos x}}$
8. $\sec x^{o}$
9. $\log\sqrt{\dfrac{1+\sin x}{1-\sin x}}$
10. $\log_e\left\{\dfrac{x+\sqrt{x^2+a^2}}{a}\right\}$
11. $\log_e\left\{\dfrac{x^2+x+1}{x^2-x+1}\right\}$
12. $\tan\left\{\log_e\sqrt{1+x^2}\right\}$
13. $a^{\tan 3x}$
14. $\log_e(\sec x+\tan x)$
15. $\sin^3 x.\sin 3x$

7.04 Derivatives of inverse trigonometrical functions

We know that inverse trigonometric functions are continuous in their domains. To differentiate these functions, we shall use the chain rule.

Illustrative Examples

Example 4. Differentiate the function $\sin^{-1} x$ for all $x\in(-1,1)$

Solution : Let $y=\sin^{-1}x$

$\Rightarrow \quad x=\sin y$

Differentiate both sides with respect to x

$$1=\cos y\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{\cos y}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{\cos(\sin^{-1}x)} \qquad \ldots (1)$$

here $\dfrac{dy}{dx}$, exists only when $\cos y\neq 0$

$\Rightarrow \quad \cos(\sin^{-1}x)\neq 0$

$\Rightarrow \quad \sin^{-1}x\neq\dfrac{-\pi}{2}$ or $\dfrac{\pi}{2}$ $\qquad \Rightarrow x\neq -1, 1 \qquad \Rightarrow x\in(-1,1)$

from (1) $\dfrac{dy}{dx}=\dfrac{1}{\sqrt{1-\sin^2 y}}=\dfrac{1}{\sqrt{1-x^2}} \qquad \because \sin y=x$

Note : Derivatives of remaining inverse trigonometric functions can be derived in the similar manner.

(i) $\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$ (ii) $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

(iii) $\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$ (iv) $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$

(v) $\frac{d}{dx}\left(\operatorname{cosec}^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}}$

Example 5. Find $\frac{dy}{dx}$:

(i) $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ (ii) $y = \sin^{-1}\sqrt{\cos x}$

(iii) $y = \sqrt{\cos^{-1}\sqrt{x}}$ (iv) $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right),\ x \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

Solution : (i) Given $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Here putting $x = \tan\theta$

$$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1} x \qquad [\because\ x = \tan\theta \Rightarrow \theta = \tan^{-1} x]$$

$$\therefore \quad \frac{dy}{dx} = \frac{2}{1+x^2}$$

(ii) Given $y = \sin^{-1}\left(\sqrt{\cos x}\right)$

Let $\sqrt{\cos x} = u$, then

$$y = \sin^{-1} u$$

$$\therefore \quad \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$$

$$\because \quad u = \sqrt{\cos x}$$

$$\therefore \quad \frac{du}{dx} = \frac{1}{2\sqrt{\cos x}}\frac{d}{dx}(\cos x) = \frac{-\sin x}{2\sqrt{\cos x}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{1}{\sqrt{1-u^2}}\left\{\frac{-\sin x}{2\sqrt{\cos x}}\right\} \qquad \text{[using (1) and (2)]}$$

putting the value of *u*,

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-\cos x}}\left\{\frac{-\sin x}{2\sqrt{\cos x}}\right\}=\frac{-\sin x}{2\sqrt{1-\cos x}\sqrt{\cos x}}$$

(iii) $$y=\sqrt{\cot^{-1}\sqrt{x}}$$

Let $\sqrt{x}=u$ and $\cot^{-1}\sqrt{x}=\cot^{-1}u=t$, then

$$y=\sqrt{t},\ t=\cot^{-1}u \ \text{ and } \ u=\sqrt{x}$$

$\therefore$ $$\frac{dy}{dt}=\frac{1}{2\sqrt{t}},\ \frac{dt}{du}=\frac{-1}{1+u^2},\ \frac{du}{dx}=\frac{1}{2\sqrt{x}}$$

$\Rightarrow$ $$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{du}.\frac{du}{dx}$$

$$=\left(\frac{1}{2\sqrt{t}}\right).\left(\frac{-1}{1+u^2}\right)\left(\frac{1}{2\sqrt{x}}\right)=\frac{-1}{4\sqrt{t}\sqrt{x}\left(1+u^2\right)}$$

$$=\frac{-1}{4\sqrt{\left(\cot^{-1}u\right)}\left(\sqrt{x}\right)\left(1+u^2\right)} \qquad [\because t=\cot^{-1}u]$$

$\therefore$ $$\frac{dy}{dx}=\frac{-1}{4\sqrt{x}(1+x)\sqrt{\cot^{-1}\sqrt{x}}} \qquad [\because u=\sqrt{x}\,]$$

(iv) Given that $$y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

Let $x=\tan\theta$

$\therefore$ $$y=\tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)=\tan^{-1}\left(\tan 3\theta\right)$$

$\because$ $$x\in\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$$

$\Rightarrow$ $-\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$ $\qquad\Rightarrow$ $-\frac{1}{\sqrt{3}}<\tan\theta<\frac{1}{\sqrt{3}}$

$\Rightarrow$ $-\frac{\pi}{6}<\theta<\frac{\pi}{6}$ $\qquad\Rightarrow$ $-\frac{3\pi}{6}<3\theta<\frac{3\pi}{6}$

$\Rightarrow$ $-\frac{\pi}{2}<3\theta<\frac{\pi}{2}$

$\Rightarrow \qquad y = \tan^{-1}(\tan 3\theta) \qquad \left(\because -\frac{\pi}{2} < 3\theta < \frac{\pi}{2}\right)$

$\Rightarrow \qquad y = 3\theta \;\Rightarrow\; y = 3\tan^{-1} x$

$\Rightarrow \qquad \frac{dy}{dx} = \frac{3}{1+x^2}$

Example 6. Differentiate the following with respect to x

(i) $\tan^{-1}\left(\sin e^x\right)$ (ii) $\sin^{-1}\left(\sqrt{\sin x^2}\right)$ (iii) $\sin^{-1}\left(\frac{a+b\cos x}{b+a\cos x}\right)$

Solution : (i) Let $y = \tan^{-1}(\sin e^x)$

Here putting $\sin e^x = u,\ e^x = v$

$$y = \tan^{-1}(u), \quad u = \sin v, \quad v = e^x$$

$\Rightarrow \qquad \frac{dy}{du} = \frac{1}{1+u^2}, \quad \frac{du}{dv} = \cos v, \quad \frac{dv}{dx} = e^x$

now, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx} = \frac{1}{1+u^2}.\cos v.e^x$

putting the values of u and v

$$\frac{dy}{dx} = \frac{1}{1+\sin^2 e^x}.\cos(e^x).e^x = \frac{e^x \cos e^x}{1+\sin^2 e^x}$$

(ii) Let $y = \sin^{-1}(\sqrt{\sin x^2})$

Here $\sqrt{\sin x^2} = u, \quad \sin x^2 = v, \quad x^2 = \omega$

$$y = \sin^{-1} u, \quad u = \sqrt{v}, \quad v = \sin\omega, \quad \omega = x^2$$

$\Rightarrow \qquad \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}, \quad \frac{du}{dv} = \frac{1}{2\sqrt{v}}, \quad \frac{dv}{d\omega} = \cos\omega, \quad \frac{d\omega}{dx} = 2x$

now, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{d\omega}\cdot\frac{d\omega}{dx} = \frac{1}{\sqrt{1-u^2}}.\frac{1}{2\sqrt{v}}.\cos\omega.2x$

Putting the values of u, v and ω

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin x^2}}.\frac{1}{2\sqrt{\sin x^2}}.(\cos x^2)(2x) = \frac{x\cos x^2}{\sqrt{\left(\sin x^2\right)\left(1-\sin x^2\right)}}$$

(iii) Let $y = \sin^{-1}\left(\frac{a+b\cos x}{b+a\cos x}\right)$

here put $\dfrac{a+b\cos x}{b+a\cos x}=u$

$$y=\sin^{-1}u, \qquad u=\frac{a+b\cos x}{b+a\cos x}$$

$$\frac{dy}{du}=\frac{1}{\sqrt{1-u^2}}\ ;\ \frac{du}{dx}=\frac{(b+a\cos x)\frac{d}{dx}(a+b\cos x)-(a+b\cos x)\frac{d}{dx}(b+a\cos x)}{(b+a\cos x)^2}$$

$$\Rightarrow \quad \frac{dy}{du}=\frac{b+a\cos x}{\sqrt{(b+a\cos x)^2-(a+b\cos x)^2}}$$

$$\frac{du}{dx}=\frac{(b+a\cos x)(-b\sin x)-(a+b\cos x)(-a\sin x)}{(b+a\cos x)^2}=\frac{(a^2-b^2)\sin x}{(b+a\cos x)^2}$$

$$\Rightarrow \quad \frac{du}{dx}=\frac{(a^2-b^2)\sin x}{(b+a\cos x)^2}$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$=\frac{b+a\cos x}{\sqrt{(b+a\cos x)^2-(a+b\cos x)^2}}\cdot\frac{\left(a^2-b^2\right)\sin x}{(b+a\cos x)^2}$$

$$=\frac{-(b^2-a^2)\sin x}{(b+a\cos x)\sqrt{(b^2-a^2)\sin^2 x}}=\frac{-\sqrt{(b^2-a^2)}}{(b+a\cos x)}$$

Example 7. Differentiate the following functions with respect to x

(i) $\tan^{-1}(\sec x+\tan x)$ (ii) $\sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ (iii) $\tan^{-1}\left(\sqrt{\dfrac{a-x}{a+x}}\right)$ (iv) $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$

Solution : (i) Let $y=\tan^{-1}(\sec x+\tan x)$

$$=\tan^{-1}\left(\frac{1+\sin x}{\cos x}\right)=\tan^{-1}\left\{\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}\right\}$$

$$=\tan^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right)=\tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}$$

$$\therefore \quad y=(\pi/4)+(x/2).$$

Differentiating with respect to x

$$\frac{dy}{dx} = 0 + \frac{1}{2} = \frac{1}{2}$$

(ii) Let $\qquad y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

here putting $\qquad x = \tan\theta$

$$y = \sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}(\cos 2\theta) = \sin^{-1}\left\{\sin\left(\frac{\pi}{2} \pm 2\theta\right)\right\}$$

$$= \frac{\pi}{2} \pm 2\theta = \frac{\pi}{2} \pm 2\tan^{-1} x$$

Differentiating with respect to x

$$\frac{dy}{dx} = 0 \pm \frac{2}{1+x^2} = \pm\frac{2}{1+x^2}.$$

(iii) Let $\qquad y = \tan^{-1}\left(\sqrt{\frac{a-x}{a+x}}\right)$

here putting $\qquad x = a\cos 2\theta$

$\therefore \qquad y = \tan^{-1}\left(\sqrt{\frac{a-a\cos 2\theta}{a+a\cos 2\theta}}\right) = \tan^{-1}\left(\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\right)$

$$= \tan^{-1}\left(\sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}\right) = \tan^{-1}(\tan\theta) = \theta$$

$$= \frac{1}{2}\cos^{-1}\left(x/a\right) \qquad [\because x = a\cos 2\theta \Rightarrow \theta = \frac{1}{2}\cos^{-1}\left(x/a\right)]$$

Differentiating with respect to x

$$\frac{dy}{dx} = -\frac{1}{2}\frac{1}{\sqrt{1-x^2/a^2}}.\frac{1}{a} = -\frac{1}{2\sqrt{a^2-x^2}}.$$

(iv) Let $\qquad y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

here putting $\qquad x = \tan\theta$

$$y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right)$$

$$= \tan^{-1}\left(\tan(\theta/2)\right) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1} x \qquad [\because x = \tan\theta]$$

Differentiating with respect to x

$$\frac{dy}{dx} = \frac{1}{2\left(1+x^2\right)}.$$

Example 8. Differentiate the following functions with respect to x

(i) $\tan^{-1}\left(\dfrac{3a^2x - x^3}{a\left(a^2 - 3x^2\right)}\right)$ (ii) $\tan^{-1}\left(\dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right)$

(iii) $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$ (iv) $\tan^{-1}\left(\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

Solution : (i) Let $y = \tan^{-1}\left(\dfrac{3a^2x - x^3}{a\left(a^2 - 3x^2\right)}\right)$

Here putting $x = a\tan\theta$

$$y = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right) = \tan^{-1}\left(\tan 3\theta\right) = 3\theta = 3\tan^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \qquad \frac{dy}{dx} = 3\frac{1}{1+x^2/a^2}\left(\frac{1}{a}\right) = \frac{3a}{x^2+a^2}.$$

(ii) Let $y = \tan^{-1}\left(\dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right)$

Here put $x = \cos\theta$

$$y = \tan^{-1}\left(\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\cos(\theta/2)+\sqrt{2}\sin(\theta/2)}{\sqrt{2}\cos(\theta/2)-\sqrt{2}\sin(\theta/2)}\right) = \tan^{-1}\left(\frac{1+\tan(\theta/2)}{1-\tan(\theta/2)}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right) = \frac{\pi}{4}+\frac{\theta}{2} = \frac{\pi}{4}+\frac{1}{2}\cdot\cos^{-1} x. \qquad [\because \cos\theta = x]$$

Differentiating with respect to x

$$\frac{dy}{dx} = 0 + \frac{1}{2}\left(-\frac{1}{\sqrt{1-x^2}}\right) = \frac{-1}{2\sqrt{1-x^2}}$$

(iii) Let $$y = \tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$$

Here put $x^2 = \cos\theta$

$$y = \tan^{-1}\left(\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\cos(\theta/2)+\sqrt{2}\sin(\theta/2)}{\sqrt{2}\cos(\theta/2)-\sqrt{2}\sin(\theta/2)}\right)$$

$$= \tan^{-1}\left(\frac{1+\tan\theta/2}{1-\tan\theta/2}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right) = \frac{\pi}{4}+\frac{\theta}{2} = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$$

$[\because x^2 = \cos\theta]$

Diferentiating with respect to x

$$\frac{dy}{dx} = 0+\frac{1}{2}\left\{-\frac{1}{\sqrt{1-x^4}}.2x\right\} = \frac{-x}{\sqrt{1-x^4}}$$

(iv) Let $$y = \tan^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}$$

$$= \tan^{-1}\left\{\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right\}$$

$$= \tan^{-1}\left\{\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right\}$$

$$= \tan^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right) = \tan^{-1}\left(\cot\frac{x}{2}\right) = \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right\} = \frac{\pi}{2}-\frac{x}{2}$$

$\therefore$ $$\frac{dy}{dx} = 0-\frac{1}{2} = -\frac{1}{2}.$$

Exercise 7.2

Differentiate the following functions with respect to x

1. (a) $\sin^{-1}\{2x\sqrt{1-x^2}\},\ -\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$ (b) $\sin^{-1}(3x-4x^3)x\in\left(-\frac{1}{2},\frac{1}{2}\right)$

2. (a) $\cos^{-1}\left(\frac{2x}{1-x^2}\right),\ x\in(-1,1)$ (b) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right),\ x\in(0,1)$

3. (a) $\cos^{-1}(4x^3-3x),\ x\in\left(\frac{1}{2},1\right)$ (b) $\cos^{-1}\left(\sqrt{\frac{1+x}{2}}\right)$ (Hint : $x=\cos\theta$)

4. (a) $\sec^{-1}\left(\frac{1}{2x^2-1}\right); x\in\left(0,\frac{1}{\sqrt{2}}\right)$ (b) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right),\ x\in(0,\ \infty)$

5. (a) $\sin^{-1}\left(\frac{1+x^2}{1-x^2}\right)+\cos^{-1}\left(\frac{1+x^2}{1-x^2}\right)$ (b) $\cos^{-1}(2x)+2\cos^{-1}\left(\sqrt{1-4x^2}\right)$

   (Hint : $\sin^{-1}\theta+\cos^{-1}\theta=\pi/2$) (Hint: $2x=\cos\theta$)

6. (a) $\tan^{-1}\left(\frac{a+x}{1-ax}\right)$ (Hint : $x=\tan\theta,\ a=\tan\alpha$) (b) $\tan^{-1}\left(\frac{2^{x+1}}{1-4^x}\right)$ (Hint : $2^x=\tan\theta$)

7. (a) $\sin\left\{2\tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)\right\}$ (Hint : $x=\cos\theta$) (b) $\cot^{-1}\left(\sqrt{1+x^2}+x\right)$ (Hint : $x=\tan\theta$)

7.05 Derivative of implicit functions

When a relationship between x and y is expressed in such a way that it is easy to solve for y and write $y=f(x)$, we say that y is given as an explicit function of x. Whereas if x cannot be expressed in terms of y (or y in terms of x) then it is called as Implicit function.

For example :

(i) Equation $x-2y-4=0$ can be expressed as $\left(y=\frac{1}{2}(x-4)\right)$. Thus this function is Explicit function.

(ii) Equation $x^3+y^3+3axy=c$ cannot be expressed independently as x in terms of y or y in terms of x, so this function is Implicit function.

Illustrative Examples

Example 9. Evaluate $\frac{dy}{dx}$

(i) $x^3+x^2y+xy^2+y^3=81$ (ii) $\sin^2 y+\cos xy=\pi$

(iii) $\sin^2 x+\cos^2 x=1$ (iv) $2x+3y=\sin x$

Solution : (i) Given $$x^3 + x^2 y + xy^2 + y^3 = 81$$

Differentiating with respect to x

$$3x^2 + x^2 \frac{dy}{dx} + y(2x) + x\left(2y\frac{dy}{dx}\right) + y^2 + 3y^2 \frac{dy}{dx} = 0$$

$\Rightarrow$ $$\left(x^2 + 2xy + 3y^2\right)\frac{dy}{dx} = -\left(3x^2 + 2xy + y^2\right)$$

$\Rightarrow$ $$\frac{dy}{dx} = -\frac{(3x^2 + 2xy + y^2)}{x^2 + 2xy + 3y^2}.$$

(ii) $\because$ $$\sin^2 y + \cos xy = \pi$$

Differentiating with respect to x

$$2\sin y \frac{d}{dx}(\sin y) + (-\sin xy)\frac{d}{dx}(xy) = 0$$

$\Rightarrow$ $$2\sin y \cos y \frac{dy}{dx} - \sin(xy)\left\{x\frac{dy}{dx} + y\right\} = 0$$

$\Rightarrow$ $$(2\sin y \cos y - x\sin xy)\frac{dy}{dx} = y\sin xy$$

$\Rightarrow$

$$\frac{dy}{dx} = \frac{y\sin xy}{2\sin y\cos y - x\sin xy} = \frac{y\sin xy}{\sin 2y - x\sin xy}.$$

(iii) $\because$ $$\sin^2 x + \cos^2 y = 1$$

Differentiating with respect to x

$$2\sin x \frac{d}{dx}(\sin x) + 2\cos y \frac{d}{dx}(\cos y) = 0$$

$\Rightarrow$ $$2\sin x\cos x + 2\cos y(-\sin y)\frac{dy}{dx} = 0$$

$\Rightarrow$ $$\sin 2x - \sin 2y \frac{dy}{dx} = 0$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{\sin 2x}{\sin 2y}$$

(iv) $\because$ $$2x + 3y = \sin x$$

Differentiating with respect to x

$$2 + 3\frac{dy}{dx} = \cos x$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{\cos x - 2}{3}.$$

Example 10. Find $\frac{dy}{dx}$:

(i) $xy + y^2 = \tan x + y$ (ii) $ax + by^2 = \cos y$

Solution : (i) $\because$ $xy + y^2 = \tan x + y$

Differentiating with respect to x

$$x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow \quad (x + 2y - 1)\frac{dy}{dx} = \sec^2 x - y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sec^2 x - y}{x + 2y - 1}$$

7.06 Logarithimic Differentiation

When the function is of the form $[f(x)]^{g(x)}$, where $f(x)$, $g(x) > 0$, then to find its derivative we take logarithm both the sides and get the results. This method is called as logarithmic differentiation. This method is applicable even if the function is algebraic.

Working method : Let $y = u^v$, where u and v, are the function of x

taking log both the sides $\log_e y = \log_e u^v$

$$\Rightarrow \quad \log_e y = v \log_e u$$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx} = v.\frac{1}{u}\frac{du}{dx} + \log_e u.\frac{dv}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left\{\frac{v}{u}\frac{du}{dx} + \log_e u\frac{dv}{dx}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = u^v\left\{\frac{v}{u}\frac{du}{dx} + \log_e u\frac{dv}{dx}\right\}$$

Illustrative Examples

Example 11. Differentiate the following functions with respect to x

(i) x^x (ii) $(\sin x)^x$ (iii) $x^{\log_e x}$ (iv) $x^{\sin x}$

Solution : (i) Let $y = x^x$

taking log both the sides

$$\log_e y = \log_e (x^x)$$

$$\Rightarrow \quad \log_e y = x \log_e x$$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + \log_e x$$

$$\Rightarrow \qquad \frac{dy}{dx} = y\{1+\log_e x\} = x^x\{1+\log_e x\} = x^x \log_e ex$$

(ii) Let $\qquad y = (\sin x)^x$

taking log both the sides $\qquad \log_e y = x \log_e \sin x$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx} = x.\frac{1}{\sin x}.\cos x + 1.\log_e \sin x$$

$$\Rightarrow \qquad \frac{dy}{dx} = y\left\{x\cot x + \log_e \sin x\right\}$$

$$\Rightarrow \qquad \frac{dy}{dx} = (\sin x)^x\left\{x\cot x + \log_e \sin x\right\}$$

(iii) Let $\qquad y = x^{\log_e x}$

taking log both the sides

$$\log_e y = \log_e x.\log_e x$$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}.\log_e x + \frac{1}{x}.\log_e x$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{2y}{x}\log_e x = \frac{2x^{\log_e x}}{x}.\log_e x = 2x^{(\log_e x - 1)}.\log_e x$$

(iv) Let $\qquad y = x^{\sin x}$

taking log both the sides

$$\log_e y = \sin x \log_e x$$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \sin x.\frac{1}{x} + (\cos x)\log_e x$$

$$\Rightarrow \qquad \frac{dy}{dx} = y\left\{\frac{\sin x}{x} + \cos x.\log_e x\right\}$$

$$= x^{\sin x}\left\{\frac{\sin x}{x} + \cos x.\log_e x\right\}$$

$$= x^{\sin x - 1}.\sin x + x^{\sin x}.\cos x.\log_e x.$$

Example 12. Differentiate the following functions with respect to x

(i) $\cos x.\cos 2x.\cos 3x$ (ii) $(\log x)^{\cos x}$ (iii) $\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Solution : (i) Let $y = \cos x.\cos 2x.\cos 3x$

taking log both sides

$$\log y = \log(\cos x) + \log(\cos 2x) + \log(\cos 3x)$$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\cos x}(-\sin x) + \frac{1}{\cos 2x}(-2\sin 2x) + \frac{1}{\cos 3x}(-3\sin 3x)$$

$$\frac{dy}{dx} = -y\{\tan x + 2\tan 2x + 3\tan 3x\}$$

$$= -\cos x.\cos 2x.\cos 3x\{\tan x + 2\tan 2x + 3\tan 3x\}$$

(ii) Let $y = (\log x)^{\cos x}$

taking log both sides

$$\log y = \cos x \log(\log x)$$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \cos x \frac{d}{dx}\{\log(\log x)\} + \log(\log x)\frac{d}{dx}(\cos x)$$

$$= \cos x.\frac{1}{\log x}.\frac{1}{x} - \sin x.\log(\log x)$$

$$\frac{dy}{dx} = y\left\{\frac{\cos x}{x\log x} - \sin x.\log(\log x)\right\}$$

$$= (\log x)^{\cos x}\left\{\frac{\cos x}{x\log x} - \sin x\log(\log x)\right\}$$

(iii) Let $y = \sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

taking log both the sides

$$\log y = \frac{1}{2}\{\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)\}$$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{(x-1)} + \frac{1}{(x-2)} - \frac{1}{(x-3)} - \frac{1}{(x-4)} - \frac{1}{(x-5)}\right]$$

$$\frac{dy}{dx}=\frac{y}{2}\left[\frac{1}{(x-1)}+\frac{1}{(x-2)}-\frac{1}{(x-3)}-\frac{1}{(x-4)}-\frac{1}{(x-5)}\right]$$

$$=\frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{(x-1)}+\frac{1}{(x-2)}-\frac{1}{(x-3)}-\frac{1}{(x-4)}-\frac{1}{(x-5)}\right]$$

Example 13. Determine $\frac{dy}{dx}$

(i) $x^y = y^x$ (ii) $y=\sqrt{x+\sqrt{x+\sqrt{x+.....\infty}}}$ (iii) $(\cos x)^y = (\sin y)^x$ (iv) $x^y.y^x = \kappa$

Solution : (i) Here $x^y = y^x$

taking log both the sides

$$y\log x = x\log y$$

Differentiating with respect to x

$$y.\frac{1}{x}+\log x.\frac{dy}{dx}=x.\frac{1}{y}\frac{dy}{dx}+\log y.1$$

$$\Rightarrow \quad \frac{dy}{dx}\left\{\log x-\frac{x}{y}\right\}=\log y-\frac{y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{y(x\log y-y)}{x(y\log x-x)}.$$

(ii) Here $y=\sqrt{x+\sqrt{x+\sqrt{x+.....\infty}}}$

$$\therefore \quad y=\sqrt{x+y}$$

$$\Rightarrow \quad y^2 = x+y$$

Differentiating with respect to x

$$2y\frac{dy}{dx}=1+\frac{dy}{dx}$$

$$(2y-1)\frac{dy}{dx}=1$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{2y-1}$$

(iii) Here $(\cos x)^y = (\sin y)^x$

taking log both the sides

$$y\log(\cos x) = x\log(\sin y)$$

Differentiating with respect to x

$$y.\frac{1}{\cos x}(-\sin x)+\log(\cos x)\frac{dy}{dx}=x.\frac{\cos y}{\sin y}\frac{dy}{dx}+\log(\sin y).1$$

$$\Rightarrow \qquad \frac{dy}{dx}=\frac{\log(\sin y)+y\tan x}{\log(\cos x)-x\cot y}$$

(iv) Here $\qquad x^y.y^x=k$

taking log both the sides

$$\log x^y+\log y^x=\log k$$

$$\Rightarrow \qquad y\log x+x\log y=\log k$$

Differentiating with respect to x

$$y.\frac{1}{x}+\log x\frac{dy}{dx}+x.\frac{1}{y}\frac{dy}{dx}+\log y.1=0$$

$$\Rightarrow \qquad \left(\log x+\frac{x}{y}\right)\frac{dy}{dx}=-\left(\log y+\frac{y}{x}\right)$$

$$\Rightarrow \qquad \frac{dy}{dx}=\frac{-y(x\log y+y)}{x(y\log x+x)}.$$

Example 14. Find $\frac{dy}{dx}$ **:**

(i) $x^a.y^b=(x+y)^{a+b}$ (ii) $\sqrt{x^2+y^2}=\log(x^2-y^2)$

(iii) $x\sqrt{1+y}+y\sqrt{1+x}=0$ (iv) $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$

Solution : (i) Here $\qquad x^a.y^b=(x+y)^{a+b}$

taking log both the sides

$$\log x^a+\log y^b=(a+b)\log(x+y)$$

$$\Rightarrow \qquad a\log x+b\log y=(a+b)\log(x+y)$$

Differentiating with respect to x

$$a.\frac{1}{x}+b.\frac{1}{y}\frac{dy}{dx}=(a+b).\frac{1}{(x+y)}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \qquad \left(\frac{b}{y}-\frac{a+b}{x+y}\right)\frac{dy}{dx}=\frac{a+b}{x+y}-\frac{a}{x}$$

$$\Rightarrow \qquad \left\{\frac{b(x+y)-y(a+b)}{y(x+y)}\right\}\frac{dy}{dx}=\frac{x(a+b)-a(x+y)}{x(x+y)}$$

$$\Rightarrow \qquad \frac{dy}{dx}=\frac{y}{x}.$$

(ii) Here $\sqrt{x^2+y^2} = \log(x^2 - y^2)$

Differentiating with respect to x

$$\frac{1}{2\sqrt{x^2+y^2}}\left(2x+2y\frac{dy}{dx}\right) = \frac{1}{\left(x^2-y^2\right)}\left(2x-2y\frac{dy}{dx}\right)$$

$$\left\{\frac{y}{\sqrt{x^2+y^2}}+\frac{2y}{x^2-y^2}\right\}\frac{dy}{dx} = \frac{2x}{x^2-y^2}-\frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{dy}{dx} = \frac{2x\sqrt{x^2+y^2}-x\left(x^2-y^2\right)}{y\left(x^2-y^2\right)+2y\sqrt{x^2+y^2}}.$$

(iii) Here $x\sqrt{1+y}+y\sqrt{1+x}=0$

$\Rightarrow$ $x\sqrt{1+y} = -y\sqrt{1+x}$

squaring both the sides

$$x^2(1+y) = y^2(1+x)$$

$\Rightarrow$ $x^2 - y^2 + x^2y - xy^2 = 0$

$\Rightarrow$ $(x-y)(x+y)+xy(x-y)=0$

$\Rightarrow$ $(x-y)(x+y+xy)=0$

If $x-y=0$ or $x=y$, which does not satisfy the given equation, $x-y \neq 0$

$\therefore$ $x+y+xy=0$

Differentiating with respect to x

$$1+\frac{dy}{dx}+1.y+x\frac{dy}{dx}=0$$

$\Rightarrow$ $(1+x)\frac{dy}{dx} = -(1+y)$

$\Rightarrow$ $\frac{dy}{dx} = -\left(\frac{1+y}{1+x}\right)$

(iv) Here $\sqrt{1-x^2}+\sqrt{1-y^2} = a(x-y)$

Here putting $x=\sin\theta,\ y=\sin\phi$

$$\sqrt{1-\sin^2\theta}+\sqrt{1-\sin^2\phi} = a(\sin\theta-\sin\phi)$$

$\Rightarrow$ $\cos\theta+\cos\phi = a(\sin\theta-\sin\phi)$

$\Rightarrow$ $2\cos\frac{\theta+\phi}{2}.\cos\frac{\theta-\phi}{2} = 2a\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}$

$$\Rightarrow \quad \cot\frac{\theta-\phi}{2}=a$$

$$\Rightarrow \quad \frac{\theta-\phi}{2}=\cot^{-1}(a)$$

$$\Rightarrow \quad \theta-\phi=2\cot^{-1}(a)$$

$$\Rightarrow \quad \sin^{-1}x-\sin^{-1}y=2\cot^{-1}a$$

Differentiating with respect to x

$$\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$$

Example 15. Find $\frac{dy}{dx}$:

(i) $y=\sqrt{\log x+\sqrt{\log x+\sqrt{\log x+\ldots\infty}}}$ (ii) $y=(\sin x)^{(\sin x)^{\ldots\infty}}$ (iii) $y=e^{x+e^{x+e^{x+\ldots\infty}}}$

Solution : (i) Here $y=\sqrt{\log x+\sqrt{\log x+\sqrt{\log x+\ldots\infty}}}$

or $y=\sqrt{\log x+y}$

Squaring both the sides $y^2=\log x+y$

Differentiating both the sides with respect to x

$$2y\cdot\frac{dy}{dx}=\frac{1}{x}+\frac{dy}{dx}$$

$$\Rightarrow \quad (2y-1)\frac{dy}{dx}=\frac{1}{x}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{x(2y-1)}.$$

(ii) Here $y=(\sin x)^{(\sin x)^{\ldots\infty}}$

$$=(\sin x)^y$$

taking log both the sides

$$\log y=y\log(\sin x)$$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx}=y.\frac{1}{\sin x}.\cos x+\log(\sin x).\frac{dy}{dx}$$

$$\left\{\frac{1}{y}-\log(\sin x)\right\}\frac{dy}{dx} = y\cot x$$

$$\frac{dy}{dx} = \frac{y^2\cot x}{1-y\log(\sin x)}.$$

(iii) Here $y = e^x + e^z + e^x + \ldots\infty = e^{x+y}$

taking log both the sides

$$\log y = (x+y)\log e$$

$\Rightarrow$ $\log y = x + y$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx} = 1+\frac{dy}{dx}$$

$$\left(\frac{1}{y}-1\right)\frac{dy}{dx} = 1$$

$\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{y}{1-y}.$

Exercise 7.3

Find $\dfrac{dy}{dx}$:

1. (a) $2x+3y=\sin y$ (b) $x^2+xy+y^2=200$
2. (a) $\sqrt{x}+\sqrt{y}=\sqrt{a}$ (b) $\tan(x+y)+\tan(x-y)=4$
3. (a) $\sin x+2\cos^2 y+xy=0$ (b) $x\sqrt{y}+y\sqrt{x}=1$
4. (a) $(x^2+y^2)^2=xy$ (b) $\sin(xy)+\dfrac{x}{y}=x^2-y$
5. (a) $x^3+y^3=3axy$ (b) $x^y+y^x=a^b$
6. (a) $y=x^y$ (b) $x^a.y^b=(x-y)^{a+b}$
7. (a) $y=e^x+e^{x^2}+....+e^{x^5}$ (b) $y=\sqrt{e^{\sqrt{x}}}, x>0$
8. (a) $y=\dfrac{\cos x}{\log x}, x>0$ (b) $y=\sqrt{x}^{\sqrt{x}^{\sqrt{x}^{\cdots\infty}}}$
9. (a) $y\sqrt{1-x^2}=\sin^{-1}x$ (b) $y\sqrt{1+x}=\sqrt{1-x}$
10. (a) $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\ldots\infty}}}$ (b) $y^x+x^y+x^x=a^b$

7.07 Derivative of parametric functions

When x and y are represented in terms of other variable like $x = f(t),\ y = \phi(t)$ then variable t is said to be parameter and equations of such type are known as parametric equations. The below given formula is also used to find $\frac{dy}{dx}$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \text{ where } \frac{dx}{dt} \neq 0.$$

Illustrative Examples

Example 16. Find $\frac{dy}{dx}$, when

(i) $x = 2at^2,\ y = at^4$ (ii) $x = \sin t,\ y = \cos 2t$ (iii) $x = 4t,\ y = \frac{4}{t}$

Solution : (i) Here $x = 2at^2 \quad \Rightarrow \frac{dx}{dt} = 4at$

and $y = at^4 \quad \Rightarrow \frac{dy}{dt} = 4at^3$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4at^3}{4at} = t^2.$$

(ii) Here $x = \sin t \quad \Rightarrow \frac{dx}{dt} = \cos t$

and $y = \cos 2t \quad \Rightarrow \frac{dy}{dt} = -2\sin 2t$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{2\sin 2t}{\cos t} = \frac{-2.2\sin t\cos t}{\cos t} = -4\sin t$$

(iii) Here $x = 4t \Rightarrow \frac{dx}{dt} = 4$

and $y = \frac{4}{t} \Rightarrow \frac{dy}{dt} = -\frac{4}{t^2}$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-4/t^2}{4} = -\frac{1}{t^2}$$

Example 17. Find $\frac{dy}{dx}$, when

(i) $x=\sin^{-1}\left(\frac{2t}{1+t^2}\right),\quad y=\cos^{-1}\left(\frac{1-t^2}{1+t^2}\right)$ (ii) $x=\frac{3at}{1+t^3},\quad y=\frac{3at^2}{1+t^3}$

(iii) $x=e^{\theta}\left(\theta+\frac{1}{\theta}\right),\quad y=e^{-\theta}\left(\theta-\frac{1}{\theta}\right)$

Solution: (i) $x=\sin^{-1}\left(\frac{2t}{1+t^2}\right),\quad y=\cos^{-1}\left(\frac{1-t^2}{1+t^2}\right)$

Here putting $t=\tan\theta$

$$x=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)=\sin^{-1}(\sin 2\theta)=2\theta \qquad \Rightarrow \frac{dx}{d\theta}=2$$

and $$y=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\cos^{-1}(\cos 2\theta)=2\theta \qquad \Rightarrow \frac{dy}{d\theta}=2$$

$\therefore$ $$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{2}{2}=1.$$

(ii) Here $x=\frac{3at}{1+t^3},\quad y=\frac{3at^2}{1+t^3}$

differentiating with respect to t

$$\frac{dx}{dt}=\frac{(1+t^3)(3a)-3at(0+3t^2)}{(1+t^3)^2}=\frac{3a-6at^3}{(1+t^3)^2}$$

and $$\frac{dy}{dt}=\frac{(1+t^3)(6at)-3at^2(0+3t^2)}{(1+t^3)^2}=\frac{6at-3at^4}{(1+t^3)^2}$$

$\therefore$ $$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{6at-3at^4}{3a-6at^3}=\frac{t(2-t^3)}{1-2t^3}$$

(iii) Here $x=e^{\theta}\left(\theta+\frac{1}{\theta}\right),\quad y=e^{-\theta}\left(\theta-\frac{1}{\theta}\right)$

differentiating with respect to θ

$$\frac{dx}{d\theta}=e^{\theta}.\left(\theta+\frac{1}{\theta}\right)+e^{\theta}\left(1-\frac{1}{\theta^2}\right)=e^{\theta}\left(\frac{\theta^2-1+\theta^3+\theta}{\theta^2}\right)$$

$$\frac{dy}{d\theta} = -e^{-\theta}\left(\theta - \frac{1}{\theta}\right) + e^{-\theta}\left(1 + \frac{1}{\theta^2}\right) = e^{-\theta}\left(\frac{\theta^2 + 1 - \theta^3 + \theta}{\theta^2}\right)$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{e^{-\theta}\left(\theta^2 + 1 - \theta^3 + \theta\right)}{e^{\theta}\left(\theta^2 - 1 + \theta^3 + \theta\right)}.$$

Example 18. If $x^2 + y^2 = t - \frac{1}{t}$ and $x^4 + y^4 = t^2 + \frac{1}{t^2}$, then prove that $x\frac{dy}{dx} + y = 0$

Solution : Given $t - \frac{1}{t} = x^2 + y^2$ and $t^2 + \frac{1}{t^2} = x^4 + y^4$

$\because$ $$\left(t - \frac{1}{t}\right)^2 = t^2 + \frac{1}{t^2} - 2$$

$\Rightarrow$ $$(x^2 + y^2)^2 = x^4 + y^4 - 2$$

$\Rightarrow$ $$x^4 + y^4 + 2x^2y^2 = x^4 + y^4 - 2$$

$\therefore$ $$x^2y^2 = -1$$

differentiating with respect to x

$$x^2.2y\frac{dy}{dx} + 2x.y^2 = 0$$

$\Rightarrow$ $$2xy\left(x\frac{dy}{dx} + y\right) = 0$$

$\Rightarrow$ $$x\frac{dy}{dx} + y = 0.$$

Exercise 7.4

Find $\frac{dy}{dx}$, when

1. (a) $x = a\sec t,\ y = b\tan t$ (b) $x = \log t + \sin t,\ y = e^t + \cos t$
2. (a) $x = \log t,\ y = e^t + \cos t$ (b) $x = a\cos\theta,\ y = b\sin\theta$
3. (a) $x = \cos\theta - \cos 2\theta,\ y = \sin\theta - \sin 2\theta$ (b) $x = \theta - \sin\theta,\ y = a(1 + \cos\theta)$
4. (a) $x = \frac{\sin^3 t}{\sqrt{\cos 2t}},\ y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$ (b) $x = a\left(\cos t + \log\tan\frac{t}{2}\right),\ y = a\sin t$
5. (a) $x = \sqrt{\sin 2\theta},\ y = \sqrt{\cos 2\theta}$ (b) $x = a\cos^3 t,\ y = a\sin^3 t$
6. If $x^3 + y^3 = t - \frac{1}{t}$ and $x^6 + y^6 = t^2 + \frac{1}{t^2}$, then prove that $x^4y^2\frac{dy}{dx} = 1$

7.08 Second Order Derivative

Let $y = f(x)$

then $\frac{dy}{dx} = f'(x)$ (1)

Now if $f'(x)$ is derivable then we can differentiate equation (1) with respect to x. Then left hand side becomes $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ which is known as second order derivative of $f(x)$ and is written as $\frac{d^2y}{dx^2}$, or $f''(x)$.

Derivatives of higher orders can also be found like this

Illustrative Examples

Example 19. Find the second order derivative of the following functions

(i) x^{20} (ii) $x^3 \log x$ (iii) $e^{6x}.\cos 3x$

(iv) $\log(\log x)$ (v) $\sin(\log x)$ (vi) $\tan^{-1} x$.

Solution : (i) Let $y = x^{20}$

$\Rightarrow \quad \frac{dy}{dx} = 20x^{19}$

$\Rightarrow \quad \frac{d^2y}{dx^2} = 20.19x^{18} = 380x^{18}$.

(ii) Let $y = x^3 \log x$

$\Rightarrow \quad \frac{dy}{dx} = x^3.\frac{1}{x} + \log x.3x^2 = x^2 + 3x^2 \log x$

$\therefore \quad \frac{d^2y}{dx^2} = 2x + 3\left\{x^2.\frac{1}{x} + \log x.2x\right\}$

$= 2x + 3(x + 2x\log x) = 5x + 6x\log x = x(5 + 6\log x)$.

(iii) Let $y = e^{6x} \cos 3x$

$\Rightarrow \quad \frac{dy}{dx} = e^{6x}(-\sin 3x).3 + \cos 3x.e^{6x}.6$

$= 6e^{6x}.\cos 3x - 3e^{6x}.\sin 3x$

$\therefore$

$\frac{d^2y}{dx^2} = 6\{e^{6x}(-\sin 3x).3 + \cos 3x.e^{6x}.6\} - 3\{e^{6x}.\cos 3x.3 + \sin 3x.e^{6x}.6\}$

$= -18e^{6x}\sin 3x + 36e^{6x}\cos 3x - 9e^{6x}\cos 3x - 18e^{6x}\sin 3x$

$= 9e^{6x}(3\cos 3x - 4\sin 3x)$.

(iv) Let $y = \log(\log x)$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\log x} \cdot \frac{1}{x}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{1}{\log x}\left(-\frac{1}{x^2}\right) + \frac{1}{x}\frac{d}{dx}\left(\frac{1}{\log x}\right)$$

$$= -\frac{1}{x^2 \log x} + \frac{1}{x}\left\{\frac{\log x.(0) - 1.\frac{1}{x}}{(\log x)^2}\right\} = -\frac{1}{x^2 \log x} + \frac{1}{x}\left(-\frac{1}{x(\log x)^2}\right)$$

$$= -\frac{1}{x^2 \log x} - \frac{1}{x^2 (\log x)^2} = -\frac{1}{x^2 \log x}\left(1 + \frac{1}{\log x}\right).$$

(v) Let $\quad y = \sin(\log x)$

$$\Rightarrow \quad \frac{dy}{dx} = \cos(\log x).\frac{1}{x}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \cos(\log x)\left(-\frac{1}{x^2}\right) + \frac{1}{x}\{-\sin(\log x)\}.\frac{1}{x}$$

$$= -\frac{\cos(\log x)}{x^2} - \frac{\sin(\log x)}{x^2} = -\frac{1}{x^2}\{\cos(\log x) + \sin(\log x)\}.$$

(vi) Let $\quad y = \tan^{-1} x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{(1+x^2)(0) - 1.(0+2x)}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2}$$

Example 20. If $y = (x + \sqrt{x^2 - 1})^m$, then prove that

$$(x^2 - 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2 y = 0.$$

Solution : Given $\quad y = (x + \sqrt{x^2 - 1})^m$

Differentiating with respect to x

$$\frac{dy}{dx} = m\left(x + \sqrt{x^2 - 1}\right)^{m-1}\left\{1 + \frac{2x}{2\sqrt{x^2 - 1}}\right\}$$

$$= m(x + \sqrt{x^2 - 1})^{m-1}\frac{(\sqrt{x^2 - 1} + x)}{\sqrt{x^2 - 1}} = \frac{m(x + \sqrt{x^2 - 1})^m}{\sqrt{x^2 - 1}} = \frac{my}{\sqrt{x^2 - 1}}$$

squaring both the sides

$$\left(x^2-1\right)\left(\frac{dy}{dx}\right)^2=m^2y^2$$

again differentiating with respect to x

$$\left(x^2-1\right)\cdot 2\left(\frac{dy}{dx}\right).\frac{d^2y}{dx^2}+2x\left(\frac{dy}{dx}\right)^2=m^2 2y\frac{dy}{dx}$$

dividing by $2\frac{dy}{dx}$

$$\left(x^2-1\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-m^2y=0.$$

Example 21. If $x^3+y^3+3ax^2=0$

$$\frac{d^2y}{dx^2}+\frac{2a^2x^2}{y^5}=0.$$

Solution : Here $\qquad x^3+y^3+3ax^2=0 \qquad (1)$

Differentiating with respect to x

$$3x^2+3y^2\frac{dy}{dx}+3a.2x=0$$

$$\Rightarrow \qquad \frac{dy}{dx}=-\left(\frac{x^2+2ax}{y^2}\right) \qquad (2)$$

Again differentiating with respect to x

$$\frac{d^2y}{dx^2}=-\left[\frac{y^2(2x+2a)-(x^2+2ax)2y\frac{dy}{dx}}{(y^2)^2}\right]$$

Substituting the value of $\frac{dy}{dx}$ from (2)

$$\frac{d^2y}{dx^2}=-\frac{1}{y^3}\left\{y\left(2x+2a\right)+\left(x^2+2ax\right)2.\frac{\left(x^2+2ax\right)}{y^2}\right\}$$

$$=-\frac{2}{y^5}\left\{y^3(x+a)+x^4+4a^2x^2+4ax^3\right\}$$

from equation (1) putting $\qquad y^3=-\left(3ax^2+x^3\right)$

$$\frac{d^2y}{dx^2}=-\frac{2}{y^5}\left\{-\left(3ax^2+x^3\right)(x+a)+x^4+4a^2x^2+4ax^3\right\}$$

$$=-\frac{2}{y^5}\left\{-3ax^3-x^4-3a^2x^2-ax^3+x^4+4a^2x^2+4ax^3\right\}$$

$$= -\frac{2}{y^5}\left(a^2x^2\right)$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0$$

Example 22. If $y = \sin\left(a\sin^{-1}x\right)$, then prove that

$$\left(1-x^2\right)y_2 - xy_1 + a^2y = 0$$

Solution : Here $y = \sin(a\sin^{-1}x)$

Differentiating with respect to x

$$y_1 = \cos\left(a\sin^{-1}x\right).\frac{a}{\sqrt{1-x^2}}$$

Squaring both the sides $(1-x^2)y_1^2 = a^2\cos^2(a\sin^{-1}x) = a^2\{1-\sin^2(a\sin^{-1}x)\}$

$$\Rightarrow \qquad (1-x^2)y_1^2 = a^2(1-y^2)$$

Again differentiating with respect to x

$$(1-x^2)2y_1y_2 - 2xy_1^2 = a^2(0-2yy_1)$$

Dividing by $2y_1$,

$$(1-x^2)y_2 - xy_1 + a^2y = 0.$$

Exercise 7.5

1. Find $\frac{d^2y}{dx^2}$, when

   (a) $y = x^3 + \tan x$ (b) $y = x^2 + 3x + 2$ (c) $y = x\cos x$

   (d) $y = 2\sin x + 3\cos x$ (e) $y = e^{-x}\cos x$ (f) $y = a\sin x - b\cos x$

2. If $y = a\sin x + b\cos x$, then prove that

$$\frac{d^2y}{dx^2} + y = 0.$$

3. If $y = \sec x + \tan x$, then prove that

$$\frac{d^2y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}.$$

4. If $y = a\cos nx + b\sin nx$, then prove that

$$\frac{d^2y}{dx^2} + n^2y = 0.$$

5. If $x = a\cos^3\theta$, $y = a\sin^3\theta$, then find $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{4}$

6. If $x^3 + y^3 - 3axy = 0$, then prove that

$$\frac{d^2y}{dx^2} = \frac{2a^2xy}{(ax-y^2)^3}.$$

7. If $y=\sin^{-1}x$, then prove that : $\left(1-x^2\right)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}=0.$

8. If $y=(\sin^{-1}x)^2$,then prove that : $\left(1-x^2\right)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}-2=0.$

7.09 Rolle's theorem

If a real valued function f is defined in the interval $[a, b]$, such that

(i) f is continuous in the closed interval in $[a, b]$

(ii) f is derivable in the open interval in (a, b)

(iii) $f(a)=f(b)$

then in the open interval (a, b) there exists a point c such that $f'(c)=0$

7.10 Geometrical meaning of Rolle's Theorem

We can define Rolle's Theorem under two conditions:

Case I: when the function f is constant then

$$f(x)=c, \quad \forall\, x\in[a, b]$$

The graph of the function will be paralel to x-axis. Thus for every point in the open interval (a, b) $f'(x)=0$ (see fig : 7.01)

Case II: When function f is not constant then

as per Rolle's theorem let f be continuous in a closed interval $[a, b]$ and derivable in the open interval (a, b), then f is derivable.That means tangents can be drawn at $x\in(a,b)$ to the curve $y=f(x)$ .Also $f(a)=f(b)$, it is clear from this that the value of the function $f(x)$ will either increase or decrease (see fig 7.02), under both the conditions there exists a point which will always be parallel to x-axis i.e. at that point $f'(x)=0$, i.e. at these points the slope of the line will be zero

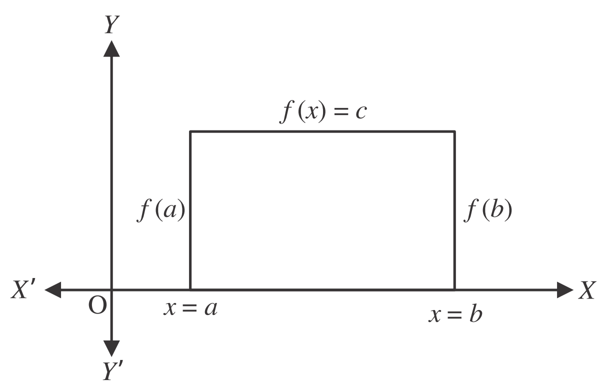
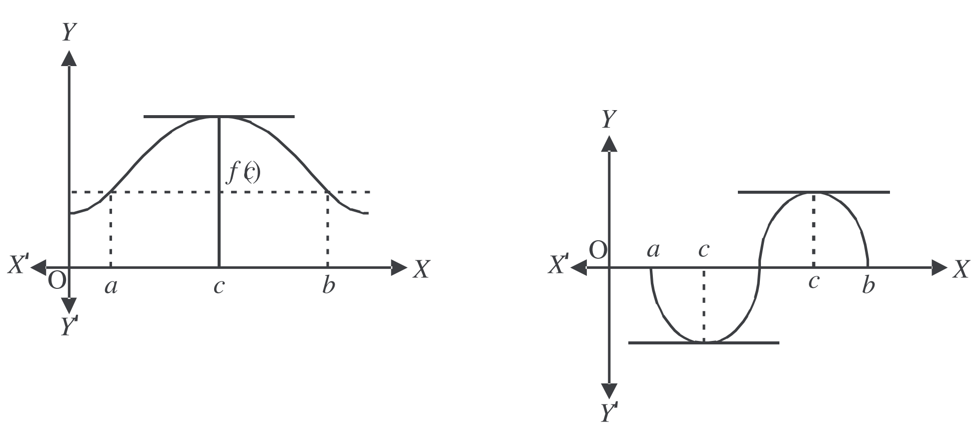


Fig. 7.01



(a) (b)

Fig 7.02

7.11 Lagrange's mean value theorem

If a real valued function f is defined in the closed interval $[a, b]$ such that

(i) f is continuous in $[a, b]$

(ii) f is differentiable in (a, b)

then there exists a point c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Note : Mean value yheorem is the extension of Rolle's theorem.

7.12 Geometrical meaning of Lagrange's mean value theorem (LMVT)

The graph of function $y = f(x)$ is shown in fig. 7.03. Also $f'(c)$ is the slope of $y = f(x)$ at point $(c, f(c))$. It is clear from the fig 7.03 that $\dfrac{f(b) - f(a)}{b - a}$ is the slope of the line drawn from the points $(a, f(a))$ and $(b, f(b))$. According to the LMVT there exists a point c in (a, b) such that the tangent drawn at point $(c, f(c))$ is parallel to the line drawn from the points $(a, f(a))$ and $(b, f(b))$.

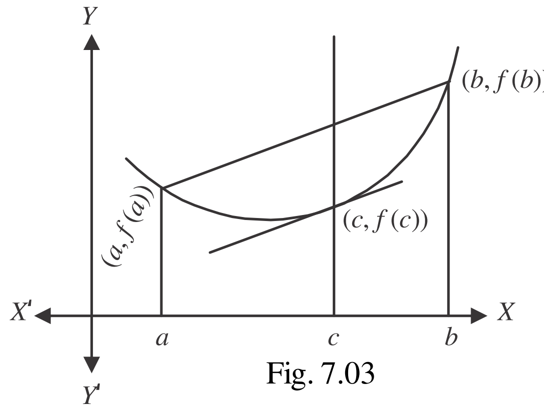


Fig. 7.03

7.13 Other form of Lagrange's mean value theorem

If we take $b = a + \hbar,\ \hbar > 0, c = a + \theta\hbar,\ 0 < \theta < 1$ and $c \in (a, b) \Rightarrow a + \theta\hbar \in (a, a + \hbar)$, in Lagrange's mean value theorem then it takes the form as shown below-

If the real valued function f is defined in the interval $[a, a + h]$

(i) f is continuous in the closed interval $[a, a + h]$

(ii) f is differentiable in the open interval $(a, a + h)$ then there exists a real number θ in the interval (0, 1) such that $f(a + h) = f(a) + hf'(a + \theta h)$

Note: For this theorem $f(a) = f(b)$ is not necessary. If $f(a) = f(b)$ then this theorem changes into Rolle's theorem.

Illustrative Examples

Example 23. Verify the Rolle's theorem for the following functions

(i) $f(x) = \sqrt{4 - x^2};\quad x \in [-2, 2]$ (ii) $f(x) = e^x \sin x;\quad x \in [0, \pi]$

Solution : (i) Clearly the function $f(x) = \sqrt{4 - x^2}$ is continuous in the interval $[-2, 2]$ and $f'(x) = \dfrac{-x}{\sqrt{4 - x^2}}$, which is defined at every point of the interval $(-2, 2)$ i.e. $f(x)$, is derivable in the interval $(-2, 2)$

$\because \quad f(-2) = 0 = f(2)$

$\Rightarrow \quad f(-2) = f(2)$

function $f(x)$, satisfies all the three conditions

Hence $\quad f'(c) = 0 \Rightarrow \dfrac{-c}{\sqrt{4 - c^2}} = 0$

$\Rightarrow \quad c = 0 \quad \because c \in (-2, 2)$

Thus Rolle's theorem is verified

(ii) $$f(x) = e^x \sin x, \ \ x \in [0, \pi]$$

Clearly the function $f(x)$, is continuous in the interval $[0, \pi]$ and $f'(x) = e^x \cos x + e^x \sin x$, which is defined at every point of the interval $(0, \pi)$ i.e. $f(x)$, is derivable in the interval $(0, \pi)$

$\because$ $$f(0) = 0 = f(\pi)$$

function $f(x)$, satisfies all the three conditions

Hence $$f'(c) = 0 \Rightarrow e^c \cos c + e^c \sin c = 0$$

$\Rightarrow$ $$e^c(\cos c + \sin c) = 0$$

$\Rightarrow$ $$\cos c + \sin c = 0$$

$\Rightarrow$ $$c = \frac{3\pi}{4} \qquad \because c \in (0, \pi)$$

Thus Rolle's theorem is verified

Example 24. Verify Rolle's theorem for the following functions

(i) $f(x) = 3 + (x-2)^{2/3}; \quad x \in [1, 3]$ (ii) $f(x) = \sin\frac{1}{x}; \quad x \in [-1, 1]$

Solution : (i) $$f(x) = 3 + (x-2)^{2/3}; \quad x \in [1, 3]$$

Clearly $f(x)$, is continuous in the interval at $[1, 3]$

$f'(x) = \dfrac{2}{3(x-2)^{1/3}}$, is infinite in the interval at $x = 2 \in (1, 3)$, $f(x)$ is not derivable.

Thus Rolle's theorem is not verified for $f(x)$ in the interval $[1, 3]$

(ii) $$f(x) = \sin\frac{1}{x}; \quad x \in [-1, 1]$$

$\because$ Function $f(x) = \sin\frac{1}{x}$ is not continuous at $x = 0$ $0 \in [-1, 1]$ thus $f(x), [-1, 1]$ is not continuous, Rolle's theorem is not verified for $f(x) = \sin\frac{1}{x}$ in the interval $[-1, 1]$.

Example 25. Examine the applicability of Lagrange's mean value theorem for following functions:

(i) $f(x) = |x|; \quad x \in [-1, 1]$ (ii) $f(x) = \frac{1}{x}; \quad x \in [-1, 1]$

(iii) $f(x) = x - \frac{1}{x}; \quad x \in [1, 3]$ (iv) $f(x) = x - 2\sin x; \quad x \in [-\pi, \pi]$

Solution : (i) $\because$ $f(x) = |x|$ is continuous everywhere hence it is continuous in the interval $[-1, 1]$ also $f(x) = |x|$ is not derivable at $x = 0$ therefore function $f(x)$, is not derivable in the interval $(-1, 1)$. Thus LMVT is not verified for $f(x)$ in the interval $[-1, 1]$

(ii) $\because f(x)=\frac{1}{x}; \quad x=0 \in [-1, 1]$ is not continuous so $f(x)$ is also not continuous in the interval [–1, 1], thus LMVT is not verified.

(iii) Here $f(x)=x-\frac{1}{x}; \quad x\in[1, 3],$ which is continuous at [1, 3] and $f'(x)=1+\frac{1}{x^2},$ which exists and finite in the interval (1, 3) thus $f(x)$ is derivable in the interval (1, 3). Hence function $f(x)$, satisfies the conditions of Lagrange's MVT

Now $$f'(c)=\frac{f(3)-f(1)}{3-1}$$

$$\Rightarrow \quad 1+\frac{1}{c^2}=\frac{3-\frac{1}{3}-\left(1-\frac{1}{1}\right)}{2}$$

$$\Rightarrow \quad 1+\frac{1}{c^2}=\frac{4}{3}$$

$$\Rightarrow \quad \frac{1}{c^2}=\frac{1}{3}$$

$$\Rightarrow \quad c=\pm\sqrt{3}$$

$$\Rightarrow \quad x=\sqrt{3}\in(1, 3)$$

Thus LMVT is verified.

(iv) Here $f(x)=x-2\sin x; \quad x\in[-\pi, \pi]$ clearly $f(x)$, is continuous and derivable in the interval $[-\pi, \pi]$ thus $f(x)$ satisifes both the conditions of MVT in the interval $[-\pi, \pi]$, hence their exists a point c in the interval $(-\pi, \pi)$ such that

$$f'(c)=\frac{f(\pi)-f(-\pi)}{\pi-(-\pi)}$$

$$\Rightarrow \quad 1-2\cos c=\frac{\pi-(-\pi)}{2\pi}=\frac{2\pi}{2\pi}=1$$

$$\Rightarrow \quad \cos c=0$$

$$\Rightarrow \quad c=\pm\frac{\pi}{2}, \pm\frac{3\pi}{2} \qquad \because c=\pm\frac{\pi}{2}\in(-\pi, \pi)$$

Thus LMVT is satisfied.

Exercise 7.6

1. Verify Rolle's theorem for the functions given below:

(a) $f(x)=e^x(\sin x-\cos x);\quad x\in[\pi/4,\ 5\pi/4]$ (b) $f(x)=(x-a)^m(x-b)^n;\quad x\in[a,b],\ m,n\in N$

(c) $f(x)=|x|;\quad x\in[-1,1]$ (d) $f(x)=x^2+2x-8;\quad x\in[-4,2]$

(e) $f(x)=\begin{cases}x^2+1 & ;\ 0\le x\le 1\\ 3-x & ;\ 1<x\le 2\end{cases}$ (f) $f(x)=[x];\quad x\in[-2,2]$

2. Verify Rolle's theorem for the functions given below :

(a) $f(x)=x^2+5x+6;\quad x\in[-3,-2]$ (b) $f(x)=e^{-x}\sin x;\quad x\in[0,\pi]$

(c) $f(x)=\sqrt{x(1-x)};\quad x\in[0,1]$ (d) $f(x)=\cos 2x;\quad x\in[0,\pi]$

3. Verify Lagrange's mean value theorem for the functions given below:

(a) $f(x)=x+\dfrac{1}{x};\quad x\in[1,3]$ (b) $f(x)=\dfrac{x^2-4}{x-1};\quad x\in[0,2]$

(c) $f(x)=x^2-3x+2;\quad x\in[-2,3]$ (d) $f(x)=\dfrac{1}{4x-1};\quad x\in[1,4]$

Miscellaneous Examples

Example 26. Find the differential coefficient of the function with respect to x

(a) $\cos x^{\circ}$ (b) $\sin\log(1+x^2)$ (c) $\log\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$

(d) $\log(x+\sqrt{x^2+a^2})$ (e) $\log_7(\log x)$

Solution : (a) Let
$$y=\cos x^{\circ}$$
$\because$
$$180^{\circ}=\pi\ \text{radian}$$
$$x^{\circ}=\frac{\pi}{180}x\ \text{radian}$$
$$y=\cos\left(\frac{\pi x}{180}\right)$$

differentiating with respect ot x
$$\frac{dy}{dx}=-\sin\left(\frac{\pi x}{180}\right)\frac{d}{dx}\left(\frac{\pi x}{180}\right)=\frac{-\pi}{180}\sin\left(\frac{\pi x}{180}\right)=\frac{-\pi}{180}\sin x^{\circ}.$$

(b) Let
$$y=\sin\log\left(1+x^2\right)$$
$\Rightarrow$
$$\frac{dy}{dx}=\cos\log\left(1+x^2\right)\frac{d}{dx}\left\{\log\left(1+x^2\right)\right\}$$

$$= \cos\left(\log\left(1+x^2\right)\right).\frac{1}{\left(1+x^2\right)}\frac{d}{dx}\left(1+x^2\right)$$

$$\frac{1}{\left(1+x^2\right)}\cos\left(\log(1+x^2)\right)(0+2x) = \frac{2x}{1+x^2}\cos\log(1+x^2)$$

(c) Let $$y = \log\tan\left(\pi/4 + x/2\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\tan(\pi/4+x/2)}\frac{d}{dx}\{\tan(\pi/4+x/2)\}$$

$$= \frac{1}{\tan(\pi/4+x/2)}\sec^2(\pi/4+x/2)\frac{d}{dx}(\pi/4+x/2)$$

$$= \frac{1}{2\sin(\pi/4+x/2)\cos(\pi/4+x/2)}$$

$$= \frac{1}{\sin 2(\pi/4+x/2)} = \frac{1}{\sin(\pi/2+x)} = \frac{1}{\cos x} = \sec x.$$

(d) Let $$y = \log(x+\sqrt{x^2+a^2})$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{(x+\sqrt{x^2+a^2})}\frac{d}{dx}(x+\sqrt{x^2+a^2})$$

$$= \frac{1}{(x+\sqrt{x^2+a^2})}.\left(1+\frac{2x}{2\sqrt{x^2+a^2}}\right)$$

$$= \frac{1}{\left(x+\sqrt{x^2+a^2}\right)}\left(\frac{\sqrt{x^2+a^2}+x}{\sqrt{\left(x^2+a^2\right)}}\right) = \frac{1}{\sqrt{x^2+a^2}}.$$

(e) Let $y = \log_7(\log x) = \dfrac{1}{\log_e 7}\{\log_e(\log x)\}$, (Base change formula)

defined for $x > 1$,

$$\frac{dy}{dx} = \frac{1}{(\log 7)}\frac{d}{dx}\{\log(\log x)\}$$

$$= \frac{1}{\log_7}.\frac{1}{\log x}\frac{d}{dx}(\log x) = \frac{1}{x\log 7.\log x}.$$

Example 27. Differentiate the following functions with respect to x

(a) $\sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$ (b) $\tan^{-1}\left(\frac{x^{1/3}+a^{1/3}}{1-(ax)^{1/3}}\right)$ (c) $\sin^{-1}(x\sqrt{1-x}-\sqrt{x}\cdot\sqrt{1-x^2})$.

Solution : (a) $y=\sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$

$$=\sin^{-1}\left(\frac{2^x\cdot 2}{1+(2^x)^2}\right)=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) \qquad [\text{let } 2^x=\tan\theta\ ]$$

$$=\sin^{-1}(\sin 2\theta)=2\theta=2\tan^{-1}(2^x)$$

$$\Rightarrow \quad \frac{dy}{dx}=2\cdot\frac{1}{1+(2^x)^2}\frac{d}{dx}(2^x)=\frac{2}{(1+4^x)}.2^x\log 2=\frac{2^{x+1}\log 2}{1+4^x}.$$

(b) Let $y=\tan^{-1}\left(\frac{x^{1/3}+a^{1/3}}{1-(ax)^{1/3}}\right)$

(Using formula, $\tan^{-1}\left(\frac{A+B}{1-AB}\right)=\tan^{-1}A+\tan^{-1}B$)

$$y=\tan^{-1}(x^{1/3})+\tan^{-1}(a^{1/3})$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{1+(x^{1/3})}\frac{d}{dx}(x^{2/3})+0$$

$$=\frac{(1/3)x^{-2/3}}{1+x^{2/3}}=\frac{1}{3x^{2/3}(1+x^{2/3})}.$$

(c) Let $y=\sin^{-1}(x\sqrt{1-x}-\sqrt{x}\cdot\sqrt{1-x^2})$

(Using $\sin^{-1}A-\sin^{-1}B=\sin^{-1}(A\sqrt{1-B^2}-B\sqrt{1-A^2})$)

$$y=\sin^{-1}(x)-\sin^{-1}(\sqrt{x})$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-(\sqrt{x})^2}}\cdot\frac{d}{dx}(\sqrt{x})$$

$$=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{2\sqrt{x}\sqrt{1-x}}.$$

Example 28. Find $\frac{dy}{dx}$ when $x=(t+1/t)^a$ and $y=a^{t+1/t}$, where a is a constant

Solution : It is clear that both x and y are functions of t and $t\neq 0$ defined for all real numbers

Now $$\frac{dx}{dt}=a\left(t+\frac{1}{t}\right)^{a-1}\frac{d}{dt}\left(t+\frac{1}{t}\right)=a\left(t+\frac{1}{t}\right)^{a-1}\left(1-\frac{1}{t^2}\right).$$

Here $\frac{dx}{dt}\neq 0$ If $1-\frac{1}{t^2}\neq 0 \quad \Rightarrow \quad t\neq \pm 1$

and $$\frac{dy}{dt}=\frac{d}{dt}(a^{t+1/t})=a^{t+1/t}.\log a\frac{d}{dt}(t+1/t)=a^{t+1/t}\cdot\log a\left(1-\frac{1}{t^2}\right)$$

now for, $t\neq\pm 1$

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{a^{(t+1/t)}\left(1-\frac{1}{t^2}\right)\log a}{a(t+1/t)^{a-1}(1-1/t^2)}=\frac{a^{(t+1/t)}\log a}{a(t+1/t)^{a-1}}.$$

Example 29. If $p^2=a^2\cos^2\theta+b^2\sin^2\theta$ then prove that $p+\frac{d^2p}{d\theta^2}=\frac{a^2b^2}{p^3}$.

Solution : Given $$p^2=a^2\cos^2\theta+b^2\sin^2\theta \tag{1}$$

differentiating with respect to θ

$$2P\frac{dp}{d\theta}=-2a^2\cos\theta\sin\theta+2b^2\sin\theta\cos\theta$$

$$=\left(b^2-a^2\right)\sin 2\theta \tag{2}$$

again differentiating with respect to θ

$$2p\frac{d^2p}{d\theta^2}+2\left(\frac{dp}{d\theta}\right)^2=2\left(b^2-a^2\right)\cos 2\theta$$

multiplying both sides with p^2

$$p^3\frac{d^2p}{d\theta^2}+p^2\left(\frac{dp}{d\theta}\right)^2=p^2\left(b^2-a^2\right)\cos 2\theta$$

adding p^4 both the sides

$$p^4+p^3\frac{d^2p}{d\theta^2}+\left(p\frac{dp}{d\theta}\right)^2=p^4+p^2\left(b^2-a^2\right)\cos 2\theta$$

putting the value from (2)

$$p^4+p^3\frac{d^2p}{d\theta^2}+\frac{\left(b^2-a^2\right)^2}{4}\cdot\sin^2 2\theta=p^4+p^2(b^2-a^2)\cos 2\theta$$

$$\Rightarrow p^4 + p^3\frac{d^2p}{d\theta^2} + (b^2 - a^2)\sin^2\theta\cos^2\theta = p^2\{p^2 + (b^2 - a^2)(\cos^2\theta - \sin^2\theta)\}$$

$$= p^2\{(a^2\cos^2\theta + b^2\sin^2\theta) + (b^2 - a^2)(\cos^2\theta - \sin^2\theta)\}\text{[from equation (1)]}$$

$$= p^2(b^2\cos^2\theta + a^2\sin^2\theta)$$

$$= (a^2\cos^2\theta + b^2\sin^2\theta)(b^2\cos^2\theta + a^2\sin^2\theta) \qquad \text{[from (1)]}$$

$$\Rightarrow p^4 + p^3\frac{d^2p}{d\theta^2} = a^2b^2(\sin^4\theta + \cos^4\theta) + a^4\sin^2\theta\cos^2\theta + b^4\sin^2\theta\cos^2\theta - (b^2 - a^2)^2\sin^2\theta\cos^2\theta$$

$$= a^2b^2(\sin^4\theta + \cos^4\theta + 2\sin^2\theta\cos^2\theta) = a^2b^2(\sin^2\theta + \cos^2\theta) = a^2b^2$$

$$\Rightarrow \qquad p + \frac{d^2p}{d\theta^2} = \frac{a^2b^2}{p^3}$$

Example 30. If $x = a\cos\theta + b\sin\theta,\quad y = a\sin\theta - b\cos\theta$ then prove that $y^2y_2 - xy_1 + y = 0$

Solution : From given equation, $x = a\cos\theta + b\sin\theta,\ y = a\sin\theta - b\cos\theta$

$$x^2 + y^2 = (a\cos\theta + b\sin\theta)^2 + (a\sin\theta - b\cos\theta)^2 = a^2 + b^2$$

differentiating with respect to x

$$\Rightarrow \qquad 2x + 2yy_1 = 0$$

$$\Rightarrow \qquad y_1 = -\frac{x}{y} \qquad (1)$$

again differentiating with respect to x

$$y_2 = -\left\{\frac{y.1 - xy_1}{y^2}\right\} = -\left\{\frac{y + x\cdot x/y}{y^2}\right\} \qquad \text{[from (1)]}$$

$$= -\frac{y^2 + x^2}{y^3} \qquad (2)$$

$$\Rightarrow \qquad y^2y_2 - xy_1 + y = y^2\left(-\frac{y^2 + x^2}{y^3}\right) - x\left(\frac{-x}{y}\right) + y$$

$$= \frac{1}{y}\{-y^2 - x^2 + x^2 + y^2\} = 0.$$

Example 31. Verify Rolle's theorem for the functions given below:

(i) $f(x)=\log\left\{\dfrac{x^2+ab}{x(a+b)}\right\}; \quad x\in[a,b], x\neq 0$ (ii) $f(x)=\tan x; \quad x\in[0,\pi]$

Solution : (i) $$f(x)=\log\left\{\frac{x^2+ab}{x(a+b)}\right\}; \quad x\in[a,b], x\neq 0$$

$$=\log(x^2+ab)-\log x-\log(a+b)$$

clearly, $f(x)$ is continuous in $[a, b]$ and logarithmic functions are derivable thus $f(x)$ is derivable in the interval (a, b) as $f'(x)=\dfrac{2x}{x^2+ab}-\dfrac{1}{x}=\dfrac{x^2-ab}{x(x^2+ab)}$

now $$f(a)=\log\left\{\frac{a^2+ab}{a(a+b)}\right\}=\log 1=0$$

$\Rightarrow$ $$f(b)=\log\left\{\frac{b^2+ab}{b(a+b}\right\}=\log 1=0$$

$\Rightarrow$ $$f(a)=f(b)$$

$f(x)$, satisfies all the three conditions of Rolle's theorem

$$\therefore \quad f'(c)=0$$

$\Rightarrow$ $$\frac{c^2-ab}{c(c^2+ab)}=0$$

$\Rightarrow$ $$c=\sqrt{ab}\in(a,b)$$

Thus Rolle's theorem is verified.

(ii) $\because$ $f(x)=\tan x, x=\pi/2$ is not continuous as $\pi/2\in[0,\pi]$ i.e $f(x)$, is not continuous in the interval $[0, \pi]$, thus for $f(x)=\tan x; \quad x\in=[0,\pi]$ Rolle's theorem is not verified.

Miscellaneous Exercise–7

Differentiate the following functions with respect to x (Q 1-8)

1. $\sin^{-1}(x\sqrt{x}); \quad 0\leq x\leq 1$

2. $\dfrac{\cos^{-1} x/2}{\sqrt{2x+7}}; \quad -2<x<2$

3. $\cot^{-1}\left\{\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}; \quad 0<x<\dfrac{\pi}{2}$

4. $x^3.e^x.\sin x$

5. $\log\left(\dfrac{x}{a^x}\right)$

6. $(x\log x)^{\log x}$

7. $x^{x^2-3}+(x-3)^{x^2}; \quad x>3$

8. $\sin^{-1} x+\sin^{-1}\sqrt{1-x^2}$

9. If $\log x = \tan^{-1}\left(\frac{y-x^2}{x^2}\right)$, then find $\frac{dy}{dx}$ 10. If $y = 12(1-\cos t)$, $x = 10(t - \sin t)$, then find $\frac{dy}{dx}$

11. If $\cos^{-1}\left(\frac{x^2-y^2}{x^2+y^2}\right) = \tan^{-1} a$, then prove that $\frac{dy}{dx} = \frac{y}{x}$

12. If $\sin y = x\sin(a+y)$, then prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$

13. If $y = (\sin x - \cos x)^{(\sin x - \cos x)}$, then find $\frac{dy}{dx}$.

14. If $y = \sin(\sin x)$, then show that

$$\frac{d^2y}{dx^2} + \tan x . \frac{dy}{dx} + y\cos^2 x = 0.$$

15. (a) If $y = e^{ax}\sin bx$, then show that

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + \left(a^2+b^2\right)y = 0.$$

(b) If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$, then prove that

$$(1-x^2)y_2 = 3ay_1 - y = 0.$$

16. Verify Rolle's theorem for the functions

(a) $f(x) = (x-2)\sqrt{x}; \quad x \in [0, 2]$ (b) $f(x) = (x-1)(x-3); \quad x \in [1, 3]$

17. Examine the applicability of Lagrange's mean value theorem for the functions given below.

(a) $f(x) = (x-1)(x-2)(x-3); \quad x \in [0, 4]$ (b) $f(x) = \begin{cases} 1+x & ; & x < 2 \\ 5-x & ; & x \geq 2 \end{cases}; \quad x \in [1, 3]$

Important Points

1. If the functions f and g are differentiable at any point c in the interval $[a, b]$ then $f \pm g, fg$ and f/g are also differentiable at point c and

(i) $D(f \pm g)(c) = f'(c) \pm g'(c)$ (ii) $D(fg)(c) = f'(c)g(c) + f(c)g'(c)$

(iii) $D(f/g)(c) = \frac{g(c)f'(c) - g'(c)f(c)}{[g(c)]^2}$; when $g(c) \neq 0$

2. If $y = f(u)$ and $u = \phi(x)$ then $\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$

3. (i) $\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$; (ii) $\frac{d}{dx}(\cos^{-1}x)=-\frac{1}{\sqrt{1-x^2}}$; (iii) $\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$

(iv) $\frac{d}{dx}(\cot^{-1}x)=-\frac{1}{1+x^2}$; (v) $\frac{d}{dx}(\sec^{-1}x)=\frac{1}{|x|\sqrt{x^2-1}}$; (vi) $\frac{d}{dx}\left(\operatorname{cosec}^{-1}x\right)=\frac{-1}{|x|\sqrt{x^2-1}}$

4. A real valued functions is continuous at a point in its domain if the limit of the function at that point equals the value of the function at that point. A function is continuous if its continuous on the whole of its domain. Every differentiable function is continuous, but the converse is not true.

5. For the function of the type $y=u^v$ solve it by taking log on both the sides.

6. $x=f(t),\ y=g(t)$ in this t is the parameter, we get $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$ where $dx/dt \neq 0$

7. If $f'(x)$ is also a continuous function of x then it can be again differentiated.

8. **Rolle's Theroem:**

If Real valued function f is defiend in the interval $[a, b]$, such that,

(i) f is continuous in the closed interval $[a, b]$

(ii) f is differentiable in the open interval (a, b)

(iii) $f(a)=f(b)$

then in the open interval (a, b) there exists a point c such that $f'(c)=0$

9. **Lagrange's mean value theorem:**

If a real valued function f is defiend in the closed interval $[a, b]$ such that

(i) f is continuous in $[a, b]$

(ii) f is differentiable in (a, b)

then there exists a point c in (a, b) such that $f'(c)=\frac{f(b)-f(a)}{b-a}$

10. **Lagrange's Mean Value Theorem:**

If we take $\mathrm{b=a+h,\ h>0,\ c=a+\theta h,\ 0<\theta<1}$ and $\mathrm{c\in(a,\ b)\Rightarrow a+\theta h\in(a,\ a+h)}$, in lagrange mean value theorem then it takes the form as shown below.

If the real valued function f is defiend in the interval $[a, a + h]$ such that

(i) f is continuous in the closed interval $[a, a+\hbar]$

(ii) f is differentiable in the open interval $(a, a+\hbar)$ then there exists a real number θ in the interval (0, 1) such that $f(a+\hbar)=f(a)+\hbar f'(a+\theta\hbar)$

ANSWERS

Exericse 7.1

1. $2x\cos x^2$ 2. $2\sec^2(2x+3)$ 3. $-2x\sin x^2\cos\left(\cos x^2\right)$ 4. $\dfrac{2\sin x}{(1+\cos x)^2}$ 5. $\dfrac{1-\sqrt{1-x^2}}{x^2\sqrt{1-x^2}}$

6. $\dfrac{\pi}{180}\cos x^{\circ}$ 7. $\cos ecx$ 8. $\dfrac{\pi}{180}\sec x^{\circ}\tan x^{\circ}$ 9. $\sec x$ 10. $\dfrac{1}{\sqrt{x^2+a^2}}$ 11. $\dfrac{2(1-x^2)}{1+x^2+x^4}$

12. $\left(\dfrac{x}{1+x^2}\right)\sec^2\left(\log\sqrt{1+x^2}\right)$ 13. $3.a^{\tan 3x}.\sec^2 3x.\log a$ 14. $\sec x$ 15. $3\sin^2 x.\sin 4x$

Exercise 7.2

1.(a) $\dfrac{2}{\sqrt{1-x^2}}$ (b) $\dfrac{3}{\sqrt{1-x^2}}$ 2. (a) $\dfrac{-2}{1+x^2}$ (b) $\dfrac{2}{1+x^2}$ 3. (a) $\dfrac{-3}{\sqrt{1-x^2}}$ (b) $\dfrac{-1}{2\sqrt{1-x^2}}$

4. (a) $\dfrac{-2}{\sqrt{1-x^2}}$ (b) $\dfrac{2}{1+x^2}$ 5. (a) 0 (b) $\dfrac{2}{\sqrt{1-4x^2}}$ 6. (a) $\dfrac{1}{1+x^2}$ (b) $\dfrac{2^{x+1}.\log 2}{1+4^x}$

7. (a) $\dfrac{-x}{\sqrt{1-x^2}}$ (b) $-\dfrac{1}{2\left(1+x^2\right)}$

Exercise 7.3

1. (a) $\dfrac{2}{\cos y-3}$ (b) $\dfrac{-(2x+y)}{x+2y}$ 2. (a) $-\sqrt{\dfrac{y}{x}}$ (b) $\dfrac{\sec^2(x+y)+\sec^2(x-y)}{\sec^2(x-y)-\sec^2(x+y)}$

3. (a) $\dfrac{\cos x+y}{2\sin 2y-x}$ (b) $\dfrac{-y}{x}\left(\dfrac{\sqrt{y}+2\sqrt{x}}{\sqrt{x}+2\sqrt{y}}\right)$ 4. (a) $\dfrac{4x^3+4xy^2-y}{x-4x^2y-4y^3}$ (b) $\dfrac{y\left\{2xy-1-y^2\cos(xy)\right\}}{\left\{y^2x\cos(xy)-x+y^2\right\}}$

5. (a) $\dfrac{ay-x^2}{y^2-ax}$ (b) $-\left\{\dfrac{yx^{y-1}+y^x\log y}{xy^{x-1}+x^y\log x}\right\}$ 6. (a) $\dfrac{y^2}{x(1-y\log x)}$ (b) $\dfrac{y}{x}$

7. (a) $e^x+2xe^{x^2}+3x^2e^{x^3}+4x^3e^{x^4}+5x^4e^{x^5}$ (b) $\dfrac{e^{\sqrt{x}}}{4\sqrt{x.e^{\sqrt{x}}}}$

8. (a) $\dfrac{x\sin x\log x+\cos x}{x\left(\log x\right)^2}$ (b) $\dfrac{y^2}{x\left(2-y\log x\right)}$ 9. (a) $\dfrac{1+xy}{1+x^2}$ (b) $\dfrac{y}{x^2-1}$

10. (a) $\dfrac{\cos x}{2y-1}$ (b) $-\left\{\dfrac{y^x.\log y+y.x^{y-1}+x^x(1+\log x)}{x.y^{x-1}+x^y\log x}\right\}$

Exercise 7.4

1. (a) $\frac{b}{a}.\cos ec\, t$ (b) $\frac{t(e^t - \sin t)}{1 + t\cos t}$ 2. (a) $t\left(e^t - \sin t\right)$ (b) $\frac{-b}{a}\cot\theta$

3. (a) $\frac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}$ (b) $-\cot\frac{\theta}{2}$ 4. (a) $\frac{\cos t(1 - 2\cos 2t)}{1 + 2\cos 2t}$ (b) $\tan t$

5. (a) $-(\tan 2\theta)^{3/2}$ (b) $-\tan t$

Exericse 7.5

1. (a) $6x + 2\sec^2 x\tan x$; (b) 2 ; (c) $-(x\cos x + 2\sin x)$; (d) $-2\sin x - 3\cos x$; (e) $2e^{-x}\sin x$;

(f) $-a\sin x + b\cos x$ 5. $\frac{4\sqrt{2}}{3a}$

Exericse 7.6

1. (a) valid (b) valid (c) invalid (d) valid (e) invalid (f) invalid
3. (a) valid (b) invalid (c) invalid (d) valid (e) valid (f) invalid

Miscellaneous Exercise – 7

1. $\frac{3}{2}\frac{\sqrt{x}}{\sqrt{1-x^3}}$ 2. $-x\left\{\frac{2x + 7 + \sqrt{4 - x^2}\cos^{-1} x/2}{\sqrt{4 - x^2}(2x+7)^{3/2}}\right\}$

3. $\frac{1}{2}$

4.. $x^3 e^x \cos x + x^3 e^x \sin x + 3x^2 e^x \sin x$

5. $\frac{1}{x} - \log a$ 6. $(x\log x)^{\log x}.\left\{\frac{\log x(1 + \log x)}{x\log x} + \frac{\log(x.\log x)}{x}\right\}$

7. $x^{x^2-3}\left\{\frac{x^2 - 3}{x} + 2x\log x\right\} + (x-3)^{x^2}\left\{\frac{x^2}{x-3} + 2x\log(x-3)\right\}$

8. 0

9. $2x\{1 + \tan(\log x)\} + x\sec^2(\log x)$

10. $\frac{6}{5}\cot\left(\frac{t}{2}\right)$

13. $(\sin x - \cos x)^{\sin x - \cos x}.(\cos x + \sin x)\{1 + \log(\sin x - \cos x)\}; \sin x > \cos x$

08

Application of Derivatives

8.01 Introduction

In previous chapter we have studied derivative of composite functions, inverse trigonometric functions, implicit functions, exponential functions and logarithemic functions. In this chapter we will study applications of the derivative in various disciplines, e.g. in engineering, science, social science and many other fields. For instance we will learn how the derivative can be used to determine rate of change of quantities or to find the equations of tangent and normal to a curve at a point.

8.02 Rate of change of quantities

Let P be a variable quantity, that changes with respect to time. Let small change in time t is δt, then corresponding change in P is δP . Then $\frac{\delta P}{\delta t}$, is average rate of change in P, and the instantaneous rate of change in P is $\frac{dP}{dt}$ where $\frac{dP}{dt} = \lim_{\delta t \to 0} \frac{\delta P}{\delta t}$.

Where, $\frac{dP}{dt}$, rate of change in P with respect to time t. Further, if two variable v and r are functions of another variable t, then

$$\frac{dv}{dt} = \frac{dv}{dr} . \frac{dr}{dt}$$

Thus, the rate of change of any one of v and r can be calculate using the rate of change in other qantity with respect to time t.

Illustrative Examples

Example 1: Find the rate of change of volume of a sphere with respect to its surface area when radius of sphere is 2 cm.

Solution : $\because$ Volume of sphere = $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$

Surface area of sphere $\quad s = 4\pi r^2 \Rightarrow \frac{ds}{dr} = 8\pi r$

$$\frac{dV}{ds} = \frac{dV/dr}{ds/dr} = \frac{4\pi r^2}{8\pi r} = \frac{r}{2}$$

$$\left(\frac{dV}{ds}\right)_{r=2} = \frac{2}{2} = 1 \text{ cm.}$$

Example 2. A ladder 10 m, long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 1.2 m / s. How fast is its height on the wall decreasing when the foot of the ladder is 6 m. away from the wall.

Solution : Let AB be position of ladder at time t

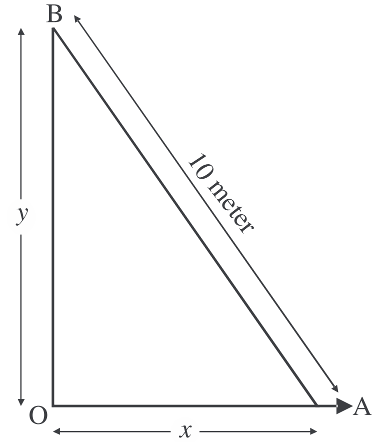


Fig. 8.01

Let $\quad$ OA = x, OB = y then $x^2 + y^2 = 10^2 \qquad (1)$

It is given $\quad \dfrac{dx}{dt} = 1.2$ m/s

Diferentiating (1) with respect to t

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \qquad (2)$$

For $x = 6$, from (1) $6^2 + y^2 = 10^2 \Rightarrow y = 8$ m.

From (2) $2 \times 6 \times 1.2 + 2 \times 8 \dfrac{dy}{dt} = 0$

$$\Rightarrow \quad \frac{dy}{dt} = -\frac{14.4}{16} = -0.9 \text{ m/s. (towards ground)}$$

Example 3. The volume of a cube is increasing at a rate of 9 cubic centimetres per second. How fast is the surface area increasing when the length of an edge is 10 centimetres?

Solution : Let x be the length of a side, V be the volume and s be the surface area of the cube. Then, $V = x^3$, $S = 6x^2$, where x is a function of time t.

Now, $\quad \dfrac{dV}{dt} = 9$ cm^3 / s.

$$\Rightarrow \quad 9 = \frac{d}{dt}(x^3) = \frac{d}{dx}(x^3)\frac{dx}{dt} = 3x^2\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{3}{x^2} \qquad (1)$$

and $\quad \dfrac{dS}{dt} = \dfrac{d}{dt}(6x^2) = \dfrac{d}{dx}(6x^2)\dfrac{dx}{dt} = 12x\left(\dfrac{3}{x^2}\right) = \dfrac{36}{x} \qquad$ [From (1) से]

$\because \quad x = 10$ cm.

$$\Rightarrow \quad \frac{dS}{dt} = \frac{36}{10} = 3.6 \text{ cm}^2 / \text{s}.$$

Example 4. The surface area of a bubble is increasing at the rate of 2 cm^2 / s. At what rate is the volume of the bubble increasing when the radius is 6 cm.

Solution : Let surface area and volume of a bubble of radius r be S and V respectively.

Then $\quad S = 4\pi r^2 \Rightarrow \dfrac{dS}{dr} = 8\pi r$

and $$V=\frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr}=4\pi r^2$$

Given that $$\frac{dS}{dt}=2\ \text{cm}^2/\text{s}$$

$\because$ $$\frac{dS}{dt}=\frac{dS}{dr}.\frac{dr}{dt}\Rightarrow 2=8\pi r\frac{dr}{dt}\Rightarrow\frac{dr}{dt}=\frac{1}{4\pi r}$$

$\because$ $$\frac{dV}{dt}=\frac{dV}{dr}.\frac{dr}{dt}=4\pi r^2.\frac{1}{4\pi r}=r$$

Hence $$\left(\frac{dV}{dt}\right)_{r=6}=6\ \ \text{cm}^3/\text{s}$$

Example 5. The length x of a rectangle is decreasing at the rate of 3 cm / minute and the width y is increasing at the rate of 2 cm / minute. When $x = 12$ and $y = 6$. Find the rate of change of the perimeter and the area of the rectange.

Solution : Since the length x is decreasing and the width y is increasing with respect to time, we have

$$\frac{dx}{dt}=-3\ \text{cm / minute},\quad \frac{dy}{dt}=2\ \ \text{cm / minute}$$

$\because$ Perimeter of rectnage $$p=2(x+y)$$

$\Rightarrow$ $$\frac{dp}{dt}=2\left(\frac{dx}{dt}+\frac{dy}{dt}\right)=2(-3+2)=-2\ \ \text{cm / minute}$$

and area of rectange $$A=x.y$$

$\Rightarrow$ $$\frac{dA}{dt}=x\frac{dy}{dt}+\frac{dx}{dt}.y$$

$$=(12)(2)+(-3).6$$

$$=24-18$$

$$=6\ \ \text{cm}^2/\ \text{minute}$$

Example 6. Water is dripping out from a conical funnel at uniform rate 4 cm^3 / s through a tiny hole at the vertex in the bottom. When the slant height of the water is 4 cm. Find the rate of the decrease of the slant height of the water, given that the semi vertical angle of the funnel is 60°.

Solution : Let volume of water at time t is V.

$\therefore$ The volume of cone of water PEF is V and slant height $PE=\ell$

$\therefore$ $$O'E=\ell\sin 60^\circ=\ell.\frac{\sqrt{3}}{2}$$

and $$O'P=\ell\cos 60^\circ=\ell.\frac{1}{2}$$

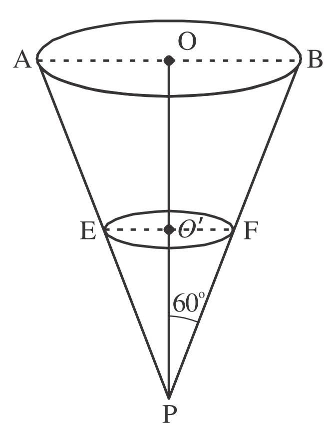


Fig. 8.02

$\because \quad V = \frac{1}{3}\pi(O'E)^2 . O'P$

$$= \frac{1}{3}\pi\left(\frac{\ell\sqrt{3}}{2}\right)^2 . \left(\frac{\ell}{2}\right)$$

$\Rightarrow \quad V = \frac{\pi \ell^3}{8}$

$\Rightarrow \quad \frac{dV}{dt} = \frac{3\pi\ell^2}{8}\frac{d\ell}{dt}$

It is given that $\frac{dV}{dt} = -4$

So $-4 = \frac{3\pi\ell^2}{8}\frac{d\ell}{dt}$

$\Rightarrow \quad \frac{d\ell}{dt} = -\frac{32}{3\pi\ell^2}$

So, at $\ell = 4$ $\quad \frac{d\ell}{dt} = \frac{-32}{3\pi(4)^2} = -\frac{2}{3\pi}$ cm / s.

Exercise 8.1

1. Find the rate of change of the area of a circle with respect to radius r, when $r = 3$ cm and $r = 4$ cm.
2. A particle is moving along the curve $y = \frac{2}{3}x^3 + 1$. Find the points on the curve at which the y-coordinate is changing twice as fast as the x coordinate.
3. A ladder 13 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, from the wall, at the rate of 1.5 m / s. How fast is its height on the wall decreasing when the foot of the ladder is 12 m away from the wall?
4. An edge of a variable cube is increasing at the rate of 3 cm / s. Fnd the rate at which the volume of the cube increasing when the edge is 10 cm long?
5. A ballon which always remains spherical on inflation, is being inflated by pumping at the rate of 900 cm^3 /s. of gas. Find the rate at whcih the radius of ballon increases when the radius is 15 cm.
6. A ballon, which always remains spherical has a variable diameter $\frac{3}{2}(2x+1)$. Find the rate at which its volume is increasing with respect to x.
7. The total cost $C(x)$ rupees, associated with the production of x units of an item is given by

   $$C(x) = 0.005\, x^3 - 0.02\, x^2 + 30\, x + 5000$$

   Find the marginal cost when 3 units are produced, here by marginal cost we mean the instantaneous rate of change of total cost at any level of output.
8. The radius of a soap bubble is increasing at the rate of 0.2 cm / s. Find the rate of increase in surface area when the radius is 7 cm. Also find the rate of change in volume when the radius is 5 cm.

9. Sand is pouring from a pipe at the rate of 12 cm^3 / s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of base. How fast is the height of the sand cone increasing when the height is 4 cm?

10. The total revenue in rupees received from the sale of x units of a product is given by

$$R(x) = 13x^2 + 26x + 15$$

Find the marginal revenue when $x = 15$.

8.03 Increasing and Decreasing Functions

In this section, we will use differentiation to find out wheather a function is increasing or decreasing.

Increasing Function : A function $f(x)$ is called an increasing function in open interval (a, b) if

$$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2), \quad \forall \quad x_1, x_2 \in (a, b)$$

Strictly Increasing Function : A function $f(x)$ is called a strictly increasing function is open interval (a, b) if

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2), \quad \forall \quad x_1, x_2 \in (a, b)$$

i.e. if x increases in open interval (a, b) then $f(x)$ will also increase.

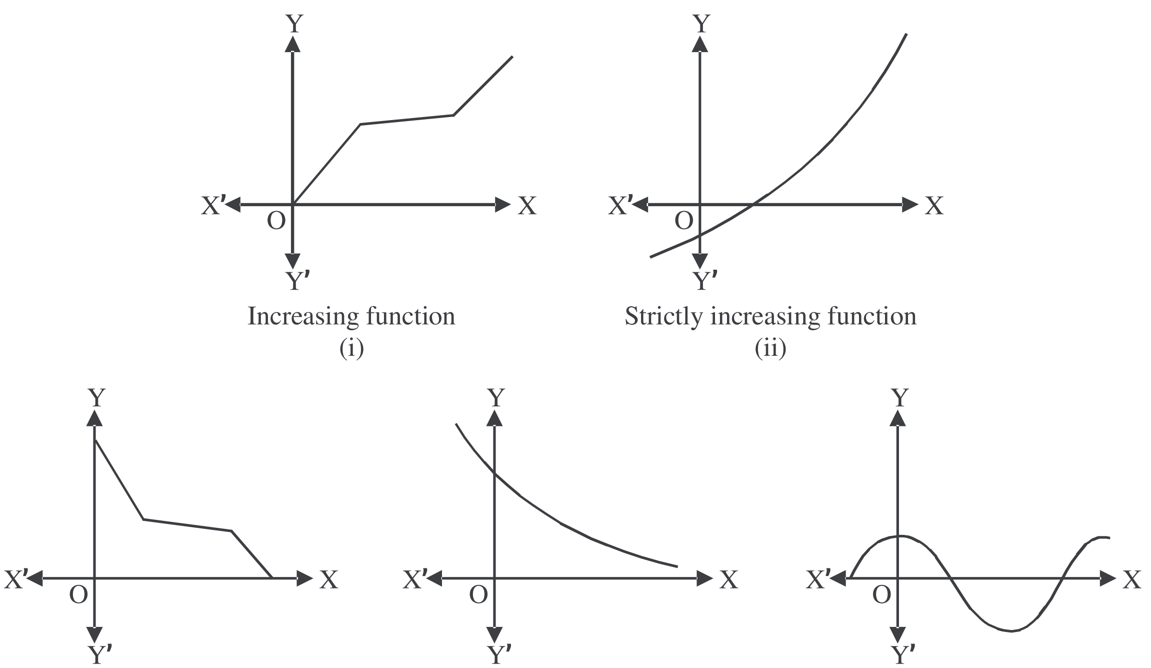
Decreasing Function : A function $f(x)$ is called a decreasing function in open interval (a, b) if

$$x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2), \quad \forall \quad x_1, x_2 \in (a, b)$$

Strictly Decreasing Function : A function $f(x)$ is called a strictly decreasing function in open interval (a, b) if

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2), \quad \forall \quad x_1, x_2 \in (a, b)$$

i.e. in open interval (a, b) when x increases $f(x)$ decreases.



Increasing function (i)

Strictly increasing function (ii)

Decreasing function (iii)

Strictly decreasing function (iv)

Neither increasing nor decreasing function (v)

Fig. 8.03

8.04 Theorem

Let f be continuous on $[a, b]$ and differentiable in the open interval (a, b). Then

(i) f is increasing in [a, b] if $f'(x) > 0$ for each $x \in [a, b]$

(ii) f is decreasing in [a, b] if $f'(x) < 0$ for each $x \in [a, b]$

(iii) f is constant function in [a, b] if $f'(x) = 0$ for each $x \in [a, b]$

Proof : (i) Let $x_1, x_2 \in [a, b]$ be such that $x_1 < x_2$

Then by Lagrange's mean value theorem there exist a point c between x_1 and x_2 such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

$\Rightarrow$ $f(x_2) - f(x_1) > 0$ $\quad (\because f'(c) > 0)$

$\Rightarrow$ $f(x_2) > f(x_1)$

So, $\forall \ x_1, x_2 \in [a, b]$

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

So, $f(x)$ is increasing function in [a, b]

Similary parts (ii) and (iii) can be proved.

Illustrative Examples

Example 7. Find the intervals in whcih the function $f(x) = 2x^3 - 9x^2 + 12x + 3$,

(a) increasing (b) Decreasing

Solution : $f(x) = 2x^3 - 9x^2 + 12x + 3$

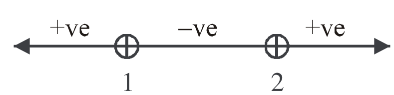


Fig. 8.04

$\Rightarrow$ $f'(x) = 6x^2 - 18x + 12$

$= 6(x^2 - 3x + 2)$

Now $f'(x) = 0 \Rightarrow 6(x^2 - 3x + 2) = 0$

$\Rightarrow$ $(x-2)(x-1) = 0$

$\Rightarrow$ $x = 1, 2$ are critical points.

(a) $f(x)$ is increasing and $f'(x) > 0$

$\Rightarrow$ $6(x^2 - 3x + 2) > 0$

$\Rightarrow$ $(x-1)(x-2) > 0$

$\Rightarrow$ $x < 1$ or $x > 2$

$\Rightarrow$ $x \in (-\infty, 1) \cup (2, \infty)$

Hence, $f(x)$ is increasing in $(-\infty, 1) \cup (2, \infty)$

(b) $f(x)$ is decreasing then $f'(x) < 0$

$\Rightarrow$ $6(x^2 - 3x + 2) < 0$

$\Rightarrow$ $(x-1)(x-2) < 0$

$\Rightarrow \quad x > 1$ or $x < 2$

$\Rightarrow \quad x \in (1, 2)$

Hence, $f(x)$ is decreasing in interval (1, 2)

Example 8. Show that the function *f* given by $f(x) = x^3 - 3x^2 + 4x,$ is strictly increasing on R.

Solution : $\because \quad f(x) = 3x^3 - 3x^2 + 4x$

$\Rightarrow \quad f'(x) = 3x^2 - 6x - 4$

$= 3(x^2 - 2x + 1) + 1$

$= 3(x-1)^2 + 1 > 0, \quad \forall \quad x \in R$

Therefore, the function *f* is strictly increasing on R

Example 9. Find the intervals is which the function $f(x) = -2x^3 + 3x^2 + 12x + 25$

(a) Increasing (b) decreasing

Solution : $\because \quad f(x) = -2x^3 + 3x^2 + 12x + 25$

$\Rightarrow \quad f'(x) = -6x^2 + 6x + 12$

$= -6(x^2 - x - 2)$

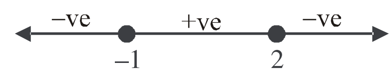


Fig. 8.05

So, $f'(x) = 0 \Rightarrow -6(x^2 - x - 2) = 0$

$\Rightarrow \quad x^2 - x - 2 = 0$

$\Rightarrow \quad (x+1)(x-2) = 0$

$\Rightarrow \quad x = -1, 2$ are critical points.

(a) If $f(x)$ is increasing then $f'(x) > 0$

$\Rightarrow \quad -6(x^2 - x - 2) > 0$

$\Rightarrow \quad x^2 - x - 2 < 0$

$\Rightarrow \quad (x+1)(x-2) < 0$

$\Rightarrow \quad x > -1$ or $x < 2$

$\Rightarrow \quad x \in (-1, 2)$

Hence $f(x)$, is increasing in $(-1, 2)$

(b) If $f(x)$ is decreasing $f'(x) < 0$

$\Rightarrow \quad -6(x^2 - x - 2) < 0$

$\Rightarrow \quad x^2 - x - 2 > 0$

$\Rightarrow \quad (x+1)(x-2) > 0$

$\Rightarrow \quad x < -1$ or $x > 2$

$\Rightarrow \quad x \in (-\infty, -1) \cup (2, \infty)$

Hence, $f(x)$ is decreasing in $(-\infty, -1) \cup (2, \infty)$

Example 10. Find the interval in which function $f(x) = \sin x - \cos x$ is increasing or decreasing.

Solution : $\because \quad f(x) = \sin x - \cos x$

$\Rightarrow \quad f'(x) = \cos x + \sin x$

$\Rightarrow \quad f'(x) = 0$

$\Rightarrow \quad \cos x + \sin x = 0$

$\Rightarrow \quad \sin(\pi/2 + x) + \sin x = 0$

$\Rightarrow \quad 2\sin(\pi/4 + x) \cdot \cos \pi/4 = 0$

$\Rightarrow \quad \sin(\pi/4 + x) = 0 = \sin \pi$

$\Rightarrow \quad \pi/4 + x = \pi$

$\Rightarrow \quad x = 3\pi/4$, which is a critical point.

when $f(x)$ is increasing then $\quad f'(x) > 0$

$\Rightarrow \quad \cos x + \sin x > 0$

$\Rightarrow \quad 2\sin(\pi/4 + x)\cos \pi/4 > 0$

$\Rightarrow \quad \sin(\pi/4 + x) > 0$

$\Rightarrow \quad \sin\{\pi - (\pi/4 + x)\} > 0$

$\Rightarrow \quad \sin(3\pi/4 - x) > 0$

$\Rightarrow \quad 3\pi/4 - x > 0$

$\Rightarrow \quad x < 3\pi/4$

$\Rightarrow \quad x \in (0,\ 3\pi/4)$

Hence $f(x)$ is increasing if $x \in (0,\ 3\pi/4)$

If $f(x)$ is decreasing then $\quad f'(x) < 0$

$\Rightarrow \quad \cos x + \sin x < 0$

$\Rightarrow \quad \sin(\pi/2 + x) + \sin x < 0$

$\Rightarrow \quad 2\sin(\pi/4 + x)\cos \pi/4 < 0$

$\Rightarrow \quad \sin(\pi/4 + x) < 0$

$\Rightarrow \quad \sin\{\pi - (\pi/4 + x)\} < 0$

$\Rightarrow \quad \sin(3\pi/4 - x) < 0$

$\Rightarrow \quad 3\pi/4 - x < 0$

$\Rightarrow \quad x > 3\pi/4 \Rightarrow x \in (3\pi/4, \pi)$

Hence $f(x)$ is decreasing if $x \in (3\pi/4, \pi)$

Example 11. Find the values of x for which $f(x)=\dfrac{x}{1+x^2}$ is increasing or decreasing?

Solution : Given $\quad f(x)=\dfrac{x}{1+x^2} \Rightarrow f'(x)=\dfrac{1-x^2}{(1+x^2)^2}$

$\therefore \quad f'(x)=0 \Rightarrow \dfrac{1-x^2}{(1+x^2)^2}=0$

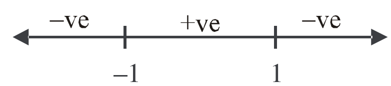


Fig. 8.06

$\Rightarrow \quad x^2-1=0$

$\Rightarrow \quad (x-1)(x+1)=0$

$\Rightarrow \quad x=-1, 1$ are critical points.

If $f(x)$ is increasing then $f'(x)>0$

$\Rightarrow \quad \dfrac{1-x^2}{(1+x^2)^2}>0$

$\Rightarrow \quad 1-x^2>0$

$\Rightarrow \quad -(x^2-1)>0$

$\Rightarrow \quad x^2-1<0$

$\Rightarrow \quad (x-1)(x+1)<0$

$\Rightarrow \quad x\in(-1,1)$

Hence $f(x)$ is increasign for $x\in(-1,1)$

If $f(x)$ is decreasign then $f'(x)<0$

$\Rightarrow \quad \dfrac{1-x^2}{(1+x^2)^2}<0$

$\Rightarrow \quad 1-x^2<0$

$\Rightarrow \quad x^2-1>0$

$\Rightarrow \quad (x-1)(x+1)>0$

$\Rightarrow \quad x\in(-\infty,-1)\cup(1,\infty)$

Hence $f(x)$ is decreasing for $x\in(-\infty,-1)\cup(1,\infty)$

Example 12. Find the intervals in whcih the following functions are increasing or decreasing

(a) x^2+2x+5 (b) $10-6x-2x^2$ (c) $(x+1)^3(x-3)^3$

Solution : (a) Let $\quad f(x)=x^2+2x+5$

$\Rightarrow \quad f'(x)=2x+2=2(x+1)$

$\therefore \quad f'(x)=0 \Rightarrow 2(x+1)=0$

$\Rightarrow \quad x=-1$

Case-I: When $x < -1$

$\Rightarrow \quad x + 1 < 0$

$\therefore \quad f'(x) = 2(-ve) = \text{Negative } < 0$

Hence $f(x)$ is decreasing in $(-\infty, -1)$

Case-II: When $x > -1$

$\Rightarrow \quad x + 1 > 0$

$\therefore \quad f'(x) = \text{Positive } > 0$

Hence $f(x)$ is increasing in $(-1, \infty)$

(b) Let $f(x) = 10 - 6x - 2x^2$

$\Rightarrow \quad f'(x) = -6 - 4x = -2(3 + 2x)$

$\therefore \quad f'(x) = 0 \Rightarrow -2(3 + 2x) = 0$

$\Rightarrow \quad x = -3/2$

Case-I: When $x < -3/2$

$\Rightarrow \quad 3 + 2x < 0$

$\Rightarrow \quad f'(x) = -2(-ve) = \text{Positive } > 0$

Hence $f(x)$ is increasing in $\left(-\infty, -3/2\right)$

Case-II: When $x > -3/2$

$\Rightarrow \quad 3 + 2x > 0$

$\Rightarrow \quad f'(x) = -2(+ve) = \text{Negative} < 0$

$\Rightarrow$ Hence $f(x)$ is decreasing in $\left(-3/2, \infty\right)$

(c) Let $f(x) = \left(x+1\right)^3 \left(x-3\right)^3$

$$\Rightarrow \quad f'(x) = 3(x+1)^2(x-3)^3 + 3(x+1)^3(x-3)^2$$

$$= 3(x+1)^2(x-3)^2\{x - 3 + x + 1\}$$

$$= 6(x+1)^2(x-3)^2(x-1)$$

If $f(x)$ is increasing then $f'(x) > 0$

$\Rightarrow \quad 6(x+1)^2(x-3)^2(x-1) > 0$

$\Rightarrow \quad x - 1 > 0 \qquad [\because 6(x+1)^2(x-3)^2 > 0]$

$\Rightarrow \quad x > 1$

Hence $f(x)$ is increasing in $(1, \infty)$

$f(x)$ is decreasing function then $f'(x) < 0$

$\Rightarrow \quad 6(x+1)^2(x-3)^2(x-1) < 0$

$\Rightarrow \quad x - 1 < 0$

$\Rightarrow \quad x < 1$

Hence $f(x)$, is decreasing in $(-\infty, 1)$

Example 13. Show that $y=\dfrac{4\sin\theta}{2+\cos\theta}-\theta$ is an increasing function of θ in $[0,\ \pi/2]$

Solution : Let

$$f(\theta)=y=\frac{4\sin\theta}{2+\cos\theta}-\theta$$

$$\Rightarrow \quad f'(\theta)=\frac{(2+\cos\theta).4\cos\theta-4\sin\theta(-\sin\theta)}{(2+\cos\theta)^2}-1$$

$$=\frac{4\cos\theta-\cos^2\theta}{(2+\cos\theta)^2}=\frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$$

$$\therefore \quad f'(\theta)=0\Rightarrow\frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}=0$$

$$\Rightarrow \quad \cos\theta=0$$

$$\Rightarrow \quad \theta=\pi/2$$

When $0<\theta<\pi/2$ then $f'(\theta)>0$

Hence $y=f(\theta)$ is increasing in $(0,\ \pi/2)$

Example 14: Prove that the function *f* given by $f(x)=x^2-x+1$ is neither increasing nor decreasing in $(-1,\ 1)$

Solution : Here

$$f(x)=x^2-x+1$$

$$\Rightarrow \quad f'(x)=2x-1$$

$$\therefore \quad f'(x)=0\Rightarrow 2x-1=0\Rightarrow x=1/2$$

Case-I: When $-1<x<1/2$ then $f'(x)<0$

Hence $f(x)$ is decreasing in $(-1,\ 1/2)$

Case-II: When $1/2<x<1$ then $f'(x)>0$

Hence $f(x)$ is increasing in $(1/2,\ 1)$

Hence $f(x)$ is neither increasing nor decreasing in $(-1,\ 1)$

Example 15: Find the value of a for which the function $f(x)=x^2+ax+1$ is increasing on [1, 2].

Solution : Given

$$f(x)=x^2+ax+1$$

$$\Rightarrow \quad f'(x)=2x+a$$

If $f(x)$ is increasing in $[1,\ 2]$ then $f'(x)>0 \quad \forall\ x\in R$

Now $\quad f'(x)=2x+a$

$$\Rightarrow \quad f''(x)=2>0,\quad \forall\ x\in R$$

$$\Rightarrow \quad f(x) \text{ is increasing at } x\in R$$

$$\Rightarrow \quad f'(x) \text{ is increasing at } [1,\ 2]$$

$\Rightarrow$ The least value of $f'(x)$ is $f'(1)$ at $[1, 2]$

$\therefore$ $f'(x) > 0 \quad \forall \; x \in [1, 2]$

$f'(1) > 0 \Rightarrow 2 + a > 0$

$\Rightarrow$ $a > -2$

$\Rightarrow$ $a \in (-2, \infty)$

Exercise 8.2

1. Show that $f(x) = x^2$ is increasing in $(0, \infty)$ and decreasing in $(-\infty, 0)$
2. Show that $f(x) = a^x, o < a < 1, R$ is decreasing in R

Prove that the following functions are increasing in given intervals.

3. $f(x) = \log \sin x, \quad x \in (0, \pi/2)$
4. $f(x) = x^{100} + \sin x + 1, \quad x \in (0, \pi/2)$
5. $f(x) = (x-1)e^x + 1, \quad x > 0$
6. $f(x) = x^3 - 6x^2 + 12x - 1, \quad x \in R$

Prove that the following functions are decreasing in given intervals

7. $f(x) = \tan^{-1} x - x, \quad x \in R$
8. $f(x) = \sin^4 x + \cos^4 x, \quad x \in (0, \pi/4)$
9. $f(x) = 3/x + 5, \quad x \in R, x \neq 0$
10. $f(x) = x^2 - 2x + 3, \quad x < 1$

Find the intervals in which the following functions are increasing or decreasing

11. $f(x) = 2x^3 - 3x^2 - 36x + 7$
12. $f(x) = x^4 - 2x^2$
13. $f(x) = 9x^3 - 9x^2 + 12x + 5$
14. $f(x) = -2x^3 + 3x^2 + 12x + 5$
15. Find the least value of a function $f(x) = x^3 + 9x + 5$, when $f(x)$ is increasing in the interval $[1, 2]$
16. Prove that the function $f(x) = \tan^{-1}(\sin x + \cos x)$, is increasing function in the interval $(0, \pi/4)$

8.05 Tangents and normals

In this section, we shall use differentiation to find the equation of the tangent line and the normal line to a curve at a given point.

The slope of the tangent to the curve $y = f(x)$ at the point (x_1, y_1) is given by $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$. So the equation of the tangent at (x_1, y_1) to the curve $y = f(x)$ is given by

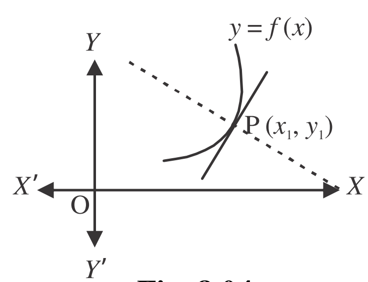


Fig. 8.04

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

Also, since the normal is perpendicular to the tangent, the slope of the normal to the curve $y = f(x)$ at (x_1, y_1) is $-\dfrac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}$, if $f'(x_1) \neq 0$

Therefore, the equation of the normal to the curve $y = f(x)$ at (x_1, y_1) is given by

$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}(x - x_1)$$

$$\Rightarrow \qquad (y - y_1)\left(\frac{dy}{dx}\right)_{(x_1, y_1)} + (x - x_1) = 0$$

Note: If a tangent line to the curve $y = f(x)$ makes an angle ψ with x-axis in the positive direction, then

$\frac{dy}{dx} = \text{slope of the tangent} = \tan\psi$

8.06 Particular cases

(i) If $\psi = 0$ means the tangent line is parallel to x-axis then $\frac{dy}{dx} = \tan 0 = 0$. In this case, the equation of the tangent at the point (x_1, y_1) is given by $y = y_1$

(ii) If $\psi = 90^\circ$, means the tangent line is perpendicular to the x-axis, i.e. parallel to the y-axis. In this case, the equation of the tangent at (x_1, y_1) is given by $x = x_1$

Illustrative Examples

Example 16. Find the equations of the tangent and normal to the curve $x^{2/3} + y^{2/3} = 2$ at (1, 1).

Solution : $\because$ $$x^{2/3} + y^{2/3} = 2$$

Differentiating with respect to x

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$$

Slope of the tangent at (1, 1) is $$\left(\frac{dy}{dx}\right)_{(1,1)} = -1$$

So, the equation of the tangent at (1, 1) is

$$y - 1 = (-1)(x - 1)$$

$$\Rightarrow \qquad x + y - 2 = 0 \qquad (1)$$

So, the equation of the tangent at (1, 1) is

$$y - 1 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(1,1)}}(x - 1)$$

$$= -\frac{1}{(-1)}(x-1) = x-1 \quad (2)$$

$$y - x = 0$$

$\Rightarrow$ (1) and (2) are required equations of tangent and normal.

Example 17. Find points on the curve $x^2 + y^2 - 2x - 3 = 0$ at whcih the tangents are

(i) Parallel to x-axis

(ii) Perpendicular to x-axis

(iii) Making equal angle with axes.

Solution : Equation of curve $x^2 + y^2 - 2x - 3 = 0$ (1)

Differentiating with respect to x

$$2x + 2y\frac{dy}{dx} - 2 = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1-x}{y}$$

(i) When tangent is parallel to x-axis, then

$$\psi = 0 \Rightarrow \frac{dy}{dx} = \tan 0 = 0$$

$$\Rightarrow \quad \frac{1-x}{y} = 0 \Rightarrow 1 - x = 0$$

$$\Rightarrow \quad x = 1$$

put $x = 1$ in (1)

$$y^2 - 4 = 0 \Rightarrow y = \pm 2$$

Hence required points are (1, 2) and (1, –2)

(ii) When tangent is perpendicualr to x-axis then

$$\psi = 90^\circ \Rightarrow \frac{dy}{dx} = \tan 90 = \infty$$

$$\Rightarrow \quad \frac{1-x}{y} = \infty$$

$$\Rightarrow \quad y = 0$$

Put $y = 0$ in (1)

$$x^2 - 2x - 3 = 0$$

$$(x-3)(x+1) = 0$$

$$\Rightarrow \quad x = 3, -1$$

Hence required points are (3, 0) and (–1, 0) .

(iii) When tangents make equal angle with axes, then $\psi = \frac{\pi}{4}$

Hence slope of tangent $\frac{dy}{dx} = \tan\frac{\pi}{4} = 1$

$\Rightarrow$ $\frac{1-x}{y} = 1 \Rightarrow y = 1-x$ (2)

Put $y = 1-x$ in (1)

$$x^2 + (1-x)^2 - 2x - 3 = 0$$

$\Rightarrow$ $x^2 - 2x - 1 = 0$

$\Rightarrow$ $x = 1 \pm \sqrt{2}$

Put this value of x in (2)

$$y = \mp\sqrt{2}$$

Hence required points are $(1+\sqrt{2}, -\sqrt{2})$ and $(1-\sqrt{2}, \sqrt{2})$.

Example 18. Find the point on the curve $y = x^3 - 11x + 5$ at which the tangent is $y = x - 11$.

Solution : Here $y = x^3 - 11x + 5$ (1)

$\Rightarrow$ $\frac{dy}{dx} = 3x^2 - 11$ (2)

Slope of tangent $y = x - 11$ is 1

From (2)

$$1 = 3x^2 - 11$$

$\Rightarrow$ $3x^2 = 12 \Rightarrow x = \pm 2$

Put $x = 2$ in equation (1)

$$y = 2^3 - 11(2) + 5 = -9$$

Put $x = -2$ in equation (1)

$$y = (-2)^3 - 11(-2) + 5 = 19$$

But point (–2, 19) does not lie on curve (1) hence the point at which the tangent is $y = x - 11$ is (–2, 9).

Example 19. Find the equation of all lines having slope zero that are tangents to the curve $y = \frac{1}{x^2 - 2x + 3}$

Solution : Here $y = \frac{1}{x^2 - 2x + 3}$ (1)

Differentiating with respect to x

$$\frac{dy}{dx} = -\frac{(2x-2)}{(x^2-2x+3)^2}$$

Here slope $= 0$

$\Rightarrow$ $$\frac{dy}{dx} = 0$$

$\Rightarrow$ $$\frac{-(2x-2)}{(x^2-2x+3)^2} = 0$$

$\Rightarrow$ $$2x-2=0$$

$\Rightarrow$ $$x=1$$

Put $x=1$ in (1)

$$y = \frac{1}{1^2-2(1)+3} = \frac{1}{2}$$

Hence at point $(1, 1/2)$ the slope of tangent $=0$ and the equation of tangent is

$y-\frac{1}{2} = 0(x-1) \Rightarrow y = \frac{1}{2}$, which is required equation of tangent.

Example 20. Find the equation of normal for the curve $2x^2 - y^2 = 14$, which is parallel to the straight line $x+3y=6$.

Solution : Let a point $P(x_1, y_1)$ on $2x^2 - y^2 = 14$, where normal is parallel to $x+3y=6$

$\therefore$ $$2x_1^2 - y_1^2 = 14 \quad (1)$$

$\because$ $$2x^2 - y^2 = 14$$

$\Rightarrow$ $$4x - 2y\frac{dy}{dx} = 0$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{4x}{2y} = \frac{2x}{y}$$

$\Rightarrow$ $$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{2x_1}{y_1}$$

$\because$ Normal at (x_1, y_1) is parallel to $x+3y=6$ hence slope of normal at (x_1, y_1) = slope of line $x+3y=6$

$\Rightarrow$ $$-\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}} = -\frac{1}{3}$$

$\Rightarrow$ $$\frac{y_1}{2x_1} = \frac{1}{3} \Rightarrow y_1 = \frac{2}{3}x_1$$

Put $y_1 = \frac{2}{3}x_1$, in (1)

$$2x_1^2 - \left(\frac{2}{3}x_1\right)^2 = 14$$

$\Rightarrow$ $$\frac{14}{9}x_1^2 = 14 \Rightarrow x_1 = \pm 3$$

$$\text{at } x_1 = 3,\ y_1 = \frac{2}{3} \times 3 = 2$$

and $$\text{at } x_1 = -3,\ y_1 = \frac{2}{3}(-3) = -2$$

Hence at (3, 2) and (–3, –2) normal is parallel to $x + 3y = 6$. Hence the equation of normal at (3, 2) is

$$y - 2 = -1/3(x-3) \Rightarrow x + 3y = 9$$

Equation of normal at (–3, –2) is

$$y + 2 = -1/3(x+3) \Rightarrow x + 3y + 9 = 0.$$

Example 21. Find the equation of the tangent to curve $y = x^2 - 2x + 7$ which is

(i) parallel to the line $2x - y + 9 = 0$

(ii) perpendicular to the line $5y - 15x = 13$

Solution : Equation of curve is $y = x^2 - 2x + 7$ (1)

$\Rightarrow$ $$\frac{dy}{dx} = 2x - 2 = 2(x-1) \quad (2)$$

(i) Slope of the straight line $2x - y + 9 = 0$ or $y = 2x + 9$ is 2

$\because$ tangent is parallel to this line, hence

$$2(x-1) = 2$$

$\Rightarrow$ $$x = 1$$

When $x = 1$, then from (1)

$$y = 1^2 - 2(1) + 7 = 6$$

Hence the equation of tangent at (1, 6) which is parallel to $2x - y + 9 = 0$ will be

$$y - 6 = 2(x-1)$$

$\Rightarrow$ $$2x - y + 4 = 0$$

(ii) Straight line $5y - 15x = 13$ or $5y = 15x + 13$

$\Rightarrow$ $y = 3x + 13/5$ Slope of line $= 3$

Slope of a line whcih is perpendicualr to $5y-15x=13$ is –1/3

$$\Rightarrow \qquad \frac{dy}{dx}=-\frac{1}{3}$$

$$\Rightarrow \qquad 2(x-1)=-1/3$$

$$\Rightarrow \qquad 6x-6=-1$$

$$\Rightarrow \qquad x=5/6$$

When $x=5/6$ then from (1)

$$y=\left(\frac{5}{6}\right)^2-2\left(\frac{5}{6}\right)+7 \quad =\frac{217}{36}$$

Hence the equation of tangent at $\left(\frac{5}{6},\frac{217}{36}\right)$ will be

$$y-\frac{217}{36}=-\frac{1}{3}\left(x-\frac{5}{6}\right)$$

$$\Rightarrow \qquad \frac{36y-217}{36}=-\frac{1}{3}\left(\frac{6x-5}{6}\right)$$

$$\Rightarrow \qquad 12x+36y-227=0$$

Which is the required equation of tangent.

Example 22. Prove that for every value of *x*, the straight line $\frac{x}{a}+\frac{y}{b}=2$, touches the curve $(x/a)^n+(y/b)^n=1$ at point (a, b).

Solution : Equation of curve $(x/a)^n+(y/b)^n=1$

Differentiating with respect to x

$$\frac{1}{a^n}nx^{n-1}+\frac{1}{b^n}ny^{n-1}\frac{dy}{dx}=0$$

$$\Rightarrow \qquad \frac{dy}{dx}=-\frac{b^n x^{n-1}}{a^n y^{n-1}}$$

$$\therefore \qquad \left(\frac{dy}{dx}\right)_{(a,b)}=-\frac{b^n\cdot a^{n-1}}{a^n b^{n-1}}=-\frac{b}{a}$$

Hence the equation of tangent at (a, b) is

$$y-b=-\frac{b}{a}(x-a)$$

$$\Rightarrow \qquad ay-ab=-bx+ab$$

$$\Rightarrow \qquad bx+ay=2ab$$

$$\Rightarrow \qquad \frac{x}{a}+\frac{y}{b}=2$$

Exericse 8.3

1. Find the slope of the tangent to the curve $y = x^3 - x$.

2. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}, x \neq 2$ at $x = 10$.

3. Find the point at which the tangent to the curve $y = \sqrt{(4x-3)} - 1$ has its slope $2/3$.

4. Find the equation of all lines having slope 2 and being tangent to the curve $y + \frac{2}{x-3} = 0$.

5. Find points on the curve $\frac{x^2}{4} + \frac{y^2}{25} = 1$ at which the tangent are

   (i) parallel to x-axis (ii) parallel to y-axis

6. Find the equation of tangent to the curve given by $x = a\sin^3 t,\ y = b\cos^3 t$ at a point where $t = \pi/2$.

7. Find the equation of normal to the curve $y = \sin^2 x$ at a point $\left(\frac{\pi}{3}, \frac{3}{4}\right)$.

8. Find the equations of the tangent and normal to the given curves at the indicated points:

   (a) $y = x^2 + 4x + 1$ at $x = 3$ (b) $y^2 = 4ax$ at $x = a$

   (c) $xy = a^2$, at $\left(at, \frac{a}{t}\right)$ (d) $y^2 = 4ax$, at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

   (e) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, at $(a\sec\theta, b\tan\theta)$ (f) $y = 2x^2 - 3x - 1$, at $(1, -2)$

   (g) $x = at^2,\ y = 2at$, at $t = 1$ (h) $x = \theta + \sin\theta,\ y = 1 - \cos\theta$, at $\theta = \pi/2$

8.07 Approximation

In this section, we will use differential to approximate values of certain quantities.

Let $y = f(x)$ be the equation of given curve. Let Δx denote a small increment is x, whereas the increment in y corresponding to the increment in x, denoted by Δy, is given by $\Delta y = f(x + \Delta x) - f(x)$. We define the followign (i) The differential of x, denoted by x is defined by $dx = \Delta x$. (ii) The differential by dy, is defined by $dy = f'(x)dx$ or $dy = \frac{dy}{dx}.\Delta x$

In case $dx = \Delta x$ is relatively small when compared with x, dy is a good approximation of Δy and we denote it by $dy \approx \Delta y$.

Illustrative Examples

Example 23. Use differential to approximate $\sqrt{26}$.

Solution : Let $y = \sqrt{x}$

Where $x = 25, \Delta x = 1$ and $x + \Delta x = 26$

$\because$ $y = \sqrt{x} = x^{1/2}$ (1)

$\Rightarrow \qquad \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$\because \qquad \Delta y = \frac{dy}{dx}.\Delta x = \frac{1}{2\sqrt{x}}\Delta x = \frac{1}{2\times 5}\times 1 = \frac{1}{10} = 0.1$

From (1) $\qquad y + \Delta y = (x+\Delta x)^{1/2}$

$\Rightarrow \qquad x^{1/2} + \Delta y = (x+\Delta x)^{1/2}$

Putting the value $\qquad (25)^{1/2} + 0.1 = (26)^{1/2}$

$\Rightarrow \qquad \sqrt{26} = 5 + 0.1 = 5.1.$

Example 24. Use diferential to approximate $(66)^{1/3}$

Solution : Let $\qquad y = x^{1/3} \qquad$ (1)

Where $x = 64, \Delta x = 2$ and $x + \Delta x = 66$

$\because \qquad y = x^{1/3}$

$\Rightarrow \qquad \frac{dy}{dx} = \frac{1}{3x^{2/3}}$

$\because \qquad \Delta y = \frac{dy}{dx}.\Delta x = \frac{1}{3x^{2/3}}.\Delta x = \frac{1}{3\times (64)^{2/3}}\times 2$

$$= \frac{1}{3\times (4)^2}\times 2 = \frac{1}{24}$$

Now From (1)

$$y + \Delta y = (x+\Delta x)^{1/3}$$

$\Rightarrow \qquad x^{1/3} + \frac{1}{24} = (66)^{1/3}$

$\Rightarrow \qquad (64)^{1/3} + \frac{1}{24} = (66)^{1/3}$

$\Rightarrow \qquad (4^3)^{1/3} + \frac{1}{24} = (66)^{1/3}$

$\Rightarrow \qquad 4 + 0.041 = (66)^{1/3}$

$\Rightarrow \qquad (66)^{1/3} = 4.041.$

Example 25. Use differential to approximate the following

(i) $\log_{10}(10.2)$ when $\log_{10} e = 0.4343$

(ii) $\log_e(4.04)$ when $\log_e 4 = 1.3863$

(iii) $\cos 61^\circ$ when $1^\circ = 0.01745$ Radian

Solution : (i) Let $y = \log_{10} x$ (1)

Where $x = 10, \Delta x = 0.2$

$\Rightarrow$ $x + \Delta x = 10.2$

$\because$ $y = \log_{10} x = \log_{10} e . \log_e x$

$\Rightarrow$ $\frac{dy}{dx} = (\log_{10} e)\frac{1}{x} = \frac{0.4343}{10}$

$\because$ $\Delta y = \frac{dy}{dx} . \Delta x = \frac{0.4343}{10} \times (0.2) = 0.008686$

From (1)

$$y + \Delta y = \log_{10}(x + \Delta x)$$

$\Rightarrow$ $\log_{10} x + \Delta y = \log_{10}(x + \Delta x)$

$\Rightarrow$ $\log_{10} 10 + 0.008686 = \log_{10}(10.2)$

$\Rightarrow$ $1 + 0.008686 = \log_{10}(10.2)$

$\Rightarrow$ $\log_{10}(10.2) = 1.008686$

(ii) Let $y = \log_e x$ (2)

Where $x = 4, \Delta x = 0.04$ and $x + \Delta x = 4.04$

$\because$ $y = \log_e x$

$\Rightarrow$ $\frac{dy}{dx} = \frac{1}{x}$

$\because$ $\Delta y = \frac{dy}{dx} . \Delta x = \frac{\Delta x}{x} = \frac{0.04}{4} = 0.01$

From (2)

$$y + \Delta y = \log_e(x + \Delta x)$$

$\Rightarrow$ $\log_e x + \Delta y = \log_e(x + \Delta x)$

Putting values

$$\log_e 4 + 0.01 = \log_e(4.04)$$

$\Rightarrow$ $\log_e(4.04) = 1.3863 + 0.01$

$$= 1.3963$$

(iii) Let $y = \cos x$ (3)

When $x = 60^\circ, \Delta x = 1^\circ = 0.01745$ radian and $x + \Delta x = 61^\circ$

$\because$ $y = \cos x$

$\Rightarrow$ $\frac{dy}{dx} = -\sin x$

$\because \quad \Delta y = \frac{dy}{dx}\Delta x = -\sin x.\Delta x$

$$= -\sin 60^\circ (0.01745)$$

$$= -0.1745 \times \frac{\sqrt{3}}{2} = -0.01511 \qquad (\because \sqrt{3} = 1.73205)$$

From (3)

$$y + \Delta y = \cos(x + \Delta x)$$

$\Rightarrow \quad \cos x + \Delta y = \cos(x + \Delta x)$

$$\cos 60^\circ + (-0.01511) = \cos(61^\circ)$$

$\Rightarrow \quad \cos 61^\circ = \frac{1}{2} - 0.01511$

$$= 0.48489.$$

Example 26. Prove that the approximation percentage error in calculating the volume of a sphere is almost three times the approximation percentage error in calculating the radius of sphere.

Solution : Let radius of sphere = r and volume = V

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$$

$\because \quad \Delta V = \frac{dV}{dr}.\Delta r$

$\Rightarrow \quad \Delta V = 4\pi r^2 \Delta r$

$\Rightarrow \quad \frac{\Delta V}{V} = \frac{4\pi r^2 \Delta r}{V} = \frac{4\pi r^2 \Delta r}{4/3\pi r^3} = 3\frac{\Delta r}{r}$

$\Rightarrow \quad \frac{\Delta V}{V} \times 100 = 3\left(\frac{\Delta r}{r} \times 100\right)$

$\Rightarrow$ Percentage error in volume = 3 (percentage error in radius).

Example 27. Find the approximate vlaue of $f(5.001)$, where $f(x) = x^3 - 7x^2 + 15$

Solution : Let $\quad y = f(x) \qquad (1)$

Where $x = 5, \Delta x = 0.001$ and $x + \Delta x = 5.001$

From (1)

$$y + \Delta y = f(x + \Delta x)$$

$\Rightarrow \quad f(x) + \frac{dy}{dx}.\Delta x = f(x + \Delta x) \qquad (2)$

$\because \quad y = f(x) = x^3 - 7x^2 + 15$

$$\Rightarrow \qquad \frac{dy}{dx} = 3x^2 - 14x$$

Using in equation (2)

$$(x^3 - 7x^2 + 15) + (3x^2 - 14x).\Delta x = f(x + \Delta x)$$

Putting the value of x

$$(5)^3 - 7(5)^2 + 15 + \{3(5)^2 - 14(5)\} \times (0.001) = f(5.001)$$

$$\Rightarrow \qquad f(5.001) = 125 - 175 + 15 + (75 - 70)(0.001)$$

$$= -34.995$$

Example 28. Find the approximate change in the volume of a cube of side x metres caused by increasing the side by 1%.

Solution : Let volume of cube is V

$$\Delta x = x \text{ of } 1\% = \frac{x}{100}$$

$$\because \qquad V = x^3 \Rightarrow \frac{dV}{dx} = 3x^2$$

Hence change in volume of cube

$$dV = \frac{dV}{dx} \Delta x$$

$$= 3x^2 \times \frac{x}{100} = \frac{3}{100} x^3$$

$$= 0.03x^3 \text{ m}^3$$

Example 29. If the radius of a sphere is measured as 7 cm with an error of 0.02 cm, then find the approximate error in calculating its volume.

Solution : Radius of sphere = 7 cm

Error in measuring radius $\Delta r = 0.02$ cm

Let the volume of sphere be V

$$V = (4/3)\pi r^3$$

$$\Rightarrow \qquad \frac{dV}{dr} = 4\pi r^2$$

$$\because \qquad dV = \frac{dV}{dr} \Delta r = 4\pi r^2.\Delta r$$

$$= 4\pi(7)^2 \times .002 = 3.92\pi$$

Exercise 8.4

Using differentials, find the approximate value of each of the following.

1. $(0.009)^{1/3}$ 2. $(0.999)^{1/10}$ 3. $\sqrt{0.0037}$ 4. $\frac{1}{(2.002)^2}$

5. $(15)^{1/4}$ 6. $\sqrt{401}$ 7. $(3.968)^{3/2}$ 8. $(32.15)^{1/5}$

9. $\sqrt{0.6}$ 10. $\log_{10}(10.1)$, when $\log_{10} e = 0.4343$

11. $\log_e(10.02)$, when $\log_e 10 = 2.3026$

12. Find the approximate change in y when $y = x^2 + 4$ as x increases from 3 to 3.1.

13. Prove that the approximation percentage error in calculating the volume of a cubical box is almost three times the approximation percentage error in claculating the edge of cube.

14. If the radius of a sphere decreases from 10 cm to 9.8 cm, find the approximate error in calculating its volume.

8.08 Maxima and Minima

In this section, we will use the concept of derivatives to calcualte the maximum or minimum values of various functions.

Let us examine the graph of a function $y = f(x)$ in the interval $[a, b]$. Observe the ordiantes of points A, P, Q, R, S and B.

The function has maximum value in some neighbourhood of points P and R which are at the top of their respective hills (ordinates) where as the function has minimum value is some neighbourhood (interval) of each of the points Q and S. Point A has least ordinate and point B has maximum ordinate. Tangents drawn to the cuve at point P, Q, R and S are parallel to x-axis, i.e., their slope $\left(\frac{dy}{dx}\right)$ are zero. The P and R are called maximum points and points Q and S are called minimum points for the function. Maximum and minimum points of a function are also regarded as extreme points.

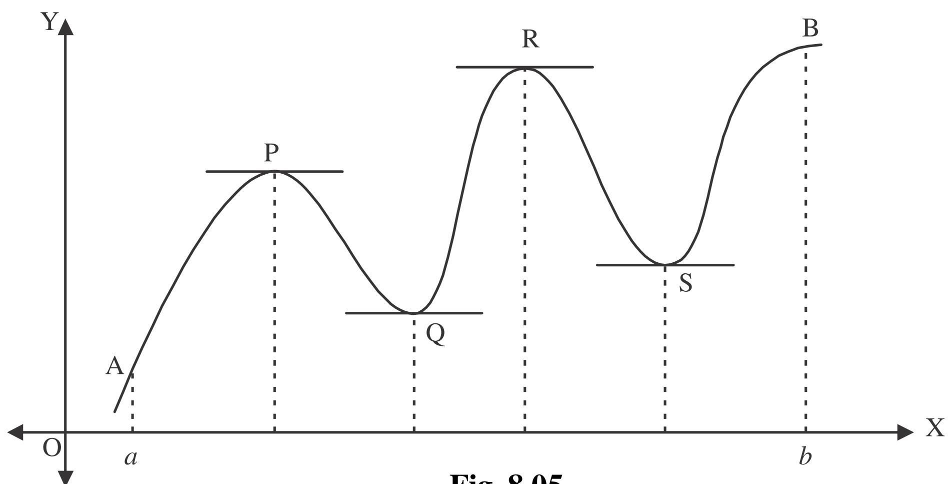


Fig. 8.05

8.09 Some Difinitions

(i) Relative maximum and minimum value

Let $f(x)$ be a real valued function and let c be an interior point he domain of f, then c is called a point in the domain of f, then c is called a point of relative maxima if there is $h > 0$ such that $f(x) \le f(c), \quad \forall \; x \in (c - \hbar, c + \hbar)$, where h is very small. The value f(c) is called the relative maximum vlaue of f.

Similarly c is called a point of relative minimum if there is $h > 0$ such that $f(x) \ge f(c), \quad \forall \; x \in (c - \hbar, c + \hbar)$ the value f(c) is called the relative minimum value of f.

(ii) Absolute maximun and minimum value

Absolute maximum value: Any function $f(x)$ has its absolute maximum value at any point $x = a$ in its domain when.

$$f(x) \leq f(c), \quad \forall \ x \in D$$

Absolute minimum value : Any function $f(x)$ has its absolute minimum value at any point $x = a$ in its domain when

$$f(x) \geq f(c), \quad \forall \ x \in D$$

Note: For a real valued function f(x) in a domain the maximum and minimum value of function may be more than one but absolute maximum and absolute minimum is only one.

8.10 Necessary condition for the extreme value of a function

Theorem : If $f(x)$ is a differentiable function then at $x = c$, necessary condition for the extreme value is $f'(c) = 0$

Note: For a function $f(x)$ at any point $x = c$, $f'(c) = 0$ is only necessary condition for maximum and minimum value of function, it is not sufficient conditiion.

For example if $f(x) = x^3$ then at $x = 0$, $f'(0) = 0$ but $f(0)$ is not extreme valeu of function because when $x > 0 \Rightarrow f(x) > f(0)$ and when $x < 0 \Rightarrow f(x) < f(0)$ and when $f(0)$ is neither minimum nor maximum.

Sufficient condition for the extreme value of a funciton

Theorem : (i) $f(x)$ will have its maximum value at $x = c$ if $f'(c) = 0$ and $f''(c) < 0$

(ii) $f(x)$ will have its minimum value at $x = c$ if $f'(c) = 0$ and $f''(c) > 0$

Note: For a function $f(x)$ at any point $x = c$, $f'(c) = 0$, $f''(c) = 0$ but $f'''(c) \neq 0$ then this point is known as inflection point.

8.11 Properties of maxima and minima of a function

If $f(x)$ is continuous function and if its graph could be drawn then we may consider the following properties.

(i) There is at least one maxima or minima between two equal values of $f(x)$.

(ii) The maxima and minima of a function always occur alternatively.

(iii) If $f'(x)$ changes sign from positive to negative as x increases then $f(x)$ passes through maxima and when $f'(x)$ changes sign from negative to psotive then $f(x)$passes through minima.

(iv) If $f'(x)$ does not change its sign then this point is called point of inflexion.

(v) At maxima and minima $f'(x) = 0$ then the line point is parallel to $x -$ axis.

8.12 Working method to find maxima and minima

1. First of all write the given function in the form of $y = f(x)$ and find $\frac{dy}{dx}$

2. Solve $\frac{dy}{dx} = 0$, let the solutions are $x = a_1, a_2, \ldots$

3. Find $\frac{d^2y}{dx^2}$ and find its value at $x = a_1, a_2, ...$

4. If $\frac{d^2y}{dx^2} < 0$ at $x = a_r$ $(r = 1, 2, . . .)$ then $x = a_r$ function $f(x)$ will have maximum value.

5. If $\frac{d^2y}{dx^2} > 0$ at $x = a_r$ $(r = 1, 2, . . .)$ then at $x = a_r$ function $f(x)$ will have minimum value. If $\frac{d^2y}{dx^2} = 0$ then we continue the process of differentiation.

6. If $\frac{d^2y}{dx^2} = 0$ $x = a_r$ $(r = 1, 2, . . .)$ then find the values of $\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}$.......until $x = a_r$ becomes zero.

   (i) If non zero differential coefficient is of odd degree like $\frac{d^3y}{dx^3}, \frac{d^5y}{dx^5}$.... then at $x = a_r$. Function has neither maxima nor minima.

   (ii) If non zero differential coefficient is of even degree like $\frac{d^4y}{dx^4}, \frac{d^6y}{dx^6}$...., then repeat the same process as $\frac{d^2y}{dx^2} \neq 0$.

8.13 Stationary point

All points on which the rate of change of $f(x)$ with respect to x is zero i.e. $f'(x) = 0$, are called stationary points.

Note: Every extreme point is a stationary point but vice versa is not always true.

Illustrative Examples

Example 30. Find maximum and minimum value of following function (if exist)

(a) $y = (2x-1)^2 + 3$ (b) $y = 9x^2 + 12x + 2$

(c) $y = -(x-1)^2 + 10$ (d) $y = x^3 + 1$

Solution : (a) Minimum value of $(2x-1)^2$ is zero hence minimum value of $(2x-1)^2 + 3$ is 3. It is clear that there is not maximum value of function.

(b) $\because$

$$y = 9x^2 + 12x + 2$$
$$= (3x+2)^2 - 2$$

$\because$ Minimum value of $(3x+2)^2$ is zero. Hence minimum value of $(3x+2)^2 - 2$ is –2 which is at $3x + 2 = 0 \Rightarrow x = -\frac{2}{3}$. It is clear that no maximum value of $y = 9x^2 + 12x + 2$ is there.

(c) It is clear that the minimum value of $-(x-1)^2$ is zero. Hence the maximum value of function $y = -(x-1)^2 + 10$ is 10. There is no minimum value of function.

(d) $\therefore$ at $x \to \infty$, $y \to \infty$

and $x \to -\infty$, $y \to -\infty$

Hence given function has neither maximum nor minimum value.

Example 31. Find the minimum and maximum value of following functions:

(a) $x^5 - 5x^4 + 5x^3 - 2$ (b) $(x-2)^6(x-3)^5$ (c) $(x-1)^2 e^x$

Solution : (a) Let $y = x^5 - 5x^4 + 5x^3 - 2$

$\Rightarrow$ $\frac{dy}{dx} = 5x^4 - 20x^3 + 15x^2$

and $\frac{d^2y}{dx^2} = 20x^3 - 60x^2 + 30x$

For extreme point of function $\frac{dy}{dx} = 0$

$\Rightarrow$ $5x^4 - 20x^3 + 15x^2 = 0$

$\Rightarrow$ $5x^2(x^2 - 4x + 3) = 0$

$\Rightarrow$ $5x^2(x-1)(x-3) = 0$

$\Rightarrow$ $x = 0, 1, 3$

Now at $x = 0$, $\frac{d^2y}{dx^2} = 0$

So, $\frac{d^3y}{dx^3} = 60x^2 - 120x + 30$

at $x = 0$, $\frac{d^3y}{dx^3} = 30 \neq 0$

So, at $x = 0$, there is no extreme value of function.

at $x = 1$ $\frac{d^2y}{dx^2} = 20(1)^3 - 60(1)^2 + 30(1) = -10 < 0$

So at $x = 1$ function has maximum value and maximum value of function is

$= (1)^5 - 5(1)^4 + 5(1)^3 - 2 = -1$

Similarity at $x = 3$, $\frac{d^2y}{dx^2} = 20(3)^3 - 60(3)^2 + 30(3)$

$= 540 - 540 + 90 = 90 > 0$

So, at $x = 3$ function has minimum value and minimum value of fucntion is

$= (3)^5 - 5(3)^4 + 5(3)^3 - 2$

$= -29$

(b) Let $$y=(x-2)^6(x-3)^5$$

$\Rightarrow$ $$\frac{dy}{dx}=6(x-2)^5(x-3)^5+(x-2)^6 5(x-3)^4$$

$$=(x-2)^5(x-3)^4\{6x-18+5x-10\}$$

$$=(x-2)^5(x-3)^4(11x-28)$$

For maxima and minima $\frac{dy}{dx}=0$

$\Rightarrow$ $(x-2)^5(x-3)^4(11x-28)=0$

$\Rightarrow$ $x=2, 3, 28/11$

at $x=2$, $\frac{dy}{dx}$ changes its sign from positive to negative ($\because$ when $x<2$ then $\frac{dy}{dx}>0$ and $x>2$ then $\frac{dy}{dx}<0$)

So at $x=2$ functions has maximum value and maximum value = 0

at $\because x=3$, $\frac{dy}{dx}$ does not change its sign ($\because$ when $x<3$ then $\frac{dy}{dx}>0$ and $x>3$ then $\frac{dy}{dx}>0$)

at $x=3$ function has neither maxima nor minima

again at $x=\frac{28}{11}$, $\frac{dy}{dx}$ changes its sign from negative to positive ($\because$ when $x<\frac{28}{11}$ then $\frac{dy}{dx}<0$ and $x>\frac{28}{11}$ then $\frac{dy}{dx}>0$)

Hence at $x=\frac{28}{11}$ function has minimum value $=\left(\frac{28}{11}-2\right)^6\left(\frac{28}{11}-3\right)^5=-\frac{6^5.5^5}{11^{11}}$

(c) Let $$y=(x-1)^2e^x$$

$\Rightarrow$ $$\frac{dy}{dx}=\{(x-1)^2+2(x-1)\}e^x$$

and $$\frac{d^2y}{dx^2}=\{(x-1)^2+4(x-1)+2\}e^x$$

For extreme value $\frac{dy}{dx}=0$

$\Rightarrow$ $\{(x-1)^2+2(x-1)\}e^x=0$

$\Rightarrow$ $(x-1)^2+2(x-1)=0$ $\{\because e^x\neq 0\}$

$\Rightarrow \qquad x^2 - 1 = 0$

$\Rightarrow \qquad x = \pm 1$

Now at $x = 1$, $\quad \frac{d^2y}{dx^2} = \{0 + 4(0) + 2\}e^1 = 2e > 0$

So at $x = 1$ function has minimum value and minimum value $= (1-1)^2 e^1 = 0$

again at $x = -1$ $\quad \frac{d^2y}{dx^2} = \{(-1-1)^2 + 4(-1-1) + 2\}e^{-1}$

$$= \{4 - 8 + 2\}e^{-1} = \frac{-2}{e} < 0$$

So, at $x = -1$ function has maximum value and maximum value is $= (-1-1)^2 e^{-1} = \frac{4}{e}$.

Exercise 32. Find the maximum value of function $(1/x)^x$

Solution : Let $\quad y = (1/x)^x$

$\Rightarrow \qquad \log y = x \log \frac{1}{x}$

$$= -x \log x = z \quad \text{Let}$$

Function y has maximum or minimum value if z has maximum or minimum value.

Now, $\quad \frac{dz}{dx} = -x.\frac{1}{x} - 1.\log x = -(1 + \log x)$

ad $\quad \frac{d^2z}{dx^2} = -\frac{1}{x}$

So, for maximum or minimum value

$$\frac{dz}{dx} = 0 \Rightarrow 1 + \log x = 0$$

$\Rightarrow \qquad \log x = -1$

$\Rightarrow \qquad x = e^{-1} = \frac{1}{e}$

at $x = 1/e$ $\quad \frac{d^2z}{dx^2} = -\frac{1}{1/e} = -e < 0$

So at $x = 1/e$, y has maximum value and maximum value $= \left[\frac{1}{1/e}\right]^{1/e} = e^{1/e}$.

Example 33. Find the shortest distance of the point (0, a) from the parabola $x^2 = y$ where $a \in [0, 5]$.

Solution : Let a point (h, k) is on the parabola, let the distance between (0, a) and (h, k) is D, then

$$D = \sqrt{(h-o)^2 + (k-c)^2} = \sqrt{h^2 + (k-c)^2} \quad (1)$$

$\because$ point (h, k) is on parabola $x^2 = y$ hence $h^2 = k$ use this in (1)

$$D = \sqrt{k + (k-c)^2}$$

$\Rightarrow$ $$D(k) = \sqrt{k + (k-c)^2}$$

$\Rightarrow$ $$D'(k) = \frac{\{1 + 2(k-c)\}}{2\sqrt{k + (k-c)^2}} \quad (2)$$

Now $$D'(k) = 0 \Rightarrow k = \frac{2c-1}{2}$$

when $k < \dfrac{2c-1}{2}$ then $2(k-c)+1 < 0$

$\Rightarrow$ $D'(k) < 0$ [from equation (2)]

and when $k > \dfrac{2c-1}{2}$ then $2(k-c)+1 < 0$

$\Rightarrow$ $D'(k) > 0$ [from equation (4)]

So at $k = \dfrac{2c-1}{2}$, D is minimum and the minimum distance

$$= \sqrt{\frac{2c-1}{2} + \left(\frac{2c-1}{2} - c\right)^2} = \frac{\sqrt{4c-1}}{2}.$$

Example 34. Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals:

(a) $f(x) = x^3, \quad x \in [-2, 2]$

(b) $f(x) = 4x - \dfrac{1}{2}x^2, \quad x \in \left[-2, 9/2\right]$

(c) $f(x) = (x-1)^2 + 3, \quad x \in [-3, 1]$

(d) $f(x) = \sin x + \cos x, \quad x \in [0, \pi]$

Solution : (a) Given $$f(x) = x^3, x \in [-2, 2]$$

$\Rightarrow$ $$f'(x) = 3x^2$$

$\therefore$ $$f'(x) = 0 \Rightarrow 3x^2 = 0 \Rightarrow x = 0 \Rightarrow f(0) = 0$$

Now $f(-2) = (-2)^3 = -8$; $f(0) = (0)^3 = 0$ and $f(2) = (2)^3 = 8$

The absolute maximum value of $f(x)$ is 8 which is obtained at $x = 2$ and absolute minimum value is -8 which is obtained at $x = -2$.

(b) Given $$f(x) = 4x - \frac{x^2}{2}$$

$\Rightarrow$ $$f'(x) = 4 - \frac{2x}{2} = 4 - x$$

for extreme value of $f(x)$ $\quad f'(x) = 0$

$\Rightarrow$ $\quad 4 - x = 0$

$\Rightarrow$ $\quad x = 4$

Now, we find the value of function at points –2, 4 and 9 / 2.

$\because$ Given function is $f(x) = 4x - \frac{x^2}{2}$ So, $f(-2) = 4(-2) - \frac{(-2)^2}{2} = -10;$ $\quad f(4) = 4(4) - \frac{(4)^2}{2} = 8$

and $\quad f(9/2) = 4(9/2) - \frac{(9/2)^2}{2} = -9/4$

Hence in the given interval [–2, 9/2] absolute maximum value = 8 and minimum value $= -10$

(c) Given function $\quad f(x) = (x-1)^2 + 3, x \in [-3, 1]$

$\Rightarrow$ $\quad f'(x) = 2(x-1)$

for extreme value of $f(x)$, $\quad f'(x) = 0$

$\Rightarrow$ $\quad 2(x-1) = 0$

$\Rightarrow$ $\quad x = 1$

The values of $f(x)$ at $x = 1, -3, 0$, and 0 are

$f(1) = (1-1)^2 + 3 = 0 + 3 = 3$; $f(-3) = (-3-1)^2 + 3 = 16 + 3 = 19$ and $f(0) = (0-1)^2 + 3 = 1 + 3 = 4$

Hence in the given interval [–3, 1] absolute maximum value is 19 which is obtained at $x = -3$ and absolute minimum value is 3 which is at $x = 1$.

(d) Given function is $\quad f(x) = \sin x + \cos x, x \in [0, \pi]$

$\Rightarrow$ $\quad f'(x) = \cos x - \sin x$

For maxima and minima of $f(x)$ $\quad f'(x) = 0$

$\Rightarrow$ $\quad \cos x - \sin x = 0$

$\Rightarrow$ $\quad \sin x = \cos x$

$\Rightarrow$ $\quad \tan x = 1$

$\Rightarrow$ $\quad x = \pi / 4$

Now $\quad f(0) = \sin 0 + \cos 0 = 0 + 1 = 1$

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

and $\quad f(\pi) = \sin \pi + \cos \pi = 0 + (-1) = -1$

Hence maximum and minimum values of $f(x)$ are $\sqrt{2}$ and –1 respectively, for given interval [0, π]

Example 35. Find two positive numbers x and y such that

(a) $x + y = 60$ and xy^3 is maximum (b) $x + y = 16$ and $x^3 + y^3$ is minimum

Solution : (a) Let $p = xy^3$

Given $x + y = 60 \Rightarrow x = 60 - y$

$\therefore$ $p = (60 - y)y^3 = 60y^3 - y^4$

$\Rightarrow$ $\frac{dp}{dt} = 180y^2 - 4y^3$

and $\frac{d^2p}{dy^2} = 360y - 12y^2$

For extreme value of p, $\frac{dp}{dy} = 0$

$\Rightarrow$ $180y^2 - 4y^3 = 0$

$\Rightarrow$ $4y^2(45 - y) = 0$

$\Rightarrow$ $y = 45$ $\{\because y = 0 \text{ is not possible } y > 0\}$

Now $\left(\frac{d^2p}{dy^2}\right)_{y=45} = 360(45) - 12(45)^2 = -8100 < 0$

So at, $y = 45$, P has maximum value.

When $y = 45$ then $x = 60 - 45 = 15$

Hence numbers are $x = 15$ and $y = 45$.

(b) Let $p = x^3 + y^3$ (1)

Given $x + y = 16$

$\Rightarrow$ $y = 16 - x$ (2)

From equation (1)

$$p = x^3 + (16 - x)^3$$

$\Rightarrow$ $\frac{dp}{dx} = 3x^2 + 3(16 - x)^2(-1)$

$= 3x^2 - 3(256 - 32x + x^2)$

$= 3(32x - 256)$ (3)

Now $\frac{dp}{dx} = 0 \Rightarrow 3(32x - 256) = 0$

$\Rightarrow$ $x = \frac{256}{32} = 8$

From equation (3) $\frac{d^2p}{dx^2} = 96 > 0$

Hence at $x = 8$, p is minimum.
Hence required positive numbers and 8 and 8.

Exercise 8.5

1. Find maximum and minimum value of following functions:

(a) $2x^3 - 15x^2 + 36x + 10$ (b) $(x-1)(x-2)(x-3)$

(c) $\sin x + \cos 2x$ (d) $x^5 - 5x^4 + 5x^3 - 1$

2. Find the maximum and minimum value, if any:

(a) $-|x+1|+3$ (b) $|x+2|-1$ (c) $|\sin 4x + 3|$ (d) $\sin 2x + 5$

3. Find the maximum and minimum value of following function if any, in the given intervals.

(a) $2x^3 - 24x + 107,\quad x \in [1, 3]$ (b) $3x^4 - 2x^3 - 6x^2 + 6x + 1,\quad x \in [0, 2]$

(c) $x + \sin 2x,\quad x \in [0, 2\pi]$ (d) $x^3 - 18x^2 + 96x,\quad x \in [0, 9]$

4. Find extreme value of following functions

(a) $\sin x \cos 2x$ (b) $a \sec x + b \operatorname{cosec} x,\ o < a < b$

(c) $x^{1/x},\ x > 0$ (d) $\frac{1}{x}.\log x,\ x \in (0, \infty)$

5. Prove that function f(x) = $\frac{x}{1 + x\tan x}$ has maximum value at $x = \cos x$.

6. Prove that $\sin^2 x(1 + \cos x)$ has maximum value at $\cos x = 1/3$

7. Prove that function $y = \sin^p \theta \cos^q \theta$ has maximum value at $\tan\theta = \sqrt{p/q}$

8.14 Applications of maxima and minima

With the help of following examples we shall use the application of derivatives in other branches as (i) Plane Geometry; (ii) Solid geometry; (iii) Mechanics; (iv) Commerce and Economics.

Illustrative Examples

Example 36. Show that of all the rectangles inscribed in a given fixed circle, the square has maximum area.

Solution : $PQRS$ is a rectangle, centre of circle is O and its radius is a

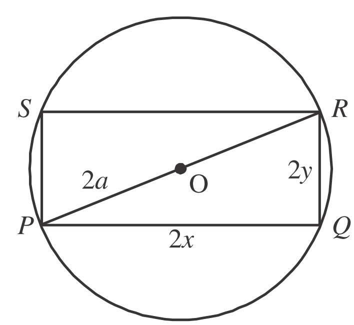


Fig. 8.06

Let $PQ = 2x,\ QR = 2y$

In right ΔPQR

$$PQ^2 + QR^2 = PR^2$$

$$\Rightarrow \quad (2x)^2 + (2y)^2 = (2a)^2$$

$$\Rightarrow \quad x^2 + y^2 = a^2$$

$$\Rightarrow \quad y = \sqrt{a^2 - x^2} \qquad (1)$$

Let area of rectangle $PQRS$ is A

$$A = (2x)\left(2\sqrt{a^2 - x^2}\right) = 4x\sqrt{a^2 - x^2}$$

$$\Rightarrow \quad \frac{dA}{dx} = 4\left\{\sqrt{a^2 - x^2} - \frac{x^2}{\sqrt{a^2 - x^2}}\right\} = \frac{4(a^2 - 2x^2)}{\sqrt{a^2 - x^2}} \quad (2)$$

For maximum or minimum value of A $\quad \frac{dA}{dx} = 0$

$$\Rightarrow \quad \frac{4(a^2 - 2x^2)}{\sqrt{a^2 - x^2}} = 0$$

$$\Rightarrow \quad a^2 - 2x^2 = 0$$

$$\Rightarrow \quad x = \frac{a}{\sqrt{2}}$$

From (2)

$$\frac{d^2A}{dx^2} = 4\left\{\frac{-4x}{\sqrt{a^2 - x^2}} - \frac{x(a^2 - 2x^2)}{(a^2 - x^2)^{3/2}}\right\}$$

at $\quad x = a/\sqrt{2}, \; \frac{d^2A}{dx^2} = -16 < 0$

So, at $\quad x = a/\sqrt{2}, A$ is maximum.

Put $\quad x = a/\sqrt{2}$, in (1) $\; y = a/\sqrt{2}$

So $\quad x = y = a/\sqrt{2}$ hence area is maximum when $x = y$

$\Rightarrow 2x = 2y$ hence rectangle is a square.

Example 37. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

Solution : Let slant height of cone = ℓ and the semi vertical angle of cone = θ

In right $\Delta OO'B$

$OO' = \ell\cos\theta = h$ (height of cone)

$O'B = \ell\sin\theta = r$ (radius of cone)

Volume of cone

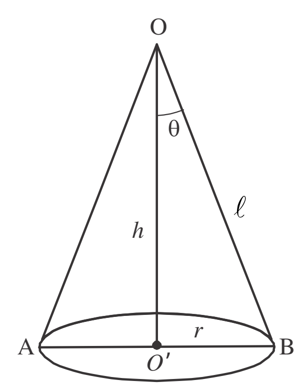


Fig. 8.07

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi \ell^2 \sin^2\theta . \ell\cos\theta$$

$$= \frac{1}{3}\pi \ell^3 \sin^2\theta\cos\theta$$

$$\therefore \quad \frac{dV}{d\theta} = \frac{1}{3}\pi\ell^3\{\sin^2\theta(-\sin\theta) + 2\sin\theta\cos\theta\cos\theta\}$$

$$= \frac{1}{3}\pi\ell^3\left(2\sin\theta\cos^2\theta - \sin^3\theta\right)$$

and $$\frac{d^2V}{d\theta^2} = \frac{1}{3}\pi\ell^3(2\cos\theta.\cos^2\theta - 4\sin\theta\cos\theta\sin\theta - 3\sin^2\theta\cos\theta)$$

$$= \frac{1}{3}\pi\ell^3(2\cos^3\theta - 7\sin^2\theta\cos\theta)$$

For maximum volume $\frac{dV}{d\theta} = 0$

$\Rightarrow$ $$\sin\theta(2\cos^2\theta - \sin^2\theta) = 0$$

$\Rightarrow$ $$\sin\theta\{2(1-\sin^2\theta) - \sin^2\theta\} = 0$$

$\Rightarrow$ $$\sin\theta\{2 - 3\sin^2\theta\} = 0$$

$\Rightarrow$ $$\sin\theta = 0, \sqrt{2/3}, -\sqrt{2/3}$$

Now $\sin\theta = \sqrt{2/3}$ or $\cos\theta = 1/\sqrt{3}$ then

$$\frac{d^2V}{d\theta^2} = \frac{1}{3}\pi\ell^3\left\{2\left(\frac{1}{\sqrt{3}}\right)^3 - 7\left(\sqrt{\frac{2}{3}}\right)^2.\frac{1}{\sqrt{3}}\right\}$$

$$= \frac{1}{3}\pi\ell^3\left\{\frac{2}{3\sqrt{3}} - \frac{14}{3\sqrt{3}}\right\} = -\frac{1}{3}\pi\ell^3\frac{12}{3\sqrt{3}} < 0$$

So, for maximum value $\sin\theta = \sqrt{2/3}$

Then $$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{2/3}}{\sqrt{1/3}} = \sqrt{2}$$

$\therefore$ Semi vertical angle $\theta = \tan^{-1}(\sqrt{2})$.

Example 38. An open tank of fixed volume has square base. If inner surface is minimum then find the ratio of depth to length of the tank.

Solution : Let the depth and height of the tank are $\hbar$ and ℓ, then

Volume of tank $$V = \ell^2\hbar \quad (1)$$

Area of Inner surface of tank $$S = \ell^2 + 4\ell\hbar$$

$\Rightarrow$ $$S = \ell^2 + 4\ell\left(\frac{V}{\ell^2}\right) \quad \text{[From (1)]}$$

$\Rightarrow$ $$S = \ell^2 + 4\frac{V}{\ell}$$

$\Rightarrow$ $$\frac{dS}{d\ell} = 2\ell - \frac{4V}{\ell^2} \text{ and } \frac{d^2S}{d\ell^2} = 2 + \frac{2.4V}{\ell^3}$$

For minimum surface area $\frac{dS}{d\ell} = 0$

$\Rightarrow$ $$2\ell - \frac{4V}{\ell^2} = 0$$

$\Rightarrow$ $$\ell^3 = 2V$$

$\Rightarrow$ $$\ell = (2V)^{1/3}$$

when $\ell = (2V)^{1/3}$ then $\frac{d^2S}{d\ell^2} = 2 + \frac{8V}{(2V)} > 0$

Hence inner surface is minimum.

From (1)

$$\hbar = \frac{V}{\ell^2} = \frac{1}{2}\frac{2V}{(2V)^{2/3}} = \frac{1}{2}(2V)^{1/3} = \frac{1}{2}.\ell$$

$\Rightarrow$ $$\frac{\hbar}{\ell} = \frac{1}{2}$$

$\therefore$ Depth of tank : Length of tank $= 1:2$

Example 39. Manufacturer can sell x items at a price if rupees $\left(5 - \frac{x}{100}\right)$ each. The cost price of x items is rupees $\left(\frac{x}{5} + 500\right)$. Find the number of items he should sell to earn maximum profit.

Solution : Let S be the selling price of x items and let c be the cost price of x items. Then, we have

$$S = \left(5 - \frac{x}{100}\right)x = 5x - \frac{x^2}{100}$$

and $$C = \frac{x}{5} + 500$$

Let profit function be p then

$$p = S - C$$

$$= 5x - \frac{x^2}{100} - \frac{x}{5} - 500$$

$$= \frac{24}{5}x - \frac{x^2}{100} - 500$$

$\Rightarrow \qquad \frac{dp}{dx} = \frac{24}{5} - \frac{x}{50}$ and $\frac{d^2p}{dx^2} = -\frac{1}{50}$

$\therefore \qquad \frac{dp}{dx} = 0 \Rightarrow \frac{24}{5} - \frac{x}{50} = 0$

$\Rightarrow \qquad x = 240$

and $\qquad \left(\frac{d^2p}{dx^2}\right)_{x=240} = -\frac{1}{50} < 0$

Hence the manufacture can earn maximum profit if he sells 240 items.

Exercise 8.6

1. Prove that the maximum area of isosceles triangle, that can be inscribed in a circle, is an equilaterla triangle.
2. The sum of perimeter of a square and circumference of a circle is given. Prove that the sum of their areas will be minimum if the side of square is equal to the diameter of cirlce.
3. A cone is made from a sphere. Prove that the volumeof cone is maximum when height of cone is two third of diameter of sphere.
4. The expense for a steamer per hour is proportional tot he cube of its velocity. If velocity of stream is x km/h then prove that the maximum velocity of steamer per hour will be (2/3) x when the steamer runs against the direction of stream.
5. The sum of the length of the hypotenuse and any side of a right angled triangle is given. Prove that the area of the triangle is maximum when the angle between is then $\pi/3$.
6. A circle of radius a is inscribed in an equilateral triangle. Prove that the minimum perimeter of triangle is $6\sqrt{3}a$.
7. A normal is drawn to a point P on an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Prove that the maximum distance from centre of ellipse to normal is $a - b$.

Miscellaneous Exercise – 8

1. The radius of a cyclinder is r and height is h then find the rate of change in surface area of cylinder with respect to radius.
2. Find the values of x and y for function $y = x^2 + 21$, where the rate of change in y is thrice the rate of change in x.
3. Prove that exponential function e^x is an increasing function.
4. Prove that the function $f(x) = \log(\sin x)$, is increasing in $(0, \pi/2)$ and decreasing in $(\pi/2, \pi)$
5. If tangents OX and OY at a point on the curve $\sqrt{x} - \sqrt{y} = \sqrt{a}$ cut the axes at P and Q, then prove that $OP + OQ = a$, where O is the origin.
6. Find the equations of tangents to the curve $y = \cos(x + y)$, $x \in [-2\pi, 2\pi]$, which is parallel to line $x + 2y = 0$.
7. If the edge of a cube is measured with an error of 5%, then find the approximate error to calculate its volume.
8. A circle disc of radius 10 cm is being heated. Due to expansion, its radius increases 2%. Find the rate at which its area is increasing.
9. Prove that the volume of the largest cone inscribed in a sphere is 8 / 27 of the volume of sphere.

10. Show that the semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}(1/3)$.

Important Points

1. If a function $f(x)$ is differentiable then at any point $x = c$ for extreme point / value it is necessary that $f'(c) = 0$
2. Function $f(x)$ will have maximum value at a point c if $f'(c) = 0$ and $f''(c) < 0$
3. Function $f(x)$ will have minimum value at a point c if $f'(c) = 0$ and $f''(c) > 0$

ANSWERS

Exercise 8.1

1. $6\pi\ cm^2/s, 8\pi\ cm^2/s$ 2. $(1, 5/3), (-1, 1/3)$ 3. $-3/10$ radian/s

4. $900\ cm^3/s$ 5. $1/\pi$ cm/s 6. $\frac{27}{8}\pi(2x+1)^2$

7. 30.02 (approx) 8. $35.2\ cm^3/sec., 20\pi\ cm^3/s$ 9. $\frac{1}{48}\pi$ 10. 126

Exercise 8.2

11. increasing in $(-\infty, -2) \cup (3, \infty)$ and decreasing in $(-2, 3)$
12. increasing in $(-1, 0) \cup (1, \infty)$ and decreasing in $(-\infty, -1) \cup (0, 1)$
13. increasing in $(-\infty, -1) \cup (0, 1)$ and decreasign in $(1, 2)$
14. increasing in $(-1, 2)$ and decreasing in $(-\infty, -1) \cup (2, \infty)$
15. -2

Exercise 8.3

1. 11 2. $-1/64$ 3. $(3, 2)$ 4. $y - 2x + 2 = 0,\ y - 2x + 10 = 0$

5. (i) $(0, 5)$ and $(0, -5)$; (ii) $(2, 0)$ and $(-2, 0)$ 6. $y = 0$ 7. $24x + 12\sqrt{3}y = 8\pi + 9\sqrt{3}$

8.	**tangent**	**Normal**
(a)	$10x - y - 8 = 0$	$x + 10y - 223 = 0$
(b)	$y - x - a = 0$	$y + x - 3a = 0$
(c)	$x + yt^2 = 2at$	$xt^3 - yt = at^4 - a$
(d)	$y - mx = \frac{a}{m}$	$my + x = 2a + \frac{a}{m^2}$
(e)	$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$	$ax\cos\theta + by\cot\theta = a^2 + b^2$
(f)	$x - y - 3 = 0$	$x + y + 1 = 0$
(g)	$x - y + a = 0$	$x + y - 3a = 0$
(h)	$2x - 2y - \pi = 0$	$2x + 2y - \pi - 4 = 0$

Exercise 8.4

1. 0.2083 2. 0.9999 3. 0.0608 4. 0.2495 5. 1.968 6. 20.025 7. 7.904
8. 2.00187 9. 0.8 10. 1.004343 11. 2.3046 12. 0-6 14. 80π cm^3

Exercise 8.5

1. (a) maximum at $x = 2$ and minimum at $x = 3$

(b) maximum at $x = \frac{6-\sqrt{3}}{3}$ and minimum at $x = \frac{6+\sqrt{3}}{3}$

(c) maximum at $x = \sin^{-1} 1/4, \pi - \sin^{-1} 1/4$ and minimum at $x = \frac{\pi}{2}, \frac{3\pi}{2}$

(d) maximum at $x = 1$ and minimum at $x = 3$

2. (a) maximum value = 3, minimum value = does not exist
   (b) maximum value = does nto exist, minimum value = -1
   (c) maximum value =4, minimum value = 2
   (d) maximum value = 6, minimum vlaue = 4
3. (a) maximum value = 160, at $x = 4$
   minimum value = 75, at $x = 2$
   (b) maximum value = 21, at $x = 0$
   minimum value = 1, at $x = 0$
   (c) maximum value = 2p, at $x = 2\pi$
   minimum value = 0, at $x = 0$
   (d) maximum value = 160, at $x = 4$
   minimum = 0, at $x = 0$
4.

	Maximum value	Minimum value
(a)	$= 1, \frac{2}{3\sqrt{6}}$,	$= -1, \frac{-2}{3\sqrt{6}}$
(b)	$= (a^{2/3} + b^{2/3})^{3/2}$	$= -(a^{2/3} + b^{2/3})^{3/2}$
(c)	$= e^{1/e}$	
(d)	$= 1/e$	

Miscellaneous Exercise – 8

1. $4\pi r + 2\pi h$ 2. $x = \pm 1, \ y = 22, \ 2$ 6. $2x + 4y + 3\pi = 0$ and $2x + 4y - \pi = 0$
7. 15% 8. 4π cm^2

09

Integration

9.01 Introduction

We have already studied how to find the derivative of a given function. As a consequence, a natural question arises : given a function say $f(x)$, can we find a function $g(x)$ such that $g'(x) = f(x)$. If such a functions $g(x)$ exist, we shall call it anti-derivative of $f(x)$ or indefinite integral of $f(x)$. Therefore, integration is an inverse process of differentiation. It is also called antiderivative or primitive.

9.02 Integration of a function

If the given function is $f(x)$ and its integral is $F(x)$, then

$$\frac{d}{dx}[F(x)] = f(x) \tag{1}$$

Here, $F(x)$ is called integration of function $f(x)$ with respect to x. In symbols, it is expressed as

$$\int f(x)dx = F(x) \tag{2}$$

where symbol $\int$ is used for integration and dx means to integrate with respect to variable x. Also, the function $f(x)$, whose integration is to be done, is called Integrand and $F(x)$ is called integral.

Since integration and differentiation are inverse process of each other. Therefore, then differentiating eq. (2) with respect to x, we get

$$\frac{d}{dx}\left[\int f(x)dx\right] = \frac{d}{dx}[F(x)]$$

or $$\frac{d}{dx}\left[\int f(x)dx\right] = f(x) \quad \text{[From (1)]}$$

For example: $\frac{d}{dx}(\sin x) = \cos x$ so $\int \cos x \, dx = \sin x$

$\frac{d}{dx}(x^2) = 2x$ so $\int 2x dx = x^2$

Remark : If $\int f(x)dx = F(x)$, then $f(x)$ is called integrand, $F(x)$ is called integral and the process of finding the integral is known as integration.

9.03 Indefinite integral and constant of integration

We know that differential coefficient of any constant is zero.

That means, $\frac{d}{dx}(c) = 0$, where c is any constant

Let $$\frac{d}{dx}[F(x)] = f(x)$$

then, $$\frac{d}{dx}[F(x)+c] = \frac{d}{dx}[F(x)] + \frac{d}{dx}(c)$$

$$= f(x) + 0$$

so $$\frac{d}{dx}[F(x)+c] = f(x)$$

On integrating both sides with respect to x,

$$\int \left[\frac{d}{dx}\{F(x)+c\} \right] dx = \int f(x)dx$$

or $$\int f(x)dx = F(x) + c\,, \qquad \text{(by definition)}$$

where c is an arbitrary constant, which is called coefficient of integration. This is independent of x. Antiderivative of any continuous function is not unique. Actually, there exist infinitely many anti-derivatives of each of these functions which can be obtained by choosing c arbitrarily from the set of real numbers. In fact, c is the parameter by varying which one gets different antiderivatives (or integrals) of the given function.

For example, $$\frac{d}{dx}(x^2+1) = 2x \Rightarrow \int 2x dx = x^2 + 1$$

$$\frac{d}{dx}(x^2+4) = 2x \Rightarrow \int 2x dx = x^2 + 4$$

but (x^2+1) and (x^2+4) are not same, they are differ by a constant.

Remark : In indefinite integration, the constant of integration should be added at the end of the process of integration.

9.04 Theorems on Integration

Theorem 1: For any constant k, $\int \kappa f(x)\,dx = \kappa \int f(x)dx$

$\therefore$ The integration of product of a constant function and variable function is equal to the product of constant function and integral of variable function.

Proof : We know by theorem of differentiation

$$\frac{d}{dx}\left[k\int f(x)dx\right] = k\frac{d}{dx}\left[\int f(x)dx\right] = k\,f(x) \qquad \text{[by definition]}$$

Integrating both sides,

$$\int \frac{d}{dx}\left[k\int f(x)dx\right]dx = \int k\,f(x)dx$$

$$k\int f(x)dx = \int k\,f(x)dx$$

or $$\int k\,f(x)dx = k\int f(x)\,dx$$

Theorem 2 : $\int [f_1(x) \pm f_2(x)]\,dx = \int f_1(x)\,dx \pm \int f_2(x)dx$

$\therefore$ The integral of sum or difference of any two variable functions is equal to the sum or difference of their integrals.

Proof : Let $\int f_1(x)dx = F_1(x)$ and $\int f_2(x)dx = F_2(x)$

$\therefore$ $\frac{d}{dx}[F_1(x)] = f_1(x)$ and $\frac{d}{dx}[F_2(x)] = f_2(x)$

Also,
$$\frac{d}{dx}[F_1(x) \pm F_2(x)] = \frac{d}{dx}[F_1(x)] \pm \frac{d}{dx}[F_2(x)]$$
$$= f_1(x) \pm f_2(x)$$

Integrating both the sides,

$$\int \frac{d}{dx}[F_1(x) \pm F_2(x)]dx = \int [f_1(x) \pm f_2(x)]\,dx$$

or,
$$F_1(x) \pm F_2(x) = \int [f_1(x) \pm f_2(x)]dx$$

or
$$\int [f_1(x) \pm f_2(x)]dx = F_1(x) \pm F_2(x)$$
$$= \int f_1(x)\,dx \pm \int f_2(x)\,dx$$

This rule can be applied for two or more terms but not necessarily applicable on sum of infinite terms.

Generalization

$$\int [k_1 f_1(x) \pm k_2 f_2(x)]\,dx = \int k_1 f_1(x)dx \pm \int k_2 f_2(x)dx$$
$$= k_1 \int f_1(x)dx \pm k_2 \int f_2(x)dx$$

9.05 Standard formulae of Integration

We already know the formulae for the derivatives of many important functions. From these formulae, we can write down the corresponding formulae for the integrals of these functiosn, as listed below which will be used to find integrals of other functions.

For example
$$\frac{d}{dx}(x^n) = nx^{n-1} (n \neq o)$$

$\Rightarrow$
$$\int nx^{n-1}dx = x^n + c$$

Putting n as $(n+1)$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c (n \neq -1)$$

Similarly following formulae can be proved

	Derivatives		**Integrals**
1.	$\frac{d}{dx}(c) = 0$	$\Rightarrow$	$\int 0 \cdot dx = c$
2.	$\frac{d}{dx}(x^n) = nx^{n-1}, \quad n \neq 0$	$\Rightarrow$	$\int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$
3.	$\frac{d}{dx}(\log\|x\|) = \frac{1}{x}, \quad x \neq 0$	$\Rightarrow$	$\int \frac{1}{x} dx = \log\|x\| + c, \quad x \neq 0$
4.	$\frac{d}{dx}(e^x) = e^x$	$\Rightarrow$	$\int e^x dx = e^x + c$
5.	$\frac{d}{dx}(a^x) = a^x \log_e a$	$\Rightarrow$	$\int a^x dx = \frac{a^x}{\log_e a} + c$
6.	$\frac{d}{dx}(\sin x) = \cos x$	$\Rightarrow$	$\int \cos x dx = \sin x + c$
7.	$\frac{d}{dx}(-\cos x) = \sin x$	$\Rightarrow$	$\int \sin x dx = -\cos x + c$
8.	$\frac{d}{dx}(\tan x) = \sec^2 x$	$\Rightarrow$	$\int \sec^2 x dx = \tan x + c$
9.	$\frac{d}{dx}(-\cot x) = \cos ec^2 x$	$\Rightarrow$	$\int \operatorname{cosec}^2 x\, dx = -\cot x + c$
10.	$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\Rightarrow$	$\int \sec x \tan x dx = \sec x + c$
11.	$\frac{d}{dx}(-\operatorname{cosec} x) = \operatorname{cosec} x \cot x$	$\Rightarrow$	$\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + c$
12.	$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, (\|x\|<1)$	$\Rightarrow$	$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$
13.	$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, (\|x\|<1)$	$\Rightarrow$	$\int \frac{-1}{\sqrt{1-x^2}} dx = \cos^{-1} x + c$
14.	$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\Rightarrow$	$\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$
15.	$\frac{d}{dx}(-\cot^{-1} x) = \frac{1}{1+x^2}$	$\Rightarrow$	$\int \frac{1}{1+x^2} dx = -\cot^{-1} x + c$
16.	$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$	$\Rightarrow$	$\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c$

17. $\frac{d}{dx}(-\text{cosec}^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$ $\Rightarrow$ $\int \frac{1}{x\sqrt{x^2-1}} = -\text{cosec}^{-1} x + c$

18. $\frac{d}{dx}|x| = \frac{|x|}{x}, (x \neq 0)$ $\Rightarrow$ $\int \frac{|x|}{x} dx = |x| + c, \quad x \neq 0$

Particularly $\frac{d}{dx}(x) = 1$ $\Rightarrow$ $\int 1.dx = x + c$

Note (a) $\frac{d}{dx}\int f(x)dx = f(x)$ (b) $\int \frac{d}{dx} f(x)dx = f(x) + c$

hence there is a difference of integral constant between differentiation of integral and integral of derivative.

Remarks :

(1) We should not conclude for formula 12 and 13 that $\sin^{-1} x = -\cos^{-1} x$ because they are differ by constant term only, because we know that $\sin^{-1} x + \cos^{-1} x = \pi/2$.

(2) In practice, we normally do not mention the interval over whcih the various functions are defined. However, in any specific problem one has to keep it in mind.

9.06 About Differentiation and Integration

(1) Both are operations on functions, the result of each is also a function.

(2) Both satisfy the property of linearity.

(3) All functions are not differentiable and integralble.

(4) The derivative fo a function, when it exist, is a unique function. The integral of a function is not so due to integral constant.

(5) We can speak of the derivastive at a point. We never speak of the integral at a point, We speak of the integral of a function over an interval on whcih the integral is defined.

(6) The derivative of a function has a geometrical meaning, namely, the slope of the tangent to the corresponding curve at a point similarly, the indefinite integral of a function represents geometrically, are of some region, or area under curve.

(7) The derivative is used for finding some physical quantities like the velocity of a moving particle, acceleration whereas integration is used for finding, centre of mass, momentum etc.

(8) The process of differentiation and integration are inverse operation of each other.

9.07 Methods of Integration

Some prominent methods to find out the integration are :

(I) Usign standard formulae

(II) Integration by substitution

(III) Integration using Partial fractions

(IV) Integration by parts

I Integration by the use of standard formulae

Here by using the standard formulae or other trigonometric formulae, We can find integral of given function. We can illustrate with the following examples.

Illustrative Examples

Example 1. Integrate the following functions with respect to x

(i) x^6 (ii) $\sqrt{x}$ (iii) $\dfrac{x^2+1}{x^4}$ (iv) $\dfrac{1}{\sqrt{x}}$

Solution : We know that $\int x^n dx = \dfrac{x^n+1}{n+1} + c, n \neq -1$

(i) Let $I = \int x^6 dx = \dfrac{x^{6+1}}{6+1} + c = \dfrac{x^7}{7} + c$

(ii) Let $I = \int \sqrt{x} dx = \int x^{1/2} dx = \dfrac{x^{1/2+1}}{(1/2)+1} + c = \dfrac{x^{3/2}}{3/2} + c = \dfrac{2}{3} x^{3/2} + c$

(iii) Let $I = \int \dfrac{x^2+1}{x^4} dx = \int \left(\dfrac{x^2}{x^4} + \dfrac{1}{x^4} \right) dx = \int \dfrac{1}{x^2} dx + \int \dfrac{1}{x^4} dx$

$$= \int x^{-2} dx + \int x^{-4} dx = \frac{x^{-2+1}}{-2+1} + \frac{x^{-4+1}}{-4+1} + c$$

$$= \frac{x^{-1}}{-1} + \frac{x^{-3}}{-3} + c = -\frac{1}{x} - \frac{1}{3x^3} + c$$

(iv) Let $I = \int \dfrac{1}{\sqrt{x}} dx = \int x^{-1/2} dx = \left[\dfrac{x^{-1/2+1}}{-1/2+1} \right] + c$

$$= \frac{x^{1/2}}{(1/2)} + c = 2\sqrt{x} + c$$

Example 2. Evaluate $\int \dfrac{ax^2+bx+c}{x} dx$

Solution : $\int \dfrac{ax^2+bx+c}{x} dx = \int \left[\dfrac{ax^2}{x} + \dfrac{bx}{x} + \dfrac{c}{x} \right] dx$

$$= \int \left(ax + b + \frac{c}{x} \right) dx$$

$$= \int ax\, dx + \int b\, dx + \int \frac{c}{x} dx$$

$$= a\int x\, dx + b\int dx + c\int \frac{1}{x} dx$$

$$= \frac{ax^2}{2} + bx + c \log|x| + k$$

Example 3. Evaluate $\int \frac{\sin^2 x}{1+\cos x} dx$

Solution :

$$\int \frac{\sin^2 x}{1+\cos x} dx = \int \frac{1-\cos^2 x}{1+\cos x} dx$$

$$= \int \frac{(1-\cos x)(1+\cos x)}{(1+\cos x)} dx$$

$$= \int (1-\cos x)\, dx = \int 1.dx - \int \cos x\, dx$$

$$= x - \sin x + c$$

Example 4. Evaluate $\int \frac{x^2}{x+1} dx$

Solution :

$$\int \frac{x^2}{x+1} dx = \int \frac{(x^2-1)+1}{(x+1)} dx$$

$$= \int \left[\frac{x^2-1}{x+1} + \frac{1}{x+1}\right] dx$$

$$= \int \left[(x-1) + \frac{1}{(x+1)}\right] dx = \int \left(x - 1 + \frac{1}{1+x}\right) dx$$

$$= \frac{x^2}{2} - x + \log|x+1| + c, (x \neq -1)$$

Example 5. Evaluate $\int \sqrt{1+\sin 2x}\, dx$

Solution :

$$\int \sqrt{1+\sin 2x}\, dx = \int \sqrt{[(\sin^2 x + \cos^2 x) + 2\sin x \cos x]}\, dx$$

$$= \int \sqrt{(\sin x + \cos x)^2}\, dx$$

$$= \int (\sin x + \cos x)\, dx$$

$$= -\cos x + \sin x + c$$

Example 6. Evaluate $\int \frac{1-\cos 2x}{1+\cos 2x} dx$

Solution :

$$\int \frac{1-\cos 2x}{1+\cos 2x} dx = \int \frac{2\sin^2 x}{2\cos^2 x} dx \qquad [\because \cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1]$$

$$= \int \tan^2 x\, dx = \int (\sec^2 x - 1) dx$$

$$= \tan x - x + c$$

Example 7. Evaluate $\int \frac{1}{1+\sin x} dx$

Solution : $$\int \frac{1}{1+\sin x} dx = \int \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x} dx$$

$$= \int \frac{1-\sin x}{1-\sin^2 x} dx = \int \frac{1-\sin x}{\cos^2 x} dx$$

$$= \int \left[\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} \right] dx$$

$$= \int (\sec^2 x - \sec x \tan x) dx$$

$$= \tan x - \sec x + c$$

Example 8. The slope of a curve is given by $\frac{dy}{dx} = 2x - \frac{3}{x^2}$. It passes through (1, 1). Find the equation of curve.

Solution : $\because \frac{dy}{dx} = 2x - \frac{3}{x^2}$

Integrating both the sides with respect tot x

$$\int \frac{dy}{dx} dx = \int \left(2x - 3x^{-2}\right) dx$$

$$\Rightarrow \quad \int dy = 2\int x dx - 3\int x^{-2} dx$$

$$\Rightarrow \quad y = \frac{2x^2}{2} - 3\frac{x^{-1}}{-1} + c$$

$$\Rightarrow \quad y = x^2 + \frac{3}{x} + c$$

$\because$ It passes through (1, 1)

$$1 = (1)^2 + \frac{3}{(1)} + c \Rightarrow c = -3$$

$\therefore$ required equation of curve

$$y = x^2 + \frac{3}{x} - 3$$

Exercise 9.1

1. Integrate the following functions with respect to x

(i) $3\sqrt{x^2}$ (ii) e^{3x} (iii) $(1/2)^x$ (iv) $a^{2\log_a x}$

Evaluate the following :

2. $\int\left(5\cos x-3\sin x+\frac{2}{\cos^2 x}\right)dx$ 3. $\int\frac{x^3-1}{x^2}dx$ 4. $\int\sec^2 x\cos ec^2 x\,dx$

5. $\int(1+x)\sqrt{x}\,dx$ 6. $\int a^x da$ 7. $\int\frac{x^2}{1+x^2}dx$

8. $\int\frac{\cos^2 x}{1+\sin x}dx$ 9. $\int\sec x(\sec x+\tan x)\,dx$

10. $\int(\sin^{-1}x+\cos^{-1}x)\,dx$ 11. $\int\frac{x^2-1}{x^2+1}dx$

12. $\int\tan^2 x\,dx$ 13. $\int\cot^2 x\,dx$ 14. $\int\frac{dx}{\sqrt{1+x}-\sqrt{x}}$

15. $\int(\tan^2 x-\cot^2 x)\,dx$ 16. $\int\frac{\sin x}{1+\sin x}dx$ 17. $\int\frac{1}{1-\cos x}dx$

18. $\int\left[1+\frac{1}{1+x^2}+\frac{3}{x\sqrt{x^2-1}}+2^x\right]dx$ 19. $\int\cot x(\tan x-\cos ec x)dx$

20. $\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx$ 21. $\int\log_x x\,dx$ 22. $\int\sqrt{1+\cos 2x}\,dx$

23. $\int\frac{\cos 2x}{\sin^2 x\cos^2 x}dx$ 24. $\int\frac{3\cos x+4}{\sin^2 x}dx$

II Integration by substitution

(a) Substitution of Variables : The given variable cna be transformed into antoher form or independent variable, then doing integrationis called integration by substitutiion.

Theorem : If x is substituted by new variable in $\int f(x)\,dx$ then $x=\phi(t)$

$$\int f(x)\,dx=\int f\phi\{t\}\phi'(t)dt,\ \text{where}\ \phi'(t)=\frac{d\phi}{dt}$$

Proof : Let $\int f(x)\,dx=F(x)$ then $\frac{d}{dx}\int f(x)dx=\frac{d}{dx}F(x)$ (From differentiation) (1)

Now if $x=\phi(t)$ then $\frac{dx}{dt}=\phi'(t)$ (2)

again $$\frac{d}{dt}F(x) = \frac{d}{dx}F(x)\,.\,\frac{dx}{dt} \quad \text{(Chain rule)}$$

$$= f(x)\,.\,\phi'(t) \qquad \text{[From (1) and (2)]}$$

$$= f\{\phi(t)\}\phi'(t)$$

Now by definition of integration

$$\int \frac{d}{dx}F(x)\,dt = \int f\{\phi(t)\}.\phi'(t)\,dt$$

Or $$F(x) = \int f\{\phi(t)\}.\phi'(t)\,dt$$

Or $$\int f(x)\,dx = \int f\{\phi(t)\}.\phi'(t)\,dt$$

Some integrands for substitution

(a) $$\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c \qquad \text{(Let } f(x) = t \text{ etc.)}$$

(b) $$\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c \qquad \text{(Let } f(x) = t \text{ etc.)}$$

(c) For linear function $f(ax+b)$

$$\int f(ax+b)\,dx = \frac{f(ax+b)}{a} + c \qquad \text{(where a, b are constants)}$$

whereas $$\int f(x)\,dx = F(x) + c$$

Formulae for linear functions

If $a \neq o$ then

(i) $$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c, \quad n \neq -1$$

(ii) $$\int \frac{1}{ax+b}dx = \frac{1}{a}\log|ax+b| + c, \ a > 0$$

(iii) $$\int e^{ax+b}dx = \frac{e^{ax+b}}{a} + c$$

(iv) $$\int \sin(ax+b)\,dx = -\frac{\cos(ax+b)}{a} + c$$

(v) $$\int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a} + c$$

Remark : There is no general ruel for substitution, it depends on the nature of integral. The success of substitution method depends that we make a substitution such that a function whose derivative also occurs in the integrand in product form.

Illustrative Examples

Example 9. Integrate the following functions with respect to x

(i) $\dfrac{\cos[\log(x)]}{x}$ (ii) $\dfrac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$ (iii) $\dfrac{\sin\sqrt{x}}{\sqrt{x}}$ (iv) $\dfrac{1}{\cos^2(5x+2)}$

Solution : (i) Let $\log x = t$ then $\dfrac{1}{x}dx = dt$

$\therefore$ $$I = \int \frac{\cos(\log x)}{x}dx = \int \cos t\, dt = \sin t + c = \sin(\log x) + c$$

(ii) Let $$I = \int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}dx$$

Let $$\sin^{-1}x = t \Rightarrow \frac{1}{\sqrt{1-x^2}}dx = dt$$

$\therefore$ $$I = \int e^t dt = e^t + c = e^{\sin^{-1}x} + c$$

(iii) $$I = \int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$$

Let $$\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}}dx = dt \Rightarrow \frac{1}{\sqrt{x}}dx = 2dt$$

$\therefore$ $$I = \int \sin t \times 2dt = 2\int \sin t\, dt$$

$$= 2\times(-\cos t) + c = -2\cos\sqrt{x} + c$$

(iv) $$I = \int \frac{1}{\cos^2(5x+2)}dx$$

$$= \int \sec^2(5x+2)\,dx$$

Let $$5x+2 = t \Rightarrow 5dx = dt \Rightarrow dx = \frac{1}{5}dt$$

$\therefore$ $$I = \int \sec^2 t \times \frac{1}{5}dt$$

$$= \frac{1}{5}\int \sec^2 t\, dt = \frac{1}{5}\tan t + c = \frac{1}{5}\tan(5x+2) + c$$

Example 10. Integrate the following functions with respect to x

(i) $\dfrac{\log[x+\sqrt{1+x^2}]}{\sqrt{1+x^2}}$ (ii) $\sec x \log(\sec x + \tan x)$ (iii) $\dfrac{1}{1+\tan x}$

Solution : (i) $$I = \int \frac{\log[x+\sqrt{1+x^2}]}{\sqrt{1+x^2}}dx$$

Let $$\log[x+\sqrt{1+x^2}] = t$$

$$\therefore \quad \frac{1}{x+\sqrt{1+x^2}} \times \left[1+\frac{2x}{2\sqrt{1+x^2}}\right]dx = dt$$

$$\Rightarrow \quad \frac{1}{[x+\sqrt{1+x^2}]} \times \frac{[\sqrt{1+x^2}+x]}{\sqrt{1+x^2}}dx = dt$$

$$\Rightarrow \quad \frac{1}{\sqrt{1+x^2}}dx = dt$$

$$\therefore \quad I = \int t\,dt$$

$$= \frac{t^2}{2} + c$$

$$= \frac{1}{2}[\log\{x+\sqrt{1+x^2}\}]^2 + c$$

(ii) $$I = \int \sec x \,.\, \log(\sec x + \tan x)dx$$

Let $$\log(\sec x + \tan x) = t$$

$$\therefore \quad \frac{1}{(\sec x + \tan x)} \times (\sec x \tan x + \sec^2 x)dx = dt$$

$$\sec x\, dx = dt$$

$$\therefore \quad I = \int t\, dt = \frac{t^2}{2} + c = \frac{1}{2}\left[\log(\sec x + \tan x)\right]^2 + c$$

(iii) $$I = \int \frac{1}{1+\tan x}dx = \int \frac{1}{1+\frac{\sin x}{\cos x}}dx = \int \frac{\cos x}{\cos x + \sin x}dx$$

$$= \frac{1}{2}\int \frac{2\cos x}{\cos x + \sin x}dx = \frac{1}{2}\int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{\cos x + \sin x}$$

$$= \frac{1}{2}\int \frac{\cos x + \sin x}{\cos x + \sin x}dx + \frac{1}{2}\int \frac{\cos x - \sin x}{\cos x + \sin x}dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{\cos x - \sin x}{\cos x + \sin x}dx$$

In second integral, Let $\cos x + \sin x = t$

$\therefore$ $(-\sin x + \cos x)dx = dt$

$\therefore$ $$I = \frac{1}{2}\int dx + \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}x + \frac{1}{2}\log|t| + c$$

$$= \frac{x}{2} + \frac{1}{2}\log|\cos x + \sin x| + c$$

(b) Integration of trigonometric functions $\tan x, \cot x, \sec x$ and $\cos ecx$

(i) Let $$I = \int \tan x\, dx = \int \frac{\sin x}{\cos x}dx$$

Let $\cos x = t \Rightarrow -\sin x\, dx = dt \Rightarrow \sin x dx = -dt$

$\therefore$ $$I = \int \frac{-dt}{t} = -\log|t| + c = -\log|\cos x| + c$$

$$= \log|\sec x| + c$$

$\therefore$ $$\int \tan x\, dx = \log|\sec x| + c = -\log|\cos x| + c$$

(ii) Let $$I = \int \cot x\, dx = \int \frac{\cos x}{\sin x}dx$$

Let $\sin x = t \Rightarrow \cos x\, dx = dt$

$\therefore$ $$I = \int \frac{dt}{t} = \log|t| = \log|\sin x| + c$$

$\therefore$ $$\int \cot t\, dx = \log|\sin x| + c$$

(iii) Let $$I = \int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)}dx$$

Let $\sec x + \tan x = t$

$\therefore$ $\left(\sec x \tan x + \sec^2 x\right)dx = dt \Rightarrow \sec x\left(\sec x + \tan x\right)dx = dt$

$\therefore$ $$I = \int \frac{dt}{t} = \log|t| + c = \log|\sec x + \tan x| + c \qquad \text{. . . (1)}$$

$$= \log\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right| + c$$

$$= \log\left|\frac{1 + \sin x}{\cos x}\right| + c$$

$$= \log\left|\frac{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}\right| + c$$

$$= \log \left| \frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)} \right| + c$$

$$= \log \left| \frac{1 + \tan x/2}{1 - \tan x/2} \right| + c$$

$$= \log \tan \left| \frac{\pi}{4} + \frac{x}{2} \right| + c$$

$\therefore$ $$\int \sec x \, dx = \log|\sec x + \tan x| + c = \log \tan \left| \frac{\pi}{4} + \frac{x}{2} \right| + c$$

(iv) Let $$I = \int \cos ecx \, dx = \int \frac{\cos ecx(\cos ecx - \cot x)}{(\cos ecx - \cot x)} dx$$

Let $$\cos ecx - \cot x = t \Rightarrow \left(-\cos ecx \cot x + \cos ec^2 x\right) dx = dt$$

$\therefore$ $$\text{cosec}x(\text{cosec}x - \cot x)\, dx = dt$$

$\therefore$ $$I = \int \frac{dt}{t} = \log|t| + c = \log|\text{cosec}x - \cot x| + \text{c}$$

$$= \log \left| \frac{1}{\sin x} - \frac{\cos x}{\sin x} \right| + c = \log \left| \frac{1 - \cos x}{\sin x} \right| + c$$

$$= \log \left| \frac{1 - 1 + 2\sin^2(x/2)}{2\sin(x/2)\cos(x/2)} \right| + c = \log \left| \tan\frac{x}{2} \right| + c$$

$\therefore$ $$\int \cos ecx \, dx = \log|\text{cosec}x - \cot x| + c = \log|\tan x/2| + c$$

$$(\because \cos ecx - \cot x = \tan x/2)$$

Example 11. Integrate $\frac{1}{\sqrt{1+\cos 2x}}$ w.r.t. x

Solution : Let $$I = \int \frac{1}{\sqrt{1+\cos 2x}} dx = \int \frac{1}{\sqrt{2\cos^2 x}} dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\cos x} dx = \frac{1}{\sqrt{2}} \int \sec x \, dx$$

$$= \frac{1}{\sqrt{2}} \log|\sec x + \tan x| + c$$

Example 12. Integrate $\sqrt{\sec x+1}$ with respect to x

Solution : Let $$I=\int\sqrt{\sec x+1}dx=\int\sqrt{\left(\frac{1}{\cos x}+1\right)}dx$$

$$=\int\sqrt{\frac{1+\cos x}{\cos x}}dx=\int\sqrt{\frac{2\cos^2 x/2}{1-2\sin^2 x/2}}dx=\int\frac{\sqrt{2}\cos x/2}{\sqrt{1-\{\sqrt{2}\sin(x/2)\}^2}}dx$$

Let $\sqrt{2}\sin(x/2)=t\Rightarrow\sqrt{2}\cos(x/2)\times 1/2dx=dt$

$\Rightarrow$ $\sqrt{2}\cos(x/2)\,dx=2dt$

$\therefore$ $$I=\int\frac{2dt}{\sqrt{1-t^2}}=2\sin^{-1}t+=2\sin^{-1}(\sqrt{2}\sin x/2)+c$$

(c) Using substitution method by trigonometric identities.

Many times when the integrand involves some trigonometric functiosn, we use some known identities to make it integrable and then find integral by suitable substitution.

Illustrative Examples

Example 13. Evaluate the following:

(i) $\int\cos 3x\cos 4x\,dx$ (ii) $\int\sin^2 x\,dx$ (iii) $\int\cos^3 x\,dx$ (iv) $\int\sin^4 x\,dx$

Solution :(i) Let $$I=\int\cos 3x\cos 4x\,dx=\frac{1}{2}\int 2\cos 4x\cos 3x\,dx$$

$$=\frac{1}{2}\int(\cos 7x+\cos x)dx=\frac{1}{2}\left[\frac{\sin 7x}{7}+\sin x\right]+c$$

(ii) Let $$I=\int\sin^2 x\,dx=\int\frac{1-\cos 2x}{2}dx=\frac{1}{2}\int(1-\cos 2x)dx$$

$$=\frac{1}{2}\left[x-\frac{\sin 2x}{2}+c\right]$$

(iii) Let $$I=\int\cos^3 x\,dx=\frac{1}{4}\int(\cos 3x+3\cos x)\,dx$$

$$\left(\because\cos 3x=4\cos^3 x-3\cos x\Rightarrow\cos^3 x=1/4(\cos 3x+3\cos x)\right)$$

$$=\frac{1}{4}\left[\frac{\sin 3x}{3}+3\sin x\right]+c$$

(iv) Let $$I=\int\sin^4 x\,dx=\int(\sin^2 x)^2dx=\int\left(\frac{1-\cos 2x}{2}\right)^2dx$$

$$=\frac{1}{4}\int(1+\cos^2 2x-2\cos 2x)\,dx$$

$$= \frac{1}{4}\int\left[1 + \frac{1+\cos 4x}{2} - 2\cos 2x\right]dx = \frac{1}{8}\int(3 + \cos 4x - 4\cos 2x)\,dx$$

$$= \frac{1}{8}\left[3x + \frac{\sin 4x}{4} - 2\sin 2x\right] + c$$

Exercise 9.2

Integrate the following functions with respect to x

1. (i) $x\sin x^2$ (ii) $x\sqrt{x^2+1}$

2. (i) $\dfrac{e^x - \sin x}{e^x + \cos x}$ (ii) $\dfrac{e^x}{\sqrt{1+e^x}}$

3. (i) $\sqrt{e^x+1}$ (ii) $\dfrac{e^{\sqrt{x}}\cos e^{\sqrt{x}}}{\sqrt{x}}$

4. (i) $\dfrac{1}{x(1+\log x)}$ (ii) $\dfrac{(1+\log x)^3}{x}$

5. (i) $\dfrac{e^{m\tan^{-1}x}}{1+x^2}$ (ii) $\dfrac{\sin^p x}{\cos^{p+2} x}$

6. (i) $\dfrac{1}{\sqrt{1+\cos 2x}}$ (ii) $\dfrac{1+\cos x}{\sin x\cos x}$

7. (i) $\sin 3x\sin 2x$ (ii) $\sqrt{1-\sin x}$

8. (i) $\cos^4 x$ (ii) $\sin^3 x$

9. (i) $\dfrac{1}{\sin x\cos^3 x}$ (ii) $\dfrac{(1+x)e^x}{\cos^2(xe^x)}$

10. (i) $\dfrac{1}{1-\tan x}$ (ii) $\dfrac{1}{1+\cot x}$

11. (i) $\dfrac{\sec^4 x}{\sqrt{\tan x}}$ (ii) $\dfrac{1-\tan x}{1+\tan x}$

12. (i) $\dfrac{\sin(x+a)}{\sin(x-a)}$ (ii) $\dfrac{\sin x}{\sin(x-a)}$

13. (i) $\dfrac{\sin 2x}{\sin 5x\sin 3x}$ (ii) $\dfrac{\sin 2x}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)}$

[Hint $= \sin 2x = \sin(5x-3x)$] [Hint $= 2x = \left(x-\frac{\pi}{6}\right)+\left(x+\frac{\pi}{6}\right)$]

14. (i) $\dfrac{1}{3\sin x+4\cos x}$ $\left[\text{Hint: } 3=r\cos\theta, 4=r\sin\theta\right]$ (ii) $\dfrac{1}{\sin(x-a)\sin(x-b)}$

15. (i) $\dfrac{\sin x\cos x}{a\cos^2 x+b\sin^2 x}$ (ii) $\dfrac{\sec x}{\sqrt{\sin(2x+\alpha)+\sin\alpha}}$

16. (i) $\dfrac{1}{\sqrt{\cos^3 x\sin(x+a)}}$ (ii) $\dfrac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}$

(d) Integration by substitution of variables by trigonometric functions.

(i) $\dfrac{1}{a^2+x^2}$ (ii) $\dfrac{1}{\sqrt{a^2-x^2}}$ (iii) $\dfrac{1}{\sqrt{x^2+a^2}}$ (iv) $\dfrac{1}{\sqrt{x^2-a^2}}$

(i) Let, $$I=\int\frac{1}{a^2+x^2}dx$$

If, $x=a\tan\theta$ then $dx=a\sec^2\theta d\theta$

Now $$I=\int\frac{a\sec^2\theta d\theta}{a^2+a^2\tan^2\theta}=\frac{1}{a}\int\frac{\sec^2\theta}{1+\tan^2\theta}d\theta$$

$$=\frac{1}{a}\int\frac{\sec^2\theta}{\sec^2\theta}d\theta=\frac{1}{a}\int d\theta=\frac{1}{a}(\theta)+c=\frac{1}{a}\tan^{-1}\frac{x}{a}+c$$

$$\int\frac{1}{a^2+x^2}dx=\frac{1}{a}\tan^{-1}\frac{x}{a}+c$$

(ii) Let $$I=\int\frac{1}{\sqrt{a^2-x^2}}dx$$

If $x=a\sin\theta$ then $dx=a\cos\theta d\theta$

$\therefore$ $$I=\int\frac{a\cos\theta d\theta}{\sqrt{a^2-a^2\sin^2\theta}}=\int\frac{a\cos\theta d\theta}{a\cos\theta}=\int d\theta=\theta+c=\sin^{-1}\frac{x}{a}+c$$

$$\int\frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\frac{x}{a}+c$$

(iii) Let $$I=\int\frac{1}{\sqrt{x^2+a^2}}dx$$

Let $x=a\tan\theta\Rightarrow dx=a\sec^2\theta d\theta$

$\therefore$ $$I=\int\frac{a\sec^2\theta d\theta}{\sqrt{a^2\tan^2\theta+a^2}}=\int\frac{a\sec^2\theta}{a\sec\theta}d\theta$$

$$=\int\sec\theta d\theta=\log|\sec\theta+\tan\theta|+c_1$$

$$= \log\left|\sqrt{1+\frac{x^2}{a^2}}+\frac{x}{a}\right|+c_1$$

$$= \log\left|\frac{\sqrt{x^2+a^2}+x}{a}\right|+c_1 = \log\left|x+\sqrt{x^2+a^2}\right|-\log a+c_1$$

$$= \log\left|x+\sqrt{x^2+a^2}\right|+c, \text{ where } c = c_1 - \log a$$

$\therefore$ $$\int\frac{1}{\sqrt{x^2+a^2}}dx = \log|x+\sqrt{x^2+a^2}|+c$$

(iv) Let $$I = \int\frac{1}{\sqrt{x^2-a^2}}dx$$

Let $x = a\sec\theta \Rightarrow dx = a\sec\theta\tan\theta\, d\theta$

$\therefore$ $$I = \int\frac{1}{\sqrt{a^2\sec^2\theta - a^2}}\times a\sec\theta\tan\theta\, d\theta = \int\frac{a\sec\theta\tan\theta}{a\tan\theta}$$

$$= \int\sec\theta\, d\theta = \log|\sec\theta+\tan\theta|+c_1$$

$$= \log\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right|+c_1 = \log\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|+c_1$$

$$= \log|x+\sqrt{x^2-a^2}|-\log a+c_1 = \log|x+\sqrt{x^2-a^2}|+c$$ (where $c = c_1 - \log a$)

$\therefore$ $$\int\frac{1}{\sqrt{x^2-a^2}}dx = \log|x+\sqrt{x^2-a^2}|+c$$

Some Suitable trigonometric substitutions

	Integrands	**Substitution**
(i)	$\sqrt{x^2+a^2}$ or $\frac{1}{\sqrt{x^2+a^2}}$	$x = a\tan\theta$
(ii)	$\sqrt{a^2-x^2}$ or $\frac{1}{\sqrt{a^2-x^2}}$	$x = a\sin\theta$ or $x = a\cos\theta$
(iii)	$\sqrt{x^2-a^2}$ or $\frac{1}{\sqrt{x^2-a^2}}$	$x = a\sec\theta$
(iv)	$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$ or $x = a\cos\theta$
(v)	$\sqrt{x+a}$	$x = a\cos 2\theta$ or $x = a\cos\theta$

(vi) $\sqrt{2ax-x^2}$	$x=2a\sin^2\theta$ or $x=a(1-\cos 2\theta)$
(vii) $\sqrt{\dfrac{a^2-x^2}{a^2+x^2}}$	$x^2=a^2\cos 2\theta$
(viii) $\sqrt{\dfrac{x+a}{x}}$ or $\sqrt{\dfrac{x}{x+a}}$	$x=a\tan^2\theta$

Illustrative Examples

Example 14. Integrate the following with respect to x

(i) $\dfrac{x}{1+x^4}$ (ii) $\dfrac{1}{\sqrt{9-25x^2}}$

Solution : (i) Let $$I=\int\frac{x}{1+x^4}dx$$

Let $$x^2=t \Rightarrow xdx=\frac{dt}{2}$$

$\therefore$ $$I=\frac{1}{2}\int\frac{dt}{1+t^2}=\frac{1}{2}\tan^{-1}(t)+c=\frac{1}{2}\tan^{-1}(x^2)+c$$

(ii) Let $$I=\int\frac{1}{\sqrt{9-25x^2}}dx=\frac{1}{5}\int\frac{1}{\sqrt{(3/5)^2-x^2}}dx$$

$$=\frac{1}{5}\sin^{-1}\left(\frac{x}{3/5}\right)+c=\frac{1}{5}\sin^{-1}\frac{5x}{3}+c$$

Example 15. Integrate $\dfrac{1}{\sqrt{x^2-4x+5}}$ with respect to x

Solution : $$I=\int\frac{1}{\sqrt{x^2-4x+5}}dx=\int\frac{1}{\sqrt{(x-2)^2+1}}dx$$

$$=\log|(x-2)+\sqrt{(x-2)^2+1}|+c$$

$$=\log|(x-2)+\sqrt{x^2-4x+5}|+c$$

Example 16. Evaluate: $\displaystyle\int\frac{1}{x^2+2x+5}dx$

Solution : Let $$I=\int\frac{1}{x^2+2x+5}dx=\int\frac{1}{(x+1)^2+(2)^2}dx$$

$$=\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+c$$

Example 17. Integrate $\frac{1}{\sqrt{5x-6-x^2}}$ with respect to x

Solution : Let $$I=\int\frac{1}{\sqrt{5x-6-x^2}}dx=\int\frac{1}{\sqrt{-6-(x^2-5x)}}dx$$

$$=\int\frac{1}{\sqrt{(25/4-6)-(x^2-5x+25/4)}}dx=\int\frac{1}{\sqrt{(1/2)^2-(x-5/2)^2}}dx$$

$$=\sin^{-1}\left[\frac{x-5/2}{1/2}\right]+c=\sin^{-1}\left(\frac{2x-5}{1}\right)+c$$

Example 18. Integrate $\frac{(1+x)^2}{x+x^3}$ with respect to x

Solution : Let $$I=\int\frac{(1+x)^2}{x+x^3}dx=\int\frac{1+x^2+2x}{x(1+x^2)}dx$$

$$=\int\left[\frac{(1+x^2)}{x(1+x^2)}+\frac{2x}{x(1+x^2)}\right]dx=\int\frac{1}{x}dx+\int\frac{2}{1+x^2}dx$$

$$=\log|x|+2\tan^{-1}x+c$$

Example 19. Integrate $\frac{\sin 2x\cos 2x}{\sqrt{9-\cos^4 2x}}$ with respect to x

Solution : Let $$I=\int\frac{\sin 2x\cos 2x}{\sqrt{9-\cos^4 2x}}dx$$

Let $$\cos^2 2x=t\Rightarrow 2\cos 2x.(-\sin 2x)2.dx=dt$$

$\Rightarrow$ $$\sin 2x\cos 2x\,dx=-\frac{dt}{4}$$

$\therefore$ $$I=-\frac{1}{4}\int\frac{dt}{\sqrt{9-t^2}}=-\frac{1}{4}\sin^{-1}\left(\frac{t}{3}\right)+c$$

$$=-\frac{1}{4}\sin^{-1}\left(\frac{\cos^2 2x}{3}\right)+c$$

Example 20. If $\int\frac{2^x}{\sqrt{1-4^x}}dx=k\sin^{-1}2^x+c$, then find the vlaue of k

Solution : Let $$I=\int\frac{2^x}{\sqrt{1-4^x}}dx=\int\frac{2^x}{\sqrt{1-(2^x)^2}}dx$$

Let $$2^x=t\Rightarrow 2^x\log_e 2dx=dt\Rightarrow 2^x dx=\frac{dt}{\log_e 2}$$

$\therefore$ $$I=\frac{1}{\log_e 2}\int\frac{dt}{\sqrt{1-t^2}}=\frac{1}{\log_e 2}\sin^{-1}(t)+c=\log_2 e.(\sin^{-1}2^x)+c$$

$\therefore$ $$\int\frac{2^x}{\sqrt{1-4^x}}dx=\log_2 e.(\sin^{-1}2^x)+c$$

but it is given that $$\int\frac{2^x}{\sqrt{1-4^x}}dx=k(\sin^{-1}2^x)+c$$

$\therefore$ On comparison, $$k=\log_2 e$$

Exercise 9.3

Integrate the following function with respect to *x*

1. (i) $\frac{1}{50+2x^2}$ (ii) $\frac{1}{\sqrt{32-2x^2}}$

2. (i) $\frac{1}{\sqrt{1-e^{2x}}}$ (ii) $\frac{1}{\sqrt{1+4x^2}}$

3. (i) $\frac{1}{\sqrt{a^2-b^2x^2}}$ (ii) $\frac{1}{\sqrt{(2-x)^2+1}}$

4. (i) $\frac{x^2}{\sqrt{x^6+4}}$ (ii) $\frac{x^4}{\sqrt{1-x^{10}}}$

5. (i) $\frac{1}{x^2+6x+8}$ (ii) $\frac{1}{\sqrt{2x^2-x+2}}$

6. (i) $\frac{e^x}{e^{2x}+2e^x\cos\propto+1}$ (ii) $\frac{1+\tan^2 x}{\sqrt{\tan^2 x+3}}$

7. (i) $\frac{1}{\sqrt{3x-2-x^2}}$ (ii) $\frac{1}{\sqrt{4+8x-5x^2}}$

8. (i) $\frac{\sin x+\cos x}{\sqrt{\sin 2x}}$ (ii) $\frac{1}{\sqrt{x^2+2ax+b^2}}$

9. (i) $\sqrt{\frac{a-x}{x}}$ (ii) $\sqrt{\frac{a+x}{a-x}}$

10. (i) $\dfrac{\sqrt{x}}{\sqrt{a^3-x^3}}$ (ii) $\dfrac{1}{(a^2+x^2)^{3/2}}$

11. (i) $\dfrac{1}{(1-x^2)^{3/2}}$ (ii) $\dfrac{x+1}{\sqrt{x^2+1}}$

12. (i) $\dfrac{1}{\sqrt{(x-\alpha)(\beta-x)}}$ (ii) $\dfrac{1}{\sqrt{2x-x^2}}$

13. (i) $\dfrac{1}{\sqrt{(x-1)(x-2)}}$ (ii) $\dfrac{\cos x}{\sqrt{4-\sin^2 x}}$

III. Integration by resolving into partial fractions

(a) Rational algebraic function

Definition : If $f(x)$ and $g(x)$ are polynomials of x then fraction $\dfrac{f(x)}{g(x)}$ is called rational algebraic function.

For example: $\dfrac{x^2-x-6}{x^3+x^2-3x+4}, \dfrac{2x+1}{2x^2+x+1}, \dfrac{x^2}{x^2+1}, \dfrac{2x^3}{(x-1)(x^2+1)}, \dfrac{x^4}{x^3+2x-4}$

Proper Rational Fraction : If in a rational algebraic fraction the power of numerator is less than the power of denominator then it is called a proper rational fraction.

Improper Rational Fraction : If in a rational algebraic fraction the power of numerator is more than or equal to the power of denominator then it is called an improper rational fraction.

For example : $\dfrac{2x+3}{3x^2+x+4}$, is a proper fraction.

For example : $\dfrac{3x^3+x^2+5x-4}{x^2+x+2}$ and $\dfrac{3x^2+x+2}{(x+1)(x+3)}$ are improper fractions.

Remark : An improper rational fraction can be expressed into a proper rational fraction by division process.

For example $\dfrac{3x^3+2x+7}{x^2+5x+9}=3(x-5)+\dfrac{50x+142}{x^2+5x+9}$

The above rational algebraci function may be expressed or convert into partial fraction and then integrate each fraction.

Partial Fraction : It is always possible to write the integrand as a sum of simpler rational functiosn by a method called partial fraction decomposition.

For example $\dfrac{2x-5}{x^2-5x+6}=\dfrac{1}{x-2}+\dfrac{1}{x-3}$

Rules of resolving a rational fraction into partial fraction

[A] First of all if the fraction is not proper then convert it into a proper fraction by using division method. So that an improper fraction will be decompose into a polynomial and proper fraction. Keep the polynomial same and decompose the real fraction into partial fraction.

[B] If denominator of proper fraction is not in the form of factors then factorize it.

[C] Now assume the constant term as equal to the power of denominator. The following indicates the types of simpler partial fraction that is associated with various kind of rational functions.

(a) If denominator contains linear factors without repetition then the form of partial fraction will be according tot he following example:

$$\frac{x}{(x-1)(x+2)(x-3)}=\frac{A}{(x-1)}+\frac{B}{(x+2)}+\frac{C}{(x-3)}$$

(b) If denominaot rcontains linear factors with repetition then the form of partial fraction will be according tot he following example:

$$\frac{x}{(x-1)^2(x+3)}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x+3)}$$

(c) If denominator contains quadratic factors then the form of partial fraction will be according to the following example:

$$\frac{x}{(x-1)(x^2+2)}=\frac{A}{(x-1)}+\frac{Bx+C}{(x^2+2)}$$

Remark : If in a partial fraction both numerator and denominator contain x^2 i.e. quadratic then x^2 must be considedred as linear and the partial fraction may be written as

$$\frac{x^2+2}{(x^2+1)(x^2+3)}=\frac{A}{x^2+1}+\frac{B}{x^2+3}$$

[D] Finding the values of constant A, B and C

(a) As discussed in [C] take LCM of denominators of partial fractions in RHS and find their sum.

(b) Fractions of both the sides are equal and denominators are also equal. hence by comparing their numerators and factors of all powers of x and constant terms find equations. The number of such equations should be same as number of unknown constants. Find the vlaues of unknown constants from equations and get the required partial fraction.

Let $$\frac{2x+3}{(x+2)(x+1)}=\frac{A}{(x+2)}+\frac{B}{(x+1)}$$

or $$\frac{2x+3}{(x+2)(x+1)}=\frac{A(x+1)+B(x+2)}{(x+2)(x+1)}$$

or $$2x+3=A(x+1)+B(x+2) \qquad (1)$$

or $$2x+3=(A+B)x+(A+2B)$$

On comparision of coefficients of equal terms

$$\left.\begin{array}{l} A+B \quad =2 \\ A+2B=3 \end{array}\right] \begin{array}{l}\text{on solving} \\ A=1, B=1\end{array}$$

so $$\frac{2x+3}{(x+2)(x+1)}=\frac{1}{(x+2)}+\frac{1}{(x+1)}$$

Alternative Methods :

(i) **Short Method :** In the above example the corresponding vlaues of x of factors $(x + 1)$ and $(x + 2)$ as $x = -1$ and $x = -2$ can be substituted in equation (1) to find the values of A and B.

(ii) **Division Method :** Division method is more sutiable for repeatign factors of denominator in fractions, in this repeathin factor may be considered as y and the division process id done so that we can get integrable terms.

For example $\frac{x^2}{(x+1)^3(x+2)}$ Let $(x+1)=y$ then

$$\frac{x^2}{(x+1)^3(x+2)}=\frac{(y-1)^2}{y^3(y+1)}=\frac{(1-2y+y^2)}{y^3(1+y)}$$

$$=\frac{1}{y^3}\left[1-3y+4y^2-\frac{4y^3}{1+y}\right]$$

$$=\frac{1}{y^3}-\frac{3}{y^2}+\frac{4}{y}-\frac{4}{1+y}$$

$$=\frac{1}{(x+1)^3}-\frac{3}{(x+1)^2}+\frac{4}{(x+1)}-\frac{4}{(x+2)}$$

which can easily be integrated

(iii) **By inspection :** If there is 1 as numerator in a real fraction and the difference of parts is a constant quantity then this methdo can be used. For this divide by difference of parts and subtract the reciprocal of bigger part from the reciprocal of smaller part.

For example $\frac{1}{(x+2)(x-3)}=\frac{1}{5}\left[\frac{1}{x-3}-\frac{1}{x+2}\right]$ here difference of parts $=(x+2)-(x-3)=5$

Some Standard Integrals

(i) $$\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c \qquad (x>a)$$

(ii) $$\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c \qquad (x<a)$$

Proof :

(i) $$\frac{1}{x^2-a^2}=\frac{1}{(x-a)(x+a)}=\frac{1}{2a}\left[\frac{1}{x-a}-\frac{1}{x+a}\right]$$ (By insepection)

$\therefore$ $$\int\frac{1}{x^2-a^2}dx=\frac{1}{2a}\int\left[\frac{1}{x-a}-\frac{1}{x+a}\right]dx\frac{1}{2a}\int\left[\frac{1}{x-a}-\frac{1}{x+a}\right]dx$$

$$=\frac{1}{2a}\int\frac{1}{x-a}dx-\frac{1}{2a}\int\frac{1}{x+a}dx$$

$$=\frac{1}{2a}\log|x-a|-\frac{1}{2a}\log|x-a|+c$$

$$=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$$

Similarly

(ii) $$\frac{1}{a^2-x^2}=\frac{1}{(a+x)(a-x)}=\frac{1}{2a}\left[\frac{1}{a+x}+\frac{1}{a-x}\right]$$

$\therefore$ $$\int\frac{1}{a^2-x^2}dx=\frac{1}{2a}\int\left[\frac{1}{a+x}+\frac{1}{a-x}\right]dx$$

$$=\frac{1}{2a}\left[\log|a+x|+\frac{\log|a-x|}{-1}\right]+c$$

$$=\frac{1}{2a}\left[\log|a+x|-\log|a-x|\right]+c$$

$$=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c$$

Remark : In some cases substittion makes the task easy. Specially when there is any power of x, Let x^{n-1} is a part of numerator and remaining fraction is a rational function of x^n then substitute $x^n=t$ and then decompose in partial fraction.

Illustrative Examples

Example 21. Integrate the following functions with respect to x

(i) $\frac{1}{16x^2-9}$ (ii) $\frac{1}{9-4x^2}$

Solution : (i) Let, $I=\int\frac{1}{16x^2-9}dx=\int\frac{1}{(4x)^2-(3)^2}dx$

Let $4x=t\Rightarrow 4dx=dt$ or $dx=\frac{1}{4}dt$

$\therefore$ $$I = \frac{1}{4}\int \frac{dt}{t^2 - 3^2} = \frac{1}{4} \times \frac{1}{2\times 3}\log\left|\frac{t-3}{t+3}\right| + c$$

$$= \frac{1}{24}\log\left|\frac{4x-3}{4x+3}\right| + c$$

Solution : (ii) Let $$I = \int \frac{1}{9-4x^2}dx = \int \frac{1}{(3)^2 - (2x)^2}dx$$

Let $$2x = t \Rightarrow dx = \frac{dt}{2}$$

$\therefore$ $$I = \frac{1}{2}\int \frac{dt}{3^2 - t^2} = \frac{1}{2} \times \frac{1}{2\times 3}\log\left|\frac{3+t}{3-t}\right| + c$$

$$= \frac{1}{12}\log\left|\frac{3+2x}{3-2x}\right| + c$$

Example 22. Integrate $\dfrac{1}{x^2 - x - 2}$ with respect to *x*.

Solution : $$\frac{1}{x^2 - x - 2} = \frac{1}{(x-2)(x+1)} = \frac{1}{3}\left[\frac{1}{x-2} - \frac{1}{x+1}\right]$$ (by mathod of inspection)

$\therefore$ $$\int \frac{1}{x^2 - x - 2}dx = \frac{1}{3}\int\left[\frac{1}{(x-2)} - \frac{1}{(x+1)}\right]dx$$

$$= \frac{1}{3}\left[\log|(x-2)| - \log|x+1|\right] + c$$

$$= \frac{1}{3}\log\left|\frac{x-2}{x+1}\right| + c$$

Example 23. Evaluate: $\displaystyle\int \frac{x^2 + x + 2}{(x-1)(x-2)}dx$

Solution : $$\frac{x^2 + x + 2}{(x-1)(x-2)} = 1 + \frac{4x}{(x-1)(x-2)}$$ (on dividing)

Let $$\frac{4x}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$$

or $$4x = A(x-2) + B(x-1) \qquad (1)$$

Now in (1)

Put $x=2$ $8 = B(2-1)$ or $B = 8$

Put $x=1$ $4 = -A$ or $A = -4$

$$\therefore \quad \frac{4x}{(x-1)(x-2)} = \frac{-4}{x-1} + \frac{8}{x-2}$$

$$\therefore \quad \frac{x^2+x+2}{(x-1)(x-2)} = 1 + \left[\frac{-4}{x-1} + \frac{8}{x-2}\right]$$

or

$$\int \frac{x^2+x+2}{(x-1)(x-2)} dx = \int \left[1 - \frac{4}{x-1} + \frac{8}{x-2}\right] dx$$

$$= x - 4\log|x-1| + 8\log|x-2| + c$$

$$= x + 4\left[2\log|x-2| - \log|x-1|\right] + c$$

$$= x + 4\log\frac{(x-2)^2}{|x-1|} + c.$$

Example 24. Integrate $\frac{1}{(x+1)^2(x^2+1)}$ with respect to x.

Solution : Let $$\frac{1}{(x+1)^2(x^2+1)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{Cx+D}{(x^2+1)}$$

$$\Rightarrow \quad 1 = A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2$$

$$\Rightarrow \quad 1 = A(x^3+x^2+x+1) + B(x^2+1) + (Cx^3+2Cx^2+Dx^2+2Dx+Cx+D)$$

$$\Rightarrow \quad 1 = x^3(A+C) + x^2(A+B+2C+D) + x(A+C+2D) + (A+B+D)$$

On comparison

$A+C=0$ (1) $\quad$ $A+B+2C+D=0$ (2)

$A+C+2D=0$ (3) $\quad$ $A+B+D=0$ (4)

From (1) and (3), $2D = O \Rightarrow D = 0$

From (1) and (2), $B+C+D=0$ $\quad$ on solving, $2C=-1 \Rightarrow C=-1/2 \therefore A=1/2$

From (1) and (4), $B-C+D=1$

From (4), $1/2+B+0=1 \Rightarrow B=1/2$

$$\therefore \quad \frac{1}{(x+1)^2(x^2+1)} = \frac{1}{2}\cdot\frac{1}{(x+1)} + \frac{1}{2}\cdot\frac{1}{(x+1)^2} - \frac{1}{2}\cdot\frac{x}{(x^2+1)}$$

$$\therefore \quad \int \frac{1}{(x+1)^2(x^2+1)}dx = \frac{1}{2}\int \frac{1}{(x+1)}dx + \frac{1}{2}\int \frac{1}{(x+1)^2}dx - \frac{1}{4}\int \frac{2x}{(x^2+1)}dx$$

$$= \frac{1}{2}\log|x+1| - \frac{1}{2}\frac{1}{(x+1)} - \frac{1}{4}\log(x^2+1) + c$$

[here $x^2+1=t \Rightarrow 2xdx = dt$]

$$= \frac{1}{2}\log|x+1| - \frac{1}{4}\log(x^2+1) - \frac{1}{2(x+1)} + c$$

Example 25. Integrate $\dfrac{x^2+x+1}{(x-1)^3}$ with respect to *x*.

Solution : Let $(x-1) = y \therefore \dfrac{x^2+x+1}{(x-1)^3} = \dfrac{(y+1)^2+(y+1)+1}{y^3}$

$$= \frac{y^2+3y+3}{y^3} = \frac{1}{y} + \frac{3}{y^2} + \frac{3}{y^3}$$

$$= \frac{1}{(x-1)} + \frac{3}{(x-1)^2} + \frac{3}{(x-1)^3}$$

$$\therefore \quad \int \frac{x^2+x+1}{(x-1)^3}dx = \int \frac{1}{(x-1)}dx + \int \frac{3}{(x-1)^2}dx + \int \frac{3}{(x-1)^3}dx$$

$$= \log|x-1| - \frac{3}{(x-1)} - \frac{3}{2(x-1)^2} + c$$

Example 26. Integrate $\dfrac{1}{\sin x + \sin 2x}$ with respect to *x*.

Solution : Let $I = \displaystyle\int \frac{1}{\sin x + \sin 2x}dx$

$$= \int \frac{1}{\sin x(1+2\cos x)}dx = \int \frac{\sin x}{\sin^2 x(1+2\cos x)}dx$$

$$= \int \frac{\sin x}{(1-\cos^2 x)(1+2\cos x)}dx$$

$$= \int \frac{-dt}{(1-t^2)(1+2t)}$$ [where $\cos x = t \Rightarrow -\sin x\, dx = dt$]

$$= -\int \frac{dt}{(1-t)(1+t)(1+2t)}$$

Again, let $\dfrac{1}{(1-t)(1+t)(1+2t)} = \dfrac{A}{(1-t)} + \dfrac{B}{(1+t)} + \dfrac{C}{(1+2t)}$

or $1 = A(1+t)(1+2t) + B(1-t)(1+2t) + C(1-t)(1+t)$

On putting on both sides,

$$\left.\begin{array}{lll} \text{put } t=1, & 1 = A(2)(3) & \Rightarrow A = 1/6 \\ \text{put } t=-1, & 1 = B(1+1)(1-2) & \Rightarrow B = -1/2 \\ \text{put } t=-1/2, & 1 = C(1+1/2)(1-1/2) & \Rightarrow C = 4/3 \end{array}\right]$$

$\therefore$ $$\frac{1}{(1-t)(1+t)(1+2t)} = \frac{1}{6}.\frac{1}{(1-t)} - \frac{1}{2}.\frac{1}{(1+t)} + \frac{4}{3}.\frac{1}{(1+2t)}$$

$\therefore$ $$I = -\int\left[\frac{1}{6}.\frac{1}{(1-t)} - \frac{1}{2}.\frac{1}{(1+t)} + \frac{4}{3}.\frac{1}{(1+2t)}\right]dt$$

$$= -\frac{1}{6}\frac{\log|1-t|}{(-1)} + \frac{1}{2}\log|1+t| - \frac{4}{3}\frac{\log|1+2t|}{2} + c$$

$$= \frac{1}{6}\log|1-\cos x| + \frac{1}{2}\log|1+\cos x| - \frac{2}{3}\log|1+2\cos x| + c$$

Example 27. Integrate $\dfrac{2x}{(x^2+1)(x^2+3)}$ with respect to x

Solution : Let $$I = \int\frac{2x}{(x^2+1)(x^2+3)}dx$$

$$= \int\frac{dt}{(t+1)(t+3)} \qquad [\text{where } x^2 = t \Rightarrow 2xdx = dt]$$

$$= \frac{1}{2}\int\left[\frac{1}{t+1} - \frac{1}{t+3}\right]dt$$

$$= \frac{1}{2}\left[\log|t+1| - \log|t+3|\right] + c$$

$$= \frac{1}{2}\log\left|\frac{t+1}{t+3}\right| + c = \frac{1}{2}\log\left(\frac{x^2+1}{x^2+3}\right) + c$$

Exampler 28. Integrate $\frac{1}{x(x^n-1)}$ with respect to x.

Solution : Let $I=\int \frac{1}{x\left(x^n-1\right)}dx$

$$=\int \frac{x^{n-1}}{x^n\left(x^n-1\right)} \quad \text{(multiplying numerator and denominator by } x^{n-1}\text{)}$$

Again, let $x^n=t \Rightarrow nx^{n-1}dx=dt \Rightarrow x^{n-1}dx=\frac{dt}{n}$

$$I=\frac{1}{n}\int \frac{dt}{t(t-1)}=\frac{1}{n}\int\left[\frac{1}{t-1}-\frac{1}{t}\right]dt=\frac{1}{n}[\log|t-1|-\log|t|]+c$$

$$=\frac{1}{n}\log\left|\frac{t-1}{t}\right|+c=\frac{1}{n}\log\left|\frac{x^n-1}{x^n}\right|+c$$

Exercise 9.4

Integrate the following functions with respect to x.

(1) $\frac{1}{16-9x^2}$ (2) $\frac{1}{x^2-36}$ (3) $\frac{3x}{(x+1)(x-2)}$ (4) $\frac{3x-2}{(x+1)^2(x+3)}$

(5) $\frac{x^2}{(x+1)(x-2)(x-3)}$ (6) $\frac{x^2}{x^4-x^2-12}$ (7) $\frac{1}{x^3-x^2-x+1}$ (8) $\frac{x^2}{(x+1)(x-2)}$

(9) $\frac{x^2}{(x^2+a^2)(x^2+b^2)}$ (10) $\frac{x+1}{x^3+x^2-6x}$ (11) $\frac{x^2+8x+4}{x^3-4x}$ (12) $\frac{1}{(x-1)^2(x+2)}$

(13) $\frac{1-3x}{1+x+x^2+x^3}$ (14) $\frac{1+x^2}{x^5-x}$ (15) $\frac{x^2+5x+3}{x^2+3x+2}$ (16) $\frac{x-1}{(x+1)(x^2+1)}$

(17) $\frac{1}{(1+e^x)(1-e^{-x})}$ (18) $\frac{1}{(e^x-1)^2}$ (19) $\frac{e^x}{e^{2x}+5e^x+6}$ (20) $\frac{\sec^2 x}{(2+\tan x)(3+\tan x)}$

(21) $\frac{1}{x(x^5+1)}$ (22) $\frac{1}{x(a+bx^n)}$ (23) $\frac{8}{(x+2)(x^2+4)}$ (24) $\frac{(1-\cos x)}{\cos x(1+\cos x)}$

(b) Integration of special forms of rational functions

(i) $\int \frac{1}{ax^2+bx+c}dx$ (ii) $\int \frac{px+q}{ax^2+bx+c}dx$

where a, b, c, p and q are constants.

Proof : (i) $ax^2+bx+c=a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right]$

$$= a\left[\left(x+\frac{b}{2a}\right)^2 - \left(\frac{b^2-4ac}{4a^2}\right)\right]$$

Case : (1) When $b^2-4ac > o$

then, $$\int\frac{dx}{ax^2+bx+c} = \frac{1}{a}\int\frac{dx}{\left(x+\frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2-4ac}{4a^2}}\right)^2}$$

$$= \frac{1}{a}\int\frac{dt}{t^2-\lambda^2} \qquad \left(\text{where } x+\frac{b}{2a}=t \text{ and } \sqrt{\frac{b^2-4ac}{4a^2}}=\lambda\right.$$

etc.)

$$= \frac{1}{a}\cdot\frac{1}{2\lambda}\log\left|\frac{t-\lambda}{t+\lambda}\right| + c$$

Case : (2) : When $b^2-4ac < o$

then $$\int\frac{dx}{ax^2+bx+c} = \frac{1}{a}\int\frac{dt}{t^2+\lambda^2}$$

$$= \frac{1}{a\lambda}\tan^{-1}\left(\frac{t}{\lambda}\right) + c$$

on again substituting the values of t and λ the required integration can be done

(ii) Let numerator $px+q=\lambda$ (differential coefficient of denominator + μ)

or $$px+q = \lambda(2ax+b)+\mu$$

On comparing the coefficients of equal terms

$$2a\lambda = p \Rightarrow \lambda = \frac{p}{2a}$$

$$b\lambda+\mu = q \Rightarrow \mu = q - \frac{bp}{2a}$$

Hence the given integral $\int\frac{px+q}{ax^2+bx+c}dx = \frac{p}{2a}\int\frac{2ax+b}{ax^2+bx+c}dx + \left(q-\frac{bp}{2a}\right)\int\frac{dx}{ax^2+bx+c}$

$$= \frac{p}{2a}\log|ax^2+bx+c| + \left(q-\frac{bp}{2a}\right)\int\frac{dx}{ax^2+bx+c}$$

Where secodn integral can be solved by method (i)

(C) Integration of irrational algebraic function

Irrational function : A function in which power of variable is fraction :

For example ; $f(x) = x^{3/2}+x+1,\ g(x) = 2\sqrt{x}+3,\ h(x) = \frac{x^2+\sqrt{x}}{1-x^{1/3}}$ etc.

Integration of standard irrational functions

(i) $\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$ (ii) $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

First Method : (i) Term $I=\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$ m there are two methods of integration.

(a) where a > o then $I=\frac{1}{\sqrt{a}}\int \frac{dx}{\sqrt{x^2+\frac{bx}{a}+\frac{c}{a}}}=\frac{1}{\sqrt{a}}\int \frac{dx}{\sqrt{\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2-4ac}{4a^2}\right)}}$

It has three steps :

(i) where $b^2-4ac>o$ then

$$I=\frac{1}{\sqrt{a}}\int \frac{dt}{\sqrt{t^2-\lambda^2}},\ \text{where } t=x+\frac{b}{2a},\lambda=\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$=\frac{1}{\sqrt{a}}\log\left|t+\sqrt{t^2-\lambda^2}\right|+c$$

(ii) when $b^2-4ac<o$ then

$$I=\frac{1}{\sqrt{a}}\int \frac{dt}{\sqrt{\left(x+\frac{b}{2a}\right)^2+\left(\frac{\sqrt{4ac-b^2}}{2a}\right)^2}}$$

$$=\frac{1}{\sqrt{a}}\int \frac{dt}{\sqrt{t^2+\lambda^2}},\ \text{where } t=x+\frac{b}{2a},\ \lambda=\frac{\sqrt{4ac-b^2}}{2a}$$

$$=\frac{1}{\sqrt{a}}.\log\left|t+\sqrt{t^2+\lambda^2}\right|+c$$

(iii) when $b^2-4ac=o$

then, $$I=\frac{1}{\sqrt{a}}\int \frac{dx}{x+\frac{b}{2a}}=\frac{1}{\sqrt{a}}\log\left|x+\frac{b}{2a}\right|+c$$

(b) when $a<o$ let $a=-\propto$

then, $$I=\int \frac{dx}{\sqrt{-\propto x^2+bx+c}}=\frac{1}{\sqrt{\propto}}\int \frac{dx}{\sqrt{\left(\frac{b^2+4c\propto}{4\alpha^2}\right)-\left(x-\frac{b}{2\propto}\right)^2}}$$

$$= \frac{1}{\sqrt{\alpha}} \int \frac{dt}{\sqrt{\lambda^2 - t^2}}, \text{ where } t = x - \frac{b}{2\alpha}, \lambda^2 = \frac{b^2 + 4c\alpha}{4\alpha^2}$$

$$= \frac{1}{\sqrt{\propto}} \sin^{-1}\left(\frac{t}{\lambda}\right) + c$$

Second method : $I = \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

Let $px + q = A\frac{d}{dx}(ax^2 + bx + c) + B$

or $px + q = A(2ax + b) + B$

on comparing and solving $A = \frac{p}{2a}, B = q - \frac{bp}{2a}$

then, $I = \frac{p}{2a}\int \frac{2ax+b}{\sqrt{ax^2+bx+c}} dx + \left(q - \frac{bp}{2a}\right)\int \frac{1}{\sqrt{ax^2+bx+c}} dx,$

where in I integral put $ax^2 + bx + c = t$ and II integral can be solved by case I discussed earlier.

Illustrative Examples

Example 29. Integrate $\frac{1}{x^2+4x+1}$ with respect to *x*.

Solution : Let $I = \int \frac{1}{x^2+4x+1} dx = \int \frac{1}{(x+2)^2 - 3} dx$

$$= \int \frac{1}{(x+2)^2 - (\sqrt{3})^2} dx = \frac{1}{2\sqrt{3}} \log\left|\frac{x+2-\sqrt{3}}{x+2+\sqrt{3}}\right| + c.$$

Example 30. Integrate $\frac{1}{1-6x-9x^2}$ with respect to *x*.

Solution : Here $1 - 6x - 9x^2 = 9\left[\frac{1}{9} - \frac{6x}{9} - x^2\right]$

$$= 9\left[\frac{2}{9} - \left(x^2 + \frac{2x}{3} + \frac{1}{9}\right)\right]$$

$$= 9\left[2/9 - (x+1/3)^2\right]$$

$\therefore$ $I = \int \frac{1}{1-6x-9x^2} dx$

$$= \frac{1}{9}\int \frac{1}{2/9 - (x+1/3)^2} dx = \frac{1}{9}\int \frac{1}{(\sqrt{2}/3)^2 - (x+1/3)^2} dx$$

$$= \frac{1}{9 \times 2 \times \frac{\sqrt{2}}{3}} \log \left| \frac{\sqrt{2}/3 + x + 1/3}{\sqrt{2}/3 - x - 1/3} \right| + c$$

$$= \frac{1}{6\sqrt{2}} \log \left| \frac{\sqrt{2} + 1 + 3x}{\sqrt{2} - 1 - 3x} \right| + c.$$

Example 31. Integrate $\frac{5x-2}{3x^2+2x+1}$ with respect to *x*.

Solution : Let $5x - 2 = A\frac{d}{dx}(3x^2 + 2x + 1) + B$

or $5x - 2 = A(6x+2) + B$

on comparing $6A = 5 \;\therefore\; A = \frac{5}{6}$ and $B = -2 - 2A = -2 - 5/3 = -11/3$

$\therefore$ $$5x - 2 = \frac{5}{6}(6x+2) - \frac{11}{3}$$

$\therefore$ $$I = \int \frac{5x-2}{3x^2+2x+1} dx$$

$$= \int \frac{5/6(6x+2) - 11/3}{3x^2+2x+1} dx = \frac{5}{6}\int \frac{6x+2}{3x^2+2x+1} dx - \frac{11}{3}\int \frac{1}{3x^2+2x+1} dx$$

$$= \frac{5}{6} \log|3x^2+2x+1| - \frac{11}{3\times 3}\int \frac{1}{x^2+2x/3+1/3} dx$$

$$= \frac{5}{6} \log|3x^2+2x+1| - \frac{11}{9}\int \frac{1}{(x+1/3)^2 + (\sqrt{2}/3)^2} dx$$

$$= \frac{5}{6} \log|3x^2+2x+1| - \frac{11}{9} \times \frac{1}{\sqrt{2}/3} \tan^{-1}\left(\frac{x+1/3}{\sqrt{2}/3}\right) + c$$

$$= \frac{5}{6} \log|3x^2+2x+1| - \frac{11}{3\sqrt{2}} \tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + c.$$

Example 32. Integrate $\frac{1}{\sqrt{x^2-8x+15}}$ with respect to *x*.

Solution : Here $$I = \int \frac{1}{\sqrt{x^2-8x+15}} dx = \int \frac{1}{\sqrt{(x-4)^2 - 1}} dx$$

$$= \log|(x-4) + \sqrt{x^2-8x+15}| + c$$

Example 33. Integrate $\dfrac{1}{\sqrt{1+3x-4x^2}}$ with respect to x

Solution : Let
$$I = \int \frac{1}{\sqrt{1+3x-4x^2}}\,dx$$
$$= \frac{1}{2}\int \frac{dx}{\sqrt{1/4+3x/4-x^2}}$$
$$= \frac{1}{2}\int \frac{dx}{\sqrt{25/64-(x^2-3x/4+9/64)}}$$
$$= \frac{1}{2}\int \frac{dx}{\sqrt{\left(\frac{5}{8}\right)^2-\left(x-\frac{3}{8}\right)^2}}$$
$$= \frac{1}{2}\sin^{-1}\left(\frac{x-3/8}{5/8}\right)+c = \frac{1}{2}\sin^{-1}\left(\frac{8x-3}{5}\right)+c$$

Example 34. Integrate $\dfrac{2x+5}{\sqrt{x^2+3x+1}}$ with respect to x

Solution : Let $2x+5=(2x+3)+2$

(On changing numerator into differential coefficient of (x^2+3x+1) by inspection)

$$\therefore \quad = \int \frac{2x+5}{\sqrt{x^2+3x+1}}\,dx = \int \frac{2x+3}{\sqrt{x^2+3x+1}}\,dx + \int \frac{2}{\sqrt{x^2+3x+1}}\,dx$$
$$= \int \frac{dt}{\sqrt{t}} + \int \frac{2}{\sqrt{(x+3/2)^2+(\sqrt{5}/2)^2}}, \quad \text{where } x^2+3x+1=t$$
$$= 2\sqrt{t}+2\log\left|(x+3/2)+\sqrt{x^2+3x+1}\right|+c$$
$$= 2\sqrt{x^2+3x+1}+2\log\left|(x+3/2)+\sqrt{x^2+3x+1}\right|+c$$

Exercise 9.5

Integrate the following functions with respect to x

(1) $\dfrac{1}{x^2+2x+10}$ (2) $\dfrac{1}{2x^2+x-1}$ (3) $\dfrac{1}{9x^2-12x+8}$ (4) $\dfrac{1}{3+2x-x^2}$

(5) $\dfrac{x}{x^4+x^2+1}$ (6) $\dfrac{\cos x}{\sin^2 x+4\sin x+5}$ (7) $\dfrac{x-3}{x^2+2x-4}$ (8) $\dfrac{3x+1}{2x^2-2x+3}$

(9) $\dfrac{x+1}{x^2+4x+5}$ (10) $\dfrac{(3\sin x-2)\cos x}{5-\cos^2 x-4\sin x}$ (11) $\dfrac{1}{2e^{2x}+3e^x+1}$ (12) $\dfrac{1}{\sqrt{4x^2-5x+1}}$

(13) $\dfrac{1}{\sqrt{5x-6-x^2}}$ (14) $\dfrac{1}{\sqrt{1-x-x^2}}$ (15) $\dfrac{1}{\sqrt{4+3x-2x^2}}$ (16) $\dfrac{x+2}{\sqrt{x^2-2x+4}}$

(17) $\dfrac{x+1}{\sqrt{x^2-x+1}}$ (18) $\dfrac{x+3}{\sqrt{x^2+2x+2}}$ (19) $\sqrt{\sec x-1}$ (20) $\sqrt{\dfrac{\sin(x-\alpha)}{\sin(x+\alpha)}}$

(21) $\dfrac{x^3}{x^2+x+1}$ (22) $\dfrac{e^x}{e^{2x}+6e^x+5}$

IV Integration of Parts:

We have studied the methods of integration by substitution, trigonometric identities and algebraic methods. But integral of some functions is either difficult or impossible with above methods. Such functions can be expressed in parts and then their integration is can be found.

Here the main functions are non algebraic functions like exponential, logarithmic and inverse trigonometric functions.

Rule of integration by parts or integration of product of functions:

Theorem : If u and v are two functions of x then

$$\int u.v\,dx = u\left(\int v\,dx\right) = \int\left[\frac{du}{dx}.\int v\,dx\right]dx$$

Proof : For any two functions $f(x)$ and $g(x)$

$$\frac{d}{dx}\{f(x).g(x)\} = f(x)\frac{d}{dx}g(x)+g(x)\frac{d}{dx}f(x)$$

integrating both sides with respect to x

$$f(x).g(x) = \int\left[f(x)\frac{d}{dx}g(x)+g(x)\frac{d}{dx}f(x)\right]dx$$

or $$\int\left[f(x)\frac{d}{dx}g(x)\right]dx = f(x)g(x)-\int\left[g(x)\frac{d}{dx}f(x)\right]dx \qquad (1)$$

Now let $$f(x)=u,\ \frac{d}{dx}[g(x)]=v \Rightarrow g(x)=\int v\,dx$$

Put this value in (1)

$\therefore$ $$\int u.v\,dx = u\int v\,dx-\int\left[\frac{du}{dx}\int v\,dx\right]dx$$

If we take u as first function and v as the second function, then this formula may be stated as follows:

"The integral of the product of two functions = (First function) × $\int$ (second function) dx– $\int$ (Differential coefficient of first function) × integral of second function dx.

Remark : The success of integration by parts method depends on selection of first and second function. Function should be selected in a manner so that the integral of second function can be done easily. Although there is no specific rule for selection of functions but following points may be kept in mind.

(i) if integrand is a product of algebraic function of x and exponential or trigonometric function them exponential or trigonometric function should be selected as second function.

(ii) In integration of single inverse trigonometric functions or logarithmic functions, unit (1) should be taken as second function.

(iii) If integral obtained in original form in right hand side then integration should be done by transposing.

(iv) Integration by parts may be used more than once in an integral as per necessity.

Note : We can select the function as they appear in word 'ILATE'

Where : I = Inverse trigonometric functions sicj as $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x$

L = Logarithmic functions such as $\log x, \log(x^2 + a^2)$

A = Algebraic functions such as $x, x+1, 2x, \sqrt{x}$

T = Trigonometric functions such as $\sin x, \cos x, \tan x$

E = Exponential function such as $a^x, e^x, 2^x, 3^{-x}$

Application of ingegration by parts

In Integral of the type $\int e^x[f(x) + f'(x)]\,dx$ and $\int [x\,f'(x) + f(x)]\,dx$

(i) Let $I = \int e^x[f(x) + f'(x)]\,dx$, where $f'(x) = \frac{d}{dx} f(x)$

$$= \int \underset{\text{II}}{e^x} \underset{\text{I}}{f(x)}\,dx + \int e^x f'(x)\,dx \quad \text{(on taking } e^x \text{ as II function)}$$

$$= f(x).e^x - \int f'(x)e^x dx + \int e^x f'(x) dx + c$$

(Integration by parts of first integral)

$$= e^x f(x) + c$$

similarly $\int e^x[f(x) + f'(x)]\,dx = e^x f(x) + c$

(ii) Let $I = \int [x\,f'(x) + f(x)]\,dx$

$$= \int \underset{\text{I}}{x} \underset{\text{II}}{f'(x)}\,dx + \int f(x)\,dx$$

put $f'(x)$ as second function in first integral and then integrating by parts

$$= x\,f(x) - \int 1 \times f(x)\,dx + \int f(x) dx$$

$$= x\,f(x) + c$$

$\therefore \quad \int [x\,f'(x) + f(x)]\,dx = x\,f(x) + c$

Illustrative Examples

Example 35. Integrate $x^2 e^x$ with respect to x

Solution : Let $I = \int \underset{\text{I}}{x^2}\, \underset{\text{II}}{e^x}\, dx$

On taking e^x as II function, Integration by parts gives

$$= x^2 e^x - \int \underset{\text{I}}{2x}\, \underset{\text{II}}{e^x}\, dx$$

$$= x^2 e^x - 2[xe^x - \int 1 \times e^x dx]$$

$$= x^2 e^x - 2xe^x + 2e^x$$

$$= e^x (x^2 - 2x + 2) + c$$

Example 36. Integrate $x \log x$ with respect to x

Solution : Let $I = \int \underset{\text{II}}{x} \underset{\text{I}}{\log x}\, dx$

On taking $\log x$ as I function and x as second function, Integration by parts gives

$$I = (\log x)\frac{x^2}{2} - \int \frac{1}{x} \times \frac{x^2}{2} dx$$

$$= \frac{x^2}{2}(\log x) - \frac{1}{2}\int x dx + c$$

$$= \frac{x^2}{2}\log x - \frac{x^2}{4} + c$$

Example 37. Integrate $x^2 \sin 2x$ with respect to x

Solution : Let $I = \int \underset{\text{I}}{x^2}\, \underset{\text{II}}{\sin 2x}\, dx$

Taking x^2 as I and $\sin 2x$ as II function respectively, Integration by parts, gives

$$I = x^2\left(\frac{-\cos 2x}{2}\right) - \int 2x \times \frac{-\cos 2x}{2} dx$$

$$= \frac{-x^2}{2}\cos 2x + \int \underset{\text{I}}{x} \cdot \underset{\text{II}}{\cos 2x}\, dx$$

Taking x as I and $\cos 2x$ as II functions respectively, again Integration by parts gives

$$= \frac{-x^2}{2}\cos 2x + x\left(\frac{\sin 2x}{2}\right) - \int 1 \times \frac{\sin 2x}{2} dx$$

$$= \frac{-x^2}{2}\cos 2x + \frac{x}{2}\sin 2x + \frac{\cos 2x}{4} + c$$

Example 38. Integrate $\log x$ with respect to x

Solution : Let $I = \int \underset{\text{II}}{1} \cdot \underset{\text{I}}{\log x}\, dx$

Taking one as second function, Integration by parts gives

$$= (\log x)(x) - \int \frac{1}{x} \times x \, dx$$

$$= x \log x - x + c$$

$$= x(\log x - 1) + c$$

$$= x[\log x - \log e] + c = x \log(x/e) + c$$

Example 39. Integrate $\tan^{-1} x$ with respect to x

Solution : Let $I = \int \tan^{-1} x \, dx$

$$I = \int \underset{\text{II}}{1} \cdot \underset{\text{I}}{\tan^{-1}} x \, dx$$

Taking $\tan^{-1} x$ as I and one as II function respectively, Integration by parts gives

$$= \left(\tan^{-1} x\right)\left(x\right) - \int \frac{1}{1+x^2} \times x \, dx$$

$$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx$$

$$= x \tan^{-1} x - \frac{1}{2} \log\left(1+x^2\right) + c \qquad \text{(where, let } 1+x^2 = t\text{)}$$

Example 40. Integrate $\cos^{-1} \sqrt{\frac{x}{a+x}} dx$ with respect to x

Solution : Let $I = \int \cos^{-1} \sqrt{\frac{x}{a+x}} dx$

Let $x = a \tan^2 \theta \Rightarrow dx = 2a \tan \theta \sec^2 \theta \, d\theta$

$$\therefore \quad I = \int \cos^{-1} \sqrt{\left(\frac{a \tan^2 \theta}{a + a \tan^2 \theta}\right)} \times 2a \tan \theta \sec^2 \theta \, d\theta$$

$$= \int \cos^{-1} \left(\frac{\tan \theta}{\sec \theta}\right) \times 2a \tan \theta \sec^2 \theta \, d\theta$$

$$= 2a \int \cos^{-1} (\sin \theta) . \tan \theta \sec^2 \theta \, d\theta$$

$$= 2a \int \cos^{-1} [\cos(\frac{\pi}{2} - \theta)] . \tan \theta \sec^2 \theta \, d\theta$$

$$= 2a \int (\frac{\pi}{2} - \theta) . \tan \theta \sec^2 \theta \, d\theta$$

Taking $(\frac{\pi}{2} - \theta)$ as I and $\tan \theta \sec^2 \theta$ as II function, integration by parts gives

$$I = 2a\left[\left(\frac{\pi}{2}-\theta\right)\frac{\tan^2\theta}{2} - \int -1\times\frac{\tan^2\theta}{2}\,d\theta\right]$$

$$\left[\because \int \tan\theta\sec^2\theta\,d\theta = \frac{\tan^2\theta}{2}\right]$$

$$= a(\frac{\pi}{2}-\theta)\tan^2\theta + \text{a}\int(\sec^2\theta - 1)d\theta$$

$$= a(\frac{\pi}{2}-\theta)\tan^2\theta + a[\tan\theta - \theta] + c$$

$$= a\left[\pi/2 - \tan^{-1}\sqrt{x/a}\right](x/a) + a\left[\sqrt{x/a} - \tan^{-1}\sqrt{x/a}\right] + c$$

$$= x\cdot\frac{\pi}{2} - x\tan^{-1}\sqrt{x/a} + \sqrt{ax} - a\tan^{-1}\sqrt{x/a} + c$$

या $$I = x\cdot\frac{\pi}{2} - (a+x)\tan^{-1}\sqrt{x/a} + \sqrt{ax} + c$$

Example 41. Evaluate $\int \log[x+\sqrt{x^2+a^2}]\,dx$

Solution : Here $$I = \int \underset{\text{II}}{1}\cdot \underset{\text{I}}{\log(x+\sqrt{x^2+a^2})}\,dx$$

Taking one as second function, integration by parts, gives

$$I = \log[x+\sqrt{x^2+a^2}].x - \int \frac{1}{[x+\sqrt{x^2+a^2}]}\times\left[1+\frac{2x}{2\sqrt{x^2+a^2}}\right]x\,dx$$

$$= x\log[x+\sqrt{x^2+a^2}] - \int \frac{1}{(x+\sqrt{x^2+a^2})}\times\frac{(\sqrt{x^2+a^2}+x)}{\sqrt{x^2+a^2}}\times x\,dx$$

$$= x\log[x+\sqrt{x^2+a^2}] - \int \frac{x}{\sqrt{x^2+a^2}}dx$$

(On putting $x^2+a^2 = t$ and solving)

$$= x\log[x+\sqrt{x^2+a^2}] - \frac{1}{2}\times 2\sqrt{x^2+a^2} + c$$

$$= x\log[x+\sqrt{x^2+a^2}] - \sqrt{x^2+a^2} + c$$

Example 42. Integrate $\dfrac{x^2}{(x\sin x+\cos x)^2}$ with respect to x

Solution : Let $$I = \int \frac{x^2}{(x\sin x+\cos x)^2}dx$$

$$= \int \frac{x}{\cos x} \cdot \frac{x \cos x}{(x \sin x + \cos x)^2} dx \qquad \left(\text{Put } x^2 = \frac{x}{\cos x} \times x \cos x \text{ in numerator}\right)$$

Taking $\frac{x}{\cos x}$ as I and remaining as II function, integration by parts gives

$$I = \frac{x}{\cos x} \times \int \frac{x \cos x}{(x \sin x + \cos x)^2} dx - \int \left[\frac{d}{dx}\left(\frac{x}{\cos x}\right) \times \int \frac{x \cos x}{(x \sin x + \cos x)^2} dx \right] dx$$

Let $x \sin x + \cos x = t \Rightarrow x \cos x \, dx = dt$

$$= \frac{x}{\cos x} \times \left[\frac{-1}{x \sin x + \cos x} \right] + \int \frac{\left[\cos x + (\sin x) x\right]}{\cos^2 x} \times \frac{1}{(x \sin x + \cos x)} dx$$

$$= \frac{-x}{\cos x (x \sin x + \cos x)} + \int \sec^2 x \, dx$$

$$= \frac{-x}{\cos x (x \sin x + \cos x)} + \tan x + c$$

$$= \frac{-x}{\cos x (x \sin x + \cos x)} + \frac{\sin x}{\cos x} + c$$

$$= \frac{-x + \sin x (x \sin x + \cos x)}{\cos x (x \sin x + \cos x)} + c$$

$$= \frac{-x + x \sin^2 x + \sin x \cos x}{\cos x (x \sin x + \cos x)} + c$$

$$= \frac{-x(1 - \sin^2 x) + \sin x \cos x}{\cos x (x \sin x + \cos x)} + c$$

$$= \frac{-x \cos^2 x + \sin x \cos x}{\cos x (x \sin x + \cos x)} + c$$

$$= \frac{\sin x - x \cos x}{x \sin x + \cos x} + c$$

Example 43. Integrate $\frac{x + \sin x}{1 + \cos x}$ with respect to x.

Solution : Let $I = \int \frac{x + \sin x}{1 + \cos x} dx = \int \frac{x + 2 \sin(x/2) \cos(x/2)}{2 \cos^2 (x/2)} dx$

$$= \frac{1}{2} \int \underset{\text{I}}{x} \underset{\text{II}}{\sec^2} (x/2) \, dx + \int \tan(x/2) \, dx$$

Taking x as I function in first integral, integration by parts gives

$$=\frac{1}{2}\left[2x\tan(x/2)-\int 1\times 2\tan(x/2)\,dx\right]+\int \tan(x/2)\,dx$$

$$=x\tan(x/2)-\int \tan(x/2)\,dx+\int \tan(x/2)\,dx$$

$$=x\tan(x/2)+c$$

Example 44. Evaluate $\int \frac{xe^x}{(x+1)^2}dx$

Solution : Let $$I=\int \frac{xe^x}{(x+1)^2}dx=\int \frac{(\overline{x+1}-1)e^x}{(x+1)^2}dx$$

$$=\int \left[\frac{1}{(x+1)}-\frac{1}{(x+1)^2}\right]e^x dx$$

$$=\int \frac{e^x}{(x+1)}dx-\int \frac{e^x}{(x+1)^2}dx$$

(Taking $\frac{1}{x+1}$ as I function in first integral, Integration by parts gives)

$$=\left[\frac{1}{(x+1)}\times e^x-\int -\frac{1}{(x+1)^2}e^x dx\right]-\int \frac{e^x}{(x+1)^2}dx$$

$$=\frac{e^x}{x+1}+\int \frac{e^x}{(x+1)^2}dx-\int \frac{e^x}{(x+1)^2}dx=\frac{e^x}{x+1}+c$$

Exercise 9.6

Integrate the following functions with respect to *x*

1. (i) $x\cos x$ (ii) $x\sec^2 x$
2. (i) x^3e^{-x} (ii) $x^3\sin x$
3. (i) $x^3(\log x)^2$ (ii) $x^3e^{x^2}$
4. (i) $e^{2x}e^{e^x}$ (ii) $(\log x)^2$
5. (i) $\cos^{-1}x$ (ii) $\operatorname{cos}ec^{-1}\sqrt{\frac{x+a}{x}}$
6. (i) $\sin^{-1}(3x-4x^3)$ (ii) $\frac{x}{1+\cos x}$
7. (i) $\tan^{-1}\sqrt{\frac{1-x}{1+x}}\,(\text{Hint}: x=\cos\theta)$ (ii) $\cos\sqrt{x}$
8. (i) $\frac{x}{1+\sin x}$ (ii) $x^2\tan^{-1}x$
9. $\frac{x\sin^{-1}x}{\sqrt{1-x^2}}$
10. $\frac{x\tan^{-1}x}{(1+x^2)^{3/2}}$
11. $e^x(\cot x+\log\sin x)$
12. $\frac{2x+\sin 2x}{1+\cos 2x}$

13. $e^x\left(\frac{1-\sin x}{1-\cos x}\right)$ 14. $e^x\left[\log x+\frac{1}{x^2}\right]$ 15. $e^x[\log(\sec x+\tan x)+\sec x]$

16. $e^x(\sin x+\cos x)\sec^2 x$ 17. $e^x\left(\frac{1}{x^2}-\frac{2}{x^3}\right)$

18. $e^x\left(\frac{1-x}{1+x^2}\right)^2\left(\text{Hint}=\left(\frac{1-x}{1+x^2}\right)^2=\frac{1}{(1+x^2)}-\frac{2x}{(1+x^2)^2}\right)$

19. $\cos 2\theta.\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)$ 20. $\frac{x^2}{(x\cos x-\sin x)^2}$

21. $\cos^{-1}(1/x)$ 22. $(\sin^{-1}x)^2$

9.08 Some special type of Integral

Many times while integrating the product of two functions, integration does not come to an end, whaterver the first or second function is. This happens in the case of exponential and trigonometric functions. In such cases using transpose we can calculate the integral.

For Example :

Integration of $e^{ax}\sin bx$ and $e^{ax}\cos bx$

Let, $I=\int \underset{\text{II}}{e^{ax}}\,\underset{\text{I}}{\sin bx}\,dx$

taking $\sin bx$ as I and e^{ax} as II function, Integration by parts, gives

$$I=\sin bx\left(\frac{e^{ax}}{a}\right)-\int b\cos bx\times\frac{e^{ax}}{a}dx$$

or $$I=\frac{1}{a}e^{ax}\sin bx-\frac{b}{a}\int \underset{\text{II}}{e^{ax}}\,\underset{\text{I}}{\cos bx}\,dx$$

Taking $\cos bx$ as I and e^{ax} as II function, Integration by parts gives.

$$I=\frac{1}{a}e^{ax}\sin bx-\frac{b}{a}\left[\cos bx.\frac{e^{ax}}{a}-\int -b\sin bx\times\frac{e^{ax}}{a}dx\right]$$

or $$I=\frac{1}{a}e^{ax}\sin bx-\frac{b}{a^2}e^{ax}\cos bx-\frac{b^2}{a^2}\int e^{ax}\sin bx\,dx$$

or $$I=\frac{1}{a}e^{ax}\sin bx-\frac{b}{a^2}e^{ax}\cos bx-\frac{b^2}{a^2}I$$

or $$I\left(1+\frac{b^2}{a^2}\right)=\frac{e^{ax}}{a^2}(a\sin bx-b\cos bx)$$ [transposing the last term]

or $$I=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+c$$

or $$\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}\left[a\sin bx-b\cos bx\right]+c$$

similarly $$\int e^{ax}\cos bx dx=\frac{e^{ax}}{a^2+b^2}\left[a\cos bx+b\sin bx\right]+c$$

9.09 Three Important Integrals

(i) $\int\sqrt{x^2+a^2}\,dx$ (ii) $\int\sqrt{x^2-a^2}\,dx$ (iii) $\int\sqrt{a^2-x^2}\,dx$

(i) Let $$I=\int\sqrt{x^2+a^2}\,dx=\int\underset{\text{I}}{\sqrt{x^2+a^2}}\cdot\underset{\text{II}}{1}\,dx$$

Here, we will take $\sqrt{a^2+x^2}$ as I and 1 as II function, Integration by parts gives

$$I=\sqrt{x^2+a^2}\times x-\int\frac{2x}{2\sqrt{x^2+a^2}}\times x\,dx$$

or $$I=x\sqrt{x^2+a^2}-\int\frac{x^2}{\sqrt{x^2+a^2}}\,dx$$

$$=x\sqrt{x^2+a^2}-\int\frac{(x^2+a^2)-a^2}{\sqrt{x^2+a^2}}\,dx$$

$$=x\sqrt{x^2+a^2}-\int\sqrt{x^2+a^2}\,dx+a^2\int\frac{1}{\sqrt{x^2+a^2}}dx$$

or $$I=x\sqrt{x^2+a^2}-I+a^2\log|x+\sqrt{x^2+a^2}|+c_1$$

or $$2I=x\sqrt{x^2+a^2}+a^2\log|x+\sqrt{x^2+a^2}|+c_1$$

or $$I=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log|x+\sqrt{x^2+a^2}|+\frac{c_1}{2}$$

or $$\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log|x+\sqrt{x^2+a^2}|+c$$ (where $c_1/2=c$)

similarly

(ii) $$\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log|x+\sqrt{x^2-a^2}|+c$$

(iii) $$\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$$

Illustrative Examples

Example 45. Integrate $e^{3x}\sin 4x$ with respec tot x

Solution : Let $I = \int \underset{\text{II}}{e^{3x}} \underset{\text{I}}{\sin 4x}\, dx$

Taking $\sin 4x$ as I and e^{3x} as II function, Integration by parts gives,

$$I = \sin 4x.\frac{e^{3x}}{3} - \int 4\cos 4x \times \frac{e^{3x}}{3} dx$$

$$= \frac{1}{3}e^{3x}\sin 4x - \frac{4}{3}\int \underset{\text{II}}{e^{3x}} \underset{\text{I}}{\cos 4x}\, dx$$

Taking $\cos 4x$ as I function, Integration by parts gives

$$I = \frac{1}{3}e^{3x}\sin 4x - \frac{4}{3}\left[\cos 4x.\frac{e^{3x}}{3} - \int -4\sin 4x \times \frac{e^{3x}}{3} dx\right]$$

or $$I = \frac{1}{3}e^{3x}\sin 4x - \frac{4}{9}e^{3x}\cos 4x - \frac{16}{9}\int e^{3x}\sin 4x\, dx$$

or $$I = \frac{e^{3x}}{9}\left[3\sin 4x - 4\cos 4x\right] - \frac{16}{9}I + c_1$$

or $$\frac{25}{9}I = \frac{1}{9}e^{3x}\left(3\sin 4x - 4\cos 4x\right) + c_1$$

or $$I = \frac{e^{3x}}{25}\left[3\sin 4x - 4\cos 4x\right] + c$$

Example 46. Evaluate $\int \frac{\sin(\log x)}{x^3} dx$

Solution : Let $I = \int \frac{\sin(\log x)}{x^3} dx$

Let $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t dt$

$$= \int \frac{(\sin t)e^t\, dt}{(e^t)^3} = \int e^{-2t}\sin t\, dt$$

$$= \frac{e^{-2t}}{(-2)^2 + (1)^2}[-2\sin t - \cos t] + c$$

$$\left[\because \int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}[a\sin bx - b\cos bx]\right]$$

$$= \frac{x^{-2}}{5}\left[-2\sin(\log x) - \cos(\log x)\right] + c$$

$$I = -\frac{1}{5x^2}\left[2\sin(\log x) + \cos(\log x)\right] + c$$

Example 47. Integrate $\frac{xe^{\sin^{-1}x}}{\sqrt{1-x^2}}$ with respect to x.

Solution : Let $I=\int\frac{xe^{\sin^{-1}x}}{\sqrt{1-x^2}}dx$

Let $\sin^{-1}x=t\Rightarrow x=\sin t\Rightarrow dx=\cos t\,dt$

$$=\int\frac{\sin t.e^t}{\cos t}\times\cos t\,dt=\int e^t\sin t\,dt$$

$$=\frac{e^t}{2}[\sin t-\cos t]+c=\frac{e^{\sin^{-1}x}}{2}\left[x-\sqrt{1-x^2}\right]+c$$

Example 48. Integrate $e^{3x}\cos(4x+5)dx$ with respect to x

Solution : Let $I=\int \underset{\text{II}}{e^{3x}}\cos(4\underset{\text{I}}{x}+5)dx$

Integration by parts gives,

$$I=\cos(4x+5).\frac{e^{3x}}{3}-\int-4\sin(4x+5)\times\frac{e^{3x}}{3}dx$$

$$=\frac{1}{3}e^{3x}\cos(4x+5)+\frac{4}{3}\int \underset{\text{II}}{e^{3x}}\sin(4\underset{\text{I}}{x}+5)dx$$

Again, Integration by parts gives,

$$I=\frac{1}{3}e^{3x}\cos(4x+5)+\frac{4}{3}\left[\sin(4x+5)\times\frac{e^{3x}}{3}-\int 4\cos(4x+5)\times\frac{e^{3x}}{3}dx\right]$$

or $$I=\frac{1}{3}e^{3x}\cos(4x+5)+\frac{4}{9}e^{3x}\sin(4x+5)-\frac{16}{9}\int e^{3x}\cos(4x+5)dx$$

or $$I=\frac{1}{9}e^{3x}\left[3\cos(4x+5)+4\sin(4x+5)\right]-\frac{16}{9}I+c_1$$

or $$\frac{25}{9}I=\frac{1}{9}e^{3x}\left[3\cos(4x+5)+4\sin(4x+5)\right]+c_1$$

or $$I=\frac{e^{3x}}{25}\left[3\cos(4x+5)+4\sin(4x+5)\right]+c$$

Example 49. Integrate the following functions with respect to x

(i) $\sqrt{x^2+2x+5}$ (ii) $\sqrt{3-2x-x^2}$ (iii) $\sqrt{x^2+8x-6}$

Solution : (i) $I=\int\sqrt{x^2+2x+5}\,dx=\int\sqrt{(x+1)^2+(2)^2}dx$

$$=\frac{(x+1)}{2}\sqrt{(x+1)^2+(2)^2}+\frac{(2)^2}{2}\log\left|(x+1)+\sqrt{(x+1)^2+2^2}\right|+c$$

$$= \frac{x+1}{2}\sqrt{x^2+2x+5} + 2\log\left|(x+1)+\sqrt{x^2+2x+5}\right| + c$$

(ii) $$I = \int \sqrt{3-2x-x^2}\,dx = \int \sqrt{4-\left(x^2+2x+1\right)}dx$$

$$= \int \sqrt{(2)^2-(x+1)^2}\,dx$$

$$= \frac{(x+1)}{2}\sqrt{(2)^2-(x+1)^2} + \frac{(2)^2}{2}\sin^{-1}\frac{(x+1)}{2} + c$$

$$= \frac{x+1}{2}\sqrt{3-2x-x^2} + 2\sin^{-1}\left(\frac{x+1}{2}\right) + c$$

(iii) Let $$I = \int \sqrt{x^2+8x-6}\ dx$$

$$= \int \sqrt{(x+4)^2-22}\,dx$$

$$= \frac{x+4}{2}\sqrt{(x+4)^2-22} - \frac{22}{2}\log\left|(x+4)+\sqrt{(x+4)^2-22}\right| + c$$

$$= \frac{(x+4)}{2}\sqrt{x^2+8x-6} - 11\log\left|(x+4)+\sqrt{x^2+8x-6}\right| + c$$

Example 50. Integrate $\sec^3 x$ with respect to x

Solution : Let $$I = \int \sec x.\sec^2 x\,dx$$

$$= \int \sqrt{1+\tan^2 x}.\sec^2 x dx$$

Let $\tan x = t \qquad \therefore \sec^2 x\,dx = dt$

$$I = \int \sqrt{1+t^2}.dt$$

$$= \frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\log\left|t+\sqrt{1+t^2}\right| + c$$

$$= \frac{\tan x}{2}\sqrt{1+\tan^2 x} + \frac{1}{2}\log\left|\tan x+\sqrt{1+\tan^2 x}\right| + c$$

$$= \frac{1}{2}\tan x\sec x + \frac{1}{2}\log\left|\tan x+\sec x\right| + c$$

Example 51. Integrate $e^{\sin x}\cos x\sqrt{4-e^{2\sin x}}\,dx$ with respect to x

Solution : Let $$I = \int e^{\sin x}\cos x\sqrt{4-e^{2\sin x}}\,dx$$

Let $e^{\sin x} = t \Rightarrow \cos x.e^{\sin x}dx = dt$

$$\therefore \qquad I = \int \sqrt{4-t^2}\,dt$$

$$= \frac{t}{2}\sqrt{4-t^2} + \frac{4}{2}\sin^{-1}\frac{t}{2} + c$$

$$= \frac{1}{2}e^{\sin x}\sqrt{4-e^{2\sin x}} + 2\sin^{-1}\left(\frac{e^{\sin x}}{2}\right) + c$$

Exercise 9.7

Integrate the following functions with respect to *x*

1. $e^{2x}\cos x$
2. $\sin(\log x)$
3. $\frac{e^{a\tan^{-1}x}}{(1+x^2)^{3/2}}$
4. $e^{x/\sqrt{2}}\cos(x+\alpha)$
5. $e^x\sin^2 x$
6. $e^{a\sin^{-1}x}$
7. $\cos(b\log x/a)$
8. $e^{4x}\cos 4x\cos 2x$
9. $\sqrt{2x-x^2}$
10. $\sqrt{x^2+4x+6}$
11. $\sqrt{x^2+6x-4}$
12. $\sqrt{2x^2+3x+4}$
13. $x^2\sqrt{a^6-x^6}$
14. $(x+1)\sqrt{x^2+1}$
15. $\sqrt{1-4x-x^2}$
16. $\sqrt{4-3x-2x^2}$

Miscellaneous Examples

Example 52. Integrate $\frac{1}{a^2\cos^2 x + b^2\sin^2 x}$ with respect to x

Solution : Let

$$I = \int \frac{1}{a^2\cos^2 x + b^2\sin^2 x}\,dx$$

Dividing numberator and denominator by $\cos^2 x$, we get

$$I = \int \frac{\sec^2 x dx}{a^2 + b^2\tan^2 x}$$

Let $\tan x = t$ then $\sec^2 x\,dx = dt$

$$\therefore \qquad I = \int \frac{dt}{a^2+b^2t^2} = \frac{1}{b^2}\int \frac{dt}{t^2+(a/b)^2}$$

$$= \frac{1}{b^2}\times\frac{1}{(a/b)}\tan^{-1}\left(\frac{t}{a/b}\right) + c$$

$$= \frac{1}{ab}\tan^{-1}\left(\frac{bt}{a}\right) + c$$

$$= \frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\tan x\right) + c$$

Example 53. Integrate $\frac{1}{x^{1/2}+x^{1/3}}$ with respect to x

Solution : Here $I = \int \frac{1}{x^{1/2} + x^{1/3}} dx$

Let $x = t^6 \Rightarrow dx = 6t^5\ dt$

$$\therefore \quad I = \int \frac{6t^5}{t^3 + t^2} dt$$

$$= \int \frac{6t^3}{t+1} dt = 6\int \left[t^2 - t + 1 - \frac{1}{t+1} \right] dt$$

$$= 6\left[\frac{t^3}{3} - \frac{t^2}{2} + t - \log|t+1| \right] + c$$

$$= 6\left[\frac{\sqrt{x}}{3} - \frac{x^{1/3}}{2} + x^{1/6} - \log(x^{1/6} + 1) \right] + c$$

Example 54. Integrate $\cos\sqrt{x}$ with respect to x

Solution : Let $I = \int \cos\sqrt{x}\ dx$

Let $\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}} dx = dt \Rightarrow dx = 2t\ dt$

$$\therefore \quad I = \int \cos t \times 2t\ dt$$

$$= 2\int \underset{\text{I}}{t} \underset{\text{II}}{\cos t}\ dt$$

$$= 2\left[t\sin t - \int 1 \times \sin t\ dt \right]$$

$$= 2[t\sin t + \cos t] + c$$

$$= 2\left[\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x} \right] + c$$

Example 55. Integrate $\frac{\sqrt{\tan x}}{\sin x \cos x} dx$ with respect to x

Solution : Let $I = \int \frac{\sqrt{\tan x}}{\sin x \cos x} dx = \int \frac{\sqrt{\tan x}}{\tan x \cos^2 x} dx$

On multiplying and dividing by $\cos x$ in denominator

$$= \int \frac{\sec^2 x}{\sqrt{\tan x}} dx \qquad \text{Let } \tan x = t \qquad \therefore \sec^2 x dx = dt$$

$$= \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + c = 2\sqrt{\tan x} + c$$

Example 56. Integrate $\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx$ with respect to x

Solution : Let $$I=\int\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx=\int\left[\frac{\sqrt{\sin x}}{\sqrt{\cos x}}+\frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right]dx$$

$$=\int\frac{\sin x+\cos x}{\sqrt{\sin x\cos x}}dx=\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{2\sin x\cos x}}dx$$

$$=\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{1-(1-2\sin x\cos x)}}dx=\sqrt{2}\int\frac{(\sin x+\cos x)}{\sqrt{1-(\sin x-\cos x)^2}}dx$$

Let $$\sin x-\cos x=t\Rightarrow(\cos x+\sin x)dx=dt$$

$\therefore$ $$I=\sqrt{2}\int\frac{dt}{\sqrt{1-t^2}}=\sqrt{2}\sin^{-1}t+c$$

$$=\sqrt{2}\sin^{-1}(\sin x-\cos x)+c$$

Example 57. Integrate $\dfrac{(x^5-x)^{1/5}}{x^6}$ with respect to x

Solution : $$I=\int\frac{(x^5-x)^{1/5}}{x^6}dx=\int\frac{x(1-1/x^4)^{1/5}}{x^6}dx$$

$$=\int\frac{(1-1/x^4)^{1/5}}{x^5}dx$$

Let $$\left(1-\frac{1}{x^4}\right)=t\Rightarrow\frac{4}{x^5}dx=dt\Rightarrow\frac{1}{x^5}dx=\frac{dt}{4}$$

$\therefore$ $$I=\frac{1}{4}\int t^{1/5}dt=\frac{1}{4}\frac{t^{1/5+1}}{(1/5+1)}+c$$

$$=\frac{1}{4}\times\frac{5}{6}t^{6/5}+c=\frac{5}{24}\left(1-\frac{1}{x^4}\right)^{6/5}+c$$

Miscellaneous Exercise 9

Integrate the following functions with respect to *x*

1. $1+2\tan x(\tan x+\sec x)$
2. $e^x\sin^3 x$
3. $x^2\log(1-x^2)$
4. $\dfrac{\sqrt{x}-\sqrt{a}}{\sqrt{(x+a)}}$ $\left[\text{Hint}: x=a\tan^2\theta\right]$
5. $\dfrac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}$
6. $\dfrac{x}{1+\sin x}$
7. $\dfrac{1}{x+\sqrt{a^2-x^2}}$
8. $\dfrac{2x-1}{(1+x)^2}$
9. $\dfrac{1}{\cos 2x+\cos 2\alpha}$
10. $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

11. $\dfrac{\sin x-\cos x}{\sqrt{\sin 2x}}$

12. $\dfrac{\sin 2x}{\sin^4 x+\cos^4 x}$

13. $\dfrac{1+x}{(2+x)^2}$

14. $\dfrac{\sin^6 x+\cos^6 x}{\sin^2 x\cos^2 x}$

15. $\dfrac{\tan^{-1} x}{x^2}$

16. $\dfrac{1}{\sin^2 x+\sin 2x}$

17. $\dfrac{1}{4x^2-4x+3}$

18. $\dfrac{1}{x[6(\log x)^2+7(\log x)+2]}$

19. $\dfrac{\sin 2x\cos 2x}{\sqrt{4-\sin^4 2x}}$

20. $\dfrac{\sin x+\cos x}{9+16\sin 2x}$

21. $\dfrac{3x-1}{(x-2)^2}$

22. $\int \dfrac{1-\cos 2x}{1+\cos 2x}dx=$

(a) $\tan x+x+c$ (b) $\cot x+x+c$ (c) $\tan x-x+c$ (d) $\cot x-x+c$

23. $\int \dfrac{1}{\sqrt{32-2x^2}}dx=$

(a) $\sin^{-1}(x/4)+c$ (b) $\dfrac{1}{\sqrt{2}}\sin^{-1}(x/4)+c$ (c) $\sin^{-1}\left(\dfrac{\sqrt{2}x}{4}\right)+c$ (d) $\cos^{-1}(x/4)+c$

24. $\int \log x\, dx=$

(a) $x\log(xe)+c$ (b) $x\log x+c$ (c) $x\log(x/e)+c$ (d) $\log x/e$

25. $\int \dfrac{1}{x(x+1)}\, dx$

(a) $\log\left(\dfrac{x}{x+1}\right)+c$ (b) $\log\left(\dfrac{x+1}{x}\right)+c$ (c) $\dfrac{1}{2}\log\left(\dfrac{x}{x+1}\right)+c$ (d) $\dfrac{1}{2}\log\left(\dfrac{x+1}{x}\right)+c$

IMPORTANT POINTS

1. If given function is $f(x)$ and its integral is $F(x)$ then by definition of integration $\dfrac{d}{dx}F(x)=f(x)$.
2. Integration is called antiderivative or primitive, it is a inverse process of differentiation.
3. For a constant k, $\int k\, f(x)dx=k\int f(x)\,dx$
4. $\int[f_1(x)\pm f_2(x)]dx=\int f_1(x)dx\pm\int f_2(x)dx$
5. Some standard formulae for integration

(i) $\int x^n dx=\dfrac{x^{n+1}}{n+1}+c,\quad n\neq -1$

(ii) $\int \dfrac{1}{x}dx=\log|x|+c$

(iii) $\int e^x dx=e^x+c$

(iv) $\int a^x dx=\dfrac{a^x}{\log a}+c$

(v) $\int \sin x dx = -\cos x + c$

(vi) $\int \cos x dx = \sin x + c$

(vii) $\int \sec^2 x\, dx = \tan x + c$

(viii) $\int \operatorname{cosec}^2 x\, dx = -\cot x + c$

(ix) $\int \sec x \tan x\, dx = \sec x + c$

(x) $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + c$

(xi) $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c = -\cos^{-1} x + c$

(xii) $\int \frac{1}{1+x^2} = \tan^{-1} x + c = -\cot^{-1} x + c$

(xiii) $\int \frac{1}{x\sqrt{x^2-1}} = \sec^{-1} x + c = -\operatorname{cosec}^{-1} x + c$

(xiv) $\int \frac{|x|}{x} dx = |x| + c,\ \ x \neq o$

(xv) $\int dx = x + c$

(xvi) $\int o\, dx = c$

6. Integration by substitution

(i) $\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$

(ii) $\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c$

(iii) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

(iv) $\int \frac{1}{ax+b} dx = \frac{1}{a} \log|ax+b| + c$

(v) $\int e^{ax+b} dx = \frac{e^{ax+b}}{a} + c$

(vi) $\int \sin(ax+b) dx = \frac{-\cos(ax+b)}{a} + c$

(vii) $\int \cos(ax+b) dx = \frac{\sin(ax+b)}{a} + c$

7. Use of substitituon method in standard formulae

(i) $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

(ii) $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} x/a + c$

(iii) $\int \frac{1}{\sqrt{x^2+a^2}} dx = \log|x + \sqrt{x^2+a^2}| + c$

(iv) $\int \frac{1}{\sqrt{x^2-a^2}} dx = \log|x + \sqrt{x^2-a^2}| + c$

8. Standard Integrals

(i) $\int \frac{1}{x^2-a^2} = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c$

(ii) $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c$

(iii) $\int \sqrt{a^2-x^2} dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$

(iv) $\int \sqrt{a^2+x^2} dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2} \log|x + \sqrt{a^2+x^2}| + c$

(v) $\int \sqrt{x^2-a^2} dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2} \log|x + \sqrt{x^2-a^2}| + c$

(vi) $\int \tan x dx = \log|\sec x| + c$

(vii) $\int \cot x dx = \log|\sin x| + c$

(viii) $\int \sec x\, dx = \log|\sec x + \tan x| + c = \log\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + c$

(ix) $\int \operatorname{cosec} x dx = \log|\operatorname{cosec} x - \cot x| + c = \log|\tan x/2| + c$

9. Integration by parts:

(i) The integral of the product of two functions = (first function) × ∫ second function dx − ∫ (differential coefficient of first function) × ∫ integral of second function dx.

i.e. $\int \underset{\text{I}}{u}\,\underset{\text{II}}{v}\, dx = u\int v\,dx - \int\left[\frac{du}{dx}\times\int v\,dx\right]dx$

(ii) $\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx] + c = \frac{e^{ax}}{a^2+b^2}\sin[bx - \tan^{-1} b/a] + c$

(iii) $\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx] + c = \frac{e^{ax}}{a^2+b^2}\cos\left[bx - \tan^{-1} b/a\right] + c$

(iv) $\int e^x[f(x)+f'(x)]dx = e^x f(x) + c$

(v) $\int [x f'(x) + f(x)]dx = x f(x) + c$

(vi) $\int [f(\log x) + f'(\log x)]dx = xf(\log x) + c$

Answer

Exercise 9.1

1. (i) $\frac{3}{5}\cdot x^{5/3} + c$ (ii) $\frac{e^{3x}}{3} + c$ (iii) $\frac{(1/2)^x}{(\log 1/2)} + c$ (iv) $\frac{x^3}{3} + c$

2. $5\sin x + 3\cos x + 2\tan x + c$

3. $x^2/2 + 1/x + c$

4. $\tan x - \cot x + c$

5. $2/3\cdot x^{3/2} + 2/5\cdot x^{5/2} + c$

6. $\frac{a^{x+1}}{x+1} + c$

7. $x - \tan^{-1} x + c$

8. $x + \cos x + c$

9. $\tan x + \sec x + c$

10. $(\pi/2)x + c$

11. $x - 2\tan^{-1} x + c$

12. $\tan x - x + c$

13. $-\cot x - x + c$

14. $\frac{2}{3}(1+x)^{3/2} + \frac{2}{3}x^{3/2} + c$

15. $\tan x + \cot x + c$

16. $x - \tan x + \sec x + c$

17. $-\cot x - \cot x \operatorname{cosec} x + c$

18. $x + \tan^{-1} x + 3sec^{-1}x + \frac{2^x}{\log 2} + c$

19. $x + \operatorname{cosec} x + c$

20. $x^2/2 + \log|x| + 2x + c$

21. $x + c$

22. $\sqrt{2}\sin x + c$

23. $-\cot x - \tan x + c$

24. $-3\operatorname{cosec} x - 4\cot x + c$

Exercise 9.2

1. (i) $(-1/2)\cos x^2 + c$ (ii) $\frac{1}{3}(x^2+1)^{3/2} + c$ 2. (i) $\log|e^x + \cos x| + c$ (ii) $2\sqrt{1+e^x} + c$

3. (i) $2\sqrt{e^x+1} + \log\left|\frac{e^x}{e^x+2}\right| + c$ (ii) $2\sin(e^{\sqrt{x}}) + c$ 4. (i) $\log|1+\log x| + c$ (ii) $\frac{1}{4}(1+\log x)^4 + c$

5. (i) $\frac{e^{m\tan^{-1}x}}{m} + c$ (ii) $\frac{(\tan x)^{p+1}}{p+1} + c$

6. (i) $\frac{1}{\sqrt{2}}\log|\sec x + \tan x| + c$; (ii) $\log|\operatorname{cosec} 2x - \cot 2x| + \log|\operatorname{cosec} x - \cot x| + c$

7. (i) $\frac{1}{2}\left[\sin x - \frac{1}{5}\sin 5x\right] + c$ (ii) $\pm 2(\sin x/2 + \cos x/2) + c$

8. (i) $\frac{1}{8}\left[3x + 2\sin 2x + \frac{1}{2}\sin 4x\right] + c$; (ii) $\frac{-3}{4}\cos x - \frac{1}{12}\cos 3x + c$

9. (i) $\log|\tan x| + \frac{1}{2}\tan^2 x + c$; (ii) $\tan(xe^x) + c$

10. (i) $\frac{1}{2}[x + \log|\sin x - \cos x|] + c$; (ii) $\frac{1}{2}[x + \log|\sin x + \cos x|] + c$

11. (i) $2\sqrt{\tan x} + \frac{2}{3}\tan^{5/2} x + c$ (ii) $\log|\sin x + \cos x| + c$

12. (i) $x\cos 2a + \sin 2a.\log|\sin(x-a)| + c$; (ii) $x\cos a + \sin a.\log|\sin(x-a)| + c$

13. (i) $\frac{1}{3}\log|\sin 3x| - \frac{1}{5}\log|\sin 5x| + c$; (ii) $\log|\sin(x+\pi/6)\sin(x-\pi/6)| + c$

14. (i) $\frac{1}{5}\log\left|\tan\left(\frac{x+\tan^{-1}(4/3)}{2}\right)\right| + c$; (ii) $\operatorname{cosec}(a-b)\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right| + c$

15. (i) $\frac{1}{2(b-a)}\log(a\cos^2 x + b\sin^2 x) + c$; (ii) $\sqrt{2}\sec\alpha\sqrt{\tan x\cos\alpha + \sin\alpha} + c$

16. (i) $\frac{2}{\cos a}\sqrt{\tan x\cos a + \sin a} + c$; (ii) $2[\sin x + x\cos\alpha] + c$

Exercise 9.3

1. (i) $\frac{1}{10}\tan^{-1}\frac{x}{5} + c$; (ii) $\frac{1}{\sqrt{2}}\sin^{-1}\frac{x}{4} + c$ 2. (i) $\log|1-\sqrt{1-e^{2x}}| + c$; (ii) $\frac{1}{2}\log\left[2x + \sqrt{4x^2+1}\right] + c$

3. (i) $\frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right)+c$; (ii) $-\log|(2-x)+\sqrt{x^2-4x+5}|+c$

4. (i) $\frac{1}{3}\log|x^3+\sqrt{x^6+4}|+c$; (ii) $\frac{1}{5}\sin^{-1}(x^5)+c$

5. (i) $\tan^{-1}(x+3)+c$; (ii) $\frac{1}{\sqrt{2}}\log\left|(x-1/4)+\sqrt{x^2-1/2x+1}\right|+c$

6. (i) $\frac{1}{\sin\propto}\tan^{-1}\left(\frac{e^x+\cos\propto}{\sin\propto}\right)+c$; (ii) $\log|\tan x+\sqrt{\tan^2 x+3}|+c$

7. (i) $\sin^{-1}(2x-3)+c$; (ii) $\frac{1}{\sqrt{5}}\sin^{-1}\left(\frac{5x-4}{6}\right)+c$

8. (i) $\sin^{-1}(\sin x-\cos x)+c$; (ii) $\log|(x+a)+\sqrt{x^2+2xa+b^2}|+c$

9. (i) $a\sin^{-1}\sqrt{x/a}+\sqrt{x}\sqrt{a-x}+c$; (ii) $-a\cos^{-1}x/a-\sqrt{a^2-x^2}+c$

10. (i) $\frac{2}{3}\sin^{-1}(x/a)^{3/2}+c$; (ii) $\frac{1}{a^2}.\frac{x}{\sqrt{x^2+a^2}}+c$

11. (i) $\frac{x}{\sqrt{1-x^2}}+c$; (ii) $\sqrt{x^2+1}+\log(x+\sqrt{x^2+1})+c$ 12. (i) $2\sin^{-1}\left(\frac{x-\propto}{\beta-x}\right)+c$; (ii) $\sin^{-1}(x-1)+c$

13. (i) $\log\left|(x-3/2)+\sqrt{x^2-3x+2}\right|+c$; (ii) $\sin^{-1}\left(\frac{\sin x}{2}\right)+c$

Exercise 9.4

1. $\frac{1}{24}\log\left|\frac{4x-3}{4x+3}\right|+c$

2. $\frac{1}{12}\log\left|\frac{x-6}{x+6}\right|+c$

3. $\log|x+1|+2\log|x-2|+c$

4. $\frac{11}{4}\log\left|\frac{x+1}{x+3}\right|+\frac{5}{2}\log\left|\frac{1}{x+1}\right|+c$

5. $-\frac{1}{6}\log|x+1|+\frac{4}{5}\log|x-2|+\frac{9}{10}\log|x+3|+c$

6. $\frac{1}{7}\log\left|\frac{x-2}{x+2}\right|+\frac{\sqrt{3}}{7}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+c$

7. $\frac{1}{4}\log\left|\frac{x+1}{x-1}\right|-\frac{1}{2(x-1)}+c$

8. $x+\frac{1}{3}\log\frac{(x-2)^4}{|x+1|}+c$

9. $\frac{1}{a^2-b^2}[a\tan^{-1}(x/a)-b\tan^{-1}(x/b)]+c$

10. $-\frac{1}{6}\log|x|+\frac{3}{10}\log|x-2|-\frac{2}{15}\log|x+3|+c$

11. $-\log|x|+3\log|x-2|-\log|x+2|+c$

12. $\frac{1}{9}\log\left|\frac{x+2}{x-1}\right|-\frac{1}{3(x-1)}+c$ 13. $\log\frac{(1+x)^2}{1+x^2}-\tan^{-1}x+c$ 14. $\log|x|-\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|+c$

15. $x+3\log|x+2|-\log|x+1|+c$ 16. $\log\frac{\sqrt{x^2+1}}{|x+1|}+c$ 17. $\frac{1}{2}\log\left|\frac{e^x-1}{e^x+1}\right|+c$

18. $\log\left|\frac{e^x}{e^x-1}\right|-\frac{1}{e^x-1}+c$ 19. $\log\left|\frac{2+e^x}{3+e^x}\right|+c$ 20. $\log\left|\left(\frac{2+\tan x}{3+\tan x}\right)\right|+c$

21. $\log|x|-\frac{1}{5}\log|x^5+1|+c$ 22. $\frac{1}{a^n}\log\left(\frac{x^n}{a+bx^n}\right)+c$

23. $\log|x+2|-\frac{1}{2}\log(x^2+4)+\tan^{-1}(x/2)+c$ 24. $\log|\sec x+\tan x|-2\tan(x/2)+c$

Exercise 9.5

1. $\frac{1}{3}\tan^{-1}\left(\frac{x^2+1}{2}\right)+c$ 2. $\frac{1}{3}\log\left|\frac{2x-1}{2x+2}\right|+c$ 3. $\frac{1}{6}\tan^{-1}\left(\frac{3x-2}{2}\right)+c$ 4. $\frac{1}{4}\log\left|\frac{x+1}{3-x}\right|+c$

5. $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+c$ 6. $\tan^{-1}[\sin(x+2)]+c$ 7. $\frac{1}{2}\log|x^2+2x-4|-\frac{2}{\sqrt{5}}\log\left|\frac{x+1-\sqrt{5}}{x+1+\sqrt{5}}\right|+c$

8. $\frac{3}{4}\log|2x^2-2x+3|+\frac{\sqrt{5}}{2}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)+c$ 9. $\frac{1}{2}\log|x^2+4x+5|-\tan^{-1}(x+2)+c$

10. $3\log|2-\sin x|+\frac{4}{2-\sin x}+c$ 11. $-\frac{1}{2}|e^{-2x}+3e^{-x}+2|+\frac{3}{2}\log\left|\frac{e^{-x}+1}{e^{-x}+2}\right|+c$

12. $\frac{1}{2}\log|(x-5/8)+\sqrt{x^2-5x/4+1/4}|+c$ 13. $\sin^{-1}(2x-5)+c$ 14. $\sin^{-1}\left|\frac{2x+1}{\sqrt{5}}\right|+c$

15. $\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{4x-3}{\sqrt{41}}\right)+c$ 16. $\sqrt{x^2-2x+4}+3\log|(x-1)+\sqrt{x^2-2x+4}|+c$

17. $\sqrt{x^2-x+1}+\frac{3}{2}\log|(x-1/2)+\sqrt{x^2-x+1}|+c$ 18. $\sqrt{x^2+2x+2}+2\log|(x+1)+\sqrt{x^2+2x+2}|+c$

19. $-\log|(\cos x+1/2)+\sqrt{\cos^2 x+\cos x}|+c$

20. $-\cos\propto\sin^{-1}\left(\frac{\cos x}{\cos\alpha}\right)-\sin\alpha.\log|\sin x+\sqrt{\sin^2 x-\sin^2\alpha}|+c$

21. $\frac{1}{2}x^2-x+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$ 22. $\frac{1}{4}\log\left|\frac{e^x+1}{e^x+5}\right|+c$

Exercise 9.6

1. (i) $x\sin x+\cos x+c$; (ii) $x\tan x-\log\sec x+c$

2. (i) $-e^{-x}(x^3+3x^2+6x+6)+c$; (ii) $-x^3\cos x+3x^2\sin x+6x\cos x-6\sin x+c$

3. (i) $\frac{x^4}{4}\left[(\log x)^2-\frac{1}{2}\log x+\frac{1}{8}\right]+c$; (ii) $\frac{1}{2}e^{x^2}(x^2-1)+c$

4. (i) $(e^x-1)e^{e^x}+c$; (ii) $x(\log x)^2-2x\log x+2x+c$

5. (i) $x\cos^{-1}x-\sqrt{1-x^2}+c$; (ii) $(x+a)\tan^{-1}\sqrt{x/a}-\sqrt{ax}+c$

6. (i) $3x\sin^{-1}x+3\sqrt{1-x^2}+c$; (ii) $x\tan x/2-2\log|\sec x/2|+c$

7. (i) $\frac{1}{2}\left[x\cos^{-1}x-\sqrt{1-x^2}\right]+c$; (ii) $2\left[\sqrt{x}\sin\sqrt{x}+\cos\sqrt{x}\right]+c$

8. (i) $\frac{-x(1-\sin x)}{\cos x}+\log(1+\sin x)+c$; (ii) $\frac{x^3}{3}\tan^{-1}x-\frac{x^6}{6}+\frac{1}{6}\log(1+x^2)+c$

9. (i) $-\sin^{-1}x.\cos(\sin^{-1}x)+x+c$

10. $\frac{-\tan^{-1}x}{\sqrt{1+x^2}}+\frac{x}{\sqrt{1+x^2}}+c$

11. $e^x\log\sin x+c$

12. $x\tan x+c$

13. $-e^x\cot x/2+c$

14. $e^x(\log x-1/x)+c$

15. $e^x\log|\sec x+\tan x|+c$

16. $e^x\sec x+\mathrm{c}$

17. $\frac{e^x}{x^2}+c$

18. $\frac{e^x}{1+x^2}+c$

19. $\frac{1}{2}\sin 2\theta\log\left|\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right|+\frac{1}{2}\log(\cos 2\theta)+c$

20. $\frac{x\sin x+\cos x}{x\cos x-\sin x}+c$

21. $x\sec^{-1}x-\log[x+\sqrt{x^2-1}]+c$

22. $x(\sin^{-1}x)^2+2\sqrt{1-x^2}(\sin^{-1}x)-2x+c$

Exercise 9.7

1. $\frac{e^{2x}}{5}[2\cos x+\sin x]+c$

2. $\frac{1}{2}x[\sin(\log x)-\cos(\log x)]+c$

3. $\frac{e^{\tan^{-1}x}}{1+a^2}\left[\frac{a+x}{\sqrt{1+x^2}}\right]+c$

4. $\frac{2}{3}e^{x/\sqrt{2}}\left[\frac{1}{\sqrt{2}}\cos(x+\propto)+\sin(x+\propto)\right]+c$

5. $\frac{e^x}{2}-\frac{e^x}{10}[\cos 2x+2\sin 2x]+c$

6. $\frac{e^{a\sin^{-1}x}}{1+a^2}[x+a\sqrt{1-x^2}]+c$

7. $\frac{x}{1+b^2}[\cos(b\log x/a)+b\sin(b\log x/a)]+c$

8. $\frac{e^{4x}}{8}\left[\frac{1}{13}(4\cos 6x+6\sin 6x)+\frac{1}{5}(4\cos 2x+2\sin 2x)\right]+c$

9. $\frac{x-1}{2}\sqrt{2x-x^2}+\frac{1}{2}\sin^{-1}(x-1)+c$

10. $\frac{x+2}{2}\sqrt{x^2+4x+6}+\log|(x+2)+\sqrt{x^2+4x+6}|+c$

11. $\dfrac{(x+3)\sqrt{x^2+6x-4}}{2}+\dfrac{13}{2}\log|(x-2)+\sqrt{x^2+6x-4}|+c$

12. $\dfrac{4x+3}{8}\sqrt{2x^2+3x+4}+\dfrac{23}{16\sqrt{2}}\log\left(\dfrac{4x+3}{4}\right)+\sqrt{(x^2+\dfrac{3}{2}x+2)}+c$

13. $\dfrac{1}{3}x^3\sqrt{a^2-x^6}+\dfrac{a^2}{2}\sin^{-1}\left(\dfrac{x^3}{a}\right)+c$

14. $\dfrac{1}{3}\left(x^2+1\right)^{3/2}+\dfrac{x}{2}\sqrt{x^2+1}+\dfrac{1}{2}\log|x+\sqrt{x^2+1}|+c$

15. $\dfrac{5}{2}\sin^{-1}\left(\dfrac{x+2}{\sqrt{5}}\right)+\dfrac{x+2}{2}\sqrt{1-4x-x^2}+c$

16. $\dfrac{(4x+3)}{8}\sqrt{4-3x-2x^2}+\dfrac{41\sqrt{2}}{32}\sin^{-1}\left(\dfrac{4x+3}{\sqrt{41}}\right)+c$

Miscellaneous Exercise – 9

1. $2(\tan x+\sec x)-x+c$

2. $\dfrac{e^x}{30}[\sin 3x-3\cos 3x+20\sin x-20\cos x]+c$

3. $\dfrac{x^3}{3}\log|1-x^2|-\dfrac{2}{3}\left(x+\dfrac{x^3}{3}\right)+\dfrac{1}{3}\log\left|\dfrac{1+x}{1-x}\right|+c$

4. $\sqrt{x^2+ax}-2\sqrt{ax+a^2}+a\log\left(\sqrt{a+x}-\sqrt{x}\right)+c$

5. $\dfrac{-\sin 2x}{2}+c$

6. $x(\tan x-\sec x)-\log|\sec x|+\log|\sec x+\tan x|+c$

7. $\dfrac{1}{2}[\sin^{-1}(x/a)+\log|x+\sqrt{a^2-x^2}\,|]+c$

8. $2\log|(1+x)|+\dfrac{2}{1+x}+c$

9. $\dfrac{1}{2}\cos ec 2\alpha\cdot\log\left|\dfrac{(x-\alpha)}{(x+\alpha)}\right|+c$

10. $2x\tan^{-1}x-\log(1+x^2)+c$

11. $-\log|(\sin x+\cos x)+\sqrt{\sin 2x}\,|+c$

12. $\tan^{-1}(\tan^2 x)+c$

13. $\log|x+2|+\dfrac{1}{2+x}+c$

14. $\tan x-\cot x-3x+c$

15. $\dfrac{-\tan^{-1}x}{x}-\dfrac{(\tan^{-1}x)^2}{2}+\log\left(\dfrac{|x|}{\sqrt{1+x^2}}\right)+c$

16. $\log\left|\dfrac{\tan x}{\tan x+2}\right|+c$

17. $\dfrac{1}{2}\tan^{-1}\left(\dfrac{2x-1}{\sqrt{2}}\right)+c$

18. $\log\left|\dfrac{2\log x+1}{3\log x+2}\right|+c$

19. $\dfrac{1}{4}\sin^{-1}\left[\dfrac{\sin^2 2x}{2}\right]+c$

20. $\dfrac{1}{40}\log\left|\dfrac{5+4(\sin x-\cos x)}{5-4(\sin x-\cos x)}\right|+c$

21. $3\log|x-2|-\dfrac{5}{x-2}+c$

22 (c)

23. (b)

24. (c)

25. (a)

10

Definite Integral

10.01 Definite Integral

The definite integral is a powerful tool in mathematics, physics, mechanics, and other disciplines. Calculation of areas bounded by curves of arc lengths, volumes, work, velocity, path length, moments of inertia and so forth reduce to the evaluations of a definite integral. The definite integral has a unique value. A definite integrals is given by a function $f(x)$ in the interval $[a, b]$ and denoted by $\int_a^b f(x)dx$ where a is called the lower limit of the integral and b is called the upper limit of the integral. The definite integral is introduced either as the limit of a sum or if it has an anti derivative F in the interval [a, b], then its value is the difference between the values of F at the end points, i.e., $F(b) - F(a)$.

(i) Definite Integral as a limit of a sum

(ii) Fundamental theorem of Integral Calculus

(iii) To find the value of common definite Integral

(iv) Basic properties of definite Integral

10.02 Definite integral as a limit of sum

In a series if the number of terms approaches to infinity and each term approches to zero, then definite integral is defined as limit of sum.

Definition : Let $f(x)$ be a continuous function defined on close interval $[a, b]$ and interval $[a, b]$ is divided into n equal parts by the points $a+h, a+2h, a+3h, ..., a+(n-1)h$ (where h is the length of each part), then

$$\int_a^b f(x)dx = \lim_{h \to 0}\left[h\{f(a)+f(a+h)+...+f(a+\overline{n-1}h)\}\right] \text{ (where } n \to \infty \text{ and } nh = b-a\text{)}$$

$$= \lim_{h \to 0}\left[h\{f(a+h)+f(a+2h)+...+f(a+nh)\}\right]$$

This method of finding the definite Integral is called ab-initio method.

Proof : Let $f(x)$ be real and continuous function in the interval $[a, b]$

Dividing the interval $[a, b]$ into n equal sub-intervals with h width $AA_n = OA_n - OA$

or $$AA_1 + A_1A_2 + A_2A_3 + ... + A_{n-1}A_n = b - a$$

or $$\underbrace{h+h+h+....+h}_{n\text{ times}} = b-a$$

or $$nh = b-a \qquad \Rightarrow h = \frac{b-a}{n}$$

let $y = f(x)$ when $x = a$, $y = f(a)$

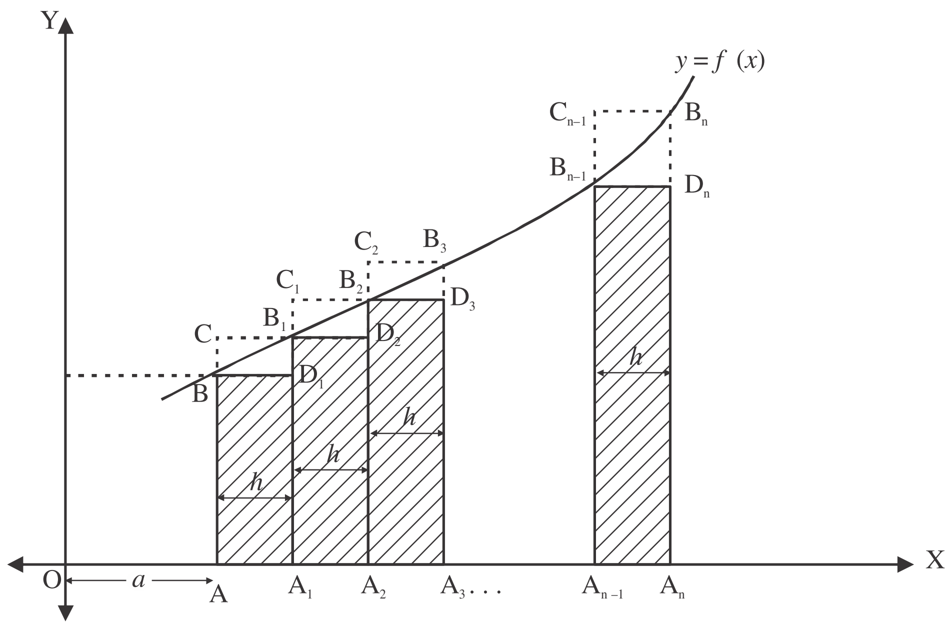
$\therefore$ According to figure, coordinates of B will be $(a, f(a))$

i.e. $AB = f(a)$

similarly

$$A_1B_1 = f(a+h),\ A_2B_2 = f(a+2h), \ldots, A_nB_n = f(a+nh)$$

Let the area of rectangular blocks below the curve in the given figure be Δ_1 then–



$$\Delta_1 = \text{Rectangle } AA_1D_1B + \text{Rectangle } A_1A_2D_2B_1 + \ldots. + \text{Rectaangle } A_{n-1}A_nD_nB_{n-1}$$

$$= AB \times AA_1 + A_1B_1 \times A_1A_2 + \ldots. + A_{n-1}B_{n-1} \times A_{n-1}A_n$$

$$= f(a) \times h + f(a+h) \times h + f(a+2h) \times h + \ldots. + f(a+\overline{n-1}h) \times h$$

$$= h\left[f(a) + f(a+h) + f(a+2h) + \ldots. + f(a+\overline{n-1}h) \right]$$

and if we denote $y = f(x)$, x - axis and two ordinates $x = a$, $x = b$ and the area bounded by AA_nB_nBA Δ then the vlaue of Δ_1 will be less then Δ again let

$$\Delta_2 = \text{Rectangle } AA_1B_1C + \text{ Rectangle } A_1A_2B_2C_1 + \ldots. + \text{ Rectangle } A_{n-1}A_nB_nC_{n-1}$$

$$= A_1B_1 \times AA_1 + A_2B_2 \times A_1A_2 + \ldots. + A_nB_n \times A_{n-1}A_n$$

$$= f(a+h) \times h + f(a+2h) \times h + \ldots. + f(a+nh) \times h$$

$$= h\left[f(a+h) + f(a+2h) + \ldots. + f(a+nh) \right]$$

This area will be greater then Δ therefore the vlaue of Δ will be greater than Δ_1 and less than Δ_2 i.e.

$$\Delta_1 < \Delta < \Delta_2$$

again $$\Delta_2 - \Delta_1 = h\,f(a+nh) - h\,f(a)$$

$$= h\left[f(b) - f(a) \right] \qquad (\because a + nh = b)$$

clearly as the rectangular strips become narrower and narrower, h will be minimum and $h \to 0$ then the value of Δ_1 and Δ_2 will be close to Δ

i.e. $\lim_{h \to 0} \Delta_1 = \lim_{h \to 0} \Delta_2 = \Delta$

$$\therefore \quad \Delta = \int_a^b f(x)\,dx = \lim_{h \to 0} h\left[f(a) + f(a+h) + ... + f(a+\overline{n-1}h)\right]$$

$$\Rightarrow \quad \Delta = \int_a^b f(x)dx = \lim_{h \to 0} h\left[f(a+h) + f(a+2h) + ... + f(a+nh)\right]$$

Conclusion : Definite Integral can be expressed as a limit of a sum

NOTE : We can define the formula as

(i) $$\int_a^b f(x)\,dx = \lim_{n \to \infty} h\left[f(a) + f(a+h) + ... + f(a+\overline{n-1}h)\right],$$

where $h = \dfrac{b-a}{n}$ clearly $n \to \infty$ then $h \to 0$

(ii) $$\int_a^b f(x)dx = \lim_{h \to 0} h\left[f(a+h) + f(a+2h) + ... + f(a+nh)\right], \text{ where } h = \frac{b-a}{n}$$

Any of the above given formula can be used to find the integration.

Some Important Results:

(i) $\sum r = 1 + 2 + 3 + ... + n = \dfrac{n(n+1)}{2}$

(ii) $\sum r^2 = 1^2 + 2^2 + 3^2 + ... + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(iii) $\sum r^3 = 1^3 + 2^3 + 3^3 + ... + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$

(v) $\sum (2r-1) = 1 + 3 + 5 + ... + (2n-1) = n^2$

(vi) $a + (a+d) + (a+2d) + ... + (a+\overline{n-1}\,d) = \dfrac{n}{2}\left[2a + (n-1)d\right]$

(vii) $a + ar + ar^2 + ... + ar^{n-1} = \dfrac{a(r^n - 1)}{(r-1)}, r \neq 1$

Illustrative Examples

Example 1. Find $\int_0^2 (2x+1)\,dx$ as the limit of a sum.

Solution : By definition $$\int_a^b f(x)\,dx = \lim_{h \to 0} h\left[f(a+h) + f(a+2h) + f(a+3h) + ... + f(a+nh)\right],$$

where $nh = b - a$

$$a=0,\ b=2,\ f(x)=2x+1,\ \ nh=2-0=2$$

$$\therefore \quad \int_0^2 (2x+1)dx = \lim_{h\to 0} h\left[f(0+h)+f(0+2h)+f(0+3h)+...+f(0+nh)\right]$$

$$= \lim_{h\to 0} h\left[f(h)+f(2h)+f(3h)+...+f(nh)\right]$$

$$= \lim_{h\to 0} h\left[(2h+1)+(4h+1)+(6h+1)+...+(2nh+1)\right]$$

$$= \lim_{h\to 0} h\left[(2h+4h+6h+...+2nh)+(1+1+1+...+n \text{ बार})\right]$$

$$= \lim_{h\to 0} h\left[2h(1+2+3+...+n)+n\right]$$

$$= \lim_{h\to 0} h\left[2h\frac{n(n+1)}{2}+n\right] = \lim_{h\to 0}\left[h^2 n(n+1)+nh\right]$$

$$= \lim_{h\to 0} h\left[nh(nh+h)+nh\right] = \lim_{h\to 0}\left[2(2+h)+2\right] \qquad (\because nh=2)$$

$$= \left[2(2+0)+2\right] = 4+2 = 6.$$

Example 2. Find $\int_{-1}^{1} e^x\, dx$ as the limit of a sum.

Solution : Here $\quad f(x)=e^x,\quad a=-1,\quad b=1 \qquad (\therefore nh=1+1=2)$

$$\int_{-1}^{1} e^x dx = \lim_{h\to 0} h\left[f(-1+h)+f(-1+2h)+f(-1+3h)+...+f(-1+nh)\right]$$

$$= \lim_{h\to 0} h\left[e^{-1+h}+e^{-1+2h}+e^{-1+3h}+...+e^{-1+nh}\right]$$

$$= \lim_{h\to 0} h\left[e^{-1}.e^{h}+e^{-1}.e^{2h}+e^{-1}.e^{3h}+...+e^{-1}.e^{nh}\right]$$

$$= \lim_{h\to 0} he^{-1}\left[e^{h}+e^{2h}+e^{3h}+...+e^{nh}\right]$$

$$= \frac{1}{e}\lim_{h\to 0} h.e^h.\frac{(e^h)^n-1}{e^n-1}$$

$$= \frac{1}{e}\lim_{h\to 0} e^h.h\frac{e^{nh}-1}{e^h-1} = \frac{1}{e}\lim_{h\to 0} he^h\frac{e^2-1}{e^h-1} \qquad [\because nh=2]$$

$$= \frac{e^2-1}{e}\lim_{h\to 0} e^h.\lim_{h\to 0}\frac{h}{e^h-1} = \left(e-1/e\right)e^{\circ}.\lim_{h\to 0}\frac{1}{\left((e^h-1)/h\right)}$$

$$= \left(e-\frac{1}{e}\right)\times 1\times\frac{1}{1} = e-\frac{1}{e}.$$

Example 3. Find $\int_0^1 x^2\,dx$ as the limit of a sum.

Solution : Here $f(x)=x^2,\ a=0,\ b=1$ $\quad\therefore\quad nh=b-a=1-0=1$

$$\therefore \quad \int_0^1 x^2\,dx=\lim_{h\to 0} h\left[f(0+h)+f(0+2h)+f(0+3h)+\ldots+f(0+nh)\right]$$

$$=\lim_{h\to 0} h\left[f(h)+f(2h)+f(3h)+\ldots+f(nh)\right]$$

$$=\lim_{h\to 0} h[h^2+4h^2+9h^2+\ldots+n^2h^2]$$

$$=\lim_{h\to 0} h.h^2[1^2+2^2+3^2+\ldots+n^2]$$

$$=\lim_{h\to 0} h^3\frac{n(n+1)(2n+1)}{6}$$

$$=\lim_{h\to 0}\frac{nh(nh+h)(2nh+h)}{6}$$

$$=\lim_{h\to 0}\frac{1(1+h)(2\times 1+h)}{6}$$

$$=\frac{(1+0)(2+0)}{6}=\frac{2}{6}=\frac{1}{3}.$$

Exercise 10.1

Evaluate the following definite integrals as a limit of sums

1. $\int_3^5 (x-2)\,dx$
2. $\int_a^b x^2\,dx$
3. $\int_1^3 (x^2+5x)\,dx$
4. $\int_a^b e^{-x}dx$
5. $\int_0^2 (x+4)\,dx$
6. $\int_1^3 (2x^2+5)\,dx$

10.03 Fundamental theorem of integral calculus

Statement : If $f(x)$ is a continuous function defined on an interval $[a, b]$ and

$\frac{d}{dx}[F(x)]=f(x)$, i.e., the anti derivative of $f(x)$ is $F(x)$ then

$$\int_a^b f(x)dx=[F(x)]_a^b=F(b)-F(a)$$

$$=\lim_{h\to 0} h\left[f(a+h)+f(a+2h)+\ldots+f(a+nh)\right],\qquad h=\frac{b-a}{n}$$

where $F(b)-F(a)$, gives the value of the definite integral and it is unique.

10.04 Definition

If $f(x)$ is a continuous function defined on an interval $[a, b]$ and the integration of $f(x)$ is $F(x)$ then

$$\int_a^b f(x)\,dx=\left[F(x)\right]_a^b=F(b)-F(a),$$

where a is called the lower limit of the integral and b is called the upper limit of the integral. The definite integrals is introduced either as the limit of a sum or if it has an anti derivative $F(x)$ in the interval $[a, b]$, then its value is the difference between the values of $F(x)$ at the end points, i.e. $F(b) - F(a)$.

10.05 To Find the value of definite integrals

To find the definite Integral, firstly we find the integration by the known method and then the limits are substituted in place of variable. The following examples show the procedure:-

(i) $$\int_0^{\pi/2} \cos x\, dx = [\sin x]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$$

(ii) $$\int_1^2 x^3\, dx = \left[\frac{x^4}{4}\right]_1^2 = \frac{2^4}{4} - \frac{1^4}{4} = 4 - \frac{1}{4} = \frac{15}{4}.$$

(iii) $$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \left[\sin^{-1} x\right]_0^1 = \sin^{-1}(1) - \sin^{-1}(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

We can find the vlaue of definite integral by the methods used to solve the indefinite integral, usually the methods are used:

(i) Using standard formula
(ii) Substitution
(iii) Partial fraction
(iv) Integration by Parts

10.06 Evaluation of definite integral by substitution

To evaluate $\int_a^b f(x)$, by substitution, the steps could be as follows:

(i) Consider the integral without limits and substitute, the independent variable (say x) with new variable t to convert the given integral to a known form.

(ii) Integrate the new integrand with respect to the new variable t without mentioning the constant of integration.

(iii) Resubstitute for the new variable and write the integration in terms of the original variable and solve it for given limit.

Illustrative Examples

Example 4. Evaluate the following definite integrals

(i) $\int_{-1}^{2} \frac{dx}{3x-2}$ (ii) $\int_{\pi/4}^{\pi/2} \frac{dx}{1-\cos 2x}$ (iii) $\int_0^{\infty} \frac{\sin(\tan^{-1} x)}{1+x^2} dx$ (iv) $\int_0^1 \frac{2x}{1+x^4} dx.$

Solution : (i) Let $$I = \int_{-1}^{2} \frac{dx}{(3x-2)} = \frac{1}{3}[\log|3x-2|]_{-1}^{2} = \frac{1}{3}[\log 4 - \log|-5|]$$

$$= \frac{1}{3}[\log 4 - \log 5] = \frac{1}{3}\log\frac{4}{5}.$$

(ii) Let $I=\int_{\pi/4}^{\pi/2}\frac{dx}{1-\cos 2x}=\int_{\pi/4}^{\pi/2}\frac{dx}{2\sin^2 x}=\frac{1}{2}\int_{\pi/4}^{\pi/2}\text{cosec}^2 x\,dx$

$$=\frac{1}{2}[-\cot x]_{\pi/4}^{\pi/2}=\frac{1}{2}[-\cot\pi/2+\cot\pi/4]=\frac{1}{2}[0+1]=\frac{1}{2}$$

(iii) Let $I=\int_0^{\infty}\frac{\sin\left(\tan^{-1}x\right)}{1+x^2}dx$

Let $\tan^{-1}x=t\Rightarrow\frac{1}{1+x^2}dx=dt$ when $x=0$ then $t=0$; $x=\infty$, $t=\pi/2$

$\therefore$ $I=\int_0^{\pi/2}\sin t\,dt=[-\cos t]_0^{\pi/2}=-\cos\pi/2+\cos 0=0+1=1.$

(iv) Let $I=\int_0^1\frac{2x}{1+x^4}dx$, Let $x^2=t\Rightarrow 2x\,dx=dt$

when $x=0$ then $t=0$; $x=1$, $t=1$

$\therefore$ $I=\int_0^1\frac{dt}{1+t^2}=[\tan^{-1}t]_0^1=\tan^{-1}(1)-\tan^{-1}(0)=\frac{\pi}{4}-0=\frac{\pi}{4}.$

Example 5. Evaluate the following definite integrals.

(i) $\int_0^{\pi/4}(2\sec^2 x+x^3+1)dx$ (ii) $\int_0^1\frac{e^x}{1+e^{2x}}dx$ (iii) $\int_0^1 x\,e^x dx$

Solution : (i) Let $I=\int_0^{\pi/4}(2\sec^2 x+x^3+1)\,dx$

$$=\left[2\tan x+\frac{x^4}{4}+x\right]_0^{\pi/4}=\left[2\tan\frac{\pi}{4}+\frac{1}{4}\left(\frac{\pi}{4}\right)^4+\frac{\pi}{4}\right]-(0+0+0)$$

$$=2\times1+\frac{1}{4}\times\frac{\pi^4}{256}+\frac{\pi}{4}=2+\frac{\pi^4}{1024}+\frac{\pi}{4}.$$

(ii) Let $I=\int_0^1\frac{e^x}{1+e^{2x}}dx$ Let $e^x=t\Rightarrow e^x dx=dt$

when $x=0$ then $t=e^{o}=1$

when $x=1$ then $t=e^1=e$

$\therefore$ $I=\int_1^e\frac{dt}{1+t^2}=[\tan^{-1}t]_1^e=\tan^{-1}e-\tan^{-1}(1)=\tan^{-1}e-\frac{\pi}{4}$

(iii) Let $I=\int_0^1\underset{\text{I}}{x}\,\underset{\text{II}}{e^x}\,dx$ (Integralsting by parts taking e^x as second function)

$$=[xe^x]_0^1-\int_0^1 1\times e^x dx=[1.e^1-0]-[e^x]_0^1$$

$$=e-[e^1-e^0]=e-e+e^0=e^0=1$$

Example 6. Evaluate the following definite integrals:

(i) $\int_0^{\pi/2} \frac{\cos x\, dx}{(1+\sin x)(2+\sin x)}$ (ii) $\int_1^e e^x\left(\frac{1+x\log x}{x}\right)dx$

Solution : (i) Let $I = \int_0^{\pi/2} \frac{\cos x\, dx}{(1+\sin x)(2+\sin x)}$ Let $\sin x = t \quad \therefore \cos x\, dx = dt$

when $x=0, t=0$ and when $x=\pi/2,\ t=1$

$$\therefore \quad I = \int_0^1 \frac{dt}{(1+t)(2+t)} = \int_0^1 \left[\frac{1}{1+t} - \frac{1}{2+t}\right]dt$$

$$= [\log|(1+t)| - \log|2+t|]_0^1$$

$$= \left[\log\left|\frac{1+t}{2+t}\right|\right]_0^1 = \log\frac{2}{3} - \log\frac{1}{2} = \log\left(\frac{2}{3}\times\frac{2}{1}\right) = \log\frac{4}{3}.$$

(ii) Let $I = \int_1^e e^x\left(\frac{1+x\log x}{x}\right)dx$

$$= \int_1^e e^x\left[\frac{1}{x} + \log x\right]dx$$

$$= \left[e^x \log x\right]_1^e \qquad \left[\because \int e^x[f(x)+f'(x)]dx = e^x f(x)\right]$$

$$= e^e \log e - e^1 \log 1 = e^e \times 1 - e\times 0 = e^e$$

Example 7. Evaluate the following definite integrals.

(i) $\int_0^{\pi/4} \frac{\sin 2x}{\sin^4 x + \cos^4 x}dx$ (ii) $\int_a^\infty \frac{dx}{x^4\sqrt{a^2+x^2}}$

Solution : (i) Let $I = \int_0^{\pi/4} \frac{\sin 2x}{\sin^4 x + \cos^4 x}dx$

$$= \int_0^{\pi/4} \frac{2\sin x\cos x}{\sin^4 x + \cos^4 x}dx$$

Dividing Nr and Dr by $\cos^4 x$, we get

$$I = \int_0^{\pi/4} \frac{2\tan x\sec^2 x}{1+\tan^4 x}dx$$

Let $\tan^2 x = t \Rightarrow 2\tan x\sec^2 x\, dx = dt$

as $x=0$ then $t=0$ and when $x=\pi/4$ then $t=1$

$$\therefore \quad I = \int_0^1 \frac{dt}{1+t^2} = [\tan^{-1} t]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

(ii) $$I = \int_a^\infty \frac{dx}{x^4\sqrt{a^2+x^2}}$$

Let $x = a\tan\theta \Rightarrow dx = a\sec^2\theta d\theta$

when $x = a$ then $\theta = \pi/4$ and $x = \infty$ then $\theta = \pi/2$

$$\therefore \quad I = \int_{\pi/4}^{\pi/2} \frac{a\sec^2\theta d\theta}{a^4\tan^4\theta\sqrt{a^2 + a^2\tan^2\theta}}$$

$$= \int_{\pi/4}^{\pi/2} \frac{a\sec^2\theta d\theta}{a^4\tan^4\theta \times a\sec\theta}$$

$$= \int_{\pi/4}^{\pi/2} \frac{\sec\theta d\theta}{a^4\tan^4\theta} = \frac{1}{a^4}\int_{\pi/4}^{\pi/2} \frac{1/\cos\theta}{\sin^4\theta/\cos^4\theta} d\theta$$

$$= \frac{1}{a^4}\int_{\pi/4}^{\pi/2} \frac{\cos^3\theta}{\sin^4\theta} d\theta = \frac{1}{a^4}\int_{\pi/4}^{\pi/2} \frac{(1-\sin^2\theta)\cos\theta d\theta}{\sin^4\theta} d\theta$$

Let $\sin\theta = t \Rightarrow \cos\theta d\theta = dt$

as $\theta = \pi/4$ then $t = 1/\sqrt{2}$ and $\theta = \pi/2$ then $t = 1$

$$\therefore \quad I = \frac{1}{a^4}\int_{1/\sqrt{2}}^{1} \frac{(1-t^2)dt}{t^4} = \frac{1}{a^4}\int_{1/\sqrt{2}}^{1}\left(\frac{1}{t^4} - \frac{1}{t^2}\right)dt$$

$$= \frac{1}{a^4}\left[-\frac{1}{3t^3} + \frac{1}{t}\right]_{1/\sqrt{2}}^{1} = \frac{1}{a^4}\left[\left(-\frac{1}{3} + \frac{1}{1}\right) - \left(-\frac{1}{3\times 1/2\sqrt{2}} + \frac{1}{1/\sqrt{2}}\right)\right]$$

$$= \frac{1}{a^4}\left[\frac{2}{3} - \left(-\frac{2\sqrt{2}}{3} + \sqrt{2}\right)\right] = \frac{1}{a^4}\left[\frac{2}{3} + \frac{2\sqrt{2}}{3} - \sqrt{2}\right]$$

$$= \frac{1}{a^4}\left[\frac{2 + 2\sqrt{2} - 3\sqrt{2}}{3}\right] = \frac{1}{a^4}\left(\frac{2-\sqrt{2}}{3}\right) = \frac{2-\sqrt{2}}{3a^4}$$

Example 8. Evaluate $\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

Solution : Let $I = \int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

Dividing Nr and Dr by $\cos^2 x$, we get

$$I = \int_0^{\pi/2} \frac{\sec^2 x\, dx}{a^2 + b^2\tan^2 x}$$

Let $b\tan x = t \Rightarrow b\sec^2 x\, dx = dt$, when $x = 0$ then $t = 0$, $x = \pi/2$ then $t = \infty$

$$\therefore \quad I = \frac{1}{b}\int_0^{\infty} \frac{dt}{a^2 + t^2} = \frac{1}{b}\times\frac{1}{a}\left[\tan^{-1}\left(\frac{t}{a}\right)\right]_0^{\infty}$$

$$= \frac{1}{ab}[\tan^{-1}\infty - \tan^{-1} 0] = \frac{1}{ab}[\pi/2 - 0] = \frac{\pi}{2ab}.$$

Example 9. Evaluate $\int_0^{\pi/2}(\sqrt{\tan x}+\sqrt{\cot x})\,dx$

Solution : Let $I=\int_0^{\pi/2}(\sqrt{\tan x}+\sqrt{\cot x})\,dx$

$$=\int_0^{\pi/2}\left[\frac{\sqrt{\sin x}}{\sqrt{\cos x}}+\frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right]dx$$

$$=\int_0^{\pi/2}\frac{\sin x+\cos x}{\sqrt{\sin x\cos x}}\,dx$$

$$=\sqrt{2}\int_0^{\pi/2}\frac{(\sin x+\cos x)\,dx}{\sqrt{2\sin x\cos x}}$$

$$=\sqrt{2}\int_0^{\pi/2}\frac{(\sin x+\cos x)dx}{\sqrt{1-(1-2\sin x\cos x)}}=\sqrt{2}\int_0^{\frac{\pi}{2}}\frac{(\sin x+\cos x)dx}{\sqrt{1-(\sin x-\cos x)^2}}$$

Let $\sin x-\cos x=t\Rightarrow(\cos x+\sin x)dx=dt$, Also when $x=0$ then $t=-1$, $x=\pi/2$ then $t=1$

$\therefore$ $$I=\sqrt{2}\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}}=\sqrt{2}\left[\sin^{-1}t\right]_{-1}^{1}$$

$$=\sqrt{2}\left[\sin^{-1}(1)-\sin^{-1}(-1)\right]=\sqrt{2}\left[\frac{\pi}{2}-\left(\frac{-\pi}{2}\right)\right]$$

$$=\sqrt{2}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)=\pi\sqrt{2}$$

Exercise 10.2

Evaluate the following definite integrals:

1. $\int_1^3(2x+1)^3\,dx$
2. $\int_0^{\pi/2}\frac{\sin x}{1+\cos^2 x}\,dx$
3. $\int_1^3\frac{\cos(\log x)}{x}\,dx$
4. $\int_0^1\frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$
5. $\int_0^{\pi/2}\sqrt{1+\sin x}$
6. $\int_o^c\frac{y}{\sqrt{y+c}}\,dy$
7. $\int_o^\infty\frac{e^{\tan^{-1}x}}{1+x^2}\,dx$
8. $\int_1^2\frac{(1+\log x)^2}{x}\,dx$
9. $\int_\alpha^\beta\frac{dx}{(x-\alpha)(\beta-x)},\ \beta>\alpha$
10. $\int_0^{\pi/4}\frac{(\sin x+\cos x)}{9+16\sin 2x}$
11. $\int_{1/e}^{e}\frac{dx}{x(\log x)^{1/3}}$
12. $\int_0^{\pi/4}\sin 2x\cos 3x\,dx$
13. $\int_e^{e^2}\left[\frac{1}{\log x}-\frac{1}{(\log x)^2}\right]dx$
14. $\int_0^1\frac{x^3}{\sqrt{1-x^2}}\,dx$
15. $\int_{\pi/2}^{\pi}\frac{1-\sin x}{1-\cos x}\,dx$

16. $\int_0^{\pi/4} \frac{dx}{4\sin^2 x + 5\cos^2 x}$ 17. $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$ 18. $\int_{-1}^{1} x \tan^{-1} x dx$

19. $\int_0^{1} \frac{x \sin^{-1} x}{\sqrt{1-x^2}}$ 20. $\int_0^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$ 21. $\int_1^{2} \log x \, dx$

22. $\int_{4/\pi}^{2/\pi} \left(-\frac{1}{x^3}\right) \cos\left(\frac{1}{x}\right) dx$ 23. $\int_0^{\pi/2} \frac{\sin x \cos x \, dx}{\cos^2 x + 3\cos x + 2}$ 24. $\int_0^{3} \sqrt{\frac{x}{3-x}} dx$

25. $\int_0^{1} \frac{x^2}{1+x^2} dx$ 26. $\int_1^{2} \frac{1}{(x+1)(x+2)} dx$

10.07 Basic properties of definite integral

Property-I If the limits are not changed then by changing the variable in definite integral the vlaue of the integral does not change.

i.e. $$\int_a^b f(x)dx = \int_a^b f(t)dt$$

Proof : Let $\int f(x)dx = F(x)$ $\therefore \int f(t)dt = F(t)$

$\therefore$ $$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

and $$\int_a^b f(t)dt = [F(t)]_a^b = F(b) - F(a) = \int_a^b f(x)dx$$

$\therefore$ $$\int_a^b f(x)dx = \int_a^b f(t)dt$$

Property-II If the limits are interchanged then the sign of the integral changes while value remain same.

i.e. $$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Proof : Let $\int f(x)dx = F(x)$

$\therefore$ $$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

and $$\int_b^a f(x)dx = [F(x)]_b^a = F(a) - F(b) = -[F(b) - F(a)] = -\int_b^a f(x)dx$$

similarly $$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Property-III If $a < c < b$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Proof : Let $\int f(x)dx = F(x)$

$\therefore$ $$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a) \quad (1)$$

again $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = [F(x)]_a^c + [F(x)]_c^b$

$$= F(c) - F(a) + F(b) - F(c)$$

$$= F(b) - F(a) \qquad (2)$$

from (1) and (2) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Generalization

If $a < c_1 < c_2 < \dots . < c_n < b$,

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \dots . + \int_{c_n}^b f(x)\,dx$$

Note: This property is used when integrand is obtained from more than one rule for given interval of integration say [a, b].

Property-IV $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Proof: $\text{LHS} = \int_a^b f(a+b-x)\,dx$

Let $a+b-x = y \Rightarrow -dx = dy$

when $x = a$ then $y = b$ and when $x = b$ then $y = a$

$\therefore$ $\text{LHS} = \int_b^a f(y)\cdot(-dy) = \int_a^b f(y)\,dy$ (by property-II)

$$= \int_a^b f(x)\,dx = \text{RHS} \qquad \text{(by property-I)}$$

i.e. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Special condition : If $a = 0$ then

$$\int_0^b f(x)\cdot dx = \int_0^b f(b-x)\,dx$$

If a fucntion $f(x)$ does not change by putting $(b-x)$ in place of x then this property is used. For using this property the lower limit has to be zero.

Illustrative Examples

Example 10. Evaluate $\int_0^{\pi/2} \frac{1}{1+\sqrt{\cot x}}\,dx$.

Solution : Let $I = \int_0^{\pi/2} \frac{1}{1+\sqrt{\cot x}}\,dx$

or, $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx \qquad (1)$

$$=\int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\,dx$$

or $$I=\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \qquad (2)$$

adding (1) and (2)

$$2I=\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx+\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

or, $$2I=\int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx=\int_0^{\pi/2} dx=[x]_0^{\pi/2}=\frac{\pi}{2}$$

$\therefore$ $$I=\frac{\pi}{4} \qquad \text{or} \quad \int_0^{\pi/2} \frac{1}{1+\sqrt{\cot x}}\,dx=\frac{\pi}{4}$$

Note: Similarly using property IV, the value of the following integrals will also be $\pi/4$.

(i) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x+\cos^n x}\,dx$ (ii) $\int_0^{\pi/2} \frac{\cos^n x}{\sin^n x+\cos^n x}\,dx$ (iii) $\int_0^{\pi/2} \frac{1}{1+\tan^n x}\,dx$

(iv) $\int_0^{\pi/2} \frac{1}{1+\cot^n x}$ (v) $\int_0^{\pi/2} \frac{\sec^n x}{\sec^n x+\operatorname{cosec}^n x}\,dx$ (vi) $\int_0^{\pi/2} \frac{\operatorname{cosec}^n x}{\sec^n x+\operatorname{cosec}^n x}\,dx$

Example 11. Prove that: $\int_{-a}^{a} f(x)\,dx=\int_{-a}^{a} f(-x)\,dx$.

Solution : Let $I=\int_{-a}^{a} f(x)\,dx$

By Property-IV, $I=\int_{-a}^{a} f(-a+a-x)dx=\int_{-a}^{a} f(-x)\,dx$

Example 12. Evaluate $\int_1^4 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx$

Solution : Let $$I=\int_1^4 \frac{\sqrt{x}dx}{\sqrt{5-x}+\sqrt{x}} \qquad (1)$$

or, $$I=\int_1^4 \frac{\sqrt{5-x}}{\sqrt{5-(5-x)}+\sqrt{5-x}}\,dx$$

or, $$I=\int_1^4 \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}\,dx \qquad (2)$$

Adding (1) and (2),

$$2I = \int_1^4 \frac{\sqrt{x}+\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}dx$$

$$= \int_1^4 dx = [x]_1^4 = 4-1=3$$

$\therefore$ $$I = 3/2.$$

Property V : $\int_o^{na} f(x)dx = n\int_o^a f(x)dx$, and $f(a+x)=f(x)$, where $f(x)$ is periodic function with period a.

Proof : By property III

$$\int_o^{na} f(x)dx = \int_o^a f(x)dx + \int_a^{2a} f(x)dx + \int_{2a}^{3a} f(x)dx + ... + \int_{(n-1)a}^{na} f(x)dx$$

Now in integral $\int_o^{2a} f(x)dx$ putting $x=a+t \Rightarrow dx = dt$ when $x=a$, $t=0$ and $x=2a$, $t=a$

$\therefore$ $$\int_a^{2a} f(x)dx = \int_o^a f(a+t)dt = \int_o^a f(a+x)dx = \int_o^a f(x)dx \qquad [\because f(a+x)=f(x)]$$

Now

$$f(x) = f(x+a) = f(x+2a) = ... = f(x+na)$$

$\therefore$ $$\int_o^{na} f(x)dx = \underbrace{\int_o^a f(x)dx + \int_o^a f(x)dx + ... + \int_o^a f(x)dx}_{n \text{ times}} = n\int_o^a f(x)dx$$

Property-VI $$\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx & ; \text{ If } f(x) \text{ is an even function i.e. } f(-x)=f(x) \\ 0 & ; \text{ If } f(x) \text{ if an odd function i.e. } f(-x)=-f(x) \end{cases}$$

Proof : By property III

$$\int_{-a}^a f(x)dx = \int_{-a}^o f(x)dx + \int_o^a f(x)dx \qquad (\because -a<0<a)$$

$$= I_1 + \int_o^a f(x)dx \qquad (1)$$

where $$I_1 = \int_{-a}^o f(x)dx$$

Let $$x=-y \Rightarrow dx = -dy$$

when $x=-a$ then $y=a$, $x=0$ then $y=0$

$\therefore$ $$I_1 = \int_a^o -f(-y)dy = \int_o^a f(-y)dy \qquad \text{(Property II)}$$

$$= \int_o^a f(-x)dx \qquad \text{(Property I)}$$

from eq. (1)

$$\int_{-a}^{a} f(x)\,dx = \int_{o}^{a} f(-x)\,dx + \int_{o}^{a} f(x)\,dx \qquad (2)$$

Case (i): when $f(x)$ is an even function if $f(-x) = f(x)$

then $$\int_{-a}^{a} f(x)\,dx = \int_{o}^{a} f(x)\,dx + \int_{o}^{a} f(x)\,dx = 2\int_{o}^{a} f(x)\,dx$$

Case (ii): when $f(x)$ is an odd function if $f(-x) = -f(x)$

then $$\int_{-a}^{a} f(x)\,dx = -\int_{o}^{a} f(x)\,dx + \int_{o}^{a} f(x)\,dx = 0$$

$$\therefore \qquad \int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx & ; \text{ If } f(x) \text{ is an even function then } f(-x) = f(x) \\ 0 & ; \text{ If } f(x) \text{ is an odd function then } f(-x) = -f(x) \end{cases}$$

Property-VII: $$\int_{0}^{2a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx & ; \text{ If } f(2a-x) = f(x) \\ 0 & ; \text{ If } f(2a-x) = -f(x) \end{cases}$$

Proof : $$\int_{o}^{2a} f(x)\,dx = \int_{o}^{a} f(x)\,dx + \int_{a}^{2a} f(x)\,dx \qquad [\text{property III } \therefore o < a < 2a]$$

$$= \int_{o}^{a} f(x)\,dx + I_1 \qquad (1)$$

here $$I_1 = \int_{a}^{2a} f(x)\,dx$$

Let $x = 2a - y \Rightarrow dx = -dy$ when $x = a$ then $y = a$ and $x = 2a$ then $y = o$

$$\therefore \qquad I_1 = \int_{a}^{o} -f(2a-y)\,dy = \int_{o}^{a} f(2a-y)\,dy \qquad (\text{property II})$$

$$= \int_{o}^{a} f(2a-x)\,dx \qquad (\text{property I})$$

substituting the vlaue of I_1 in (1)

$$\int_{o}^{2a} f(x)\,dx = \int_{o}^{a} f(x)\,dx + \int_{o}^{a} f(2a-x)\,dx$$

Case (i): when $$f(2a-x) = f(x)$$

then $$\int_{o}^{2a} f(x)\,dx = \int_{o}^{a} f(x)\,dx + \int_{o}^{a} f(x)\,dx = 2\int_{o}^{a} f(x)\,dx$$

Case (ii): when $$f(2a-x) = -f(x)$$

then $$\int_{o}^{2a} f(x)\,dx = \int_{o}^{a} f(x)\,dx - \int_{o}^{a} f(x)\,dx = 0$$

$$\therefore \qquad \int_{0}^{2a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx & ; \text{ If } f(2a-x) = f(x) \\ 0 & ; \text{ If } f(2a-x) = -f(x) \end{cases}$$

Note: (i) when $f(2a-x)=f(x)$ then $f(x)$ should not be considered as even function $f(x)$ is even function only when $f(-x)= f(x)$.
(ii) If the lower limit is zero then we use property-IV i.e. we substitute x with $f(a+b-x)$ but some time $f(x)$ doesn't change then we use property VII.

10.08 Special property (Eliminating x)

If $f(a+b-x)=f(x)$ then eliminating x from $\int_a^b x\, f(x)\,dx$

$$\int_a^b x\, f(x)\,dx = \frac{a+b}{2}\int_a^b f(x)\,dx$$

Proof : Let $\qquad I=\int_a^b f(x)\,dx$

Using Property IV

$$\int_a^b (a+b-x)\, f(a+b-x)\,dx$$

but given $\qquad f(a+b-x)=f(x)$

$\therefore \qquad I=\int_a^b (a+b-x)\, f(x)\,dx$

$$=(a+b)\int_a^b f(x)dx-\int_a^b x\, f(x)\,dx$$

or $\qquad I=(a+b)\int_a^b f(x)dx-I$

or $\qquad 2I=(a+b)\int_a^b f(x)\,dx \Rightarrow I=\frac{a+b}{2}\int_a^b f(x)\,dx$

Illustrative Examples

Example 13. Evaluate $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$

Solution : Let $\qquad I=\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$

or, $\qquad I=\int_0^\pi x.\left(\frac{\sin x}{1+\cos^2 x}\right)dx$

here, $\quad f(x)=\frac{\sin x}{1+\cos^2 x}$

$$\therefore \quad f(\pi-x)=\frac{\sin(\pi-x)}{1+\cos^2(\pi-x)}=\frac{\sin x}{1+\cos^2 x}=f(x)$$

$\therefore$ Eliminating x,

$$I=\frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}dx$$

Let $\cos x = t \Rightarrow \sin x\,dx = -dt$ $x = 0$ then $t = 1$ and $x = \pi$ then $t = -1$

$\therefore$
$$I = \frac{\pi}{2}\int_{1}^{-1}\frac{-dt}{1+t^2} = \frac{\pi}{2}\int_{-1}^{1}\frac{1}{1+t^2}dt = \frac{\pi}{2}(\tan^{-1} t)_{-1}^{1}$$

$$= \frac{\pi}{2}\left[\tan^{-1}(1) - \tan^{-1}(-1)\right] = \frac{\pi}{2}\left[\frac{\pi}{4} - \left(\frac{-\pi}{4}\right)\right] = \frac{\pi}{2}\left[\frac{\pi}{2}\right] = \frac{\pi^2}{4}.$$

Important standard integral

$$I = \int_0^{\pi/2} \log \sin x\,dx = -\frac{\pi}{2}\log 2 = \int_0^{\pi/2} \log \cos x\,dx$$

Solution : Let $I = \int_0^{\pi/2} \log \sin x\,dx$ (1)

Using property IV,

$$I = \int_0^{\pi/2} \log\left[\sin\left(\pi/2 - x\right)\right]dx$$

or $I = \int_0^{\pi/2} \log \cos x\,dx$ (2)

Adding (1) and (2)

$$2I = \int_0^{\pi/2} [\log \sin x + \log \cos x]\,dx$$

$$= \int_0^{\pi/2} \log(\sin x \cos x)dx$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right)dx = \int_0^{\pi/2} (\log \sin 2x - \log 2)dx$$

$$= \int_0^{\pi/2} \log \sin 2x\,dx - \log 2\int_0^{\pi/2} dx$$

$$= \int_0^{\pi/2} \log \sin 2x\,dx - (\log 2)[x]_0^{\pi/2}$$

or $2I = I_1 - \frac{\pi}{2}(\log 2)$ (3)

when $I_1 = \int_0^{\pi/2} \log \sin 2x\,dx$

Let $2x = t \Rightarrow dx = \frac{dt}{2}$

when $x = 0$ then $t = 0$ and $x = \pi/2$ then $t = \pi$

$\therefore$ $I_1 = \frac{1}{2}\int_0^{\pi} \log(\sin t)\,dt = \frac{1}{2}\times 2\int_0^{\pi/2} \log \sin t\,dt$ (Property VII)

$= \int_0^{\pi/2} \log \sin x\,dx$ (Property I) (Using equation (1))

$\therefore$ from equation (3) $\quad 2I = I - \frac{\pi}{2}\log_e 2 \Rightarrow I = -\frac{\pi}{2}(\log_e 2)$

or $\quad \int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \cos x \, dx = -\frac{\pi}{2}\log 2$.

$$\int_0^{\pi/2} \log \operatorname{cosec} x \, dx = \int_0^{\pi/2} \log \sec x \, dx = \frac{\pi}{2}\log 2.$$

Illustrative Examples

Example 14. Evaluate the following definite Integrals

(i) $\int_1^4 f(x)\,dx$ when $f(x) = \begin{bmatrix} 4x+3, 1 \le x \le 2 \\ 3x+5, 2 \le x \le 4 \end{bmatrix}$ (ii) $\int_0^2 |1-x|\,dx$ (iii) $\int_{-1}^1 e^{|x|}dx$

Solution : (i) $\int_1^4 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^4 f(x)\,dx$

$$= \int_1^2 (4x+3)\,dx + \int_2^4 (3x+5)dx \left[\because f(x) = \begin{bmatrix} 4x+3 & ; & 1 \le x \le 2 \\ 3x+5 & ; & 2 \le x \le 4 \end{bmatrix}\right]$$

$$= \left[2x^2 + 3x\right]_1^2 + \left[\frac{3x^2}{2} + 5x\right]_2^4$$

$$= \left[(8+6)-(2+3)\right] + \left[(24+20)-(6+10)\right] = 9 + 28 = 37.$$

(ii) $\int_0^2 |1-x|\,dx = \int_0^1 |1-x|\,dx + \int_1^2 |1-x|\,dx$

$$\left[\because \ |1-x| = \begin{cases} 1-x & x<1 \\ -(1-x), & x>1 \end{cases}\right]$$

$$= \int_0^1 (1-x)\,dx + \int_1^2 (1-x)\,dx$$

$$= \left[x - x^2/2\right]_0^1 - \left[x - x^2/2\right]_1^2$$

$$= \left[(1-1/2)-0\right] - \left[(2-2)-(1-1/2)\right] = (1/2) + (1/2) = 1.$$

(iii) $\int_{-1}^1 e^{|x|}dx = \int_{-1}^0 e^{|x|}dx + \int_0^1 e^{|x|}dx \quad \left[\because |x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}\right]$

$$= \int_{-1}^0 e^{-x}dx + \int_0^1 e^x dx$$

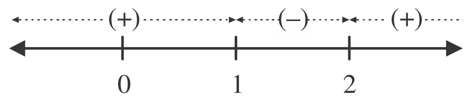
$$= [-e^{-x}]_{-1}^0 + [e^x]_0^1 = (-e^o + e^1) + (e - e^o) = 2e - 2.$$

Example 15. Evaluate the following definite integrals

(i) $\int_0^2 \left|x^2-3x+2\right| dx$ (ii) $\int_{1/e}^{e} \left|\log_e x\right| dx$ (iii) $\int_0^{\pi} \left|\cos x\right| dx$

Solution : (i) Here $x^2-3x+2=(x-1)(x-2)$

The sign of x^2-3x+2 will be different for various vlaues of x



$$\therefore \qquad |x^2-3x+2|=\begin{cases} x^2-3x+2, & 0\le x\le 1 \\ -(x^2-3x+2), & 1\le x\le 2\end{cases}$$

$$\therefore \qquad \int_0^2 |x^2-3x+2|\,dx=\int_0^1 |x^2-3x+2|\,dx+\int_1^2 |x^2-3x+2|\,dx$$

$$=\int_0^1 (x^2-3x+2)\,dx+\int_1^2 -(x^2-3x+2)\,dx$$

$$=\left[\frac{x^3}{3}-\frac{3x^2}{2}+2x\right]_0^1-\left[\frac{x^3}{3}-\frac{3x^2}{2}+2x\right]_1^2$$

$$=\left[(1/3-3/2+2)-(0)\right]-\left[(8/3-6+4)-(1/3-3/2+2)\right]$$

$$=\frac{5}{6}-\frac{2}{3}+\frac{5}{6}=\frac{5}{3}-\frac{2}{3}=1$$

(ii) $$\int_{1/e}^{e}|\log_e x|\,dx=\int_{1/e}^{1}|\log_e x|\,dx+\int_1^e|\log_e x|\,dx$$

$$=\int_{1/e}^{1}-\log_e x\,dx+\int_1^e \log_e x\,dx \qquad \left[\because |\log_e x|=\begin{cases}-\log_e x, & \text{If } \ 1/e<x<1 \\ \log_e x, & \text{If } \ 1\le x<e\end{cases}\right]$$

$$=-\left[x(\log_e x-1)\right]_{1/e}^{1}+\left[x(\log_e x-1)\right]_1^e \qquad \left[\because \int \log_e x\,dx=x(\log_e x-1)\right]$$

$$=-\left[(0-1)-1/e(-1-1)\right]+\left[e(1-1)-(0-1)\right]$$

$$=1-2/e+1=2-2/e$$

(iii) $$\int_0^{\pi}|\cos x|\,dx=\int_0^{\pi/2}|\cos x|\,dx+\int_{\pi/2}^{\pi}|\cos x|\,dx$$

$$=\int_0^{\pi/2}\cos x\,dx+\int_{\pi/2}^{\pi}(-\cos x)\,dx \qquad \because |\cos x|=\begin{cases}\cos x & ; \ 0<x\le \pi/2 \\ -\cos x & ; \ \pi/2<x\le \pi\end{cases}$$

$$=[\sin x]_0^{\pi/2}-[\sin x]_{\pi/2}^{\pi}$$

$$=(\sin\pi/2-\sin 0)-(\sin\pi-\sin\pi/2)=(1-0)-(0-1)=2$$

Example 16. Evaluate the following definite integrals:

(i) $\int_0^{\pi/2} \log \cot x \, dx$ (ii) $\int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x \cos x} dx$

Solution : (i) Let $I = \int_0^{\pi/2} \log \cot x \, dx$ (1)

or, $I = \int_0^{\pi/2} \log\left[\cot(\pi/2 - x)\right] dx$ (using property IV)

or, $I = \int_0^{\pi/2} \log \tan x \, dx$ (2)

adding (1) and (2)

$$2I = \int_0^{\pi/2} \log \cot x \, dx + \int_0^{\pi/2} \log \tan x \, dx$$

$$= \int_0^{\pi/2} \left[\log\left(\cot x\right) + \log\left(\tan x\right)\right] dx$$

$$= \int_0^{\pi/2} \log\left(\cot x \times \tan x\right) dx$$

$$= \int_0^{\pi/2} \log(1)\, dx = \int_0^{\pi/2} (0)\, dx$$

or, $2I = 0$ $\therefore I = 0$

(ii) Let $I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x \cos x} dx$ (1)

using property IV

$$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)} dx$$

or, $I = \int_0^{\pi/2} \frac{\cos x - \sin x}{1+\sin x \cos x} dx$ (2)

adding (1) and (2)

$$2I = 0 \Rightarrow I = 0$$

Example 17. Evaluate the following definite Integrals:

(i) $\int_0^8 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{8-x}} dx$ (ii) $\int_0^a \frac{dx}{x+\sqrt{a^2-x^2}}$

Solution : (i) Let $I = \int_0^8 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{8-x}} dx$ (1)

using property IV,

$$I = \int_0^8 \frac{\sqrt{8-x}}{\sqrt{8-x}+\sqrt{8-(8-x)}} dx$$

or $$I=\int_0^8 \frac{\sqrt{8-x}}{\sqrt{8-x}+\sqrt{x}}dx \qquad (2)$$

adding (1) and (2)

$$2I=\int_0^8 \frac{\sqrt{x}+\sqrt{8-x}}{\sqrt{8-x}+\sqrt{x}}dx=\int_0^8 dx=[x]_0^8=8, \qquad \therefore \quad I=4$$

(ii) $$I=\int_0^a \frac{dx}{x+\sqrt{a^2-x^2}}$$

Let $x=a\sin\theta \Rightarrow dx=a\cos\theta\, d\theta$

when $x=0$ then $\theta=0$ and $x=a$ then $\theta=\pi/2$

$\therefore$ $$I=\int_0^{\pi/2}\frac{a\cos\theta\, d\theta}{a\sin\theta+a\cos\theta}=\int_0^{\pi/2}\frac{\cos\theta}{\sin\theta+\cos\theta}d\theta \qquad (1)$$

Property-(IV) $$I=\int_0^{\pi/2}\frac{\cos(\frac{\pi}{2}-\theta)\,d\theta}{\sin(\frac{\pi}{2}-\theta)+\cos(\frac{\pi}{2}-\theta)}$$

$$I=\int_0^{\pi/2}\frac{\sin\theta\, d\theta}{\cos\theta+\sin\theta} \qquad (2)$$

adding (1) and (2)

$$2I=\int_0^{\pi/2}\left(\frac{\sin\theta+\cos\theta}{\sin\theta+\cos\theta}\right)d\theta$$

$$=\int_0^{\pi/2}d\theta=[\theta]_0^{\pi/2}=\frac{\pi}{2}-0$$

$\therefore$ $$I=\frac{\pi}{4}$$

Example 18. Evaluate $\displaystyle\int_0^{\pi/2}\frac{\sin^2 x}{\sin x+\cos x}dx$

Solution : Let $$I=\int_0^{\pi/2}\frac{\sin^2 x}{\sin x+\cos x}dx \qquad (1)$$

using property IV $$I=\int_0^{\pi/2}\frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx$$

or, $$I=\int_0^{\pi/2}\frac{\cos^2 x}{\cos x+\sin x}dx \qquad (2)$$

Adding (1) and (2),

$$2I=\int_0^{\pi/2}\frac{\sin^2 x+\cos^2 x}{\sin x+\cos x}dx$$

$$I=\frac{1}{2}\int_0^{\pi/2}\frac{1}{\sin x+\cos x}dx$$

$$=\frac{1}{2}\int_0^{\pi/2}\frac{1}{\left(\frac{2\tan(x/2)}{1+\tan^2(x/2)}\right)+\left(\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}\right)}$$

(converting $\sin x$ and $\cos x$ into $\tan x/2$)

$$=\frac{1}{2}\int_0^{\pi/2}\frac{1+\tan^2(x/2)}{2\tan(x/2)+1-\tan^2(x/2)}dx$$

or $$I=\frac{1}{2}\int_0^{\pi/2}\frac{\sec^2(x/2)}{1+2\tan(x/2)-\tan^2(x/2)}dx$$

Let $\tan\frac{x}{2}=t\Rightarrow\frac{1}{2}\sec^2\frac{x}{2}dx=dt$

when $x=0$ then $t=0$; when $x=\pi/2$ then $t=1$

$\therefore$ $$I=\int_0^1\frac{dt}{1+2t-t^2}=\int_0^1\frac{dt}{2-(t-1)^2}$$

$$=\frac{1}{2\sqrt{2}}\left[\log\left|\frac{\sqrt{2}+(t-1)}{\sqrt{2}-(t-1)}\right|\right]_0^1$$

$$=\frac{1}{2\sqrt{2}}\left[\log\frac{\sqrt{2}}{\sqrt{2}}-\log\frac{\sqrt{2}-1}{\sqrt{2}+1}\right]$$

$$=\frac{1}{2\sqrt{2}}\left[0+\log\frac{\sqrt{2}+1}{\sqrt{2}-1}\right]=\frac{1}{2\sqrt{2}}\log\left[\frac{(\sqrt{2}+1)}{(\sqrt{2}-1)}\times\frac{(\sqrt{2}+1)}{(\sqrt{2}+1)}\right]$$

$$=\frac{1}{2\sqrt{2}}\log\frac{(\sqrt{2}+1)^2}{(2-1)}=\frac{2}{2\sqrt{2}}\log(\sqrt{2}+1)=\frac{1}{\sqrt{2}}\log(\sqrt{2}+1).$$

Example 19. Evaluate the followng Integral

$$\int_{-a}^{a}\sqrt{\frac{a-x}{a+x}}dx$$

Solution : Let $I=\int_{-a}^{a}\sqrt{\frac{a-x}{a+x}}dx$

$$=\int_{-a}^{a}\frac{a-x}{\sqrt{a^2-x^2}}dx$$

$$=\int_{-a}^{a}\frac{a}{\sqrt{a^2-x^2}}dx-\int_{-a}^{a}\frac{x}{\sqrt{a^2-x^2}}dx \qquad (1)$$

or, $=I_1-I_2$

where $I_1=\int_{-a}^{a}\frac{a}{\sqrt{a^2-x^2}}dx=2a\int_{0}^{a}\frac{1}{\sqrt{a^2-x^2}}dx$ ($\because f(x)$ is an even function)

using property VI

$$=2a\left[\sin^{-1}x/a\right]_0^a=2a\left(\sin^{-1}(1)-\sin^{-1}(0)\right)=2a\times\left(\pi/2-0\right)=\pi a$$

and $I_2=\int_{-a}^{a}\frac{x}{\sqrt{a^2-x^2}}dx=0$

(property VI when $f(x)$ is an odd function $\int_{-a}^{a}f(x)dx=0$)

$\therefore$ from (1), $I=\pi a-0=\pi a$

Example 20. Prove that:

$$\int_{0}^{\pi/4}\log_e(1+\tan x)dx=\frac{\pi}{8}\log_e 2.$$

Solution : Let $I=\int_{0}^{\pi/4}\log_e(1+\tan x)\,dx$

Using Property IV,

$$I=\int_{0}^{\pi/4}\log_e\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx$$

$$=\int_{0}^{\pi/4}\log_e\left[1+\frac{\tan(\pi/4)-\tan x}{1+\tan(\pi/4)\tan x}\right]dx$$

$$=\int_{0}^{\frac{\pi}{4}}\log_e\left[1+\frac{1-\tan x}{1+\tan x}\right]dx$$

$$=\int_{0}^{\pi/4}\log_e\left(\frac{2}{1+\tan x}\right)dx$$

$$= \int_0^{\pi/4} [\log_e 2 - \log_e (1+\tan x)]\, dx$$

$$= \int_0^{\pi/4} (\log_e 2)\, dx - \int_0^{\pi/4} \log_e (1+\tan x)\, dx$$

or $$I = (\log_e 2)[x]_0^{\pi/4} - I$$

or $$2I = \frac{\pi}{4}\log_e 2 \Rightarrow I = \frac{\pi}{8}\log_e 2$$

$\Rightarrow$ $$\int_0^{\pi/4} \log(1+\tan x)\, dx = \frac{\pi}{8}\log_e 2, \quad \text{Hence proved.}$$

Example 21. Prove that: $I = \int_0^{\pi} \log(1+\cos x)\, dx = \pi \log_e (1/2)$.

Solution : Let $$I = \int_0^{\pi} \log(1+\cos x)\, dx \quad \text{. . . (1)}$$

Using property IV,

$$I = \int_0^{\pi} \log[1+\cos(\pi - x)]\, dx$$

or, $$I = \int_0^{\pi} \log(1-\cos x)\, dx \quad \text{. . . (2)}$$

Adding (1) and (2),

$$2I = \int_0^{\pi} \log(1+\cos x) + \log(1-\cos x)\, dx$$

$$= \int_0^{\pi} \log\{(1+\cos x)(1-\cos x)\}\, dx$$

$$= \int_0^{\pi} \log(1-\cos^2 x)\, dx$$

or $$2I = \int_0^{\pi} \log \sin^2 x\, dx = 2\int_0^{\pi} \log \sin x\, dx$$

or $$I = \int_0^{\pi} \log \sin x\, dx$$

or $$I = 2\int_0^{\pi/2} \log \sin x\, dx \quad \text{(property VII)}$$

or $$I = 2I_1, \text{ and } I_1 = \int_0^{\pi/2} \log \sin x\, dx \quad \text{. . . (3)}$$

or $$I_1 = \int_0^{\pi/2} \log \cos x\, dx \quad \text{(Using property IV)} \quad \text{. . . (4)}$$

Adding equations (3) and (4),

$$2I_1 = \int_0^{\pi/2} (\log \sin x + \log \cos x)\, dx$$

$$= \int_0^{\pi/2} \log(\sin x \cos x)\,dx$$

or $$2I_1 = \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right)dx$$

or $$2I_1 = \int_0^{\pi/2} \log(\sin 2x)\,dx - \int_0^{\pi/2} (\log 2)\,dx$$

or $$2I_1 = I_2 - (\log 2)[x]_0^{\pi/2}$$

or $$2I_1 = I_2 - \frac{\pi}{2}\log 2 \qquad \ldots (5)$$

where $$I_2 = \int_0^{\pi/2} \log(\sin 2x)\,dx$$

Let $2x = t \Rightarrow 2dx = dt$ and when $x = 0$ then $t = 0$, when $x = \pi/2$ then $t = \pi$

$$\therefore \quad I_2 = \frac{1}{2}\int_0^{\pi} \log(\sin t)\,dt = \frac{1}{2}\int_0^{\pi} \log(\sin x)\,dx \qquad \text{(property I)}$$

or, $$I_2 = \frac{1}{2} \times 2\int_0^{\pi/2} \log(\sin x)\,dx \qquad \text{(property VII)}$$

or, $$I_2 = \int_0^{\pi/2} \log \sin x\, dx = I_1$$

putting the value of I_2 in equation (5)

$$2I_1 = I_1 - \frac{\pi}{2}\log 2$$

or $$I_1 = \frac{\pi}{2}\log\frac{1}{2}$$

$$\therefore \quad I = 2I_1 = 2 \times \frac{\pi}{2}\log\frac{1}{2} = \pi \log\frac{1}{2}$$

or $$\int_0^{\pi/2} \log(1 + \cos x)\,dx = \pi \log\frac{1}{2}$$

Example 22. Prove that

$$\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\,dx = \pi\left[(\pi/2) - 1\right]$$

Solution : $$\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\,dx = \int_0^{\pi} x.\left(\frac{\sin x}{1 + \sin x}\right)dx$$

Here, $$f(x) = \frac{\sin x}{1 + \sin x}$$

then, $f(\pi - x) = \dfrac{\sin(\pi - x)}{1+\sin(\pi - x)} = \dfrac{\sin x}{1+\sin x} = f(x)$

$\therefore$ Eliminating x rule, $\int_a^b xf(x)\,dx = \dfrac{a+b}{2}\int_a^b f(x)\,dx$

$$\int_0^\pi \frac{x\tan x}{\sec x + \tan x}dx = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\sin x}dx$$

$$= \frac{\pi}{2}\int_0^\pi \left(1 - \frac{1}{1+\sin x}\right)dx = \frac{\pi}{2}\int_0^\pi \left(1 - \frac{1-\sin x}{\cos^2 x}\right)dx$$

$$= \frac{\pi}{2}\int_0^\pi \left(1 - \sec^2 x + \sec x \tan x\right)dx$$

$$= \frac{\pi}{2}\left[x - \tan x + \sec x\right]_0^\pi = \frac{\pi}{2}\left[(\pi - 0 - 1) - (0 - 0 + 1)\right]$$

$$= \frac{\pi}{2}[\pi - 2] = \pi(\pi/2 - 1).$$ Hence Proved

Exercise 10.3

Evaluate the following definite integrals:

1. $\int_{-2}^{2} |2x+3|\,dx$

2. $\int_{-2}^{2} |1-x^2|\,dx$

3. $\int_1^4 f(x)\,dx$, where $f(x) = \begin{cases} 7x+3 & ; \quad 1 \le x \le 3 \\ 8x & ; \quad 3 \le x \le 4 \end{cases}$

4. $\int_0^3 [x]\,dx$ when $[.]$ is the greatest integer function

5. $\int_{-\pi/4}^{\pi/4} x^5 \cos^2 x\,dx$

6. $\int_{-\pi}^{\pi} \dfrac{\sin x \cos x}{1+\cos^2 x}dx$

7. $\int_{-\pi/4}^{3\pi/4} \dfrac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$

8. $\int_0^\pi \dfrac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}dx$

9. $\int_0^{\pi/2} \sin 2x . \log \tan x\,dx$

10. $\int_{-1}^{1} \log\left[\dfrac{2-x}{2+x}\right]dx$

11. $\int_0^1 \log\left(\dfrac{1}{x} - 1\right)dx$

12. $\int_{\pi/6}^{\pi/3} \dfrac{dx}{1+\sqrt{\tan x}}$

13. $\int_0^{\pi/2} \dfrac{\sin x}{\sin x + \cos x}dx$

14. $\int_0^{\pi/2} \log \sin 2x\,dx$

15. $\int_{-\pi/4}^{\pi/4} \dfrac{\left(x+\dfrac{\pi}{4}\right)}{2-\cos 2x}dx$

16. $\int_0^\pi \log(1-\cos x)\,dx$

17. $\int_{-\pi/4}^{\pi/4} \sin^2 x\, dx$

18. $\int_0^{\pi} \frac{x}{1+\sin x} dx$

19. $\int_0^{\pi} x \sin^3 x\, dx$

20.- $\int_0^{\pi/2} \log(\tan x + \cot x) dx$

21. $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x} dx$

22. $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)} dx$

Miscellaneous Examples

Example 23. Prove that:

$$\int_0^{\pi} \frac{x\, dx}{1+\cos\alpha \sin x} = \frac{\pi\alpha}{\sin\alpha}$$

Solution : Let $f(x) = \frac{1}{1+\cos\alpha \sin x}$

$\therefore$ $$f(\pi - x) = \frac{1}{1+\cos\alpha \sin(\pi - x)} = \frac{1}{1+\cos\alpha \sin x} = f(x)$$

eliminating x rule

$$\int_0^{\pi} \frac{x}{1+\cos\alpha \sin x} dx = \frac{\pi}{2} \int_0^{\pi} \frac{1}{1+\cos\alpha \sin x} dx$$

$$= \frac{\pi}{2} \int_0^{\pi} \frac{1}{1+\cos\alpha \left(\frac{2\tan(x/2)}{1+\tan^2(x/2)} \right)} dx$$

$$= \frac{\pi}{2} \int_0^{\pi} \frac{\sec^2(x/2)}{1+\tan^2(x/2)+2\cos\alpha \tan(x/2)} dx$$

Let $\tan(x/2) = t \Rightarrow \frac{1}{2}\sec^2(x/2) \cdot dx = dt$

when $x = 0$ then $t = 0$ and when $x = \pi$ then $t = \infty$

$\therefore$ $$\int_0^{\pi} \frac{x}{1+\cos\alpha \sin x} dx = \frac{\pi}{2} \int_0^{\infty} \frac{2}{1+t^2+2t\cos x} dx$$

$$= \pi \int_0^{\infty} \frac{dt}{(t+\cos\alpha)^2 + (\sin\alpha)^2}$$

$$= \pi \times \frac{1}{\sin\alpha} \left[\tan^{-1}\left(\frac{t+\cos\alpha}{\sin\alpha} \right) \right]_0^{\infty}$$

$$= \frac{\pi}{\sin\alpha}\left[\tan^{-1}(\infty) - \tan^{-1}(\cot\alpha)\right]$$

$$= \frac{\pi}{\sin\alpha}\left[\pi/2 - (\pi/2 - \alpha)\right] \qquad \left[\because \cot\alpha = \tan(\pi/2-\alpha)\right]$$

$$= \frac{\pi}{\sin\alpha}(\alpha) = \frac{\pi\alpha}{\sin\alpha}$$

Example 24. Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$

Solution : Let $I = \int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$

$$= \frac{1}{a^2-b^2}\int_0^\infty \left(\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2}\right)dx \qquad \text{(Partial fractions)}$$

$$= \frac{1}{(a^2-b^2)}\left[\frac{1}{b}\tan^{-1}\frac{x}{b} - \frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^\infty$$

$$= \frac{1}{(a^2-b^2)}\left[\left(\frac{1}{b}\tan^{-1}\infty - \frac{1}{a}\tan^{-1}\infty\right) - (0-0)\right]$$

$$= \frac{1}{(a^2-b^2)}\left[\frac{1}{b}.\frac{\pi}{2} - \frac{1}{a}.\frac{\pi}{2}\right]$$

$$= \frac{\pi}{2(a^2-b^2)}\left(\frac{a-b}{ab}\right) = \frac{\pi}{2(a+b)(a-b)} \times \frac{(a-b)}{ab} = \frac{\pi}{2ab(a+b)}$$

Example 25. Evaluate $\int_{\pi/4}^{\pi/2} \cos 2x \log \sin x \, dx$

Solution : Let $I = \int_{\pi/4}^{\pi/2} \underset{\text{II}}{\cos 2x} \underset{\text{I}}{\log \sin x} \, dx$

$$= \left[\log \sin x . \frac{\sin 2x}{2}\right]_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2} \cot x \times \frac{\sin 2x}{2} dx$$

$$= \left[0 - \frac{1}{2}\log\frac{1}{\sqrt{2}}\right] - \int_{\pi/4}^{\pi/2} \cos^2 x \, dx$$

$$= -\frac{1}{2}\log\frac{1}{\sqrt{2}} - \frac{1}{2}\int_{\pi/4}^{\pi/2}(1+\cos 2x)\,dx$$

$$=\frac{1}{4}\log 2-\frac{1}{2}\left[x+\frac{\sin 2x}{2}\right]_{\pi/4}^{\pi/2}$$

$$=\frac{1}{4}\log 2-\frac{1}{2}\left[\left(\frac{\pi}{2}+\frac{\sin\pi}{2}\right)-\left(\frac{\pi}{4}+\frac{\sin\pi/2}{2}\right)\right]$$

$$=\frac{1}{4}\log 2-\frac{1}{2}\left[\frac{\pi}{4}-\frac{1}{2}\right]$$

$$=\frac{1}{4}\log 2-\frac{\pi}{8}+\frac{1}{4}.$$

Example 26. Evaluate $\int_0^{\infty}\frac{\log(1+x^2)}{1+x^2}dx$.

Solution : Let $x=\tan\theta \Rightarrow dx=\sec^2\theta\, d\theta$

when $x=0$ then $\theta=0$ and $x=\infty$ then $\theta=\pi/2$

$\therefore$ $$I=\int_0^{\pi/2}\frac{\log\left(1+\tan^2\theta\right)}{\left(1+\tan^2\theta\right)}\sec^2\theta\, d\theta$$

$$=\int_0^{\pi/2}\log\left(1+\tan^2\theta\right)d\theta=\int_0^{\pi/2}\log\sec^2\theta\, d\theta$$

$$=2\int_0^{\pi/2}\log\sec\theta\, d\theta=-2\int_0^{\pi/2}\log\cos\theta\, d\theta$$

$$=-2\int_0^{\pi/2}\log\cos\left(\pi/2-\theta\right)d\theta \quad \text{(Property IV)}$$

$$=2\int_0^{\pi/2}\log\sin\theta\, d\theta=-2\left(-\pi/2\log 2\right) \quad \text{(standard integral)}$$

$$=\pi\log_e 2$$

Miscellaneous Exericse –10

1. The value of $\int_0^{\pi/4}\sqrt{1+\sin 2x}\, dx$ is

   (A) $2\int_0^{a}\sin^3 x.x\, dx$ (B) 0 (C) a^2 (D) 1

2. The value of $\int_2^{5}\frac{\sqrt{x}}{\sqrt{x}+\sqrt{7-x}}dx$ is

   (A) 3 (B) 2 (C) 3 / 2 (D) 1 / 2

3. The value of $\int_{a-c}^{b-c} f(x+c)\,dx$ is

(A) $\int_a^b f(x+c)\,dx$ (B) $\int_a^b f(x)\,dx$ (C) $\int_{a-2c}^{b-2c} f(x)\,dx$ (D) $\int_a^b f(x+2c)dx$

4. If $A(x)=\int_0^x \theta^2 d\theta$, then the value of $A(3)$ is

(A) 9 (B) 27 (C) 3 (D) 81

Evaluate the following definite integrals:-

5. $\int_1^2 \frac{(x+3)}{x(x+2)}dx$

6. $\int_1^2 \frac{xe^x}{(1+x)^2}dx$

7. $\int_0^{\pi/2} e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$

8. $\int_{1/3}^1 \frac{(x-x^3)^{1/3}}{x^4}dx$

9. $\int_0^{\pi/2} x^2\cos^2 x\,dx$

10. $\int_0^1 \tan^{-1} x\,dx$

11. $\int_0^{\pi/4} \sin 3x \sin 2x$

12. $\int_{-2}^2 |1-x^2|\,dx$

13. $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{(1+\cos^2 x)}dx$

14. $\int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1-x^2)^{3/2}}dx$

15. $\int_0^1 (\cos^{-1} x)^2 dx$

16. $\int_0^{\pi} \frac{dx}{1-2a\cos x+a^2}, a>1$

17. Prove that $\int_0^{\pi} \frac{xdx}{a^2\cos^2 x+b^2\sin^2 x}=\frac{\pi^2}{2ab}$

IMPORTANT POINTS

1. The vlaue of definite integral is unique.

2. (i) $\int_a^b k\,f(x)dx=k\int_a^b f(x)\,dx$ (ii) $\int_a^b [f(x)\pm\phi(x)]\,dx=\int_a^b f(x)dx\pm\int_a^b \phi(x)dx$

(iii) $\int_a^a f(x)dx=0$

3. (i) $\int_a^{\infty} f(x)dx=\lim_{b\to\infty}\int_a^b f(x)dx$ (ii) $\int_{-\infty}^b f(x)dx=\lim_{a\to\infty}\int_{-a}^b f(x)dx$

(iii) $\int_{-\infty}^{\infty} f(x)dx=\lim_{a\to\infty}\int_{-a}^a f(x)dx$

4. **Properties of definite integral:**

(i) $\int_a^b f(x)\,dx=\int_a^b f(t)\,dt$ (ii) $\int_a^b f(x)\,dx=-\int_b^a f(x)dx$

(iii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)dx,$ where $a<c<b$

Generalisation: $a < c_1 < c_2 < c_3 < \ldots < c_n < b$

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx + \ldots + \int_{c_n}^{b} f(x)\,dx$$

(iv) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx \ \therefore\ \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(v) $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$ if $f(a+x) = f(x)$ [$f(x)$ is a periodic function of period a]

(vi) $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{If is an even function i.e. } f(-x) = f(x) \\ 0, & \text{If } f(x) \text{ is an odd function i.e. } f(-x) = -f(x) \end{cases}$

(vii) $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{If } f(2a-x) = f(x) \\ 0, & \text{If } f(2a-x) = -f(x) \end{cases}$

5. **Rule of eliminating *x*** If $f(a+b-x) = f(x)$ then

$$\int_a^b x\,f(x)dx = \frac{a+b}{2}\int_a^b f(x)dx$$

6. $$\int_0^{\pi/2} \log\sin x\,dx = -\frac{\pi}{2}\log 2 = \int_0^{\pi/2} \log\cos x\,dx$$

and $$\int_0^{\pi/2} \log\operatorname{cosec} x\,dx = \frac{\pi}{2}\log 2 = \int_0^{\pi/2} \log\sec x\,dx$$

7. **Definite Integral as a limit of sum :** If $f(x)$ is continuous function in given interval $[a, b]$ then divide interval $[a, b]$ in n equal parts having width h .
To evaluate definite integral from this is called "Integration from first principal".

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a+h) + f(a+2h) + \ldots + f(a+nh)\right], \text{where } n\to\infty,\ nh = b-a$$

Answers

Exericse 10.1

1. 4	2. $\frac{1}{3}\left(b^3 - a^3\right)$	3. 86 / 3	4. $e^{-a} - e^{-b}$
5. 10		6. 82 / 3	

Exercise 10.2

1. 290	2. $\pi/4$	3. $\sin(\log 3)$	4. $2(e-1)$
5. 2	6. $\frac{2}{3}\left(2-\sqrt{2}\right)c^{3/2}$	7. $e^{\pi/2} - 1$	8. $\frac{1}{3}(1+\log 2)^3 - \frac{1}{3}$

9. π

10. $\frac{1}{20}\log_e 3$

11. 0

12. $\frac{3\sqrt{2}-4}{10}$

13. $\left(e^2/2\right)-e$

14. 2 / 3

15. $\log\left(e/2\right)$

16. $\frac{1}{2\sqrt{5}}\tan^{-1}\frac{2}{\sqrt{5}}$

17. $\pi/4$

18. $\frac{\pi-2}{2}$

19. 1

20. $\frac{\pi}{2\left(a+b\right)}$

21. $\log\left(4/e\right)$

22. $\frac{\pi}{2}-\frac{\pi}{4\sqrt{2}}-\frac{1}{\sqrt{2}}$

23. $\log\left(9/8\right)$

24. $3\pi/2$

25. $1-\pi/4$

26. $\log\left(9/8\right)$

Exercise 10.3

1. 25 / 2
2. 4
3. 62
4. 3
5. 0
6. 0
7. p / 2
8. p / 2
9. 0
10. 0
11. 0
12. p / 12
13. p / 4
14. $\frac{\pi}{2}\log\frac{1}{2}$
15. $\frac{\pi^2}{6\sqrt{3}}$
16. $\pi\log\frac{1}{2}$
17. $\frac{\pi}{4}-\frac{1}{2}$
18. π
19. $\frac{2\pi}{3}$
20. $\pi\log 2$
21. 1
22. $\frac{b-a}{2}$

Miscellaneous Exercise 10

1. (B)
2. (C)
3. (B)
4. (A)
5. $\frac{1}{2}\log 6$
6. $\frac{e}{6}\left(2e-3\right)$
7. $e^{\pi/2}$
8. 4
9. $\frac{\pi}{48}\left(\pi^2-6\right)$
10. $\frac{\pi}{4}-\frac{1}{2}\log 2$
11. $\frac{3\sqrt{2}}{10}$
12. 4
13. π^2
14. $\frac{\pi}{4}-\frac{1}{2}\log 2$
15. $\pi - 2$
16. $\frac{\pi}{a^2-1}, a>1$

11

Application of Integrals : Quadrature

11.01 Introduction

Quadrature means the process of finding out the area bounded by a given curve.

11.02 Area under a curve

Theorem : The area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and x-axis is expressed by definite integral $\int_a^b f(x)\,dx = \int_a^b y\,dx$

Proof : Let the equation of curve PQ be $y = f(x)$ where $f(x)$ is single valued real and continuous function of x in domain $[a, b]$. According to figure, we need to find the area of figure $PRSQP$.

Let $E(x, y)$ is any point on curve and $F(x+\delta x,\ y+\delta y)$ is a point in the neighbourhood. EA and FB are ordinates of E and F respectively.

Draw a perpendicular EC from E to FB and a perpendicualr FD from F to extended AE

$$AB = OB - OA = (x+\delta x) - x = \delta x$$

$$FC = FB - CB = (y+\delta y) - y = \delta y$$

Let area $RAEPR = A$

Now if the increment in x is δx and the increment in corresponding area is δA, then

$$\delta A = \text{area } ABFEA$$

$\therefore$ From figure, (area of rectangle $ABCE$) < area ($ABFEA$) < (area of rectangle $ABFD$)

$\Rightarrow$ $y\delta x < \delta A < (y+\delta y).\delta x$

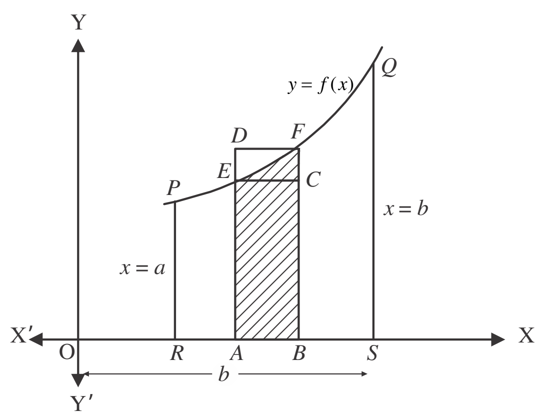


Fig. 11.01

$\Rightarrow$ $y < \dfrac{\delta A}{\delta x} < y+\delta y$

When, $F \to E$ then $\delta x \to o$ and $y+\delta y \to y$

$\Rightarrow$ $\lim\limits_{\delta x\to 0} y \le \lim\limits_{\delta x\to 0} \dfrac{\delta A}{\delta x} \le \lim\limits_{\delta x\to 0} (y+\delta y)$

$\Rightarrow$ $y \le \dfrac{dA}{dx} \le y$

$\Rightarrow$ $\dfrac{dA}{dx} = y \Rightarrow dA = ydx \Rightarrow dA = f(x)\,dx$

Integrating both the sides with respect to x and within the limits $x = a$ and $x = b$.

$$\int_a^b dA = \int_a^b f(x)dx$$

or
$$[A]_a^b = \int_a^b f(x)\,dx$$

or (area A when $x = b$) – (area A when $x = a$) $= \int_a^b f(x)\,dx$

or area $PRSQP - 0 = \int_a^b f(x)dx$

or area $PRSQP = \int_a^b f(x)dx$ or $\int_a^b y\,dx$

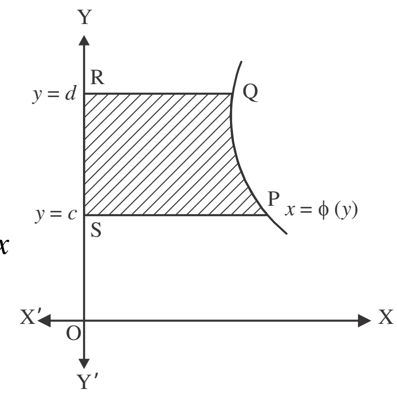


Fig. 11.02

The area of curve $y = f(x)$, under ordinates $x = a$ and $x = b$

and x-axis is $= \int_a^b f(x)dx$ or $\int_a^b y\,dx$

Similarly, the area between curve $x = \phi(y)$, y-axis and the abscissa $y = c$, $y = d$ is given by

$$= \int_c^d \phi(y)dy \text{ or } = \int_c^d x\,dy$$

Remark : To find out the area of figure, a rough sketch should be made so that it is easy to determine the limits of curve and symmetry of curve with respect to axes.

11.03 Symmetrical Area

If the curve is symmetrical with respect to any axis or any straight line, then find the area of one symmetrical part and then by multiplying with number of symmetrical parts in order to get area.

For example : Find the area enclosed by circle $x^2 + y^2 = a^2$

Solution : Clearly the centre of circle is (0, 0) and radius is a and it is also symmetrical about both the axes.

Total area of circle = 4 × [area of OABO in first quandrant]

= 4 × [Area bounded by circle $y = \sqrt{a^2 - x^2}$, x-axis $x = 0$ and $x = a$]

$$= 4\int_a^b y\,dx = 4\int_a^b \sqrt{a^2 - x^2}dx$$

$$= 4\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4\left[\left(o + \frac{a^2}{2}.\frac{\pi}{2}\right) - (o + o)\right] = \pi a^2$$

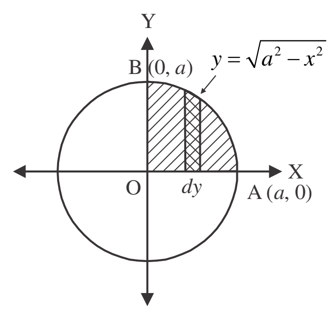


Fig. 11.03

11.04 Area of a curve around x-axis

Area is always considered as positive. It may happen that some is below the x-axis (which will be negative). Therefore the total area can be calcualted by adding up the numerical values of both the areas.

For Example : Find the area enclosed by the curve $y = \cos x$ and x-axis when $o \le x \le \pi$.

Solution : It is clear from the graph that the required area's portion is above x-axis and some portion is below x-axis.

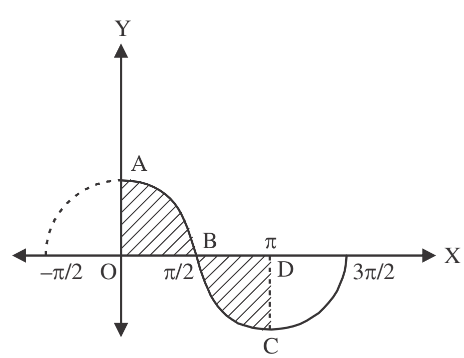


Fig. 11.04

So required area $= \int_0^{\pi/2} \cos x\, dx + \left|\int_{\pi/2}^{\pi} \cos x\, dx\right|$

$$= [\sin x]_o^{\pi/2} + \left|[\sin x]_{\pi/2}^{\pi}\right| = (1-0) + |0-1|$$

$$= 1 + 1 = 2 \text{ sq. units}$$

Illustrative Examples

Example 1. Find the area bounded by the parabola $y^2 = 4x$ and line $x = 3$.

Solution : On tracing the given parabola and line

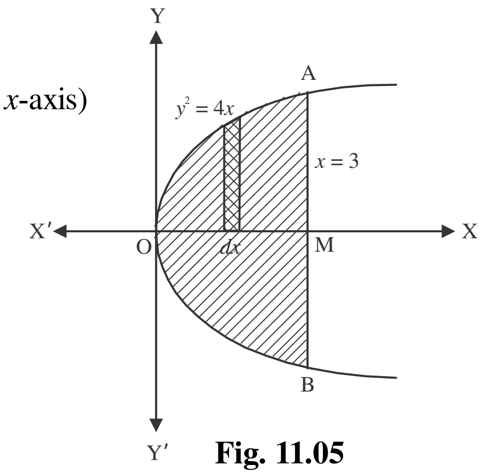


Fig. 11.05

Required area = area AOBMA

= 2 × area AOMA (∵ Parabola is symmetrical about x-axis)

$$= 2\int_0^3 y\, dx$$

$$= 2\int_0^3 \sqrt{4x}\, dx = 2 \times 2\int_0^3 \sqrt{x}\, dx$$

$$= 4 \times \left[\frac{2}{3} x^{3/2}\right]_0^3 = \frac{8}{3}\left[3^{3/2} - 0\right]$$

$$= \frac{8}{3} \times 3\sqrt{3} = 8\sqrt{3} \text{ Sq. units.}$$

Example 2. Find the area enclosed above x-axis by curve $y = 2\sqrt{1-x^2}$ and x-axis.

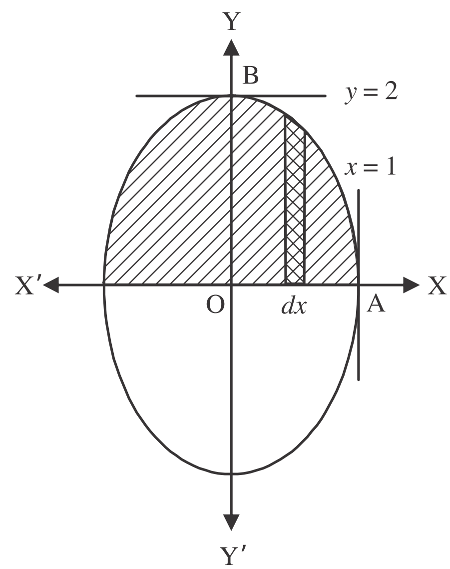


Fig. 11.06

Solution : On simplifying $y = 2\sqrt{1-x^2}$

$$y^2 = 4(1-x^2) \quad \text{or} \quad \frac{x^2}{1} + \frac{y^2}{4} = 1 \qquad (1)$$

Clearly, curve $y = 2\sqrt{1-x^2}$ is upper part of ellipse (1) so according to figure, we have to find out the area of shaded region.

∴ required area = 2 × area OABO

$$= 2\int_0^1 y\, dx = 2\int_0^1 2\sqrt{1-x^2}\, dx$$

$$= 4\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right]_0^1$$

$$= 4\left[\left(0 + \frac{1}{2}.\frac{\pi}{2}\right) - (o + o)\right] = \pi \text{ sq. unit.}$$

Example 3. Find the area enclosed by $y^2 = 4ax$, x-axis, line $x = 2a$ and latus rectum.

Solution : We have know that the equation of latus rectum of parabola $y^2 = 4ax$ is $x = a$. This is presented by LSL' in figure and line PMQ is $x = 2a$.

So required area = area $SMPL$

$$= \int_a^{2a} y\, dx = \int_a^{2a} \sqrt{4ax}\, dx$$

$$= 2\sqrt{a}\int_a^{2a} \sqrt{x}\, dx = 2\sqrt{a}\left[\frac{2}{3}x^{3/2}\right]_a^{2a}$$

$$= 2\sqrt{a}\left[\frac{2}{3}\times(2a)^{3/2} - \frac{2}{3}a^{3/2}\right]$$

$$= 2\sqrt{a}\left[\frac{4\sqrt{2}}{3}a\sqrt{a} - \frac{2}{3}a\sqrt{a}\right]$$

$$= \frac{4a^2}{3}\left[2\sqrt{2} - 1\right].$$

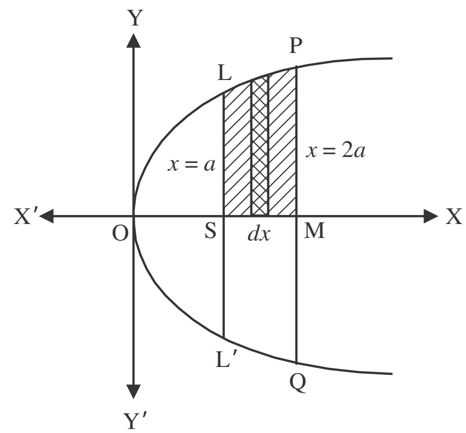


Fig. 11.07

Example 4. Find the area enclosed by parabola $y = 4x^2$ and lines, $y = 1$ and $y = 4$.

Solution : Parabola $y = 4x^2$ so $x^2 = \frac{1}{4}y$ and lines $y = 1$ and $y = 4$ will be traced as followed.

So, required area = area $PQRSP$

$= 2 \times$ area $RQLM$

$$= 2\int_1^4 x\, dy$$

$$= 2\int_1^4 \frac{1}{2}\sqrt{y}\, dy = \int_1^4 \sqrt{y}\, dy$$

$$= \frac{2}{3}\left[(y)^{3/2}\right]_1^4 = \frac{2}{3}\left[4^{3/2} - 1^{3/2}\right]$$

$$= \frac{2}{3}[8-1] = \frac{14}{3} \text{ sq. units.}$$

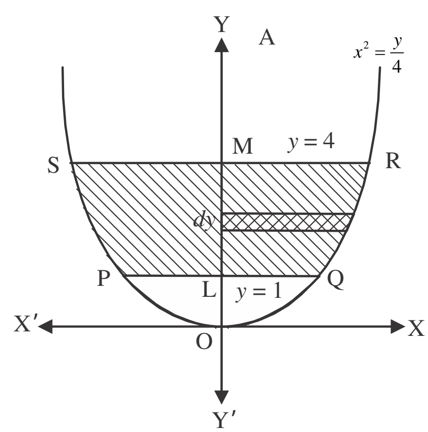


Fig. 11.08

Example 5. Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the ordinates $x = 0$ and $x = ae$, where $b^2 = a^2(1-e^2), e < 1$.

Solution : The required area of the region $BPSQB'OB$ is enclosed by the ellipse and the lines $x = o$ and $x = ae$

The area is symmetrical about x-axis. So

$$\text{required area } BPSQB'OB = 2\int_o^{ae} y\, dx$$

So, by the equation of ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{y^2}{b^2}=1-\frac{x^2}{a^2} \text{ or } \frac{y^2}{b^2}=\frac{a^2-x^2}{a^2}$$

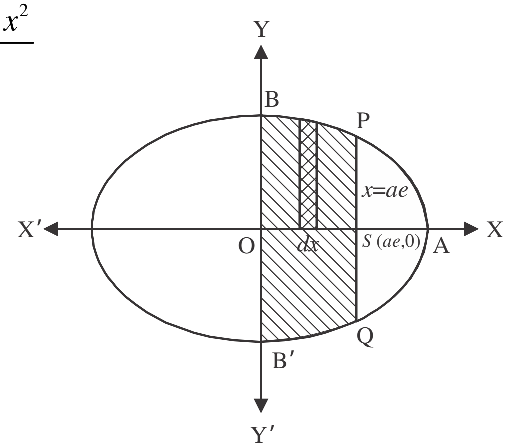


Fig. 11.09

or $$y^2=\frac{b^2}{a^2}\left(a^2-x^2\right) \text{ or } y=\frac{b}{a}\sqrt{a^2-x^2}$$

So, required area $$=2\int_o^{ae}\frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$=\frac{2b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_o^{ae}$$

$$=\frac{2b}{a}\left[\left(\frac{ae}{2}\sqrt{a^2-a^2e^2}+\frac{a^2}{2}\sin^{-1}\frac{ae}{a}\right)-(o+o)\right]$$

$$=\frac{2b}{a}\left[\frac{ae}{2}.a\sqrt{1-e^2}+\frac{a^2}{2}\sin^{-1}(e)\right]$$

$$=\frac{2a^2b}{2a}\left[e\sqrt{1-e^2}+\sin^{-1}e\right]$$

$$=ab\left[e\sqrt{1-e^2}+\sin^{-1}e\right] \text{ sq. units.}$$

Example 6. Find the area of the region in the first quadrant enclosed by x-axis, line $x=\sqrt{2}y$ and the circle $x^2+y^2=9$.

Solution : The centre of circle $x^2+y^2=9$ is (o, o) and radius is 3 unit. Straight line $x=\sqrt{2}y$ passes through origin and cuts the circle at P. On solving the equations of circle and line.

$$x^2+\frac{x^2}{2}=9 \Rightarrow x^2=6 \Rightarrow x=\pm\sqrt{6} \text{ then } y=\pm\sqrt{3}$$

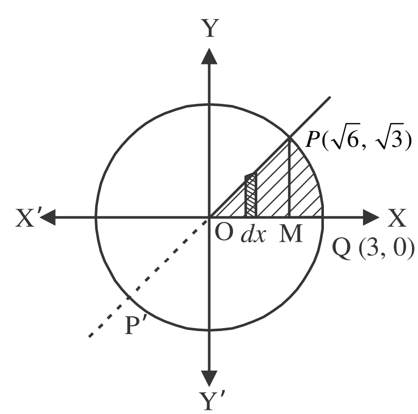
$\therefore$ Coordinates of P $(\sqrt{6},\sqrt{3})$, Q $(3, 0)$ and M $(\sqrt{6},0)$.

required area = area OMPO + area PMQP

$$=\int_{0(y\text{ from line})}^{\sqrt{6}} y\,dx+\int_{\sqrt{6}(y\text{ from circle})}^{3} y\,dx$$

$$=\int_0^{\sqrt{6}}\frac{x}{\sqrt{2}}\,dx+\int_{\sqrt{6}}^{3}\sqrt{9-x^2}\,dx$$

$$=\left[\frac{x^2}{2\sqrt{2}}\right]_0^{\sqrt{6}}+\left[\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\frac{x}{3}\right]_{\sqrt{6}}^{3}$$



Fi.g 11.10

$$=\left(\frac{3}{\sqrt{2}}+0\right)+\left[\left(0+\frac{9}{2}\sin^{-1}(1)\right)-\left(\frac{\sqrt{6}}{2}\sqrt{3}+\frac{9}{2}\sin^{-1}\frac{\sqrt{6}}{3}\right)\right]$$

$$=\frac{3}{\sqrt{2}}+\frac{9\pi}{4}-\frac{3}{\sqrt{2}}-\frac{9}{2}\sin^{-1}\frac{\sqrt{2}}{\sqrt{3}}=\frac{9}{4}\left(\pi-2\sin^{-1}\frac{\sqrt{2}}{\sqrt{3}}\right) \text{ sq. units.}$$

Example 7. Find the area of the smaller part of the circle $x^2+y^2=a^2$ cut by the line $x=\frac{a}{\sqrt{2}}$.

Solution : On solving the equations of circle and line,

$$\frac{a^2}{2}+y^2=a^2 \Rightarrow y^2=\frac{a^2}{2} \Rightarrow y=\frac{a}{\sqrt{2}}$$

$\therefore$ Coordinates of P $\left(a/\sqrt{2},\ a/\sqrt{2}\right)$

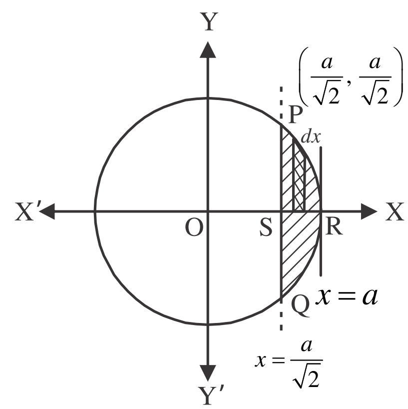


Fig. 11.11

required area = area $PSQRP$

= 2 × area PSRP

$$=2\int_{a/\sqrt{2}}^{a} y\,dx=2\int_{a/\sqrt{2}}^{a}\sqrt{a^2-x^2}\,dx$$

$$=2\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{a/\sqrt{2}}^{a}$$

$$=2\left[\left(\frac{a}{2}\sqrt{a^2-a^2}+\frac{a^2}{2}\sin^{-1}\frac{a}{a}\right)-\left(\frac{a}{2\sqrt{2}}\sqrt{a^2-\frac{a^2}{2}}+\frac{a^2}{2}\sin^{-1}\frac{a}{a\sqrt{2}}\right)\right]$$

$$=2\left[o+\frac{a^2}{2}.\frac{\pi}{2}-\frac{a^2}{4}-\frac{a^2}{2}.\frac{\pi}{4}\right]$$

$$=2\left(\frac{\pi a^2}{4}-\frac{a^2}{4}-\frac{\pi a^2}{8}\right)=2\left(\frac{\pi a^2}{8}-\frac{a^2}{4}\right)=\frac{\pi a^2}{4}-\frac{a^2}{2}$$

$$=\frac{a^2}{4}(\pi-2) \text{ sq. units}$$

Example 8. Find the area of the smaller part of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, cut by line $y=c$, when $c<b$.

Solution : According to figure the area bounded between ellipse and line is shaded.

required area = area $BQPRB$

= 2 × area $BQPRB$

$$= 2\int_c^b x\,dy$$

$$= 2\int_c^b \frac{a}{b}\sqrt{b^2 - y^2}\,dy$$

$$= 2\frac{a}{b}\left[\frac{y}{2}\sqrt{b^2 - y^2} + \frac{b^2}{2}\sin^{-1}\left(\frac{y}{b}\right)\right]_c^b$$

$$= \frac{2a}{b}\left[0 + \frac{b^2}{2}\sin^{-1}(1) - \frac{c}{2}\sqrt{a^2 - c^2} - \frac{b^2}{2}\sin^{-1}\left(\frac{c}{b}\right)\right] \text{ sq. units}$$

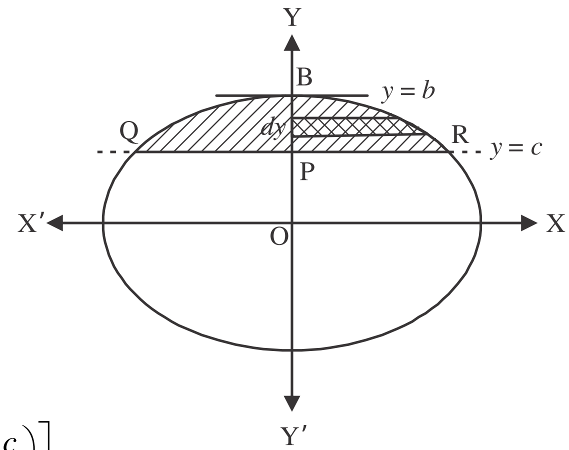


Fig. 11.12

Example 9. Find the area bounded by line $2x + y = 4$, x-axis and ordinates $x = 0$ and $x = 3$.

Solution : According to figure, line $2x + y = 4$, meets x-axis at $x = 2$ and y-axis at $y = 4$. When x is from 0 to 2, then the graph is above x-axis and when x is from 2 and 3, then graph is below x-axis.

So, required area = area OABO + area ALMA

$$= \int_0^2 y\,dx + \left|\int_2^3 y\,dx\right|$$

$$= \int_0^2 (4 - 2x)\,dx + \left|\int_2^3 (4 - 2x)\,dx\right|$$

$$= \left[4x - x^2\right]_0^2 + \left|\left[4x - x^2\right]_2^3\right|$$

$$= \left[(8-4) - (0-0)\right] + \left|(12-9) - (8-4)\right|$$

$$= 4 + |3 - 4| = 4 + 1 = 5$$

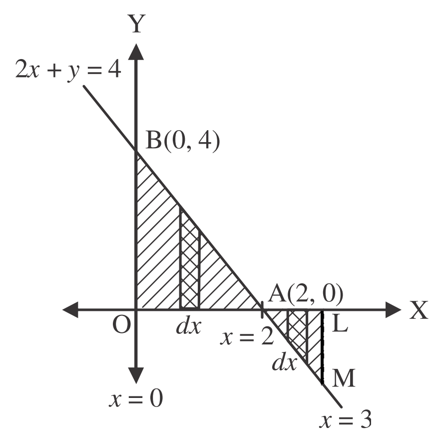


Fig. 11.13

Exercise 11.1

1. Find the area enclosed by parabola $y^2 = 4ax$ and its latus rectum.
2. Find the area bounded by circle $x^2 + y^2 = 4$, y-axis and $x = 1$.
3. Find the area enclosed by $y = \sin x$ and x-axis, when $0 \le x \le 2\pi$.
4. Find the area enclosed by $y = 2\sqrt{x}$ and between $x = 0, x = 1$.
5. Find the area enclosed by $y = |x|, x = -3, x = 1$ and x-axis.
6. Find the area enclosed by $x^2 = 4ay$, x-axis and line $x = 2$.
7. Find the area enclosed by ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ above x –axis.
8. Find the total area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
9. Find the area enclosed by line $\frac{x}{a} - \frac{y}{b} = 2$ and both axes.

10. Find the area bounded by lines $x + 2y = 8$, $x = 2$, $x = 4$ and x-axis.

11. Find the area bounded by $y = x^2$, x-axis and ordinates $x = 1$, $x = 2$.

12. Find the area bounded by $y = 4x^2$ (in first quadrant), $x = 0$, $y = 1$ and $y = 4$.

11.05 Area between two Curves

Theorem : The area between two curves $y = f(x)$ and $y = g(x)$ and between the ordinates $x = a$ and $x = b$ is $= \int_a^b \left[f(x) - g(x) \right] dx$

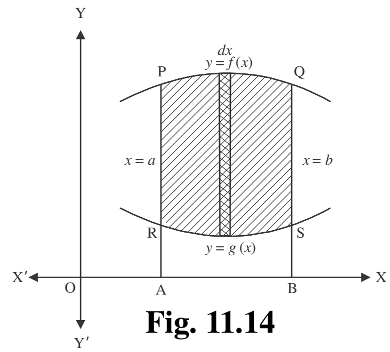


Fig. 11.14

Proof : In the figure the shaded region represents the area between the curves $y = f(x)$ and $y = g(x)$ and two lines $x = a$ and $x = b$.

The area of this region = area $PQBAP$ – area $RSBAR$

$$= \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$$

$$= \int_a^b \left[f(x) - g(x) \right] dx$$

or $$\int_{a\,(y=f(x))}^{b} y\, dx - \int_{a\,(y=g(x))}^{b} y\, dx$$

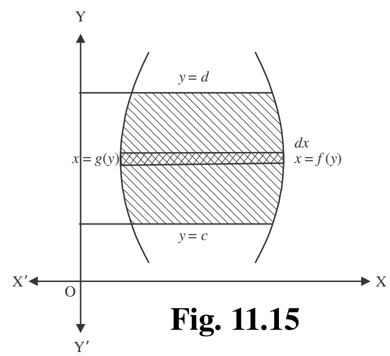


Fig. 11.15

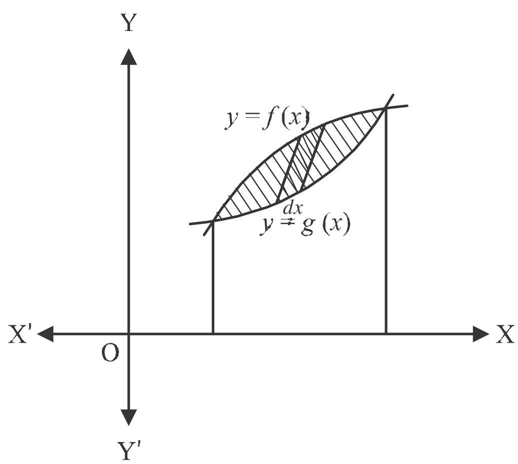
Remark : The area between two curves $x = f(y)$ and $x = g(y)$ and lines $y = c$ and $y = d$ is

$$= \int_a^b \left[f(y) - g(y) \right] dy$$

Special Cases :

Case-I : If two curves intersect each other at two points then area fo common region is

$$= \int_a^b \left[f(x) - g(x) \right] dx$$



Fig, 11.16

Case-II: If two curves intersect at one point and the area between them is bounded by x–axis then,

required area $= \int_a^c f(x)\, dx + \int_c^b g(x)\, dx$

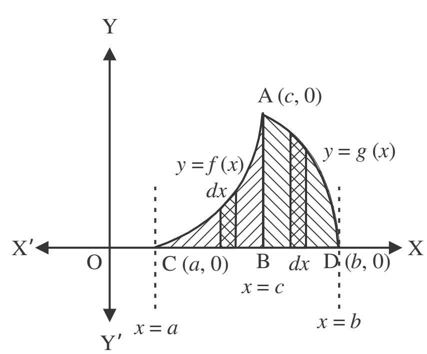


Fig. 11.17

[where both curves intersect each other at $A(C, O)$]

Case-III: If two curves, intersect each other at more than two points.

In the interval $[a, b]$, two curves $y = f(x)$ and $y = g(x)$, intersect each other at A, B and C. Clearly in $[a, c]$ $f(x) \geq g(x)$ and $g(x) \geq f(x)$ in $[c, d]$.

required area = area $APBQA$ + area $BECDB$

$$= \int_a^c \left[f(x) - g(x)\right]dx + \int_c^b \left[g(x) - f(x)\right]dx$$

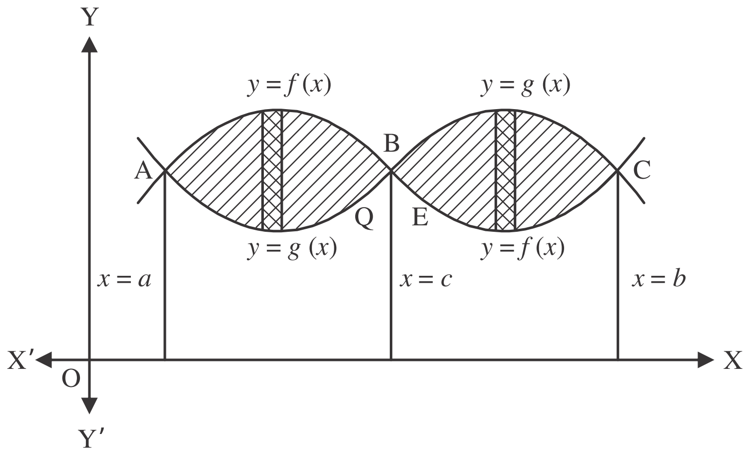


Fig. 11.18

Illustrative Examples

Example 10. Find the area bounded by parabola $y^2 = 4ax$ and line $y = x$ in first qudrant.

Solution : On solving the equations of parabola and line

$$y^2 = 4ax \quad \text{or} \quad x(x - 4a) = 0 \Rightarrow x = 0, 4a \quad \therefore y = 0, 4a$$

So, the line cuts the parabola at $0\ (0, 0)$ and $A(4a, 4a)$ so the area between parabola and lines is

$$= \int_{0(y \text{ from parabola})}^{4a} y\, dx - \int_{0(y \text{ from line})}^{4a} y\, dx$$

$$= \int_0^{4a} \sqrt{4ax}\, dx - \int_0^{4a} x\, dx = 2\sqrt{a}\int_0^{4a} \sqrt{x}\, dx - \int_0^{4a} x\, dx$$

$$= 2\sqrt{a} \times \frac{2}{3}\left[x^{3/2}\right]_0^{4a} - \left[\frac{x^2}{2}\right]_0^{4a}$$

$$= \frac{4\sqrt{a}}{3}\left[(4a)^{3/2} - 0\right] - \left[\frac{(4a)^2}{2} - 0\right]$$

$$= \frac{32a^2}{3} - 8a^2 = \frac{8a^2}{3} \text{ Sq. units.}$$

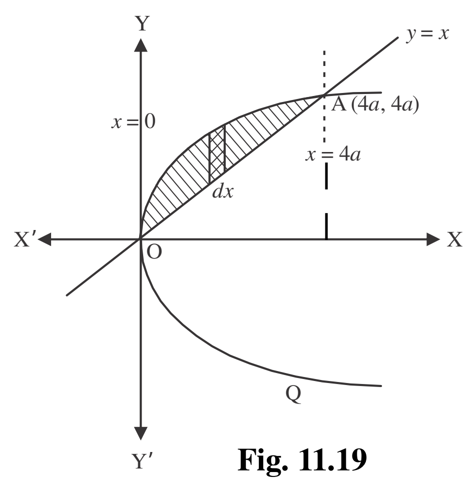


Fig. 11.19

Example 11. Find the area bounded by circle $x^2+y^2=a^2$ and curve $y=|x|$.

Solution : The lines represented by curve $y=|x|$ are $y=x$ and $y=-x$. They intersect the circle at points A and B whose coordinates are $(a/\sqrt{2}, a/\sqrt{2})$ and $(-a/\sqrt{2}, a/\sqrt{2})$. Required area is shaded in Fig.

$\therefore$ required area = area $AOBCA$

= 2 × area $AOCA$

$$=2\int_0^{a/\sqrt{2}}(\sqrt{a^2-x^2}-x)\,dx$$

where $f(x)$ in taken from circle and $g(x)$ is taken from line $y=x$

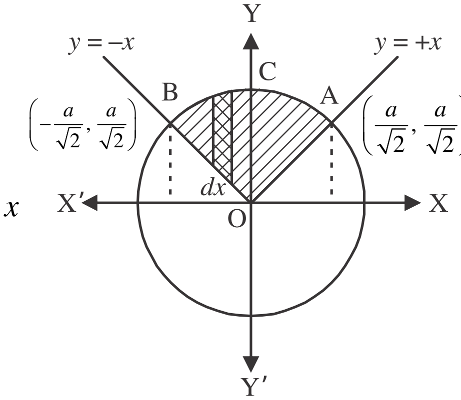


Fig. 11.20

$$=2\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}-\frac{x^2}{2}\right]_0^{a/\sqrt{2}}$$

$$=2\left[\frac{a}{2\sqrt{2}}\sqrt{a^2-\frac{a^2}{2}}+\frac{a^2}{2}\sin^{-1}\frac{a}{a\sqrt{2}}-\frac{a^2}{2\times 2}\right]-2[0+0-0]$$

$$=2\left[\frac{a}{2\sqrt{2}}\times\frac{a}{\sqrt{2}}+\frac{a^2}{2}\times\frac{\pi}{4}-\frac{a^2}{4}\right]=2\left[\frac{a^2}{4}+\frac{\pi a^2}{8}-\frac{a^2}{4}\right]$$

$$=\frac{\pi a^2}{4}\text{ sq. units.}$$

Example 12. Find the area between the parabolas $y^2=4ax$ and $x^2=4by$.

Solution : The equations of given paraboals are

$$y^2=4ax \quad\text{and}\quad x^2=4by$$

On solving both the equations

$$\left(x^2/4b\right)^2=4ax \quad\text{or}\quad x^4=64ab^2x$$

or $\quad x(x^3-64ab^2)=0 \Rightarrow x=0,\ 4(ab^2)^{1/3}$

So both the curves will intersect x-axis at $x=0$ and $x=4(ab^2)^{1/3}$

On tracing the curves we get the fig. 11.21

Hence the area between the curves is $OCABO$

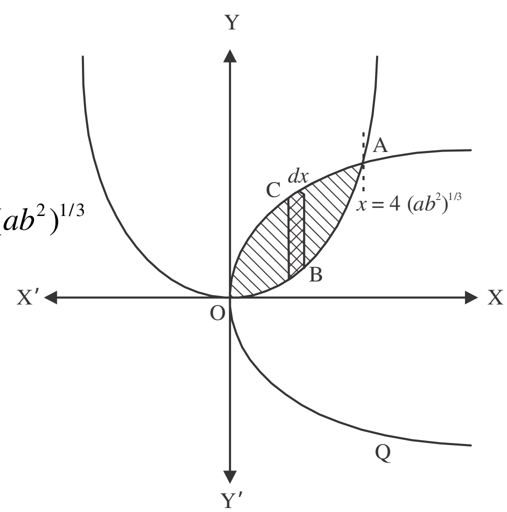


Fig. 11.21

$$=\int_{y\,(\text{From};\ y^2=4ax)}^{4(ab^2)^{1/3}} y\,dx-\int_{y\,(\text{From};\ x^2=4by)}^{4(ab^2)^{1/3}} y\,dx$$

$$=\int_0^{4(ab^2)^{1/3}}\sqrt{4ax}\,dx-\int_0^{4(ab^2)^{1/3}}\frac{x^2}{4b}\,dx$$

$$=2\sqrt{a}.\frac{2}{3}[x^{3/2}]_0^{4(ab^2)^{1/3}}-\frac{1}{4b}\left[\frac{x^3}{3}\right]_0^{4(ab^2)^{1/3}}$$

$$= \frac{4}{3}\sqrt{a}\left[\left[4(ab^2)^{1/3}\right]^{3/2} - 0\right] - \frac{1}{12b}\left[\left\{4(ab^2)^{1/3}\right\}^3 - 0\right]$$

$$= \frac{4}{3}\sqrt{a}\left[8(ab^2)^{1/2}\right] - \frac{1}{12b}[64\,ab^2]$$

$$= \frac{32\sqrt{a}}{3}\sqrt{a}\,b - \frac{1}{12b} \times 64\,ab^2$$

$$= \frac{32}{3}ab - \frac{16ab}{3} = \frac{16ab}{3} \text{ Sq. units.}$$

Example 13. Find the area of smaller region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and line $\frac{x}{a} + \frac{y}{b} = 1$.

Solution : As per diagram, the smaller region between the ellipse and line is represented by shaded region. Clearly the line cuts the ellipse at $A(a, 0)$ and $B(0, \text{b})$. So required area $ACBDA$

$$= \int_{0\,(y \text{ from ellipse})}^{a} y\,dx - \int_{0\,(y \text{ from line})}^{a} y\,dx$$

$$= \int_{o}^{a} \frac{b}{a}\sqrt{a^2 - x^2}\,dx - \int_{o}^{a} \frac{b}{a}(a - x)\,dx$$

$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{o}^{a} - \frac{b}{a}\left(ax - \frac{x^2}{2}\right)_{o}^{a}$$

$$= \frac{b}{a}\left[\left(o + \frac{a^2}{2} \times \frac{\pi}{2}\right) - (o + o)\right] - \frac{b}{a}\left[\left(a^2 - \frac{a^2}{2}\right) - (o - o)\right]$$

$$= \frac{\pi ab}{4} - \frac{ab}{2} = \frac{ab}{4}(\pi - 2) \text{ sq. units.}$$

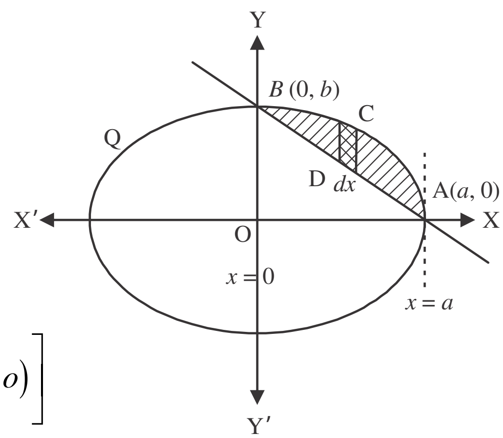


Fig. 11.22

Example 14. Find the area between the parabola $x^2 = 4y$ and line $x = 4y - 2$.

Solution : On solving the equations of parabola and straight line

$$x = x^2 - 2 \text{ or } x - x^2 - 2 = 0 \Rightarrow (x - 2)(x + 1) = 0 \Rightarrow x = 2, -1$$

Clearly, the line cuts the parabola at $x = 2$ and $x = -1$.

So, required area $ABOA$ $= \int_{-1\,(y,\text{ from line})}^{2} y\,dx - \int_{-1\,(y,\text{ from parabola})}^{2} y\,dx$

$$= \int_{-1}^{2} \frac{x + 2}{4}\,dx - \int_{-1}^{2} \frac{x^2}{4}\,dx$$

$$= \frac{1}{4}\left(\frac{x^2}{2} + 2x\right)_{-1}^{2} - \left[\frac{x^3}{12}\right]_{-1}^{2}$$

$$= \frac{1}{4}\left[\left(\frac{4}{2} + 4\right) - \left(\frac{1}{2} - 2\right)\right] - \left[\frac{8}{12} - \left(\frac{-1}{12}\right)\right]$$

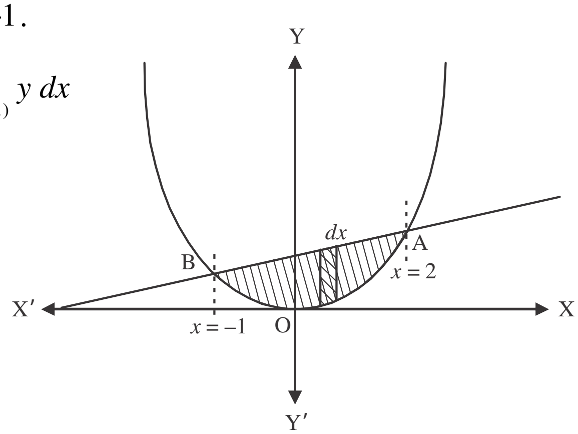


Fig. 11.23

$$= \frac{1}{4}\left[6+\frac{3}{2}\right]-\frac{9}{12}=\frac{1}{4}\times\frac{15}{2}-\frac{9}{12}=\frac{15}{8}-\frac{3}{4}=\frac{9}{8} \text{ sq. units.}$$

Example 15. Find the area of smaller region between $x^2+y^2=2$ and $x=y^2$.

Solution : The area of smaller region between circle $x^2+y^2=2$ and parabola $x=y^2$ is presented by shaded region, to find out the points of intersection. On solving the equation.

$$x^2+x=2 \Rightarrow x^2+x-2=0$$

$$\Rightarrow (x+2)(x-1)=0$$

$$\Rightarrow x=-2,1 \text{ when } x=1 \text{ then } y=\pm 1$$

So both the curves intersect each other at $A(1, 1)$ and $B(1, -1)$.

So required area = area $AOBCO$ = 2 × area $AODCA$

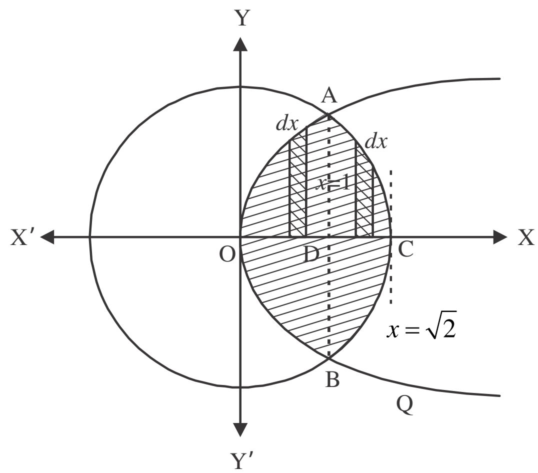


Fig. 11.24

$$= 2[AODA + ADCA]$$

$$= 2\left[\int_{0(y \text{ from parabola})}^{1} y\,dx+\int_{1\,(y \text{ from circle})}^{\sqrt{2}} y\,dx\right]$$

$$= 2\left[\int_0^1 \sqrt{x}\,dx+\int_1^{\sqrt{2}}\sqrt{2-x^2}\,dx\right]$$

$$= 2\left[\frac{2}{3}\left\{x^{3/2}\right\}_0^1+\left\{\frac{x}{2}\sqrt{2-x^2}+\frac{2}{2}\sin^{-1}\frac{x}{\sqrt{2}}\right\}_1^{\sqrt{2}}\right]$$

$$= 2\left[\frac{2}{3}\times(1-0)+\left(0+\sin^{-1}1\right)-\left(\frac{1}{2}+\sin^{-1}\frac{1}{\sqrt{2}}\right)\right]$$

$$= 2\left[\frac{2}{3}+\frac{\pi}{2}-\frac{1}{2}-\frac{\pi}{4}\right]=2\left[\frac{1}{6}+\frac{\pi}{4}\right]=\left[\frac{1}{3}+\frac{\pi}{2}\right] \text{ sq. units}$$

Example 16. Using integration, find the area of region bounded by the triangle whose vertices are (–1, 1), (0, 5) and (3, 2).

Soluton : Let $A(-1, 1)$, $B(0, 5)$ and $C(3, 2)$ are vertices of triangle.

Equation of line AB

$$y-1=\frac{5-1}{0+1}(x+1)$$

or $$y-1=4x+4$$

or $$4x-y+5=0 \qquad (1)$$

equation of line BC

$$y-5=\frac{2-5}{3-0}(x-0)$$

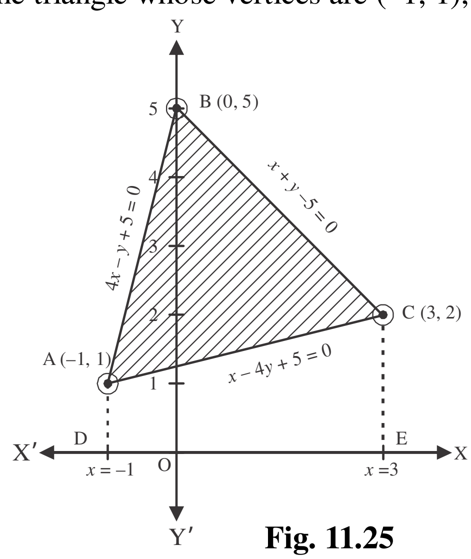


Fig. 11.25

or $\quad 3y-15=-3x$

or $\quad x+y-5=0 \qquad (2)$

equation of line CA

$$y-1=\frac{2-1}{3+1}(x+1)$$

or $\quad 4y-4=x+1$

or $\quad x-4y+5=0 \qquad (3)$

So, area of ΔABC = area of trapezium $ABOD$ + area of trapezium $BOEC$ – area of trapezium $ACED$

$$=\int_{-1(\text{from line AB})}^{0} y\,dx+\int_{0(\text{from line BC})}^{3} y\,dx-\int_{-1(\text{from line CA})}^{3} y\,dx$$

$$=\int_{-1}^{0}(4x+5)\,dx+\int_{0}^{3}(5-x)\,dx-\int_{-1}^{3}\frac{x+5}{4}dx$$

$$=\left[2x^2+5x\right]_{-1}^{0}+\left[5x-\frac{x^2}{2}\right]_{0}^{3}-\frac{1}{4}\left[\frac{x^2}{2}+5x\right]_{-1}^{3}$$

$$=\left[(0+0)-(2-5)\right]+\left[(15-9/2)-(0-0)\right]-\frac{1}{4}\left[(9/2+15)-(1/2-5)\right]$$

$$=[3]+[21/2]-\frac{1}{4}(39/2+9/2)$$

$$=3+\frac{21}{2}-6=\frac{21}{2}-3=\frac{15}{2} \text{ sq. units.}$$

Exercise 11.3

1. Find the area between parabola $y^2=2x$ and circle $x^2+y^2=8$.
2. Find the area between parabola $4y=3x^2$ and line $3x-2y+12=0$.
3. Find the area between curves $y=\sqrt{4-x^2}$, $x=\sqrt{3}y$ and x-axis.
4. Find the the area between circle $x^2+y^2=16$ and line $y=x$ in first quadrant.
5. Find the common area between parabolas $y^2=4x$ and $x^2=4y$.
6. Find the area between $x^2+y^2=1$ and $x+y=1$ in first quadrant.
7. Find the area between $y^2=4ax$, line $y=2a$ and y-axs.
8. Find the area of circle $x^2+y^2=16$ whcih is exterior to the parabola $y^2=6x$.
9. Using integration, find the area of region bounded by triangle whose vertices are $A(2, 0)$, $B(4, 5)$ and $C(6, 3)$.
10. Using integration, find the area of triangular region whose sides have the equations $3x-2y+3=0$, $x+2y-7=0$ and $x-2y+1=0$.

Miscellaneous Examples

Example 17. Find the area in first quadrant bounded by curves $x^2+y^2=\pi^2$ and $y=\sin x$.

Solution : The area bounded by $x^2+y^2=\pi^2$ and $y=\sin x$ in first qudrant is shaded in figure.

Required area = $OCABO$

$$=\int_{0(y\text{ from circle})}^{\pi} y\,dx-\int_{0(y\text{ from } y=\sin x)}^{\pi} y\,dx$$

$$=\int_0^{\pi}\sqrt{\pi^2-x^2}\,dx-\int_0^{\pi}\sin x\,dx$$

$$=\left[\frac{x}{2}\sqrt{\pi^2-x^2}+\frac{\pi^2}{2}\sin^{-1}\frac{x}{\pi}\right]_0^{\pi}-\left[-\cos x\right]_0^{\pi}$$

$$=\left[\left\{0+\frac{\pi^2}{2}\sin^{-1}(1)\right\}-\{0+0\}\right]+\left[\cos\pi-\cos 0\right]$$

$$=\frac{\pi^2}{2}\times\frac{\pi}{2}+(-1)-1=\frac{\pi^3}{4}-2=\frac{\pi^3-8}{4}\text{ sq. units}$$

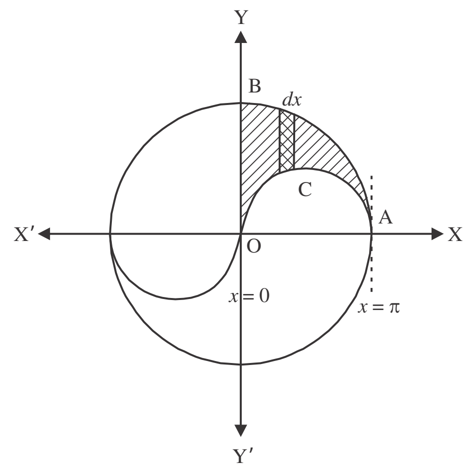


Fig. 11.26

Example 18. Find the area between the circles $x^2+y^2=1$ and $(x-1)^2+y^2=1$.

Solution : Given circles are

$$x^2+y^2=1 \qquad (1)$$

$$(x-1)^2+y^2=1 \qquad (2)$$

Centres of circles (1) and (2) are (0, 0) and (1, 0) respectively and the radii of both circles are 1. On solving the equations of circles (1) and (2).

$$x^2-(x-1)^2=0$$

or $\quad x^2-x^2+2x-1=0$

$\Rightarrow \quad x=1/2 \quad \Rightarrow \quad y=\pm\sqrt{3}/2$

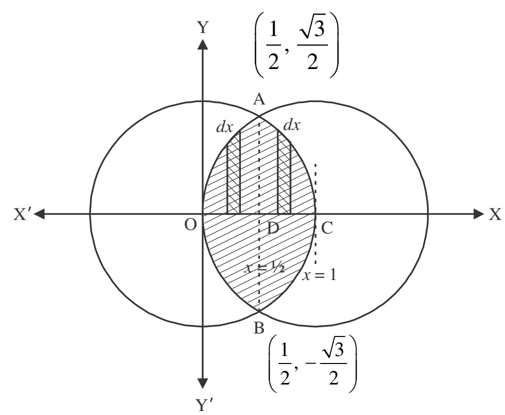


Fig. 11.27

$\therefore$ Coordinates of $A=(1/2,\ \sqrt{3}/2)$ and coordinates of B $(1/2,\ -\sqrt{3}/2)$

where A and B are point of intersection of both the circles

So required area = area $OACBO$

= 2 × area $OACDO$

= 2 [area $OADO$ + area $ADCA$]

$$=2\left[\int_{0\,(y,\text{from circle }(2))}^{1/2} y\,dx+\int_{1/2\,(y,\text{ from circle }(1))}^{1} y\,dx\right]$$

$$=2\left[\int_0^{1/2}\sqrt{1-(x-1)^2}\,dx+\int_{1/2}^{1}\sqrt{1-x^2}\,dx\right]$$

$$= 2\left[\frac{x-1}{2}\sqrt{1-(x-1)^2}+\frac{1}{2}\sin^{-1}(x-1)\right]_0^{1/2}+2\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x\right]_{1/2}^{1}$$

$$= 2\left[\left\{-\frac{1}{4}\sqrt{1-\frac{1}{4}}+\frac{1}{2}\sin^{-1}\left(\frac{-1}{2}\right)\right\}-\left\{\frac{-1}{2}\times 0+\frac{1}{2}\sin^{-1}(-1)\right\}\right]$$

$$+2\left[\left\{0+\frac{1}{2}\sin^{-1}(1)\right\}-\left\{\frac{1}{4}\sqrt{1-\frac{1}{4}}+\frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right)\right\}\right]$$

$$= 2\left[-\frac{\sqrt{3}}{8}+\frac{1}{2}\times\left(-\frac{\pi}{6}\right)-\frac{1}{2}\times\left(-\frac{\pi}{2}\right)\right]+2\left[\frac{1}{2}\times\frac{\pi}{2}-\frac{\sqrt{3}}{8}-\frac{1}{2}\times\left(\frac{\pi}{6}\right)\right]$$

$$= -\frac{\sqrt{3}}{4}-\frac{\pi}{6}+\frac{\pi}{2}+\frac{\pi}{2}-\frac{\sqrt{3}}{4}-\frac{\pi}{6}=\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right) \text{ sq. units}$$

Example 19. Find the area between the curves $y = \sin x$, $y = \cos x$, y-axis and $o \le x \le \pi/2$.

Solution : On solving $y = \sin x$ and $y = \cos x$, $\sin x = \cos x \Rightarrow \tan x = 1$

$\Rightarrow \qquad x = \pi/4$

Hence both intersect at $x = \pi/4$

So at B $x = \pi/4$ hence

required area = area of $AOBA$

$= \text{area } ABEO - \text{area } OBEO$

$$= \int_{0\,(y,\text{ From } y=\cos x)}^{\pi/4} y\cdot dx - \int_{0\,(y,\text{ from } y=\sin x)}^{\pi/4} y\cdot dx$$

$$= \int_0^{\pi/4}\cos x\,dx - \int_0^{\pi/4}\sin x\,dx$$

$$= [\sin x]_0^{\pi/4} - [\cos x]_0^{\pi/4}$$

$$= \sin\frac{\pi}{4} - 0 + \left(\cos\frac{\pi}{4} - \cos 0\right)$$

$$= \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-1=\frac{2}{\sqrt{2}}-1=(\sqrt{2}-1) \text{ sq. units.}$$

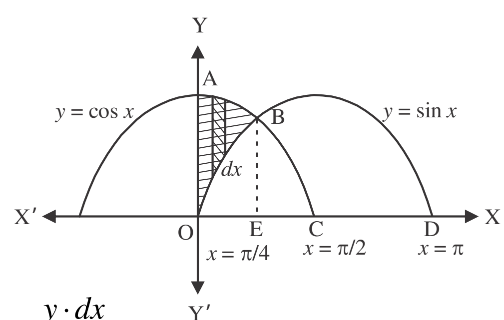


Fig. 11.28

Example 20. Find the area of region $\{(x, y) \mid x^2 \le y \le x\}$.

Solution : Given that:

$$y = x^2 \qquad (1)$$

and $$y = x \qquad (2)$$

Curve (1) is upward parabola and line $y = x$ passes through origin. The region between parabola and lines has been shaded. On solving equ. (1) and (2).

$$x^2 = x \Rightarrow x(x-1) = 0$$

$$\Rightarrow x = 0, 1$$

$\therefore$ $$y = 0, 1$$

Hence parabola and line intersect each other at (0, 0) and (1, 1)

$\therefore$ Required area = area *OCABO*

$$= \int_{0(y \text{ from line})}^{1} y\,dx - \int_{0(y \text{ from parabola})}^{1} y\,dx$$

$$= \int_0^1 x\,dx - \int_0^1 x^2\,dx$$

$$= \left[x^2/2\right]_0^1 - \left[x^3/3\right]_0^1$$

$$= (1/2 - 0) - (1/3 - 0) = 1/6 \text{ sq. units}$$

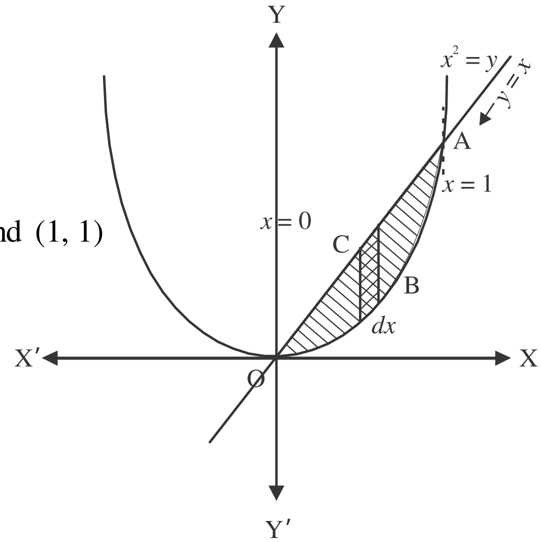


Fig. 11.29

Example 21. Find the area bounded by the $y = x^2 + 2$, lines $y = x, x = 0$ and $x = 3$.

Solution : Curve $y = x^2 + 2$ is a parabola whose vertex (0, 2) is on y–axis. $y = x$ is a line passes through origin. The required area bounded by curve $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$ is shaded in figure. In the figure the coordinates of point Q are (3, 11) which is a point of intersection of $x = 3$ and $y = x^2 + 2$.

Required area = area *OPQRO*

$$= \int_{0(y \text{ from parabola})}^{3} y\,dx - \int_{0(y \text{ from line})}^{3} y\,dx$$

$$= \int_0^3 (x^2 + 2)dx - \int_0^3 x\,dx$$

$$= \left[(x^3/3) + 2x\right]_0^3 - \left[x^2/2\right]_0^3$$

$$= \{(27/3 + 6)\} - (0 + 0) - [(9/2) - 0]$$

$$= 9 + 6 - (9/2) = 21/2 \text{ sq. units.}$$

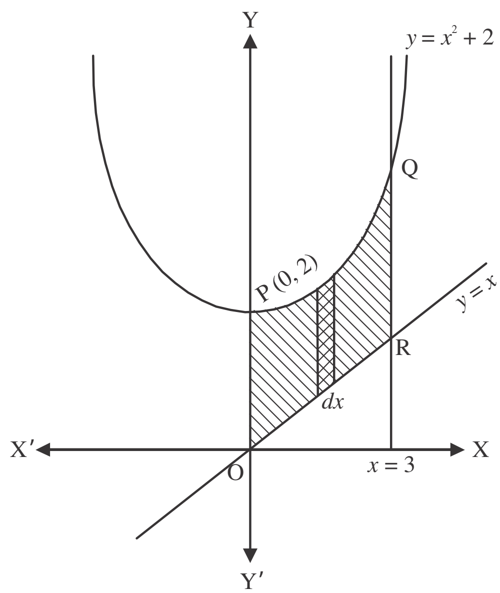


Fig. 11.30

Miscellaneous Exercise – 11

1. The area bounded by curve $y = \sqrt{x}$ and $y = x$ is (in sq. units)

   (a) 1 (b) 1 / 9 (c) 1 / 6 (d) 2 / 3

2. The area (in sq. units) bounded by curves $y^2 = x$ and $x^2 = y$ is

   (a) 1 / 3 (b) 1 (c) 1 / 2 (d) 2

3. The area (in sq. units) bounded by parabola $x^2 = 4y$ and its latus rectum is

(a) 5 / 3 (b) 2 / 3 (c) 4 / 3 (d) 8 / 3

4. The area (in sq. units) bounded by $y = \sin x, \frac{\pi}{2} \le x \le \frac{3\pi}{2}$ and x-axis is

(a) 1 (b) 2 (c) 1 / 2 (d) 4

5. The area (in sq. units) bounded by $y^2 = 2x$ and circle $x^2 + y^2 = 8$ is

(a) $(2\pi + 4/3)$ (b) $(\pi + 2/3)$ (c) $(4\pi + 4/3)$ (d) $(\pi + 4/3)$

6. Find the area between parabola $y^2 = x$ and line $x + y = 2$.

7. Find the area between $y^2 = 2ax - x^2$ and $y^2 = ax$ in first quadrant.

8. Find the area between parabola $y = x^2$ and $y = |x|$.

9. Find the common area between circle $x^2 + y^2 = 16$ and parabola $y^2 = 6x$.

10. Find the area bounded by $x^2 + y^2 = 1$ and $x + y \ge 1$.

11. Using integration find the area of a triangle whose vertices are (–1, 0), (1, 3) and (3, 2).

12. Find the area bounded by line $y = 3x + 2$, x-axis and ordinates $x = -1$ and $x = 1$.

13. Find the area between $y^2 = 2x$, $y = 4x - 1$ and $y \ge o$.

14. Find the area between $y^2 = 4x$, y-axis and line $y = 3$.

15. Find the area between the two circles $x^2 + y^2 = 4$ and $(x-2)^2 + y^2 = 4$.

IMPORTANT POINTS

1. The area bounded by curve $y = f(x)$, x-axis and ordinates $x = a$ and $x = b$ is given by definite integral $\int_a^b f(x)\,dx$ or $\int_a^b y\,dx$ i.e. area $= \int_a^b f(x)\,dx = \int_a^b y\,dx$.

2. The area of the region bounded by the curve $x = \phi(y)$, y-axis and the lines $y = c$, $y = d$ is given by the formula : Area $= \int_c^d \phi(y)\,dy = \int_c^d x\,dy$.

3. If the curve is symmetrical about any principal axis or any straight line, then the total area may be calculated by multiplying the area of one symmetrical part by number of symmetrical parts.

4. Quadrature is always considered as positive. So if some portion of area is above x-axis and some portion is below x-axis then calculate the required area as a sum of individual parts of both areas.

5. The area of the region enclosed between two curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ is given by the formula.

   Area $= \int_a^b [f(x) - g(x)]\,dx$, where $f(x) \ge g(x)$ in [a, b]

6. The area of the region enclosed between two curves $x=\phi(y)$ and $x=\psi(y)$ and $y=c$ and $y=d$ is given by the formula $=\int_c^d[\phi(y)-\psi(y)]dy$

ANSWERS

Exercise 11.1

1. $8/3\ a^2$ sq. units 2. $(\sqrt{3}+2\pi/3)$ sq. units 3. 4 sq. units

4. 4 / 3 sq. units 5. 5 sq. units 6. 2 / *3a* sq. units 7. 3*p* sq. units

8. πab sq. units 9. 2*ab* sq. units 10. 5 sq. units 11. 7 / 3 sq. units

12. 7 / 3 sq. units

Exercise 11.2

1. $(2\pi+4/3)$ sq. units 2. 27 sq. units 3. $\pi/3$ sq. units

4. 2π sq. units 5. 16 / 3 sq. units 6. $\pi-2/4$ sq. units 7. $2\ a^2/3$ sq. units

8. 9 / 2 sq. units 9. 7 sq. units 10. 4 sq. units

Miscellaneous Exercise – 11

1. (c) 2. (a) 3. (d) 4. (b) 5 (a)

6. 9 / 2 sq. units 7. $a^2(\pi/4-2/3)$ sq. units 8. 1 / 3 sq. units

9. $4/3(\sqrt{3}+4\pi)$ sq. units 10. $\pi-2/4$ sq. units 11. 4 sq. units

12. 13 / 3 sq. units 13. 1 / 3 sq. units 14. 9 / 4 sq. units

15. $(8\pi/3-2\sqrt{3})$ sq. units

12

Differential Equations

12.01 Introduction

Most of the problems in science and engineering are solved by finding how one quantitiy is related or depends upon one or more quantities. In many problems, it is easier to find a relation between the rate of changes in the variables than between the variables themselves. The study of this relationship gives rise to differential equations. Therefore, an equation involving dependent variable, independent variable and derivative of the dependent variable with respect to independent variable is called a differential equation.

Differential equations which involve only one independant variable are called ordinary differential equations. If the differential equation involves more than one independents variable, then it is called a partial different equations. Here we shall confine ourselves to the study of ordinary differential equations only. Now onward, we will use the term 'differential equation' for ordinary differential equation.

For example : $\frac{dy}{dx} = x^2 y, \ \frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \sin x$,

Where x is independent variable and y is dependent variable.

12.02 Order and Degree of a Differential Equation

Order of differential equation: Order of a differential equation is defined as the order of the highest order derivative of the dependent variable with respect to the independent variable involved into the given differential equation.

For example :

(i) Differential equation $\frac{dy}{dx} = e^x$ is the order one because in this equation the dependent variable y has maximum one differentiation.

(ii) Differential equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 2y = \sin\theta$, because in this equation the dependent variable y has maximum two times differentiation.

(iii) Differential equation $\left(\frac{dy}{dx}\right)^3 + \frac{dy}{dx} + 3y = 0$ the of order one because the dependent variable y has maximum one differentiation.

Degree of a Differential Equation :

The degree of a differential equation is the degree of the highest order derivatives, when differential cofficients are made free from redicals and fractions.

(i) The degree of $\left(\frac{d^3y}{dx^3}\right)^2 + \frac{dy}{dx} - 3y = 0$ is two because the highest order derivative is $\frac{d^3y}{dx^3}$ whose power is 2.

(ii) The degree of $\frac{d^2y}{dx^2}+\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{2/3}=0$ is three, because on rationalization it becomes $\left(\frac{d^2y}{dx^2}\right)^3=-\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^2$ and the power of highest derivative is 3.

(iii) The degree of differential equation $\frac{dy}{dx}=\frac{x^2+y^2}{xy}$ is one.

Remark : Order and degree (if defined) of a differential equation are always positive integrals.

Illustrative Examples

Example 1. Find the order and degree of following differential equations.

(i) $\frac{dy}{dx}-\cos x=0$ (ii) $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=e^x$ (iii) $\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^4=\cos x$

(iv) $y=x\frac{dy}{dx}+\frac{a^2}{dy/dx}$ (v) $\frac{d^4y}{dx^4}+\sin\left(\frac{d^3y}{dx^3}\right)=0$

Solution :

(i) The highest order derivative of y in this differential equation is $\frac{dy}{dx}$ so its order is 1 and the highest power of $\frac{dy}{dx}$ is 1, so its degree is 1.

(ii) The highest order derivative of y in the given differential equation is $\frac{d^2y}{dx^2}$ so its order is 2 and the highest power of $\frac{d^2y}{dx^2}$ is 1, so its degree is 1.

(iii) The highest order derivative of y in the given differential equation is $\frac{d^2y}{dx^2}$, so its order is 2 and the highest power of $\frac{d^2y}{dx^2}$ is one, so its degree is 1.

(iv) On simplification we see that the given differential equation is $x\left(\frac{dy}{dx}\right)^2+a^2=y\frac{dy}{dx}$, hence order is 1 and degree is 2.

(v) The highest derivative of y in the given differential equation is $\frac{d^4y}{dx^4}$, so its order is 4, also the given differential equation is not a polynomial in context with diffrential coefficients. So the degree of equation is not defined.

Exercise 12.1

Find the order and degree of following differential equations.

1. $\frac{dy}{dx}=\sin 2x+\cos 2x$

2. $\frac{d^2y}{dx^2}=\sin x+\cos x$

3. $\left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{dy}{dx}\right)=0$

4. $\left(\frac{dy}{dx}\right)^3+\frac{1}{dy/dx}=2$

5. $a\frac{d^2y}{dx^2}=\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$

6. $xdx+ydy=0$

7. $\left(\frac{d^2y}{dx^2}\right)^3+y\left(\frac{dy}{dx}\right)^2+y^5=0$

8. $x\frac{dy}{dx}+\frac{3}{(dy/dx)}=y^2$

12.03 Formation of differential equation

If the given family f of curves depends on only one constant parameter then it is represented by an equation of the form

$$f(x,\ y,\ a)=0 \qquad (1)$$

Differentiating equatioin (1) with respect to x

$$\phi(x,\ y,\ y',\ a)=0 \qquad \left[\text{where } y'=\frac{dy}{dx}\right] \qquad (2)$$

The required differential equation is then obtained by eliminating a from equation (1) and (2) as

$$f(x,\ y,\ y')=0$$

This is called the required differential equation of family of curves. Similarly if the given equation has two arbitrary constants then differentiatign twice and by eliminating the arbitrary constants, we get the equation of family of curves.

Illustrative Examples

Example 2. Find the differential equation of family of straight lines which passes through orgin.

Solution : The equation of straight line passing through origin is

$$y=mx,\ \text{where } m \text{ is arbitary.} \qquad (1)$$

On differentiating equation (1)

$$\frac{dy}{dx}=m \qquad (2)$$

On eliminating m from (1) and (2)

$$x\frac{dy}{dx}=y$$, which is the required differential equation.

Example 3. Find the differential equation of family of $y = ae^{2x} + be^{-x}$

Solution : $$y = ae^{2x} + be^{-x} \quad (1)$$

Differentiating eq. (1) with respect to x

$$\frac{dy}{dx} = 2ae^{2x} - be^{-x} \quad (2)$$

Again differentiating

$$\frac{d^2y}{dx^2} = 4ae^{2x} + be^{-x} \quad (3)$$

From (2) and (3)

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2ae^{2x} + 2be^{-x} = 2\left(ae^{2x} + be^{-x}\right)$$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2y. \quad \text{(From eq. (1))}$$

This is the required differential equation.

Example 4. Find the differential equation of family of curves for $y = e^x\left[A\sin x + B\cos x\right]$

Solution : $y = e^x\left[A\sin x + B\cos x\right]$ Differentiatign with respect to x (1)

$$\frac{dy}{dx} = e^x[A\sin x + B\cos x] + e^x[A\cos x - B\sin x]$$

$$\Rightarrow \quad \frac{dy}{dx} = y + e^x[A\cos x - B\sin x] \quad (2)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x\left[A\cos x - B\sin x\right] + e^x\left[-A\sin x - B\cos x\right]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} = \frac{dy}{dx} - y - y \quad \text{(From (2))}$$

or $$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0.$$

This is the required differential equation.

Exercise 12.2

1. Find the differential equation of family of curves for $y = ax + \frac{b}{x}$.
2. Find the differential equation of family of curves for $x^2 + y^2 = a^2$.
3. Find the differential equation of family of curves for $y = Ae^{3x} + Be^{5x}$.

4. Find the differential equation of family of curves for $y = e^x\left[A\cos x + B\sin x\right]$.

5. Find the differential equation of family of curves for $y = a\cos(x+b)$, where a and b are arbitrary variables.

12.04 Solution of a Differential Equation

The solution to the differential equation used in the equation refers to a relationship in the independent and dependent variables which does not contain any differential coefficient and the given differential equation is satisfied for derivative obtained.

The solution of a differential equation is also called its primitive because the differential equation is a relation derived from it.

General, particular and singular solution

(i) **General solution :** In the solution of a differential equation if number of arbitrary constant are equal to the order of it then that solution is called general solution. This is also called total solution or total integral or total primitive.

For Example : $y = A\cos x + B\sin x$ is a general solution of differential equation $\frac{d^2y}{dx^2} + y = 0$ because arbitrary variables present in the solution are equal to the order 2 of the equation.

(ii) **Particular solution :** The solution of a differential equation obtained by assigning particular values of the arbitrary constants in the general solution is called 'particular solution'.

For Example : $y = 3\cos x + 2\sin x$ is a particular solution of differential equation $\frac{d^2y}{dx^2} + y = 0$

(iii) **Singular solution :** Singular solutions of a differential equation are those where arbitrary constants are not present and fails to have a particular solution of general solution.

Remark : Singular solution is not there in syllabus. Hence we will not discuss it here in detail.

Illustrative Examples

Example 5: Prove that $y = cx + \frac{a}{c}$ is a solution of differential equation $y = x\frac{dy}{dx} + \frac{a}{dy/dx}$.

Solution : Given equation is $y = cx + (a/c)$. (1)

differentiating with respect to x

$$\frac{dy}{dx} = c \qquad (2)$$

On eliminating c from (1) and (2)

$$y = x\left(\frac{dy}{dx}\right) + \frac{a}{(dy/dx)}$$

Hence $y = cx + a/c$ is solution of given differential equation.

Example 6. Prove that $y = a\sin 2x$ is solution of given differential equation $\frac{d^2y}{dx^2} + 4y = 0$.

Solution : Given equation is $y = a\sin 2x$. (1)

differentiating with respect to x

$$\frac{dy}{dx} = 2a\cos 2x \qquad (2)$$

again differentiating with respect to x

$$\frac{d^2y}{dx^2} = -4a\sin 2x \qquad (3)$$

$$\frac{d^2y}{dx^2} + 4a\sin 2x = 0$$

and $$\frac{d^2y}{dx^2} + 4y = 0 \qquad \text{[From Eq. (1)]}$$

Hence $y = a\sin 2x$ is a solution of given differential equation.

Example 7. Prove that $y + x + 1 = 0$ is solution of differential equation $(y-x)\,dy - (y^2 - x^2)\,dx = 0$.

Solution : Given equation is

$\because$ $$y + x + 1 = 0$$

$\therefore$ $$y = -(x+1) \Rightarrow dy = -dx \qquad (1)$$

LHS of given differential equation

$$= (y-x)\,dy - (y^2 - x^2)\,dx$$

$$= (y-x)(-dx) - (y-x)(y+x)\,dx \qquad [\because \text{From eq. (1)}]$$

$$= -(y-x)(1+x+y)\,dx$$

$$= 0$$

$$= \text{RHS}$$

Hence $y + x + 1 = 0$ is a solution of differential equation.

Exercise 12.3

1. Prove that $y^2 = 4a(x+a)$ is a solution of differential equation $y = \left[1 - \left(\frac{dy}{dx}\right)^2\right] = 2x\frac{dy}{dx}$.

2. Prove that $y = ae^{-2x} + be^{x}$ is a solution of differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$.

3. Prove that $y=\dfrac{c-x}{1+cx}$ is a solution of differential equation $\left(1+x^2\right)\dfrac{dy}{dx}+\left(1+y^2\right)=0$.

4. Prove that $y=a\cos\left(\log x\right)+b\sin\left(\log x\right)$ is a solution of differential equation $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y=0$.

5. Prove that $xy=\log y+c$ is a solution of differential equation $\dfrac{dy}{dx}=\dfrac{y^2}{1-xy}\left(xy\neq 1\right)$.

12.05 Differential Equation of First Order and First Degree

There exists a dependent variable x, an independent variable y and $\dfrac{dy}{dx}$ in an differential equation of first order and first degree. hence the equation may be written as

$$\frac{dy}{dx}=f\left(x,\ y\right),\ \text{where}\ f\left(x,\ y\right)\ \text{is a function of}\ x\ \text{and}\ y$$

or
$$\frac{dy}{dx}=\frac{f\left(x,\ y\right)}{g\left(x,\ y\right)}$$

or
$$f(x,\ y)\,dx+g(x,\ y)\,dy=0$$

As it is not possible to integrate every function similarly it is not possible to find solution of every differential equation. But if the differential equation is in standard form of any one out of below mentioned then it is possible to have solution of such differential equations.

(A) Differential equation in which variable separation is possible.

(B) Variable separation is possible by substitution.

(C) Homogeneous differential equations.

(D) Differential equation are reducible to homogeneous form.

(E) Linear differential equation.

(F) Differential equation are reducible to linear differential equation.

Remark : Apart from above discussed methods in some situation the solution of differential equation is possible by finding integral multiple, but as not a part of syllabus, the studies of such cases is not porvided here.

(A) Variable separable form

In the equation $M(x,\ y)\,dx+N(x,\ y)\,dy=0$ on separating the variables and writing in the form of

$$f(x)\,dx+g(y)\,dy=0 \qquad (1)$$

here the variables are separated hence on integrating the each term of equation (1) following solution is obtained.

$$\int f(x)\,dx+\int g(y)\,dy=C,\ \text{where}\ C\ \text{is any arbitrary cosntant.}$$

Illustrative Examples

Example 8. Solve $\frac{dy}{dx}=e^{x+y}$.

Solution : Given equation is $\frac{dy}{dx}=e^{x}\cdot e^{y}$

now on separating the variables $e^{x}dx=e^{-y}dy$

integrating both the sides $\int e^{x}dx=\int e^{-y}dy$

$\Rightarrow$ $e^{x}=-e^{-y}+C$

or $e^{x}+e^{-y}=C$, where C is integral constant.

This is the required solution.

Example 9. Solve $\frac{dy}{dx}=\sin x-x$.

Solution : Given equation is $\frac{dy}{dx}=\sin x-x$

on separating the variables, $dy=(\sin x-x)dx$

integrating both the sides, $\int dy=\int(\sin x-x)dx$

or $y=-\cos x-\frac{x^{2}}{2}+C$, where C is integral constant.

This is the required solution.

Example 10. Solve $x\cos^{2}y\,dx=y\cos^{2}x\,dy$.

Solution : Given equation is $x\cos^{2}y\,dx=y\cos^{2}x\,dy$

or $\frac{dy}{dx}=\frac{x\cos^{2}y}{y\cos^{2}x}=\frac{x\sec^{2}x}{y\sec^{2}y}$

On separating the variables

or $y\sec^{2}y\,dy=x\sec^{2}x\,dx$

integrating both the sides $\int y\sec^{2}y\,dy=\int x\sec^{2}x\,dx$

on integrating by parts

$y\tan y-\log\sec y=x\tan x-\log\sec x+C$, where C is integral constant.

This is the required solution.

Example 8. Solve: $\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0.$

Solution : Given equation is $$\frac{dy}{dx}=-\sqrt{\frac{1-y^2}{1-x^2}}$$

Now on separating the variables $$\frac{dx}{\sqrt{1-x^2}}=-\frac{dy}{\sqrt{1-y^2}}$$

integrating both the sides $$\int\frac{dx}{\sqrt{1-x^2}}=-\int\frac{dy}{\sqrt{1-y^2}}$$

$\sin^{-1}x=-\sin^{-1}y+C_1$ (First form) where C_1 is integral constant

If we take C_1 as $\sin^{-1}C$ then

$$\sin^{-1}x+\sin^{-1}y=\sin^{-1}C$$

by inverse circular formula $$\left[\sin^{-1}x+\sin^{-1}y=\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}\right]$$

$$\sin^{-1}\left[x\sqrt{1-y^2}+y\sqrt{1-x^2}\right]=\sin^{-1}C$$

or $$x\sqrt{1-y^2}+y\sqrt{1-x^2}=C$$

This is the required solution.

Exercise 12.4

Solve the following differential equations.

1. $(e^y+1)\cos x\,dx+e^y\sin x\,dy=0$
2. $(1+x^2)dy=(1+y^2)dx$
3. $(x+1)\frac{dy}{dx}=2xy$
4. $\frac{dy}{dx}=e^{x-y}+x^2e^{-y}$
5. $(\sin x+\cos x)dy+(\cos x-\sin x)dx=0$
6. $\frac{dy}{dx}=\frac{3e^{2x}+3e^{4x}}{e^x+e^{-x}}$
7. $\sec^2 x\tan y\,dy+\sec^2 y\tan x\,dx=0$
8. $\frac{dy}{dx}=\frac{x(2\log x+1)}{\sin y+y\cos y}$
9. $(1+\cos x)dy=(1-\cos x)dx$
10. $\sqrt{1-x^6}\,dy=x^2dx$

(B) Differential equation reducable to variable separable

In this method the given differential equation may be reduced to variable separable form by suitable substitution and by getting its solution and again substituting required solution can be obtained. Following examples will explain the above method.

Illustrative Examples

Example 12. Solve $\frac{dy}{dx}=(4x+y+1)^2$.

Solution : Let $4x+y+1=t$

On differentiating with respect to x,

$$4+\frac{dy}{dx}=\frac{dt}{dx}\Rightarrow\frac{dy}{dx}=\frac{dt}{dx}-4$$

by substitution in given equation

$$\frac{dt}{dx}-4=t^2$$

$\Rightarrow$ $$\frac{dt}{dx}=t^2+4$$

$\Rightarrow$ $$\frac{1}{t^2+4}dt=dx \quad \text{(seperation of variables)}$$

on integration $$\int\frac{1}{t^2+(2)^2}dt=\int dx$$

or $$\frac{1}{2}\tan^{-1}(t/2)=x+C\text{, where C is integral cosntant}$$

or $$\tan^{-1}t/2=2x+2C$$

or $$t=2\tan(2x+C_1),\quad \text{where } C_1=2C$$

putting the vlaue of t the desired solution is

$$4x+y+1=2\tan(2x+C_1)$$

Example 13. Solve: $(x-y)^2\frac{dy}{dx}=a^2$.

Solution : On writing the given equation in the following form

$$\frac{dy}{dx}=\frac{a^2}{(x-y)^2} \tag{1}$$

Let $$x-y=t\Rightarrow 1-\frac{dy}{dx}=\frac{dt}{dx}$$

So from eq. (1) $$1-\frac{dt}{dx}=\frac{a^2}{t^2}$$

on simplification $$\frac{dt}{dx}=1-\frac{a^2}{t^2}=\frac{t^2-a^2}{t^2}$$

so $$dx = \left[1 + \frac{a^2}{(t^2 - a^2)}\right] dt$$

On integration $$\int dx = \int \left[1 + \frac{a^2}{t^2 - a^2}\right] dt$$

or $$x = t + a^2 \frac{1}{2a} \log\left(\frac{t-a}{t+a}\right) + C,$$ where C is integral constant.

putting the value of t the required solution is

$$y = \frac{a}{2} \log\left\{\frac{x-y-a}{x-y+a}\right\} + C.$$

Example 14. Solve: $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$.

Solution : Let $x + y = t$, on differentiating with respect to x

$$1 + \frac{dy}{dx} = \frac{dt}{dx}$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{dt}{dx} - 1$$

on substitution in given equation

$$\frac{dt}{dx} - 1 = \sin t + \cos t$$

or $$\frac{dt}{dx} = 1 + \sin t + \cos t$$

or $$\frac{dt}{(\sin t + \cos t + 1)} = dx$$ [separation of variable]

or $$\frac{(1/2)\sec^2(t/2)}{1 + \tan\ (t/2)} dt = dx$$

on integration $$\int \frac{(1/2)\sec^2(t/2)}{1 + \tan(t/2)} dt = \int dx$$

$$\log\left[1 + \tan\frac{t}{2}\right] = x + C,$$ where C is integral constant

or $$\log\left[1 + \tan\frac{(x+y)}{2}\right] = x + C.$$ [$\because$ on putting $t = x + y$]

Example 15. Solve $\left[\frac{x+y-a}{x+y-b}\right]\frac{dy}{dx}=\frac{x+y+a}{x+y+b}$

Solution : From the given equation

$$\frac{dy}{dx}=\frac{(x+y+a)(x+y-b)}{(x+y-a)(x+y+b)} \quad (1)$$

Let $x+y=t \Rightarrow 1+\frac{dy}{dx}=\frac{dt}{dx}$ (on differentiation)

So, $$\frac{dt}{dx}=\frac{(t+a)(t-b)}{(t-a)(t+b)}+1$$

on simplifying $$\frac{dt}{dx}=\frac{2(t^2-ab)}{(t-a)(t+b)}$$

or $$2dx=\left[1+\frac{t(b-a)}{t^2-ab}\right]dt$$

on integration $$\int 2dx=\int\left[1+\frac{t(b-a)}{t^2-ab}\right]dt$$

$$2x=t+\frac{b-a}{2}\log(t^2-ab)+C,$$ where C is the integral constant

on putting the vlaue of t, the requried solution is

$$x-y=\frac{b-a}{2}\log\left[(x+y)^2-ab\right]+C.$$

Exercise 12.5

Solve the following differential equations.

1. $(x+y)^2\frac{dy}{dx}=a^2$
2. $\frac{dy}{dx}=\frac{1}{x+y+1}$
3. $\cos(x+y)dy=dx$
4. $e^{x+y}=1+\frac{dy}{dx}$
5. $(x+y)(dx-dy)=dx+dy$
6. $\frac{dy}{dx}=\frac{x+y+1}{x+y}$
7. $x+y=\sin^{-1}\left(\frac{dy}{dx}\right)$
8. $\frac{dy}{dx}=\frac{1}{x-y}+1$
9. $\frac{dy}{dx}=\sec(x+y)$
10. $\frac{dy}{dx}=\frac{(x-y)+3}{2(x-y)+5}$

(C) Homogeneous differential equation

Differential equation $f(x, y)\,dx + g(x, y)\,dy = 0$ is called homogeneous differential equation if it could be expressed in following form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \qquad (1)$$

i.e. in $f(x, y)$ and $g(x, y)$ the sum of degrees of x and y in every term always remains same.

Let, $$y = vx \qquad (2)$$

differentiating with respect to x

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad (3)$$

Using (2) and (3) in (1)

$$v + x\frac{dv}{dx} = F(v)$$

or $$x\frac{dv}{dx} = F(v) - v$$

or $$\frac{1}{F(v) - v}dv = \frac{dx}{x}$$ [separationof variable]

on integration $$\int \frac{1}{F(v) - v}dv = \int \frac{1}{x}dx = \log x + C$$, where C is integral constant.

On solving LHS and putting $v = \frac{y}{x}$, gives the required solution of differential equation.

Remark : If the homogeneous differential equation is of the form $\frac{dx}{dy} = f(x, y)$, where $f(x, y)$ is a homogeneous function of degree zero, then put $x = vy$ and find $\frac{dx}{dy}$ and put the value of $\frac{dx}{dy} = f(x, y)$ and find the general solution of differeneation equation.

Illustrative Examples

Example 16. Solve, $\frac{dy}{dx} = \frac{3xy + y^2}{3x^2}$

Solution : Given equation $$\frac{dy}{dx} = \frac{3xy + y^2}{3x^2} \qquad (1)$$

Given equation is homogeneous differential equation

so let $$y = vx \qquad (2)$$

$\Rightarrow$ $$\frac{dy}{dx} = v + \frac{xdv}{dx} \qquad (3)$$

using equation (2) and (3) in (1)

$$v + x\frac{dv}{dx} = \frac{3vx^2 + v^2x^2}{3x^2} = \frac{3v + v^2}{3}$$

or $$x\frac{dv}{dx} = \frac{3v + v^2}{3} - v = \frac{v^2}{3}$$

or $$\frac{1}{v^2}dv = \frac{1}{3x}dx$$ [on separating the varaibles]

or $$-\frac{1}{v} = \frac{1}{3}\log|x| + C,$$ where C is integral constant

or $$-\frac{x}{y} = \frac{1}{3}\log|x| + C.$$ $\left[\because v = \frac{y}{x}\right]$

this is the required solution.

Example 17. Solve : $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$.

Solution : $$\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right) \quad (1)$$

This given equation is homogeneous differential equation

So, let $$y = vx$$

$\Rightarrow$ $$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

now from (1) $$v + x\frac{dv}{dx} = v + \tan v$$

or $$\frac{1}{x}dx = \cot v\, dv$$ [by separating the variables]

on integrating $\log|x| = \log\sin v + \log C$, where $\log C$ is integral constant.

or $$x = C\sin v$$

on putting the value of v required solution is

$$x = C\sin\left(\frac{y}{x}\right).$$

Example 18. Solve $x\sin\left(\frac{y}{x}\right)\frac{dy}{dx} = y\sin\left(\frac{y}{x}\right) - x$

Solution : From the given equation

$$\frac{dy}{dx} = \frac{y\sin(y/x) - x}{x\sin(y/x)} \quad (1)$$

Given equation is homogeneous differential equation

So, let $$y = vx \qquad (2)$$

$$\therefore \qquad \frac{dy}{dx} = v + x\frac{dv}{dx} \qquad (3)$$

so by eq. (1) $$v + x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v}$$

or $$v + x\frac{dv}{dx} = v - \operatorname{cosec} v$$

or $$\frac{1}{x}dx = -\sin v dv$$ [by separating the variables]

$\log(x/c) = \cos v$, where C is integral constant

or $$x = Ce^{\cos v}$$

on putting the vlaue of v required solution is

$$x = ce^{\cos(y/x)}$$

Example 19. Solve : $x\dfrac{dy}{dx} = y(\log y - \log x + 1)$

Solution : From given equation $$\frac{dy}{dx} = \frac{y}{x}\left[\log\frac{y}{x} + 1\right] \qquad (1)$$

equation (1) is homogeneous equation

So, let $$y = vx \qquad (2)$$

$$\therefore \qquad \frac{dy}{dx} = v + x\frac{dv}{dx} \qquad (3)$$

using equation (2) and (3) in equation (1)

$$v + x\frac{dv}{dx} = v(\log v + 1)$$

or $$x\frac{dv}{dx} = v\log v$$

or $$\frac{1}{v\log v}dv = \frac{1}{x}dx$$ [by separating the variables]

on integration $$\int\frac{(1/v)}{\log v}dv = \int\frac{1}{x}dx$$

or $\log(\log v) = \log x + \log C$, where $\log C$ is integral constant

or $$\log v = Cx$$

or $$\log\frac{y}{x}=Cx \qquad [\because\ v=y/x]$$

This is the required solution

Exercise 12.6

Solve the following differential equations.

1. $x^2ydx-\left(x^3+y^3\right)dy=0$
2. $\frac{dy}{dx}=\frac{y}{x}+\sin\left(\frac{y}{x}\right)$
3. $x\frac{dy}{dx}+\frac{y^2}{x}=y$
4. $x\sin\left[\frac{y}{x}\right]\frac{dy}{dx}=y\sin\left[\frac{y}{x}\right]-x$
5. $xdy-ydx=\sqrt{x^2+y^2}\,dx$
6. $\left(x^2+y^2\right)dy=2xydx$
7. $\left(1+e^{x/y}\right)dx+e^{x/y}\left(1-\frac{x}{y}\right)dy=0$
8. $\left(3xy+y^2\right)dx+\left(x^2+xy\right)dy=0$
9. $x^2\frac{dy}{dx}=x^2+xy+y^2$
10. $x\left(x-y\right)dy=y\left(x+y\right)dx$

(D) Differential Equation Reducible to Homogeneous Form

When differential equation is of the form $\frac{dy}{dx}=\frac{ax+by+c}{a'x+b'y+c'}$, where $\frac{a}{a'}\neq\frac{b}{b'}$ (1)

where c and c' are constants then this may be reduced to a homogeneous eq. by substitution $x=X+h$ and $y=Y+k$ we may get the required solutions

so, let $X=x-h$; $Y=y-k$

$dx=dX$; $dy=dY$

so by eq. (1)
$$\frac{dY}{dX}=\frac{a(X+h)+b(Y+k)+c}{a'(X+h)+b'(Y+k)+c'}$$

or
$$\frac{dY}{dX}=\frac{(aX+bY)+(ah+bk+c)}{(a'X+b'Y)+(a'h+b'k+c')} \qquad (2)$$

In order to make equation (2) a homogeneous, the constants h and k are selected such that

$$\left.\begin{aligned}ah+bk+c=0\\ a'h+b'k+c'=0\end{aligned}\right\} \qquad (3)$$

on solving them the values of h and k are found now using equation (3) in equation (2)

$$\frac{dY}{dX}=\frac{aX+bY}{a'X+b'Y} \qquad (4)$$

which is homogeneous, hence solve (4) by homogeneous method and at last put $X=x-h$ and $Y=y-k$ and get the required solutions.

Remark : The above methods fails when $\frac{a}{b}=\frac{a'}{b'}$ because then the vlaues of h and k will be either infinite or not defined, in such case let $\frac{a}{a'}=\frac{b}{b'}=\frac{1}{m}$ then the equation (1) will be of form.

$$\frac{dy}{dx}=\frac{ax+by+c}{m[ax+by]+c'} \tag{5}$$

Now solving eq. (5) by substitution $ax+by=v$

$$\frac{dv}{dx}=a+b\left(\frac{v+c}{mv+c'}\right)$$

which can be solved by method of separation of variables.

Illustrative Examples

Example 20. Solve : $\frac{dy}{dx}=\frac{7x-3y-7}{7y-3x+3}$.

Solution : Given equation is reducible to homogeneous differential equation because $\frac{a}{a'}\neq\frac{b}{b'}$

so put $\quad x=X+h,\ y=Y+k$

$$\frac{dY}{dX}=\frac{7X-3Y+(7h-3k-7)}{-3X+7Y+(7K-3h+3)} \tag{1}$$

Select h and k such that

$$7h-3k-7=0$$

and $\quad 7k-3h+3=0$

on solving these, $h=1$ and $k=0$

So, from equation (1) $\quad \frac{dY}{dX}=\frac{7X-3Y}{-3X+7Y} \quad (2)$

which is homogeneous, so put $Y=vX$

$$\frac{dY}{dX}=v+X\frac{dv}{dX}$$

so, from (2) $\quad v+X\frac{dv}{dx}=\frac{7-3v}{-3+7v}$

$\Rightarrow \quad X\frac{dv}{dX}=\frac{7-3v}{-3+7v}-v$

or $\quad -7\frac{dX}{X}=\frac{7v-3}{v^2-1}dv \quad$ [separation of variable]

or $\quad -7\frac{dX}{X}=\frac{7}{2}\left(\frac{2v}{v^2-1}\right)dv-\frac{3}{v^2-1}dv$

On integration $-7\log X = \frac{7}{2}\log\left(v^2-1\right) - \frac{3}{2}\log\left(\frac{v-1}{v+1}\right) - \log C$, where log C is integral constant

$$\therefore \qquad \log X^7 + \log\frac{(v^2-1)^{7/2}(v+1)^{3/2}}{(v-1)^{3/2}} = \log C$$

$$\log\left[(v+1)^5(v-1)^2\right]X^7 = \log C$$

putting the value of v $\qquad \log\left[\left(\frac{Y}{X}+1\right)^5\left(\frac{Y}{X}-1\right)^2\right]X^7 = \log C$

or $\qquad \left(Y+X\right)^5\left(Y-X\right)^2 = C$

now put $\qquad X = x-1$ and $Y = y$

$$\left(y+x-1\right)^5\left(y-x+1\right)^2 = C$$

This is the required solution.

Example 21. Solve : $\frac{dy}{dx} = \frac{x+y+1}{x+y-1}$.

Solution : The given differential eq. is not reducible to homogeneous form because here $\frac{a}{a'} = \frac{b}{b'}$

So, to solve such equation we will substitute.

$$x+y = v$$

or $\qquad 1+\frac{dy}{dx} = \frac{dv}{dx}$

so $\qquad \frac{dy}{dx} = \frac{dv}{dx} - 1 = \frac{v+1}{v-1}$ $\qquad$ [From given eq.]

or $\qquad \frac{dv}{dx} = \frac{2v}{v-1}$

or $\qquad 2dx = \frac{(v-1)}{v}dv$

or $\qquad 2dx = \left(1-\frac{1}{v}\right)dv$

on integration, $\qquad \int 2dx = \int\left(1-\frac{1}{v}\right)dv$

$2x = v - \log v + C$, where C is integral constant

on putting the value of v, $\qquad 2x = x+y-\log\left(x+y\right)+C$

or, $x-y+\log(x+y)=C$

This is the required solution.

Example 22. Solve $\dfrac{dy}{dx}=\dfrac{x+y+1}{2x+2y+3}$.

Solution : Given equation $\dfrac{dy}{dx}=\dfrac{x+y+1}{2x+2y+3}, \dfrac{a}{a'}=\dfrac{b}{c'}$ is of the form

so let $x+y=v \Rightarrow \dfrac{dy}{dx}=\dfrac{dv}{dx}-1$

or $$\frac{dv}{dx}-1=\frac{v+1}{2v+3}$$

or $$\frac{dv}{dx}=\frac{v+1}{2v+3}+1=\frac{3v+4}{2v+3}$$

or $$\frac{2v+3}{3v+4}dv=dx \qquad \text{[Separation of variable]}$$

on integration $$\int\left[\frac{2}{3}+\frac{1}{3}\left(\frac{1}{3v+4}\right)\right]dv=\int dx$$

$\dfrac{2}{3}v+\dfrac{1}{9}\log(3v+4)=x+C$, where C is integral constant.

$$6v+\log(3v+4)=9x+C_1 \qquad \text{(where, } C_1 = 9\,C\text{)}$$

or $$6(x+y)+\log(3x+3y+4)=9x+C_1 \qquad \text{(on putting the value of } v\text{)}$$

or $$6y-3x+\log(3x+3y+4)=C_1$$

This is the required solution.

Exericse 12.7

Solve the following differential equations.

1. $\dfrac{dy}{dx}+\dfrac{3x+2y-5}{2x+3y-5}=0$
2. $\dfrac{dy}{dx}=\dfrac{x-y+3}{2x+2y+5}$
3. $(2x+y+1)dx+(4x+2y-1)dy=0$
4. $\dfrac{dy}{dx}=\dfrac{1-3x-3y}{2(x+y)}$
5. $\dfrac{dy}{dx}=\dfrac{6x-2y-7}{2x+3y-6}$

(E) Linear Differential Equation

A differential equation in the form

$$\frac{dy}{dx}+Py=Q, \qquad (1)$$

Where P and Q are cosntants or functions of x only, is known as first order linear differential equation. Another form of first order linear differential equation is

$$\frac{dx}{dy} + p_1 x = Q_1 \qquad (2)$$

where P_1 and Q_1 are constants or functions of y only.

Solution of linear differential equation (1) : Multiplying both sides of (1) by $e^{\int P dx}$

$$e^{\int P dx}\left[\frac{dy}{dx} + Py\right] = e^{\int P dx} Q$$

or
$$\frac{d}{dx}\left[y e^{\int P dx}\right] = e^{\int P dx} Q$$

integrating both the sides

$y \cdot e^{\int P dx} = \int Q e^{\int P dx} dy + C$, where C is integral constant.

or
$$y = e^{-\int P dx}\{\int Q e^{\int P dx} dx + C\}$$

Whcih is the required solution of (1).

Remarks:

(i) $e^{\int P dx}$ is called as integrating factor of eq. (1), which is abreviated as I.F. .

(ii) Before solving the differential equation the coefficient of derivative should be always one.

(iii) In linear differential eq $\left(\frac{dx}{dy} + P_1 x = Q_1\right)$ the integrating factor is $e^{\int P_1 dy}$ and its solution is given by

$$x = e^{-\int P_1 dy}\left\{\int Q_1 e^{\int P_1 dy} dy + C\right\}$$

Illustrative Examples

Example 23. Solve $\left(1 - x^2\right)\frac{dy}{dx} - xy = 1$.

Solution : On writing the given equation in standard form

$$\frac{dy}{dx} + \left(-\frac{x}{(1-x^2)}\right) y = \frac{1}{(1-x^2)}$$

here
$$P = -\frac{x}{\left(1-x^2\right)},\ Q = \frac{1}{\left(1-x^2\right)}$$

So integrating factor
$$\text{(I.F.)} = e^{\int P dx} = e^{-\frac{1}{2}\int \frac{2x}{1-x^2} dx} = e^{\frac{1}{2}\log\left(1-x^2\right)} = \sqrt{1-x^2}$$

so, solution will be $y\,(\text{I.F.}) = \int (\text{I.F.})\,Q\,dx + C$, where C is integral constant

$\therefore$ $$y\sqrt{1-x^2} = \int \sqrt{1-x^2}\cdot\frac{1}{(1-x^2)}\,dx$$

$$= \int \frac{1}{\sqrt{1-x^2}}\,dx$$

or $$y\sqrt{1-x^2} = \sin^{-1}x + C.$$

This is the required solution.

Example 24. Solve : $\sec x\dfrac{dy}{dx} = y + \sin x$.

Solution : On writing the given equation in standard form

$$\frac{dy}{dx} - y\cos x = \sin x\cos x,$$

here $P = -\cos x,\ Q = \sin x\cos x$

So integrating factor $(\text{I.F.}) = e^{\int P dx} = e^{-\int \cos dx} = e^{-\sin x}$

so, solution is $y\cdot e^{-\sin x} = \int \sin x\cos x e^{-\sin x}dx + C$, where C is integral constant

$$= \int te^{-t}dt + C \qquad [\text{here } t = \sin x,\ \therefore dt = \cos x dx\ ]$$

$$= -e^{-t}(1+t) + C \qquad [\text{integration by parts}]$$

$$= -e^{-\sin x}(1+\sin x) + C \qquad (\because\ t = \sin x)$$

or $$y = Ce^{\sin x} - (1+\sin x)$$

This is the required solution.

Example 25. Solve : $x\log x\dfrac{dy}{dx} + y = 2\log x$

Solution : On writing the given equation in standard form

$$\frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x},$$

where $P = \dfrac{1}{x\log x},\ Q = \dfrac{2}{x}$

Integrating factor $(\text{I.F.}) = e^{\int P dx} = e^{\int \frac{1}{x\log x}dx} = e^{\log(\log x)} = \log x$

Integrating factor $y\log x = \int \dfrac{2}{x}\log x dx + C$, Where C is integral constant

$$= 2\frac{(\log x)^2}{2} + C$$

or $$y = (\log x) + \frac{C}{(\log x)}.$$

This is the required solution.

Example 26. Solve : $\left(1+y^2\right)dx = \left(\tan^{-1} y - x\right)dy$.

Solution : From given equation

$$\frac{dx}{dy} + \frac{1}{\left(1+y^2\right)}x = \frac{\tan^{-1} y}{1+y^2},$$

here $$P_1 = \frac{1}{1+y^2},\ Q_1 = \frac{\tan^{-1} y}{1+y^2}$$

so integrating factor $$(\text{I.F.}) = e^{\int P_1 dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1} y}$$

so, the solution is $$xe^{\tan^{-1} y} = \int e^{\tan^{-1} y}\left(\frac{\tan^{-1} y}{1+y^2}\right)dy + C$$, where C is integral constant

$$= \int te^t dt + C \qquad [\text{where } \tan^{-1} y = t\,]$$

$$= (t-1)e^t + C$$

on putting the vlaue of t, the required solution of equation is

$$x = (\tan^{-1} y - 1) + ce^{-\tan^{-1} y}.$$

Exercise 12.8

Solve the following differential equations.

1. $\dfrac{dy}{dx} + 2y = 4x$
2. $\cos^2 x\dfrac{dy}{dx} + y = \tan x$
3. $\left(1+x^2\right)\dfrac{dy}{dx} + 2yx = 4x^2$
4. $\left(2x-10y^3\right)\dfrac{dy}{dx} + y = 0$
5. $\dfrac{dy}{dx} + y\cot x = \sin x$
6. $\left(1-x^2\right)\dfrac{dy}{dx} + 2xy = x\sqrt{1-x^2}$
7. $\sin^{-1}\left[\dfrac{dy}{dx} + \dfrac{2}{x}y\right] = x$
8. $x\dfrac{dy}{dx} + 2y = x^2\log x$
9. $dx + xdy = e^{-y}\sec^2 ydy$
10. $\left(1+y^2\right) + \left(x - e^{\tan^{-1} y}\right)\dfrac{dy}{dx} = 0$

(F) Differential Equation Reducible to Linear Differential Equation

Bernoulli's equation

$$\frac{dy}{dx} + Py = Qy^n \qquad (1)$$

Above equation may be transformed in linear differential equation by dividing the differntial equation by y^n so dividing by y^n to both sides

$$y^{-n}\frac{dy}{dx}+Py^{1-n}=Q \tag{2}$$

Let $$y^{1-n}=v$$

$$(1-n)\,y^{-n}\frac{dy}{dx}=\frac{dv}{dx}$$

$$y^{-n}\frac{dy}{dx}=\frac{1}{(1-n)}\frac{dv}{dx}$$

putting the above value in equation (2)

$$\frac{1}{(1-n)}\frac{dv}{dx}+Pv=Q$$

or $$\frac{dv}{dx}+(1-n)Pv=(1-n)Q$$

which is a linear differential equation and can be solved by the method discussed in article (E).

Illustrative Examples

Example 27. Solve : $x\frac{dy}{dx}+y=x^3y^6$.

Solution : On dividing both sides of equation by xy^6

$$\frac{1}{y^6}\frac{dy}{dx}+\frac{1}{xy^5}=x^2 \tag{1}$$

Let $$\frac{1}{y^5}=v \Rightarrow \frac{-5}{y^6}\frac{dy}{dx}=\frac{dv}{dx}$$

so, transformed form of (1) is $-\frac{1}{5}\frac{dv}{dx}+\frac{1}{x}v=x^2$

or $\frac{dv}{dx}-\frac{5}{x}v=-5x^2$, which is linear differential equation (2)

so, integrating factor $$(I.F.)=e^{\int P dx}=e^{-5\int\frac{1}{x}dx}=e^{-5\log x}=\frac{1}{x^5}$$

so, the solution of equation (2) $$v\frac{1}{x^5}=\int\frac{1}{x^5}(-5x^2)dx+C$$

or $$\frac{v}{x^5}=-5\int x^{-3}dx+C=\frac{5}{2x^2}+C$$

so, putting the value of v, the required solution is

$$y^{-5} = \frac{5}{2}x^3 + 6x^5.$$

Example 28. Solve : $\frac{dy}{dx} = \frac{e^y}{x^2} - \frac{1}{x}$.

Solution : From given equation $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$

dividing by e^y $$e^{-y}\frac{dy}{dx} + \frac{e^{-y}}{x} = \frac{1}{x^2} \qquad (1)$$

Let $$e^{-y} = v \Rightarrow -e^{-y}\frac{dy}{dx} = \frac{dv}{dx}$$

so, transformed form of (1) is $-\frac{dv}{dx} + \frac{1}{x}v = \frac{1}{x^2}$

or $$\frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x^2} \qquad (2)$$

which is linear differential equation.

so integrating factor $$(I.F.) = e^{\int P dx} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$$

so, the solution of (2) will be $$v \cdot \frac{1}{x} = \int \frac{1}{x}\left(-\frac{1}{x^2}\right)dx$$

or $$\frac{v}{x} = \frac{1}{2x^2} + C$$

on putting the vlaue of v the required solution is

$$2xe^{-y} - 1 = 2x^2C.$$

Example 29. Solve : $\frac{dy}{dx} + \left(2x\tan^{-1}y - x^3\right)\left(1+y^2\right) = 0$.

Solution : Given equation is $\frac{dy}{dx} + \left(2x\tan^{-1}y - x^3\right)\left(1+y^2\right) = 0$

or $$\frac{1}{\left(1+y^2\right)}\frac{dy}{dx} = -\left(2x\tan^{-1}y - x^3\right)$$

or $$\frac{1}{\left(1+y^2\right)}\frac{dy}{dx} + 2x\tan^{-1}y = x^3 \qquad (1)$$

Let $$\tan^{-1}y = v \Rightarrow \frac{1}{\left(1+y^2\right)}\frac{dy}{dx} = \frac{dv}{dx}$$

so, from eq. (1) $$\frac{dv}{dx} + 2xv = x^3$$

which is a linear differential equation, where $P = 2x,\ Q = x^3$

$\therefore$ Integrating factor $(I.F.) = e^{2\int x dx} = e^{x^2}$

so required solution is
$$v \cdot e^{x^2} = \int x^3 e^{x^2} dx + C$$
$$= \frac{1}{2}\int x^2 (2x) e^{x^2} dx + C$$
$$= \frac{1}{2}\int te^t dt + C, \qquad \text{[where } t = x^2, \therefore dt = 2xdx \text{]}$$
$$= \frac{1}{2}e^t (t-1) + C \qquad \text{[Integration by parts]}$$
$$= \frac{1}{2}e^{x^2}(x^2 - 1) + C, \qquad [\because t = x^2]$$

again substituting the vlaue of v

$$(\tan^{-1} y)e^{x^2} = \frac{1}{2}e^{x^2}(x^2 - 1) + C$$
$$\tan^{-1} y = \frac{1}{2}(x^2 - 1) + ce^{-x^2}.$$

This is the required solution.

Example 30. Find the particular solution of differential equation $\frac{dy}{dx} + 2y\tan x = \sin x$ If $x = \pi/3$ and $y = 0$.

Solution : Given differential equation is

$$\frac{dy}{dx} + 2y\tan x = \sin x \qquad (1)$$

Here $$P = 2\tan x,\ Q = \sin x$$
$$\text{I.F.} = e^{2\int \tan x dx} = e^{2\log\sec x} = e^{\log\sec^2 x} = \sec^2 x$$

General solution of differnetial equation is

$$y \times \text{I.F.} = \int (\text{I.F.}) \times Q dx$$

or $$y \cdot \sec^2 x = \int \sec^2 x \times \sin x dx$$

or $$y \cdot \sec^2 x = \int \sec x \tan x\, dx$$

or $$y \cdot \sec^2 x = \sec x + C \qquad (2)$$

when $x = \pi/3,\ y = 0$ put in eq. (2)

$$0 = \sec \pi/3 + C$$

or $$C = -2$$

put $C = -2$ in equation (2)

$$y \sec^2 x = \sec x - 2$$

or $$y = \cos x - 2\cos^2 x$$

Which is the required solution.

Exercise 12.9

Solve the following differential equations.

1. $\dfrac{dy}{dx} + xy = x^3 y^3$

2. $\dfrac{dy}{dx} = e^{x-y}\left(e^x - e^y\right)$

3. $\dfrac{dy}{dx} - y \tan x = -y^2 \sec x$

4. $\tan x \cos y \dfrac{dy}{dx} + \sin y + e^{\sin x} = 0$

5. $\dfrac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

6. $\dfrac{dy}{dx} + \dfrac{y}{x} \log y = \dfrac{y}{x^2} (\log y)^2$

7. $(1 + x^2)\dfrac{dy}{dx} + 2xy = \dfrac{1}{1 + x^2}$; where $x = 1,\ y = 0$

Miscellaneous Exercise 12

1. Solution of $\left(x^2 + 1\right)\dfrac{dy}{dx} = 1$ is

(a) $y = \cot^{-1} x + C$ (b) $y = \tan^{-1} x + C$ (c) $y = \sin^{-1} x + C$ (d) $y = \cos^{-1} x + C$

2. Solution of $\dfrac{dy}{dx} + 2x = e^{3x}$ is

(a) $y + x^2 = \dfrac{1}{3} e^{3x} + C$ (b) $y - x^2 = \dfrac{1}{3} e^{3x} + C$ (c) $y + x^2 = e^{3x} + C$ (d) $y - x^2 = e^{3x} + C$

3. Solution of $\dfrac{dy}{dx} + \cos x \tan y = 0$ is

(a) $\log \sin y + \sin x + C$ (b) $\log \sin x \sin y = C$

(c) $\sin y + \log \sin x + C$ (d) $\sin x \sin y = C$

4. Solution $\dfrac{dy}{dx} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$ is

(a) $y = \log\left(e^x + e^{-x}\right) + C$ (b) $y = \log\left(e^x - e^{-x}\right) + C$

(c) $y = \log\left(e^x + 1\right) + C$ (d) $y = \log\left(1 - e^{-x}\right) + C$

5. Solution of $e^{-x+y}\dfrac{dy}{dx}=1$ is

(a) $e^{y}=e^{x}+C$ (b) $e^{y}=e^{-x}+C$ (c) $e^{-y}=e^{-x}+C$ (d) $e^{-y}=e^{x}+C$

6. Solution of $\dfrac{dy}{dx}+\dfrac{1}{y}+y=0$ is

(a) $x+\dfrac{1}{2}\log\left(1+y\right)=C$ (b) $x+\dfrac{1}{2}\log\left(1+y^{2}\right)=C$

(c) $x+\log\left(1+y\right)=C$ (d) $x+\log\left(1+y^{2}\right)=C$

7. Solution of $\dfrac{dy}{dx}=\cos^{2}y$ is

(a) $x+\tan y=C$ (b) $\tan y=x+C$ (c) $\sin y+x=C$ (d) $\sin y-x=C$

8. Solution of $\dfrac{dy}{dx}=e^{y+x}+e^{y}x^{2}$ is

(a) $e^{x}+e^{y}=\dfrac{x^{3}}{3}+C$ (b) $e^{-x}+e^{y}+\dfrac{x^{3}}{3}=C$ (c) $e^{-x}+{}^{-y}=\dfrac{x^{3}}{3}+C$ (d) $e^{x}+e^{-y}+\dfrac{x^{3}}{3}=C$

9. By what substitution will the differential equation $\dfrac{dy}{dx}+\dfrac{y}{x}=\dfrac{y^{2}}{x^{2}}$ change in the linear equation

(a) $y=t$ (b) $y^{2}=t$ (c) $\dfrac{1}{y}=t$ (d) $\dfrac{1}{y^{2}}=t$

10. By what substitution will the differential equation $\dfrac{dy}{dx}+xy=e^{-x}y^{3}$ change in the linear equation

(a) $\dfrac{1}{y}=v$ (b) $y^{-2}=v$ (c) $y^{-3}=v$ (d) $y^{3}=v$

11. Find the general solution of differential equation $\dfrac{dy}{dx}+2x=e^{2x}$.

12. Find integrating factor of differential equation $\dfrac{dy}{dx}+y\tan x=\sin x$.

13. Find integrating factor of differential equation $\dfrac{dy}{dx}+\dfrac{1}{\sin x}y=e^{x}$.

14. Differential equation $\cos\left(x+y\right)\dfrac{dy}{dx}=1$ is of which form?

15. Differential equation $\dfrac{dy}{dx}-y\tan x=e^{x}\sec x$ is of which form?

Find general solution of following equations.

16. $\frac{dy}{dx}=\frac{4x+3y+1}{3x+2y+1}$

17. $\frac{dy}{dx}=\frac{y}{x}\left\{\log\left(\frac{y}{x}\right)+1\right\}$

18. $x\frac{dy}{dx}=y+2\sqrt{y^2-x^2}$

19. $\frac{dy}{dx}=e^{x-y}\left(e^y-e^x\right)$

20. $\frac{dy}{dx}+x\sin 2y=x^3\cos^2 y$

IMPORTANT POINTS

1. An equation involving derivatives independent variable, dependent variable and derivative of the dependent variable with respect to independent variable is known as a differential equation. Differential equations are of two types:
   (i) Ordinary differentaial equation
   (ii) Partial differential equation
2. Order of a differential equationis the order of highest order derivative occuring in the differential equation.
3. Degree of a differential equationis the degree of the highest order derivative, when differential cofficients are made free from redicals and functions.
4. **Solution of differential equation:**
   The solution to the differential equation used in the equation refers to a relationship in the independent and dependent variables which does not contain any differential coefficient and the given differential equatiion is satisifed from derivative obtained.
   The solution of a differential equation is also called its primitive because the differential equation is a relation derived from it.
   (i) **General or total solution :** In the solutioin of a differential equation if there are arbitarary constants equal tot he order of it then that solution is called general solution. This is also called total solutiion or total integral or total primitive.
   (ii) **Particular solution :** The solution of a differential equation obtained by assigning particular values to the arbitrary cosntants in the general solution is called particualr solution.
   (iii) **Singular solution :** The solutions of a differential equation where arbitrary constants are not present and fail to have a particualr solutiion of general solution.
5. **Differential methods to solve differential equation of first order and first degree:**
   (A) Variable Sepaerable Method : Differential equations with variable separable on wriing the equation in general form $f(x)\,dx+g(y)\,dy=0$ and then on integrating, the required solution may be accurid.
   (B) Varable separation by substitution : The given differential equation may be reduced to variable, separable form by suitable substitution and by getting its solution and again substituting required solution can be obtained.

(C) Homogeneous differential equation : If the general form of differential equation may be written in the form of $\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)} = \frac{ax+by}{cx+dy}$ where $f_1(x, y)$ and $f_2(x, y), x$ are homogeneous functions of *x* and *y* then to reduce in variable separable equation use substitution $y = vx$.

(D) Equation reducible to homogeneous form

(i) form $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$, where $\frac{a}{a'} \neq \frac{b}{b'}$

to reduce into homogeneous use $x = X + h,\ y = Y + k$ cosntants *h* and *k* are selected such that $ah+bk+c=0$ and $a'h+b'k+c'=0$ on solving them the vlaues of *h* and *k* are found. At last put $X = x - h$ and $Y = y - k$ and get the required solutions.

(ii) when $\frac{a}{a'} = \frac{b}{b'}$ then put $ax + by = v$ and reduce the equationto variable separatio form and then get the solution.

(E) Linear differential equation

(i) Generalrm $\frac{dy}{dx} + Py = Q$ where *P* nd *Q*, are constants or function of *x*

Integrating factor $(I.F.) = e^{\int P dx}$

Soution : $y(\text{I.F.}) = \int (\text{I.F.}) \times Q dx + C$

(ii) General form $\frac{dx}{dy} + P_1 x = Q_1$ where P_1 and Q_1 are constants or function of *y*

then integratign factor $(I.F.) = e^{\int P_1 dy}$

Solution $x \times \text{I.F.} = \int I.F. \times Q_1 dy + C$

6. Differential equation reducible to linear differential equation (Bernoulli's equation) $\frac{dy}{dx} + Py = Qy^n$, where *P* and *Q*, are constants or function of *x*, to reduce it into a linear differential equation divide by y^n, then put $\frac{1}{y^n} = t$ and solve. At last put $t = y^{-n}$ to get required solution.

Answers

Exercise 12.1

1. order 1 degree 1
2. order 2 degree 1
3. order 2 degree 2
4. order 1 degree 4
5. order 2 degree 2
6. order 1 degree 1
7. order 2 degree 3
8. order 1 degree 2

Exercise 12.2

1. $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=0$
2. $x+y\dfrac{dy}{dx}=0$
3. $\dfrac{d^2y}{dx^2}-8\dfrac{dy}{dx}+15y=0$
4. $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+2y=0$
5. $\dfrac{d^2y}{dx^2}+y=0$

Exercise 12.4

1. $\sin x\left(e^y+1\right)=C$
2. $y-x=C(1+xy)$
3. $\log y=2\left[x-\log(x+1)\right]+C$
4. $e^y=e^x+\dfrac{1}{3}x^3+C$
5. $e^y\left(\sin x+\cos x\right)=C$
6. $y=e^{3x}+C$
7. $\sin^2 x+\sin^2 y=C$
8. $y\sin y=x^2\log x+C$
9. $y=2\tan\dfrac{x}{2}-x+C$
10. $y=\dfrac{1}{3}\sin^{-1}x^3+C$

Exercise 12.5

1. $x+y=a\tan\left(\dfrac{y-C}{a}\right)$
2. $x+y+2=ce^y$
3. $y=\tan\left(\dfrac{x+y}{2}\right)+C$
4. $x+e^{-(x+y)}=C$
5. $x-y+c=\log(x+y)$
6. $2(y-x)=\log(1+2x+2y)+C_1$
7. $x=\tan(x+y)-\sec(x+y)+C$
8. $2x+(x-y)^2=0$
9. $y=\tan\left(\dfrac{x+y}{2}\right)+C$
10. $2(x-y)+\log(x-y+2)=x+c$

Exercise 12.6

1. $y=Ce^{x^3/3y^3}$
2. $\tan\dfrac{y}{2x}=Cx$
3. $(x+cy)=y\log x$
4. $x=Ce^{\cos(y/x)}$
5. $y+\sqrt{x^2+y^2}=Cx^2$
6. $y=C\left(x^2-y^2\right)$
7. $x+ye^{x/y}=C$
8. $x^2y^2+2x^3y=C$
9. $\tan^{-1}\left(\dfrac{y}{x}\right)=\log x+C$
10. $\dfrac{x}{y}+\log(xy)=0$

Exercise 12.7

1. $3\left(x^2+y^2\right)+4xy-10(x+y-1)=C$
2. $x-2y+\log(x-y+2)=C$
3. $x+2y+\log(2x+y-1)=C$
4. $3x+2y+C+2\log(1-x-y)=0$

5. $3(y-1)^2+4\left(x-\frac{3}{2}\right)(y-1)-6\left(x-\frac{3}{2}\right)^2=C$

Exercise 12.8

1. $y=2x-1+Ce^{-2x}$ 2. $y=\tan x-1+Ce^{-\tan x}$ 3. $y=\frac{4x^3}{3(1+x^2)}+\frac{C}{(1+x^2)}$ 4. $xy^2=2y^5+C$

5. $y\sin x=\frac{1}{2}x-\frac{1}{4}\sin 2x+C$ 6. $y=\sqrt{1-x^2}+C(1-x^2)$

7. $x^2y=C+(2-x^2)\cos x+2x\sin x$ 8. $16x^2y=4x^4\log x-x^4+C$

9. $xe^y=\tan y+C$ 10. $x=\frac{1}{2}e^{\tan^{-1}y}+Ce^{-\tan^{-1}y}$

Exercise 12.9

1. $y^{-2}=1+x^2+Ce^{x^2}$ 2. $e^y=e^x-1+Ce^{-e^x}$ 3. $\frac{1}{y}-\sin x+C\cos x=0$

4. $\sin x\sin y=C+e^{\sin x}$ 5. $\tan y=\frac{1}{2}(x^2-1)+Ce^{-x^2}$ 6. $\frac{1}{\log y}=\frac{1}{2x}+Cx$

7. $y(1+x^2)=\tan^{-1}x-\pi/4$

Miscellaneous Exercise 12

1. (b)	2 (a)	3. (a)	4. (b)
5. (a)	6. (b)	7. (b)	8. (d)
9. (c)	10. (b)	11. $y+x^2=\frac{1}{2}e^{2x}+C$	12. $\sec x$

13. $\tan x/2$ 14. Equation reducible to variable separation 15. Linear equation

16. $2x^2+3xy+y^2+x+y=0$ 17. $\log\left(\frac{y}{x}\right)=Cx$ 18. $y+\sqrt{y^2-x^2}=Cx^3$

19. $e^y=e^x+1+Ce^{e^x}$ 20. $e^{e^2}\tan y=\frac{1}{2}(x^2-1)e^{x/2}+C$

13

Vector

13.01 Introduction

As we knows that many useful physical quantities in nature are of two types, scalars and vectors. Scalars are those quantities which are completely determined by a single real number when the units of measurement of that quantity are given. Scalars are not related or assigned to any particular direction in space. For example, mass, volume, temperature, density etc are scalars. Scalars depend only on the points in space but not on any particular choice of the coordinate system. Vectors are those quantities which are completely determined if their lengths (also called magnitude) and their directions in space are given. For example displacement, velocity, acceleration, force, weight, momentum, electric field intensity etc. are vectors.

In this chapter, we will study basic concepts about vectors, various operations on vectors and their algebraic and geometric properties.

13.02 Basic Concepts

Let L be any straight line in plane or three dimensional space. This line can be given two directions by means of arrow heads. A line with one of these directions prescribed is called a *directed line*. Now observe that if we restrict the line *L* to the line segment AB, then a magnitude is prescribed on the line L with one of the two directions, so that we obtain a *directed line segment* (Fig). Thus, a directed line segment has magnitude as well as direction.

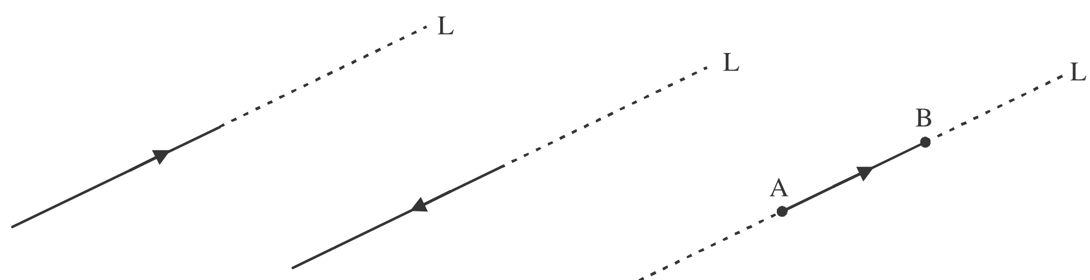


Fig. 13.01

Each directed line segment has following properties:

(i) Length: The length of directed line segment $\vec{AB}$ is the length of line segment represented by AB or $|\vec{AB}|$

(ii) Support: The base of a directed line segment $\vec{AB}$ is a line L whose segment is AB

(iii) Sense: the point A from where the vector $\vec{AB}$ starts is called its *initial point,* and the point B where it ends is called its *terminal point*. A directed line segment $B\vec{A}$ is from A to B where as for it is from B to A

Note: Although $\vec{AB}$ and $\vec{BA}$ have same length and base yet they are different vectors as $\vec{AB}$ and $\vec{BA}$ are opposite senses.

Vector Quantity : A quantity that has magnitude as well as directionis called a vector notice that directed line segment is a vector, denoted as $\vec{AB}$ or simply as $\vec{a}$, and read as vector $\vec{AB}$ or vector $\vec{a}$

Magnitude of the Vector: The distance between initial and terminal points of a vector is called the *magnitude* (or length) of the vector, denoted as $|\vec{a}|$ or $|\overrightarrow{AB}|$ where $\vec{a}$ thus the magnitude of vector $=|\vec{a}|=a$

Note : $|\vec{a}| \geq 0$

13.03 Various Types of Vectors

(1) Unit vector : A vector whose magnitude is unity (i.e. 1 unit) is called a unit vector. The unit vector in the direction of a given vector $\hat{a}$. We denote the unit vector in the direction of vector *a*, *b*, *c* as $\hat{a}, \hat{b}, \hat{c}$ and it is given by

$$\hat{a}=\frac{\vec{a}}{|\vec{a}|}, \hat{b}=\frac{\vec{b}}{|\vec{b}|}, \hat{c}=\frac{\vec{c}}{|\vec{c}|}$$

$\hat{a}$ is read as *a* cap.

(2) Zero or Null Vector: A vector whose initial and terminal points coincide, is called a zero vector (or null vector), and denoted as $\vec{O}$. Zero vector can not be assigned a definite direction as it has zero magnitude. Or alternatively otherwise, it may be regarded as having any direction. The vectors $\overrightarrow{AA}$, $\overrightarrow{BB}$ represent the zero vector.

also $|\vec{a}|=0$

i.e. if $|\overrightarrow{AB}|=0$

then A and B coincides.

(3) Like Vectors: If two vectors have same direction or senses then they are called Like Vectors.

(4) Equal Vectors: Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal, if they have the same magnitude and direction regardless of the positions of their initial points, and written as $\vec{a}, \vec{b}$.

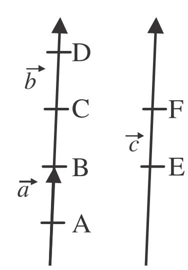


Fig. 13.02

In the fig : (13.02) the initial and terminal points of vectors $\overrightarrow{AB}, \overrightarrow{CD}, \overrightarrow{EF}$ represented by $\vec{a}, \vec{b}$ and $\vec{c}$ are different but their length is same therefore they are equal vectors.

i.e. $\overrightarrow{AB}=\overrightarrow{CD}=\overrightarrow{EF}$

If $\vec{a}$ and $\vec{b}$ are equal vectors then we write them as $\vec{a}=\vec{b}$.

(5) Unlike Vectors: If the direction of the vectors are opposite then they are called unlike vectors.

(6) Negative Vector: A vector whose magnitude is the same as that of a given vector (say, $\overrightarrow{BA}$ is negative of the vector $\overrightarrow{AB}$, and written as $\overrightarrow{BA} = -\overrightarrow{AB}$

$\therefore$ If $\vec{a} = \overrightarrow{AB}$ then $\overrightarrow{BA} = -\vec{a}$

Position Vector

From a rectangular coordinate system consider a point P, having coordinates (x, y) with respect to the origin O(0, 0). Then the vector $\overrightarrow{OP}$ having O and P as its initial and terminal points, respectively, is called the *position vector* of the point P with respect to O. Using distance formula (from Class XI), the magnitude of $\overrightarrow{OP}$ (or $\vec{r}$) is given by

$$\left|\overrightarrow{OP}\right| = \sqrt{x^2 + y^2}$$

For Example : Represent graphically a displacement of 40 km, 30° east of North.

Solution : The vector $\overrightarrow{OP}$, represents the required displacement (Fig: 13.03)

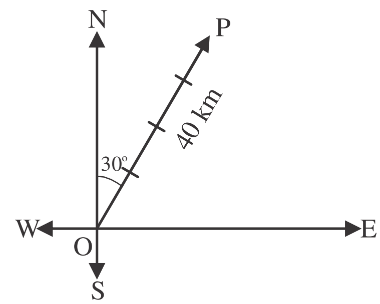


Fig. 13.03

13.04 Addition of Vectors

(A): Addition of Two Vectors

If there are two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ in a plane whcih are denoted by $\vec{a}$ and $\vec{b}$ then we add the two vectors by two methods.

I. Triangle law of Vector Addition: A vector $\overrightarrow{OE}$ simply means the displacement from a point E to the point F. Now consider a situation that a girl moves from O to E and then from E to F (Fig. 13.04). The net displacement made by the girl from point o to the point F is given by the vector $\overrightarrow{OF}$ and expressed as

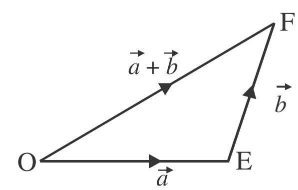


Fig. 13.04

$$\overrightarrow{OE} + \overrightarrow{EF} = \overrightarrow{OF}$$

$$\vec{a} + \vec{b} = \overrightarrow{OF} \text{ where } \overrightarrow{OE} = \vec{a} \text{ and } \overrightarrow{EF} = \vec{b}$$

This is known as the *triangle law of vector addition*. In general, if we have two vectors $\vec{a}$ and $\vec{b}$ (Fig. 13.04), then to add them, they are positioned so that the initial point of one coincides with the terminal point of the other. According to this law, "If two vectors in same order represents the two sides of a triangle then their sum is represented by the third side of triangle in opposite order".

II. Parallelogram law of Vector Addition: We have two vectors $\vec{a}$ and $\vec{b}$ represented by the two adjacent sides of a parallelogram in magnitude and direction (Fig. 13.05), then their sum $\vec{a} + \vec{b}$ is represented in magnitude and direction by the diagonal of the parallelogram through their common point.

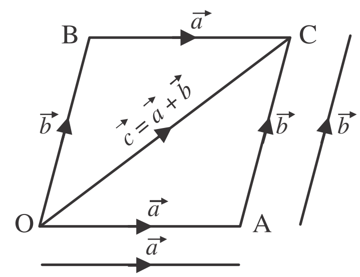


Fig. 13.05

Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$

Now OACB is a Parallelogram and OC is the digonal of OACB Here! $\overrightarrow{OA} = \overrightarrow{BC} = \vec{a}$ and $\overrightarrow{OB} = \overrightarrow{AC} = \vec{b}$.

In triangle OAC using triangle law of addition $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \vec{a} + \vec{b}$

So, if two vectors are represented in magnitude and direction by two adjacent sides of a parallelogram, then their sum is represented by diagonal of parallelogram which is cointial with the given vectors. This is known as 'parallelogram law of vector addition'.

(B) Addition of more than two Vectors:

For addition of more or more than two vectors the triangle law of addition can be used. This addition of vectors is known as Polygon law of vector addition.

Example : Suppose we have to add vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$. Let us take point O in a plane. Draw $\overrightarrow{OA} = \vec{a}$ s, also draw $\overrightarrow{AB} = \vec{b}$ similarly draw $\overrightarrow{BC} = \vec{c}$. Now by triangle law of vector addition we have

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB} \Rightarrow \vec{a} + \vec{b} = \overrightarrow{OB}$$

$$\overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OC} \Rightarrow \vec{a} + \vec{b} + \vec{c} = \overrightarrow{OC}$$

and $$\overrightarrow{OC} + \overrightarrow{CD} = \overrightarrow{OD} \Rightarrow \vec{a} + \vec{b} + \vec{c} + \vec{d} = \overrightarrow{OD}$$

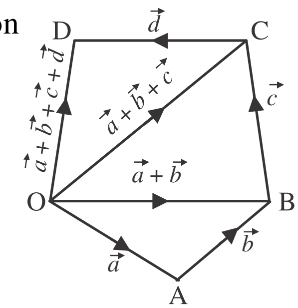


Fig. 13.06

sNow vector $\overrightarrow{OD}$ denotes the sum of vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$. Polygon OABCD is called as Polygon of vectors.

Note : If the initial point of first vector and terminal point of last vector coincides then the sum of the vectors is always zero.

13.05 Properties of Vector Addition:

Vector addition has the following properties:

(i) Commutativity: Addition of vectors follows the commutative law i.e. for any two vectors $\vec{a}$ and $\vec{b}$

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

Proof : Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{AB} = \vec{b}$

By Triangle law of addition we have

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{a} + \vec{b} \qquad \text{. . . (i)}$$

Complete the parallelogram OABC, such that

$$\overrightarrow{CB} = \overrightarrow{OA} = \vec{a}$$

and $$\overrightarrow{OC} = \overrightarrow{AB} = \vec{b}$$

In triangle OCB,

$$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB} = \vec{b} + \vec{a} \qquad \text{. . . (ii)}$$

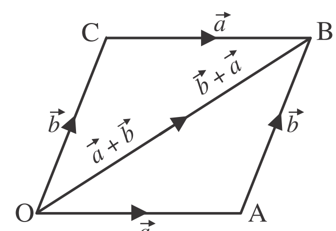


Fig. 13.07

From equation (i) and (ii),

$$\vec{a}+\vec{b}=\vec{b}+\vec{a}$$

Thus addition of vectors is commutative.

(ii) Associativity: Addition of vectors obeys the associative law i.e. let $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors then

$$\left(\vec{a}+\vec{b}\right)+\vec{c}=\vec{a}+\left(\vec{b}+\vec{c}\right)$$

Proof : Let vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are denoted by $\overrightarrow{OA}, \overrightarrow{AB}$ and $\overrightarrow{BC}$, thus $\overrightarrow{OA}=\vec{a}, \overrightarrow{AB}=\vec{b}$ and $\overrightarrow{BC}=\vec{c}$. Using triangle law of vector addition in triangle OAB and OBC

$$\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}=\vec{a}+\vec{b}$$

and $\overrightarrow{OC}=\overrightarrow{OB}+\overrightarrow{BC}=\left(\vec{a}+\vec{b}\right)+\vec{c}$ (1)

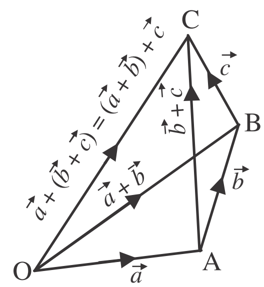


Fig. 13.08

Similarly triangle law of vector addition in triangles ABC and OAC

$$\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}=\vec{b}+\vec{c}$$

and $\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{AC}=\vec{a}+\left(\vec{b}+\vec{c}\right)$ (2)

from equation (1) and (2)

$$\left(\vec{a}+\vec{b}\right)+\vec{c}=\vec{a}+\left(\vec{b}+\vec{c}\right)$$

Thus the addition of vectors is associative.

Note: It is clear from the above rule that addition of vectors $\vec{a},\vec{b},\vec{c}$ does not depend in the order in which they are added. Thus the above addition can be expressed as $\vec{a}+\vec{b}+\vec{c}$.

(iii) Identity:

For every vector $\vec{a}$, $\vec{a}+\vec{0}=\vec{a}=\vec{0}+\vec{a}$, where $\vec{0}$ is a zero vector is known as identity vector for addition

Proof : From definition of addition of vectors

$$\overrightarrow{OA}=\overrightarrow{OA}+\overrightarrow{AA}=\vec{a}+\vec{0}$$

$\therefore$ $\vec{a}=\vec{a}+\vec{0}$

similarly $\vec{a}=\vec{0}+\vec{a}$

(iv) Additive inverse : For every vector $\vec{a}$, there corresponds a vector $-\vec{a}$ such that $\vec{a}+(-\vec{a})=(-\vec{a})+\vec{a}=\vec{0}$

Proof : Let vector $\vec{a}=\overrightarrow{OP}$ then by definition of Negative Vector, $\left(-\vec{a}\right)$ will be denoted by $\overrightarrow{PO}$

Now $\vec{a}+\left(-\vec{a}\right)=\overrightarrow{OP}+\overrightarrow{PO}=\overrightarrow{OO}=\vec{O}$

similarly $\left(-\vec{a}\right)+\vec{a}=\overrightarrow{PO}+\overrightarrow{OP}=\overrightarrow{PP}=\vec{O}$

Fig. 13.10

thus from (1) and (2) $\vec{a}+\left(-\vec{a}\right)=\left(-\vec{a}\right)+\vec{a}=\vec{O}$

13.06 Subtraction of Vectors

Let $\vec{a}$ and $\vec{b}$ are two vector quantities and let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$. Now if we have to find $\vec{a} - \vec{b}$ then at point B draw a line BD opposite in direction and equal in length to BC which represents the directed line segment as $\overrightarrow{BD} = -\vec{b}$

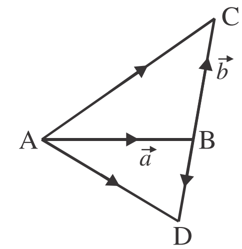


Fig. 13.10

Join A and D. Now using triangle law of addition in triangle ABD

$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = \vec{a} + (-\vec{b}) = \vec{a} - \vec{b}$$

Similarly if we have to subtract $\vec{a}$ from $\vec{b}$ i.e. we have to find $(\vec{a} - \vec{b})$ then add the negative of vector $\vec{a}$ i.e. $(-\vec{a})$ to vector $\vec{b}$

13.07 Multiplication of a Vector by a Scalar

Let $\vec{a}$ be a given vector and λ a scalar. Then the product of the vector $\vec{a}$ by the scalar λ is denoted as $\lambda\vec{a}$, is called the multiplication of vector $\vec{a}$ by the scalar λ Note that, $\lambda\vec{a}$ is also a vector, collinear tot he vector $\vec{a}$. The vector, $\lambda\vec{a}$ has the direction same (or opposite) to that of vector $\vec{a}$ according as the value of λ is positive (or negative). Also, the magntiude of vector $\lambda\vec{a}$ is $|\lambda|$ times the magnitude of the vector $\vec{a}$, i.e.,

$$|\lambda\vec{a}| = |\lambda||\vec{a}|$$

A geometric visualization of multiplication of a vector by a scalar is given in Fig. 13.10,

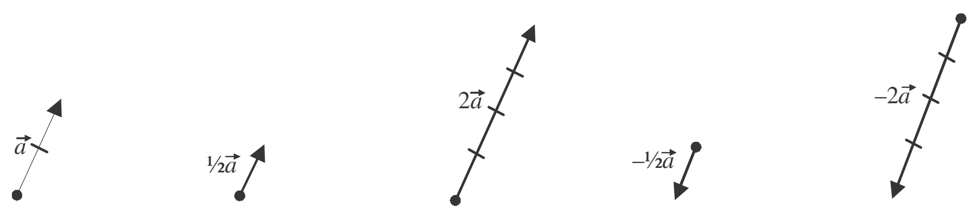


Fig. 13.11

What $\lambda = -1$, then $\lambda\vec{a} = -\vec{a}$ which is a vector having magnitude equal to the magnitude of $\vec{a}$. The vector $-\vec{a}$ called the *negative* (or *additive inverse*) *of vector* $\vec{a}$ and we always have $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{O}$.

Also, if $\lambda = \frac{1}{|\vec{a}|}$, provided $\vec{a} \neq 0$, i.e. $\vec{a}$ is not a null vector, then

$$|\lambda\vec{a}| = |\lambda||\vec{a}| = \frac{1}{|\vec{a}|}|\vec{a}| = 1$$

So, $\lambda\vec{a}$ represents the unit vector in the direction of $\vec{a}$

$$\hat{a} = \frac{1}{|\vec{a}|}\vec{a}$$

13.08 Components of a Vector

Let us take the points A (1, 0 , 0), B(0, 1, 0) and C (0, 0, 1) on the x-axis, y-axis and z-axis respectively. Then, clearly

$$|\vec{OA}|=1, |\vec{OB}|=1 \text{ and } |\vec{OC}|=1$$

The vectors $\vec{OA}, \vec{OB}$ and $\vec{OC}$, each having magnitude 1, are called *unit vectors along the axes* OX, OY and OZ respectively and denoted by $\hat{i}, \hat{j}$ and $\hat{k}$ respectively

Let P (x, y,. z) is a point whose position vector is $\vec{OP}$. Therefore

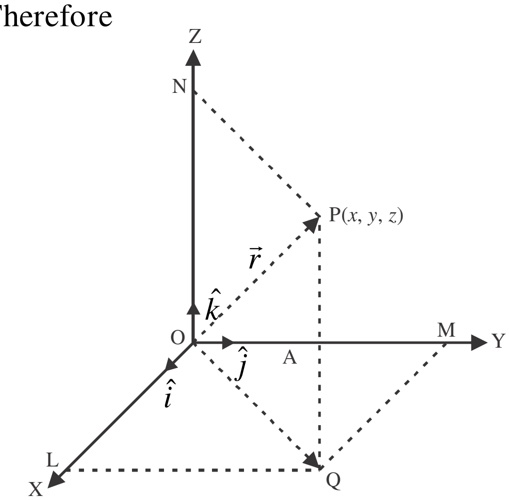


Fig. 13.12

$$\vec{OL}=x\hat{i}$$

$$\vec{OM}=\vec{LQ}=y\hat{j}$$

$$\therefore \quad \vec{OQ}=\vec{OL}+\vec{LQ}$$

$$=x\hat{i}+y\hat{j}$$

again $\vec{OP}=\vec{OQ}+\vec{QP}$

$$=\left(x\hat{i}+y\hat{j}\right)+z\hat{k}$$

$$=x\hat{i}+y\hat{j}+z\hat{k}$$

Thus with respect to O we get the position vector of P i.e. $\vec{OP}=x\hat{i}+y\hat{j}+z\hat{k}$.

This is known as the component form of the vector where x, y and z *are the scalar components of* $\overrightarrow{OP}$ and $x\hat{i}, y\hat{j}$ and $z\hat{k}$ are the vector components of $\overrightarrow{OP}$ Some times x, y and z are also termed as *rectangular components*.

If $\quad \overrightarrow{OP}=\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ then

$$\overrightarrow{OP}=|\vec{r}|=\sqrt{x^2+y^2+z^2}$$

13.09 Vector joining two points

If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are any two points, then the vector joining P_1 and P_2 is the vector $\overrightarrow{P_1P_2}$ (Fig. 13.12). Joining the points P_1 and P_2 with the origin O, and applying triangle law, from the triangle OP_1P_2, $\overrightarrow{OP_1}+\overrightarrow{P_1P_2}=\overrightarrow{OP_2}$ we have

Using the properties of vector addition, the above equation becomes

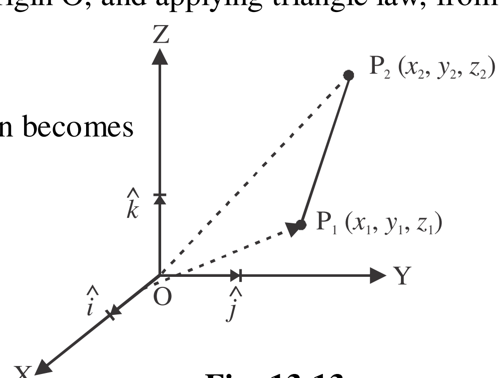


Fig. 13.13

$$\overrightarrow{P_1P_2}=\overrightarrow{OP_2}-\overrightarrow{OP_1}$$

i.e. $$\overrightarrow{P_1P_2}=\left(x_2\hat{i}+y_2\hat{j}+z_2\hat{k}\right)-\left(x_2\hat{i}+y_1\hat{j}+z_1\hat{k}\right)$$

$$=\left(x_2-x_1\right)\hat{i}+\left(y_2-y_1\right)\hat{j}+\left(z_2-z_1\right)\hat{k}$$

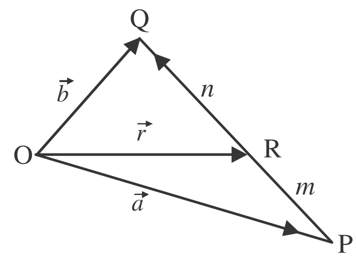
The magnitude of vector $\overrightarrow{P_1P_2}$ is given by

$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

13.10 Section Formula

Let P and O be two points represented by the position vectors $\overline{OP}$ and $\overline{OQ}$ with respect to the origin O. Then the line segment joining the points P and Q may be divided by a third point, say R, in two ways-internally and externally. (Fig. 13.10 (a) and Fig. 13.10 (b)). Here, we intend to find the position vector $\overline{OR}$ for the point R with respect tot he origin O. We take the two cases one by one.

Case& I: When R divides PQ internally



Fi.g 13.14 (a)

Let R, divides $\overline{PQ}$ internally in the ratio $m : n$ (Fig. 13.13(a))

$$\frac{PR}{RQ} = \frac{m}{n}$$

$\Rightarrow$ $$nPR = mRQ$$

$\Rightarrow$ $$n\overrightarrow{PR} = m\overrightarrow{RQ}$$

$\Rightarrow$ n (*position vector of R* –*position vector of* P) = m (*position vector of Q* –*position vector of* R)

$\Rightarrow$ $$n(\vec{r}-\vec{a}) = m(\vec{b}-\vec{r})$$

$\Rightarrow$ $$(m+n)\vec{r} = m\vec{b}+n\vec{a}$$

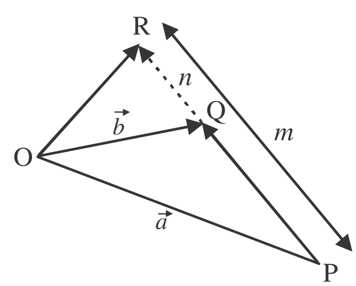
$\Rightarrow$ $$\vec{r} = \frac{m\vec{b}+n\vec{a}}{m+n}$$

Here, the position vector of the point R which divides P and Q internally in the ratio of $m : n$ is given by

$$\overrightarrow{OR} = \frac{m\vec{b}+n\vec{a}}{m+n}$$

Case& II: When R, divides PQ externally:

Let the position vector of the point R which divides the line segment PQ externally in the ratio $m : n$ (Fig. 13.14(b) then



Fi.g 13.14 (b)

$$\frac{PR}{QR} = \frac{m}{n}$$

$\Rightarrow$ $$nPR = mQR$$

$\Rightarrow$ $$n\overrightarrow{PR} = m\overrightarrow{QR}$$

$\Rightarrow n$ (*Position vector of R* –*Position vector of* P) = m (*Position vector of R* – *Position vector of* Q)

$\Rightarrow$ $$n(\vec{r}-\vec{a}) = m(\vec{r}-\vec{b})$$

$\Rightarrow$ $$m\vec{b}-n\vec{a} = m\vec{r}-n\vec{r}$$

$\Rightarrow$ $$\vec{r} = \frac{m\vec{b}-n\vec{a}}{m-n}$$

Note: if R, is the midpoint of PQ, then $m : n$. And therefore, from Case I, the midpoint R of $\overrightarrow{PQ}$, will have its position vector as $\overrightarrow{OR} = \frac{\vec{a}+\vec{b}}{2}$.

Illustrative Examples

Example 1. Find the sum of the vectors $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.

Solution : The sum of the vectors $= \vec{a} + \vec{b} + \vec{c}$

$$= \left(\hat{i} - 2\hat{j} + \hat{k}\right) + \left(-2\hat{i} + 4\hat{j} + 5\hat{k}\right) + \left(\hat{i} - 6\hat{j} - 7\hat{k}\right)$$

$$= \left(\hat{i} - 2\hat{j} + \hat{k}\right) + \left(-2\hat{i} + 4\hat{j} - 6\hat{k}\right) + \left(\hat{i} + 5\hat{j} - 7\hat{k}\right)$$

$$= 0 - \hat{i} - 4\hat{j} - \hat{k} = -4\hat{j} - \hat{k}$$

Example 2. If vectors $\vec{a} = x\hat{i} + 2\hat{j} + z\hat{k}$ and $\vec{b} = 2\hat{i} + y\hat{j} + \hat{k}$ are equal then find the value of x, y and z.

Solution : Two vectors are equal if their scalar components are equal.

Thus if $\vec{a}$ and $\vec{b}$ are equal if $x = 2,\ y = 2,\ z = 1$

Example 3. Let $\vec{a} = \hat{i} + 2\hat{j}$ and $\vec{b} = 2\hat{i} + \hat{j}$ then is $|\vec{a}| = |\vec{b}|$? Are vector $\vec{a}$ and $\vec{b}$ equal?

Solution : Here $|\vec{a}| = \sqrt{1^2 + 2^2} = \sqrt{5}$ and $|\vec{b}| = \sqrt{2^2 + 1^2} = \sqrt{5}$

Therefore $|\vec{a}| = |\vec{b}|$ But the given vectors are not equal becase their corresponding components are not equal.

Example 4. Find the unit vector in the direction of the vector $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$.

Solution : The unit vector along vector $\vec{a}$ is $\hat{a} = \frac{1}{|\vec{a}|}\vec{a}$.

now $|\vec{a}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$

therefore $\hat{a} = \frac{1}{\sqrt{14}}\left(2\hat{i} + 3\hat{j} + \hat{k}\right) = \frac{2}{\sqrt{14}}\hat{i} + \frac{3}{\sqrt{14}}\hat{j} + \frac{1}{\sqrt{14}}\hat{k}$

Example 5. Find a vector in the direction of vector $\vec{a} = \hat{i} - 2\hat{j}$ which has magnitude 7 units.

Solution : The unit vector along vector $\vec{a}$ is $\hat{a} = \frac{1}{|\vec{a}|}\vec{a} = \frac{1}{\sqrt{5}}\left(\hat{i} - 2\hat{j}\right) = \frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{j}$

therefore the vector along $\vec{a}$ having magnitude 7 unit $7\hat{a} = 7\left(\frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{j}\right) = \frac{7}{\sqrt{5}}\hat{i} - \frac{14}{\sqrt{5}}\hat{j}$

Example 6. Find the unit vector in the direction of the vector $\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} + 3\hat{k}$.

Solution : The sum of the given vectors

$$\vec{a} + \vec{b} = \vec{c} \text{ (let) } \therefore \vec{c} = 4\hat{i} + 3\hat{j} - 2\hat{k}$$

and $$|\vec{c}| = \sqrt{4^2 + 3^2 + (-2)^2} = \sqrt{29}$$

Required unit vector

$$\hat{c} = \frac{1}{|\vec{c}|}\vec{c} = \frac{1}{\sqrt{29}}\left(4\hat{i} + 3\hat{j} - 2\hat{k}\right) = \frac{4}{\sqrt{29}}\hat{i} + \frac{3}{\sqrt{29}}\hat{j} - \frac{2}{\sqrt{29}}\hat{k}$$

Example 7. Find the vector directed from point P to Q joining the points P(2, 3, 0) and Q(–1, –2, –4).

Solution: As P is the initial point and Q is the terminal point, therefore

$$\overrightarrow{PQ} = \text{Position vector of Q} - \text{Position vector of P}$$

$$\overrightarrow{PQ} = -i - 2j - 4k - (2i + 3j)$$

$$\overrightarrow{PQ} = (-1-2)\hat{i} + (-2-3)\hat{j} + (-4-0)\hat{k}$$

$$\Rightarrow \quad \overrightarrow{PQ} = -3\hat{i} - 5\hat{j} - 4\hat{k}$$

Example 8. Find the position vector of a point R which divides the line joining two points P and Q in ratio 2 : 1 whose position vectors are $\overrightarrow{OP} = 3\vec{a} - 2\vec{b}$ and $\overrightarrow{OQ} = \vec{a} + \vec{b}$.

Solution : (i) the position vector of a point R which divides the line joining two points P and Q in the ratio 2 : 1 internally is

$$\overrightarrow{OR} = \frac{2\left(\vec{a} + \vec{b}\right) + \left(3\vec{a} - 2\vec{b}\right)}{3} = \frac{5\vec{a}}{3}$$

(ii) the position vector of a point R whcih divides the line joining two points P and Q in the ratio 2 : 1 externally is

$$\overrightarrow{OR} = \frac{2\left(\vec{a} + \vec{b}\right) - \left(3\vec{a} - 2\vec{b}\right)}{2-1} = 4\vec{b} - \vec{a}$$

Example 9. Show that the points $A\left(2\hat{i} - \hat{j} + \hat{k}\right)$, $B\left(\hat{i} - 3\hat{j} - 5\hat{k}\right)$, $C\left(3\hat{i} - 4\hat{j} - 4\hat{k}\right)$ are the vertices of a right angled triangle.

Solution : We have $\overrightarrow{AB} = (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k}$

$$\overrightarrow{BC} = (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$$

and $$\overrightarrow{CA} = (2-3)\hat{i} + (-1+4)\hat{j} + (1+4)\hat{k} = -\hat{i} + 3\hat{j} + 5\hat{k}$$

Further, note that $|\overrightarrow{AB}|^2 = 41 = 6 + 35 = |\overrightarrow{BC}|^2 + |\overrightarrow{CA}|^2$

Here, the triangle is a right angled triangle.

Exercise 13.1

1. Compute the magnitude of the following vectors:

$$\vec{a} = \hat{i} + \hat{j} + \hat{k};\ \vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k};\ \vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$$

2. Write two different vectors having same magnitude.
3. Write two different vectors having same direction.
4. Find the vlaues of x and y so that the vectors $2\hat{i} + 3\hat{j}$ and $x\hat{i} + y\hat{j}$ are equal.
5. Find the scalar and vector components of the vector with initial point (2, 1) and terminal point (–5, 7)
6. Find the sum of the vectors $\vec{a} = \hat{i} - 2\hat{j} + \hat{k};\ \vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.
7. Find the unit vector in the direction of the vector $\vec{c} = \hat{i} + \hat{j} + 2\hat{k}$.
8. Find the unit vector in the direction of vector $\overrightarrow{PQ}$ where P and Q are the points (1, 2, 3) and (4, 5, 6), respectively.
9. For given vectors, $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$, find the unit vector in the direction of the vector $\vec{a} + \vec{b}$.
10. Find a vector in the direction of vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 8 units.
11. Show that the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $-4\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear.
12. Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $P\left(\hat{i} + 2\hat{j} - \hat{k}\right)$ and $Q\left(-\hat{i} + \hat{j} + \hat{k}\right)$ respectively, in the ratio 2 : 1

    (i) internally (ii) externally.
13. Find the position vector of the mid point of the vector joining the points P(2, 3, 4) and Q(4, 1, –2).
14. Show that the points A, B and C with position vectors, $\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}, \vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$ respectively form the vertices of a right angled triangle.

13.11 Product of Two Vectors

So far we have studied about addition and subtraction of vectors. An other algebraic operation which we intend to discuss regarding vectors is their product. We may recall that product of two numbers is a number, product of two matrices is again a matrix. But in case of functions, we may multiplication of two vectors is also defined in two ways, namely, scalar (or dot) product where the result is a scalar, and vector (or vectors is also defined in two ways, namely, scalar (or dot) product where the result is a scalar, and vector (or cross) product where the result is a vector.

(I) Scalar product: In this the product of two vectors is a Scalar.

(II) Vector product: In this the product of two vectors is a vector.

13.12 Scalar or dot Product

Definition : If product of two vectors is a scalar quantity then it is called 'scalar or dot-product of vector'.

The scalar product of two non zero vectors $\vec{a}$ and $\vec{b}$ denoted by $\vec{a}\cdot\vec{b}$ (read as $\vec{a}$ dot $\vec{b}$) is defined as:

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = ab\cos\theta$$

$$(|\vec{a}| = a \text{ and } |\vec{b}| = b \text{ are the magnitudes of } \vec{a} \text{ and } \vec{b})$$

Note: When both the vectors are Unit vectors, i.e. $|\vec{a}|=1$, $|\vec{b}|=1$

$$\hat{a}.\hat{b} = (1)(1)\cos\theta = \cos\theta$$

13.13 Geometrical interpretation of Scalar Product

Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$ are two vectors, inclined at an angle θ, the scalar product is given by

$$\vec{a}\cdot\vec{b} = ab\cos\theta$$

$$= |\vec{a}||\vec{b}|\cos\theta \qquad (1)$$

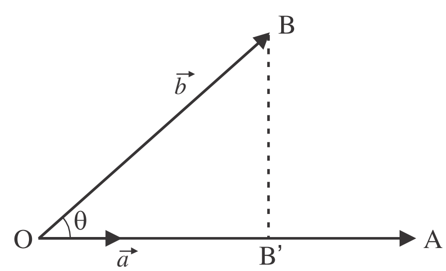


Fig. 13.15

Now from point A and B drop perpendicualr AM and BN on OB and OA then from ΔOMA and ΔONB

$OM = OA\cos\theta$ i.e. projection of $\overrightarrow{OM}$ in the direction of $\overrightarrow{OB}$

$ON = OB\cos\theta$ i.e projection of $\overrightarrow{OB}$ in the direction of $\overrightarrow{OA}$

From equation (1)

$$\vec{a}\cdot\vec{b} = |\vec{a}|(|\vec{b}|\cos\theta) = |\vec{a}|(ON)$$

$$= (\text{magnitude of } \vec{a})\ (\text{projection of } \vec{b} \text{ on } \vec{a}) \qquad (2)$$

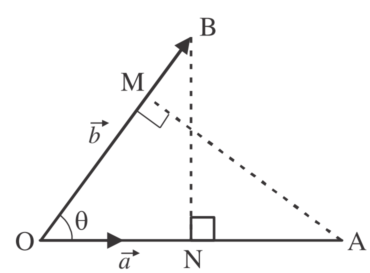


Fig. 13.16

Similarly from equation (1)

$$\vec{a}\cdot\vec{b} = |\vec{b}|(|\vec{a}|\cos\theta) = |\vec{b}|(OM)$$

$$= (\text{magnitude of } \vec{b})\ (\text{projection of } \vec{a} \text{ on } \vec{b}) \qquad (3)$$

Thus the scalar product of two vectors is the product of modulus of either vector and the project of the other in its direction.

Note: from (2) Projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \dfrac{\vec{a}}{|\vec{a}|}\cdot\vec{b} = \hat{a}\cdot\vec{b}$

and from (3) Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \vec{a}\cdot\dfrac{\vec{b}}{|\vec{b}|} = \vec{a}\cdot\hat{b}$

13.14 Some Important Deductions from Scalar Product of Vectros

We know that

$$\vec{a}\cdot\vec{b} = ab\cos\theta \qquad (1)$$

Observeations:

(i) When vectors $\vec{a}$ and $\vec{b}$ are parallel: In this condition the value of $\theta = 0^{\circ}$, thus from (1)

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos 0^{\circ} = |\vec{a}||\vec{b}| = ab$$

(ii) When vectors $\vec{a}$ and $\vec{b}$ coincides: In this condition the angle between the two vectors is zero i.e. $\theta = 0^{\circ}$, thus from (1)

$$\vec{a}\cdot\vec{a} = |\vec{a}||\vec{a}|\cos 0^{\circ} = |\vec{a}||\vec{a}| = aa = a^2$$

(iii) When vectors $\vec{a}$ and $\vec{b}$ are linear: In this condition the angle between the two vectors is 180° i.e. $\theta = 180^{\circ}$ thus from (1)

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos 180^{\circ} = ab(-1) = -ab$$

(iv) When vectors $\vec{a}$ and $\vec{b}$ are mutually perpendicualr: In this condition the angle between the two vectors is 90° i.e. $\theta = \pi/2$ thus from (1)

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\frac{\pi}{2} = |\vec{a}||\vec{b}|0 = 0$$

thus if two vectors are perpendicualr then

$$\vec{a}\cdot\vec{b} = 0$$

Converse: If the scalar product of two non-zero vectors $\vec{a}$ and $\vec{b}$ is zero then the vectors are perpendicular let

let $\quad \vec{a}\cdot\vec{b} = 0$

$\Rightarrow \quad |\vec{a}||\vec{b}|\cos\theta = 0$

$\Rightarrow \quad \cos\theta = 0 \qquad \left[\because |\vec{a}| \neq 0, |\vec{b}| \neq 0\right]$

$\Rightarrow \quad \theta = \pi/2 \quad \Rightarrow \quad \vec{a} \perp \vec{b}$

So $\quad \vec{a}\cdot\vec{b} = 0 \quad \Leftrightarrow \quad \vec{a} \perp \vec{b}$

Note: In view of the observations, for mutually perpendicualr unit vectors $\hat{i}, \hat{j}, \hat{k}$ we have

$$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$$

and $\quad \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

The above result can be expressed in the form of a table also

$\cdot$	i	j	k
i	1	0	0
j	0	1	0
k	0	0	1

13.15. Properties of Scalar Product

(i) **Commutativity:** Thus scalar product of two vector sis commutative.

Proof : We know that

$$\vec{a}\cdot\vec{b} = ab\cos\theta$$
$$= b\,a\cos\theta \qquad (\because ab = ba,)$$
$$= \vec{b}\cdot\vec{a}$$

(ii) **Associativity:** If $\vec{a}$ and $\vec{b}$ are two vectors then let m be any scalar

$$(m\vec{a})\cdot\vec{b} = \vec{a}\cdot(m\vec{b}) = m(\vec{a}\cdot\vec{b})$$

(iii) **Distributivity:** If $\vec{a},\vec{b}$ and $\vec{c}$ are three vectors then

$$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$$

similarly $\qquad (\vec{b}+\vec{c})\cdot\vec{a} = \vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a}$

13.16 Scalar Product of Two Vectors in terms of the Components

Let $\qquad \vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, are two vectors

$$\vec{a}\cdot\vec{b} = \left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)\cdot\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)$$
$$= a_1b_1(\hat{i}\cdot\hat{i})+a_1b_2(\hat{i}\cdot\hat{j})+a_1b_3(\hat{i}\cdot\hat{k})+a_2b_1(\hat{j}\cdot\hat{i})+a_2b_2(\hat{j}\cdot\hat{j})$$
$$+a_2b_3(\hat{j}\cdot\hat{k})+a_3b_1(\hat{k}\cdot\hat{i})+a_3b_2(\hat{k}\cdot\hat{j})+a_3b_3(\hat{k}\cdot\hat{k}) \quad \text{(from property (ii) and (iii))}$$
$$= a_1b_1+a_2b_2+a_3b_3 \qquad \text{(Article 13.15)}$$

$\therefore \qquad \vec{a}\cdot\vec{b} = a_1b_1+a_2b_2+a_3b_3$

Note: $\qquad \vec{a}\cdot\vec{a} = \left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)\cdot\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)$

$$= a_1a_1+a_2a_2+a_3a_3 = a_1^{\,2}+a_2^{\,2}+a_3^{\,2} = a^2$$

$\therefore \qquad (\vec{a})^2 = a^2$

13.17 Angle Between two Vectors:

We know by the definition of scalar product

$$\vec{a}\cdot\vec{b} = a\,b\cos\theta$$

or $\quad \cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{ab} = \left(\dfrac{\vec{a}}{a}\right)\cdot\left(\dfrac{\vec{b}}{b}\right) = \hat{a}\cdot\hat{b}$, where $\hat{a},\hat{b}$ are the unit vectors in the direction of $\vec{a}$ and $\vec{b}$

again if $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then

$$\vec{a}\cdot\vec{b} = (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\cdot(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})$$
$$= a_1b_1+a_2b_2+a_3b_3 \qquad \text{(Article 13.16)}$$

$$\therefore \qquad \cos\theta = \frac{\vec{a}\cdot\vec{b}}{ab} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

Note: if vectors $\vec{a}$ and $\vec{b}$ are mutually perpendicualr then $a_1b_1 + a_2b_2 + a_3b_3 = 0$

13.18 Components of any Vector $\vec{b}$ along and perpendicular to a Vector $\vec{a}$

Let $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}$ and $BM \perp OA$.

$\therefore$ by triangle law of addition in ΔOBM $\vec{b} = \overrightarrow{OB} = \overrightarrow{OM} + \overrightarrow{MB}$, where $\overrightarrow{OM}$ and $\overrightarrow{MB}$ are the perpendicualr vectors of vector $\vec{b}$ along vector $\vec{a}$

Now $\overrightarrow{OM} = (OM)\hat{a} = (b\cos\theta)\hat{a}$

$$= b\left(\frac{\vec{a}\cdot\vec{b}}{ab}\right)\hat{a} \qquad \text{(Article 13.17)}$$

$$= \left(\frac{\vec{a}\cdot\vec{b}}{a}\right)\hat{a} = \left(\frac{\vec{a}\cdot\vec{b}}{a^2}\right)\vec{a} \qquad \left[\because \hat{a} = \frac{\vec{a}}{a}\right]$$

and $\overrightarrow{MB} = \overrightarrow{OB} - \overrightarrow{OM}$

$$= \vec{b} - \left(\frac{\vec{a}\cdot\vec{b}}{a^2}\right)\vec{a}$$

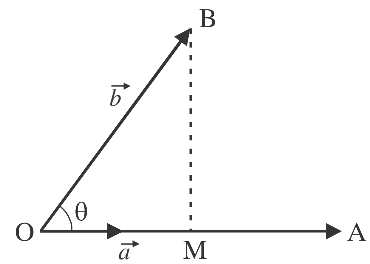


Fig. 13.17

Thus components of vector $\vec{b}$ in the direction of vector $\vec{a}$ and perpendicular along $\vec{a}$ are $\left(\frac{\vec{a}\cdot\vec{b}}{a^2}\right)\vec{a}$ and $\vec{b} - \left(\frac{\vec{a}\cdot\vec{b}}{a^2}\right)\vec{a}$

Illustrative Examples

Example 10. If $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{j} + \hat{k}$ then find the value of $\vec{a}\cdot\vec{b}$.

Solution: $\vec{a}\cdot\vec{b} = (\hat{i} + 2\hat{j} + 3\hat{k})\cdot(3\hat{i} + 2\hat{j} + \hat{k})$

$$= (1)(3) + (2)(2) + (3)(1) = 3 + 4 + 3 = 10$$

Thus the value of $\vec{a}\cdot\vec{b}$ is 10.

Example 11. For what value of λ are the vectors $2\hat{i} + \lambda\hat{j} + 5\hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ mutually perpendicular?

Solution: the vectors are perpendicular if their product is zero

$$(2\hat{i} + \lambda\hat{j} + 5\hat{k})\cdot(-\hat{i} + \hat{j} + \hat{k}) = 0$$

or $(2)(-1) + (\lambda)(1) + (5)(1) = 0$

or $2 + \lambda + 5 = 0$

or $\lambda = -3$

Thus at $\lambda = -3$ the vectors are perpendicular to each other.

Example 12. Find the angle between the vectors $3\hat{i}+\hat{j}+3\hat{k}$ and $2\hat{i}+2\hat{j}-\hat{k}$.

Solution: Let $\vec{a}=3\hat{i}+\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}+2\hat{j}-\hat{k}$ and let θ be the angle between $\vec{a}$ and $\vec{b}$.

$$\vec{a}\cdot\vec{b}=ab\cos\theta$$

$$\Rightarrow \quad \cos\theta=\frac{\vec{a}\cdot\vec{b}}{ab}=\frac{(3\hat{i}+\hat{j}+3\hat{k})\cdot(2\hat{i}+2\hat{j}-\hat{k})}{\sqrt{9+1+9}\sqrt{4+4+1}}$$

$$=\frac{(3)(2)+(1)(2)+(3)(-1)}{\sqrt{19}\sqrt{9}}=\frac{5}{3\sqrt{19}}$$

$$\Rightarrow \quad \cos^{-1}\left(\frac{5}{3\sqrt{19}}\right)$$

Example 13. Show that-

(i) $(\vec{a}+\vec{b})=a^2+2\vec{a}\cdot\vec{b}+b^2$

and (ii) $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=a^2-b^2$

Solution: (i) $(\vec{a}+\vec{b})^2=(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$

$$=\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}$$

$$=a^2+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{b}+b^2 \qquad [\because \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}]$$

$$=a^2+2\vec{a}\cdot\vec{b}+b^2$$

(ii) $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=\vec{a}\cdot\vec{a}-\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}-\vec{b}\cdot\vec{b}$

$$=a^2-\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{b}-b^2$$

$$=a^2-b^2 \qquad [\because \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}]$$

Example 14. If θ is the angle between the two vectors $\hat{a}$ and $\hat{b}$ then prove that

$$\sin(\theta/2)=\frac{1}{2}|\hat{a}-\hat{b}|$$

Solution : $|\hat{a}-\hat{b}|^2=(\hat{a}-\hat{b})\cdot(\hat{a}-\hat{b})$

$$=\hat{a}.\hat{a}-\hat{a}.\hat{b}-\hat{b}.\hat{a}+\hat{b}\hat{b}$$

$$=|\hat{a}|^2-2\hat{a}.\hat{b}+|\hat{b}|^2 \qquad \left[\because \hat{a}.\hat{b}=\hat{b}.\hat{a}\right]$$

$$=1-2\hat{a}\cdot\hat{b}+1 \qquad \left[\because |\hat{a}|=1=|\hat{b}|\right]$$

$$=2-2(1)(1)\cos\theta=2(1-\cos\theta)$$

$$=2\cdot\left(2\sin^2\frac{\theta}{2}\right)$$

$\Rightarrow \quad \left|\hat{a}-\hat{b}\right| = 2\sin\frac{\theta}{2}$ or $\sin\frac{\theta}{2} = \frac{1}{2}\left|\hat{a}-\hat{b}\right|$

Example 15. (i) If $\vec{a},\vec{b},\vec{c}$ are mutually perpendicualr vectors with equal magnitudes, then prove that vector $\vec{a}+\vec{b}+\vec{c}$ makes equal angle with vectors $\vec{a},\vec{b}$ and $\vec{c}$.

(ii) $\vec{a},\vec{b},\vec{c}$ are the vectors of magnitude 3, 4, 5 resp. If every vector is perpendicular on the sum of the other two then find the magnitude of vector $\vec{a}+\vec{b}+\vec{c}$.

Solution: (i) $\vec{a},\vec{b},\vec{c}$ are mutually perpendicualr therefore $\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$

again the magnitude of vectors $\vec{a},\vec{b},\vec{c}$ are equal $a = b = c$

and $\quad \left(\vec{a}+\vec{b}+\vec{c}\right)^2 = \left(\vec{a}+\vec{b}+\vec{c}\right).\left(\vec{a}+\vec{b}+\vec{c}\right)$

$$= \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+\vec{c}\cdot\vec{c}$$

$$= a^2+b^2+c^2 = 3a^2 \qquad \left[\because a = b = c \text{ तथा } \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0 \text{ इत्यादि}\right]$$

$$\Rightarrow \quad \left|\vec{a}+\vec{b}+\vec{c}\right| = \sqrt{3}a$$

$$\because \quad (\vec{a}+\vec{b}+\vec{c})\cdot\vec{a} = \vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a} = a^2$$

Let θ_1 be the angle between $\vec{a}+\vec{b}+\vec{c}$ and $\vec{a}$

$$\therefore \quad (\vec{a}+\vec{b}+\vec{c})\cdot\vec{a} = |\vec{a}+\vec{b}+\vec{c}||\vec{a}|\cos\theta_1$$

$$\Rightarrow \quad a^2 = (\sqrt{3}a)(a)\cos\theta_1$$

$$\Rightarrow \quad \cos\theta_1 = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \theta_1 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Similarly if vector $\vec{a}+\vec{b}+\vec{c}$ makes angle θ_2 and θ_3 with $\vec{b}$ and $\vec{c}$ then it can be proved that $\theta_2 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ and $\theta_3 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

i.e. vector $\vec{a}+\vec{b}+\vec{c}$ makes equal angle with the vectors $\vec{a},\vec{b}$ and $\vec{c}$

(ii) $\quad \vec{a}\cdot\left(\vec{b}+\vec{c}\right) = 0,\ \vec{b}\cdot\left(\vec{a}+\vec{c}\right) = 0$ and $\vec{c}\cdot\left(\vec{a}+\vec{b}\right) = 0$

adding all the three $2\left(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}\right) = 0$

and $\quad \vec{a}\cdot\vec{a} = a^2 = 9,\ b^2 = 16,\ c^2 = 25$

$$\left(\vec{a}+\vec{b}+\vec{c}\right)\cdot\left(\vec{a}+\vec{b}+\vec{c}\right)=\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{c}\cdot\vec{c}+2\left(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}\right)$$

$$\Rightarrow \qquad |\vec{a}+\vec{b}+\vec{c}|^2=9+16+25+0=50$$

$$\Rightarrow \qquad \left|\vec{a}+\vec{b}+\vec{c}\right|=\sqrt{50}=5\sqrt{2} \text{ units}$$

Exercise 13.2

1. If the magnitude of two vectors is 4 and 5 units then find their scalar product if the angle between the two vectors is

   (i) 60^o (ii) 90^o (iii) 30^o

2. Find the value of $\vec{a}.\vec{b}$ if $\vec{a}$ and $\vec{b}$ respectively are

   (i) $2\hat{i}+5\hat{j}; 3\hat{i}-2\hat{j}$ (ii) $4\hat{i}+3\hat{k}; \hat{i}-\hat{j}+\hat{k}$ (iii) $5\hat{i}+\hat{j}-2\hat{k}; 2\hat{i}-3\hat{j}$

3. Prove that $(\vec{a}\cdot\vec{b})^2 \le |\vec{a}|^2\,|\vec{b}|^2$

4. If the coordinates of P and Q are (3, 4) and (12, 9) respectively. Find the value of $\angle POQ$ where O is the origin.

5. For what value of λ are the vectors $\vec{a}$ and $\vec{b}$ mutually perpendicular.

   (i) $\vec{a}=2\hat{i}+\lambda\hat{j}+\hat{k}; \quad \vec{b}=4\hat{i}-2\hat{j}-2\hat{k}$ (ii) $\vec{a}=2\hat{i}+3\hat{j}+4\hat{k}; \quad \vec{b}=3\hat{i}+2\hat{j}-\lambda\hat{k}$

6. Find the projection of vector $4\hat{i}-2\hat{j}+\hat{k}$ on the vector $3\hat{i}+6\hat{j}-2\hat{k}$.

7. If $\vec{a}=2\hat{i}-16\hat{j}+5\hat{k}$ and $\vec{b}=3\hat{i}+\hat{j}+2\hat{k}$ then find a vector $\vec{c}$ where $\vec{a},\vec{b},\vec{c}$ denote the sides of right angle triangle.

8. If $\left|\vec{a}+\vec{b}\right|=\left|\vec{a}-\vec{b}\right|$, then prove that $\vec{a}$ and $\vec{b}$ are mutually perpendicular to each other.

9. If the coordinates of the points A, B, C and D are $(3, 2, 4)$, $(4, 5, -1)$, $(6, 3, 2)$ and $(2, 1, 0)$ respectively Then prove that lines AB and CD are mutually perpendicular.

10. For any vector $\vec{a}$ prove that $\vec{a}=(\vec{a}\cdot\hat{i})\hat{i}+(\vec{a}\cdot\hat{j})\hat{j}+(\vec{a}\cdot\hat{k})\hat{k}$

11. Using the vector method Prove that sum of the diagonals of the parallelogram is equal to the sum of square of its sides.

13.19 Vector or Cross Product of two Vectors

Definition : The vector product of two non zero vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\times\vec{b}$ and defined as

$$\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\hat{n}\,, \tag{1}$$

If the angle between $\vec{a}$ and $\vec{b}$ is $\theta\left(0\le\theta\le\pi\right)$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a},\vec{b}$ and $\hat{n}$ form a right handed screw system i.e., the right handed screw system rotated from $\vec{a}$ to $\vec{b}$ moves in the direction of $\hat{n}$.

In terms of vector product, the angle between two vectors $\vec{a}$ and $\vec{b}$ may be given as

$$|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\theta \quad\Rightarrow\quad \sin\theta=\frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} \qquad (2)$$

from (1) $$\hat{n}=\frac{\vec{a}\times\vec{b}}{|\vec{a}||\vec{b}|\sin\theta}=\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$$

thus the unit vector perpendicualr to vector $\vec{a}$ and $\vec{b}$ is $=\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$ (3)

13.20 Geometrical Interpretation of Vector Product

Let $\overrightarrow{OA}=\vec{a}, \overrightarrow{OB}=\vec{b}$ are two non parallel and non-zero vectors, the angle between them is θ and $\hat{n}$ is the unit vector perpendicular to vectors $\vec{a}$ and $\vec{b}$ then,

$$\left|\vec{a}\times\vec{b}\right|=\left|\vec{a}\right|\left|\vec{b}\right|\sin\theta$$

$$=(OA)(OB)\sin\theta \qquad (1)$$

Area of $OACB$

Cosnidering OA and OB as the sides of the parallelogram $OACB$,
Area of $OACB$ = 2 (Area of ΔOAB)

$$=2\left(\frac{1}{2}OA\cdot OB\sin\theta\right)=OA\cdot OB\sin\theta \qquad (2)$$

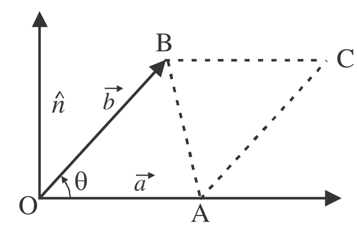


Fig. 13.18

from (1) and (2) the magnitude of $\vec{a}\times\vec{b}=\left|\vec{a}\times\vec{b}\right|$

13.21 Some Important Deductions from Vector Product

(i) The product of two parallel vectors is always zero:

Proof : If $\vec{a}$ and $\vec{b}$ are two parallel vectors and let θ be the angle between them then $\theta=0^\circ$ or $\theta=\pi^\circ$ thus in both the situations the value of the $\sin\theta$ will be zero.

$\therefore \quad \vec{a}\times\vec{b}=ab\sin\theta\hat{n}=(0)\hat{n}=\vec{O}$ [zero vector]

Converse : If the product of two vectors is zero then the vectors are parallel as

$\vec{a}\times\vec{b}=\vec{O}, \Rightarrow ab\sin\theta\hat{n}=\vec{O} \Rightarrow \sin\theta=0$ $[\because a\neq 0, b\neq 0]$

$\Rightarrow \quad \theta=0$ या $\theta=\pi$

i.e. $\vec{a}$ and $\vec{b}$ are parallel vectors

Note: (i) $\vec{a}\times\vec{a}=\vec{O}$, (ii) $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=\vec{O}$

(ii) The magnitude of product of two vectors is equal to the product of the two magnitude of the two vectors.

Proof : If $\vec{a}$ and $\vec{b}$ are two perpendicualr vectors then $\theta = 90^\circ$.

$$\therefore \quad \vec{a}\times\vec{b} = \left(ab\sin 90^\circ\right)\hat{n}$$

$$= \left(ab\right)\hat{n}$$

$$\Rightarrow \quad \left|\vec{a}\times\vec{b}\right| = ab$$

Magnitude of vectors $\vec{a}\times\vec{b}$ = (magnitude of $\vec{a}$) (magnitude of $\vec{b}$ u)] Here $\hat{n}$, is a unit vector along $\vec{a}$ and $\vec{b}$ and obeys the left hand rule.

Special Condition :

$$\hat{i}\times\hat{j} = (1)(1)\sin 90^\circ \hat{k} = \hat{k}$$

similarly $\hat{j}\times\hat{k} = \hat{i}$ and $\hat{k}\times\hat{i} = \hat{j}$

again $\hat{j}\times\hat{i} = -\hat{k}$ (opposite to $\hat{i}\times\hat{j}$)

similarly $\hat{k}\times\hat{j} = -\hat{i}$ and $\hat{i}\times\hat{k} = -\hat{j}$

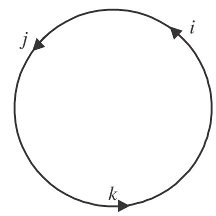


Fig. 13.19

This can be understood by the fig. 13.19.

13.22 Algebraic Properties of Vector Product

(i) Commutativity: Vector product is not commutative i.e.

$$\vec{a}\times\vec{b} \neq \vec{b}\times\vec{a}$$

(ii) Associativity: Vector product is associative with respect to any scalar *m* i.e.

$$m(\vec{a}\times\vec{b}) = (m\vec{a})\times\vec{b} = \vec{a}\times(m\vec{b})$$

(iii) Distributivity: Vector product obyes the distributive law:

$$\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$$

13.23 Vector Product of two Vectors in Terms of Components

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are two vectors then

$$\vec{a}\times\vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\times(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$= a_1b_1(\hat{i}\times\hat{i}) + a_1b_2(\hat{i}\times\hat{j}) + a_1b_3(\hat{i}\times\hat{k}) + a_2b_1(\hat{j}\times\hat{i})$$

$$+a_2b_2(\hat{j}\times\hat{j}) + a_2b_3(\hat{j}\times\hat{k}) + a_3b_1(\hat{k}\times\hat{i}) + a_3b_2(\hat{k}\times\hat{j}) + a_3b_3(\hat{k}\times\hat{k})$$

$$= a_1b_1(\vec{0}) + a_1b_2(\vec{k}) + a_1b_3(-\hat{j}) + a_2b_1(-\hat{k}) + a_2b_2(\vec{0}) + a_2b_3(\hat{i}) + a_3b_1(\hat{j}) + a_3b_2(-\hat{i}) + a_3b_3(\vec{0})$$

$$= (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

$$\therefore \quad \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

which is a determinant form of $\vec{a}\times\vec{b}$.

13.24 Angle between two Vectors

If θ is the angle between $\vec{a}$ and $\vec{b}$

$$\vec{a}\times\vec{b} = ab\sin\theta\hat{n}$$

$$\Rightarrow \quad |\vec{a}\times\vec{b}| = |ab\sin\theta||\hat{n}| = ab|\sin\theta||\hat{n}|$$

$$\Rightarrow \quad \sin^2\theta = \frac{|\vec{a}\times\vec{b}|^2}{(a^2)(b^2)}$$

$$= \frac{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}{(a_1^{\,2} + a_2^{\,2} + a_3^{\,2})(b_1^{\,2} + b_2^{\,2} + b_3^{\,2})}$$

13.25 Vector area of a Triangle

(i) If $\vec{a}$ and $\vec{b}$ are the sides of the triangle

Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$ then $\vec{a}\times\vec{b} = ab\sin\theta\hat{n}$

Now area of (ΔOAB) $= \frac{1}{2}ab\sin\theta\hat{n} = \frac{1}{2}(\vec{a}\times\vec{b})$,

here $\hat{n}$ is the unit vector

Note: Now area of $\Delta OBA = \frac{1}{2}\left(\vec{b}\times\vec{a}\right) = -\frac{1}{2}\left(\vec{a}\times\vec{b}\right)$

(ii) If the position vectors $\vec{a}, \vec{b}$ and $\vec{c}$ of triangle ABC are given

The sides of ΔABC, AB and AC

$$\overrightarrow{AB} = \vec{b} - \vec{a} \quad \text{and} \quad \overrightarrow{AC} = \vec{c} - \vec{a}$$

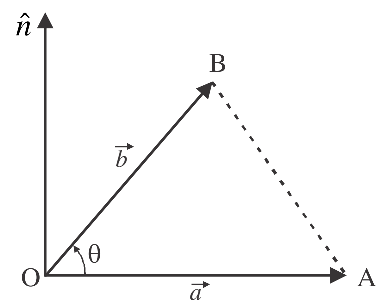


Fig. 13.20

$\therefore$ Area of triangle ΔABC $= \frac{1}{2}\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)$

$$= \frac{1}{2}\left[\left(\vec{b}-\vec{a}\right)\times\left(\vec{c}-\vec{a}\right)\right]$$

$$= \frac{1}{2}\left[\vec{b}\times\vec{c} - \vec{b}\times\vec{a} - \vec{a}\times\vec{c} + \vec{a}\times\vec{a}\right]$$

$$= \frac{1}{2}\left[\vec{b}\times\vec{c} + \vec{a}\times\vec{b} + \vec{c}\times\vec{a}\right] \qquad \left[\because \vec{a}\times\vec{a} = \vec{O}\right]$$

$$= \frac{1}{2}\left[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}\right]$$

13.26 Condition of Collinearity of Three points

If points A, B and C are collinear then the Area of triangle will be zero.

Let the position vectors of ΔABC are $\vec{a}, \vec{b}$ and $\vec{c}$, therefore area fo ΔABC $= 0$

$$\Rightarrow \quad \frac{1}{2}\left(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right)=0$$

$$\Rightarrow \quad \vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=0$$

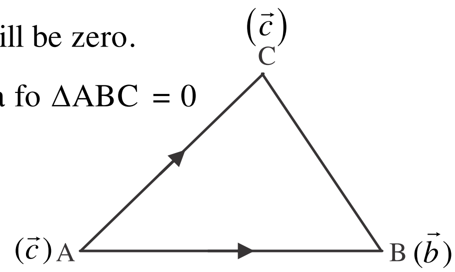


Fig. 13.21

Illustrative Examples

Example 16. Find the value of $(2\hat{i}-3\hat{j}+4\hat{k})\times(3\hat{i}+4\hat{j}-4\hat{k})$.

Solution : $(2\hat{i}-3\hat{j}+4\hat{k})\times(3\hat{i}+4\hat{j}-4\hat{k})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -3 & 4\\ 3 & 4 & -4\end{vmatrix}$

$$=(12-16)\hat{i}+(12+8)\hat{j}+(8+9)\hat{k} = -4\hat{i}+20\hat{j}+17\hat{k}$$

thus required value $-4\hat{i}+20\hat{j}+17\hat{k}$

Example 17. If $\vec{a}=3\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}-2\hat{j}+2\hat{k}$ then find the unit vector $\hat{n}$ perpendicular to vectors $\vec{a}$ and $\vec{b}$.

Solution : By the definition of vector product

$$\hat{n}=\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$$

$$=\frac{(3\hat{i}+\hat{j}+2\hat{k})\times(2\hat{i}-2\hat{j}+2\hat{k})}{\left|(3\hat{i}+\hat{j}+2\hat{k})\times(2\hat{i}-2\hat{j}+2\hat{k})\right|}$$

again $\quad (3\hat{i}+\hat{j}+2\hat{k})\times(2\hat{i}-2\hat{j}+2\hat{k})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 1 & 2\\ 2 & -2 & 2\end{vmatrix}$

$$=(2+4)\hat{i}+(4-6)\hat{j}+(-6-2)\hat{k}$$

$$=6\hat{i}-2\hat{j}-8\hat{k}$$

$$\hat{n}=\frac{6\hat{i}-2\hat{j}-8\hat{k}}{\left|6\hat{i}-2\hat{j}-8\hat{k}\right|}$$

$$=\frac{6\hat{i}-2\hat{j}-8\hat{k}}{\sqrt{36+4+64}}=\frac{6\hat{i}-2\hat{j}-8\hat{k}}{\sqrt{104}}$$

$$=\frac{3\hat{i}-\hat{j}-4\hat{k}}{\sqrt{26}}$$, which is the required solution

Thus the required perpendicular unit vector is $\frac{1}{\sqrt{26}}(3\hat{i}-\hat{j}-4\hat{k})$.

Example 18. If $\vec{a}\times\vec{b}=\vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c}=\vec{b}\times\vec{d}$, then Prove that $\vec{a}-\vec{d}$ and $\vec{b}-\vec{c}$ are parallel.

Solution :

$$(\vec{a}-\vec{d})\times(\vec{b}-\vec{c})=(\vec{a}\times\vec{b}-\vec{a}\times\vec{c})-(\vec{d}\times\vec{b}-\vec{d}\times\vec{c})$$

$$=\vec{a}\times\vec{b}-\vec{a}\times\vec{c}+\vec{b}\times\vec{d}+(-\vec{c})\times\vec{d}$$

$$=(\vec{a}\times\vec{b}-\vec{c}\times\vec{d})+(\vec{b}\times\vec{d}-\vec{a}\times\vec{c})$$

$$=\vec{O}+\vec{O}=\vec{O}$$

$\therefore$ $\vec{a}-\vec{d}$ and $\vec{b}-\vec{c}$ are parallel vectors

Example 19. If $\vec{a}\times\vec{b}=\vec{c}\times\vec{b}$ then Prove that $\vec{a}-\vec{c}=\lambda\vec{b}$, where λ is a scalar

Solution: $\vec{a}\times\vec{b}=\vec{c}\times\vec{b}$

$\Rightarrow$ $\vec{a}\times\vec{b}-\vec{c}\times\vec{b}=0$

$\Rightarrow$ $(\vec{a}-\vec{c})\times\vec{b}=0$

$\therefore$ $\vec{a}-\vec{c}$ and $\vec{b}$ are parallel therefore $\vec{a}-\vec{c}=\lambda\vec{b}$, where λ is a scalar

Note: (i) If $\vec{a}-\vec{c}$ and $\vec{b}$ are in the same direction then λ is positive

(ii) If $\vec{a}-\vec{c}$ and $\vec{b}$ are opposite then λ is negative

Example 20. If $A(1, 2, 2)$, $B(2, -1, 1)$ and $C(-1, -2, 3)$ are any three points in a plane then find a vector perpendicualr to the plane ABC whose magnitude is 5 units.

Solution :

$$\overrightarrow{AB}=\text{(position vector of B)}-\text{(position vector of A)}$$

$$=\left(2\hat{i}-\hat{j}+\hat{k}\right)-\left(\hat{i}+2\hat{j}+2\hat{k}\right)$$

$$=\hat{i}-3\hat{j}-\hat{k}$$

and

$$\overrightarrow{AC}=\text{(position vector of C)}-\text{(position vector of A)}$$

$$=\left(-\hat{i}-2\hat{j}+3\hat{k}\right)-\left(\hat{i}+2\hat{j}+2\hat{k}\right)$$

$$=-2\hat{i}-4\hat{j}+\hat{k}$$

$\because$ $\overrightarrow{AB}$ and $\overrightarrow{AC}$ both are in plane ABC thus vectors $\overrightarrow{AB}\times\overrightarrow{AC}$ is perpendicualr to the plane

therefore $\overrightarrow{AB}\times\overrightarrow{AC}=\left(\hat{i}-3\hat{j}-\hat{k}\right)\times\left(-2\hat{i}-4\hat{j}+\hat{k}\right)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & -1 \\ -2 & -4 & 1 \end{vmatrix}$$

$$= -7\hat{i} + \hat{j} - 10\hat{k}$$

Unit vector perpendicualr to the plane ABC

$$\hat{n} = \frac{-7\hat{i} + \hat{j} - 10\hat{k}}{\sqrt{49+1+100}} = \frac{-1}{\sqrt{150}}\left(7\hat{i} - \hat{j} + 10\hat{k}\right)$$

magnitude of the vector with 5 in the direction perpendicualr to it is

$$= 5\left[\frac{-1}{\sqrt{150}}\left(7\hat{i} - \hat{j} + 10\hat{k}\right)\right] = \frac{-1}{\sqrt{6}}\left(7\hat{i} - \hat{j} + 10\hat{k}\right)$$

Example 21. Prove that the Area of rectangle $ABCD$ is $\frac{1}{2}\overrightarrow{AC} \times \overrightarrow{BD}$ where AC and BD are the diagonals.

Solution: Area of rectangle $ABCD$ = Area of ΔACD + Area of ΔABC

$$= \frac{1}{2}\overrightarrow{AC} \times \overrightarrow{AD} + \frac{1}{2}\overrightarrow{AB} \times \overrightarrow{AC}$$

$$= \frac{1}{2}\left[\overrightarrow{AC} \times \overrightarrow{AD} - \overrightarrow{AC} \times \overrightarrow{AB}\right]$$

$$= \frac{1}{2}\left[\overrightarrow{AC} \times \left(\overrightarrow{AD} - \overrightarrow{AB}\right)\right] = \frac{1}{2}\overrightarrow{AC} \times \overrightarrow{BD}$$

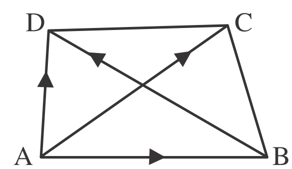


Fig. 13.22

Thus Area of Rectangle $= \frac{1}{2}\left|\overrightarrow{AC} \times \overrightarrow{BD}\right|$

Exercise 13.3

1. Find the vector product of $3\hat{i} + \hat{j} - \hat{k}$ and $2\hat{i} + 3\hat{j} + \hat{k}$.

2. Find the unit vector perpendicualr to the vectors $\hat{i} - 2\hat{j} + \hat{k}$ and $2\hat{i} + \hat{j} - 3\hat{k}$.

3. For vectors $\vec{a}$ and $\vec{b}$ Prove that $\left(\vec{a} \times \vec{b}\right)^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$

4. Prove that $\vec{a} \times \left(\vec{b} + \vec{c}\right) + \vec{b} \times \left(\vec{c} + \vec{a}\right) + \vec{c} \times \left(\vec{a} + \vec{b}\right) = 0$.

5. If $\hat{a}, \hat{b}, \hat{c}$ are the unit vectors such that $\hat{a} \cdot \hat{b} = \hat{a} \cdot \hat{c} = 0$ and the angle between $\hat{b}$ and $\hat{c}$ is $\pi/6$ then prove that $\hat{a} = \pm 2\left(\vec{b} \times \vec{c}\right)$.

6. Find the value of $|\vec{a}\times\vec{b}|$ if $|\vec{a}|=10, |\vec{b}|=2$ and $\vec{a}\cdot\vec{b}=12$.

7. Find the vector with magnitude 9 units whcih is perpendicular to the vectors $4\hat{i}-\hat{j}+3\hat{k}$ and $-2\hat{i}+\hat{j}-2\hat{k}$.

8. Show that $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})=2(\vec{a}\times\vec{b})$. Also explain geometrically.

9. For any vector $\vec{a}$ prove that $|\vec{a}\times\hat{i}|^2+|\vec{a}\times\hat{j}|^2+|\vec{a}\times\hat{k}|^2=2|\vec{a}|^2$.

10. If the two sides of the triangle are given by $\hat{i}+2\hat{j}+2\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$ then find the area of the triangle.

13.27 Product of Three Vectors

The product of three vectors can have the following six conditions:

(i) $\vec{a}(\vec{b}\cdot\vec{c})$ (ii) $\vec{a}\cdot(\vec{b}\cdot\vec{c})$ (iii) $\vec{a}\times(\vec{b}\cdot\vec{c})$

(iv) $\vec{a}(\vec{b}\times\vec{c})$ (v) $\vec{a}\cdot(\vec{b}\times\vec{c})$ (vi) $\vec{a}\times(\vec{b}\times\vec{c})$

By observation the following facts are to be considered

(i) $\vec{a}(\vec{b}\cdot\vec{c})$ is meaningless, because $\vec{b}\cdot\vec{c}$ is a scalar quantity, thus here $\vec{a}$ is a vector whose magnitude is a product of $(\vec{b}\cdot\vec{c})$, but this condition does not specify the product of three vectors.

(ii) $\vec{a}\cdot(\vec{b}\cdot\vec{c})$ is meaningless, because $\vec{b}\cdot\vec{c}$ is a scalar whereas to find the scalar product with $\vec{a}$ a vector term is required.

(iii) $\vec{a}\times(\vec{b}.\vec{c})$ is meaningless, because $\vec{b}\cdot\vec{c}$ is a scalar and to get the vector product with $\vec{a}$, a vector term is required.

(iv) $\vec{a}(\vec{b}\times\vec{c})$ is meaningless, because $\vec{b}\times\vec{c}$ is a vector term and $\vec{a}$ is also a vector, but there is no sign of $(\cdot)$ or $(\times)$ so nothing can be predicted about the result.

(v) $\vec{a}\cdot(\vec{b}\times\vec{c})$ is meaningful, because $\vec{b}\times\vec{c}$ is a vector and $\vec{a}$ is also a vector and the product of these two vectors is possible and the result is a scalar. This is known as the scalar triple product.

(vi) $\vec{a}\times(\vec{b}\times\vec{c})$ is meaningful, because $\vec{b}\times\vec{c}$ is a vector and $\vec{a}$ is also a vector, the vector product of these terms is possible and the result is also a vector, this is called as vector triple product.

Thus from the above analysis only the product of two types of vectors is possible.

13.28 Scalar Triple Product

Definition: If the vector product of two vector quantities is again multipled with the scalar quantity then this product is known as scalar triple product.

As both vector and scalar product are found in this triple products so it is also known as mixed product.

If $\vec{a}, \vec{b}, \vec{c}$ are any three vectors then $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)$ is known as scalar triple product of vectors $\vec{a}, \vec{b}, \vec{c}$ and is also written as $[\vec{a}\ \vec{b}\ \vec{c}]$, also $[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c})$ and $[\vec{b}\ \vec{a}\ \vec{c}] = \vec{b}\cdot(\vec{a}\times\vec{c})$ ।

Note: It is also known as Box Product, it is to be noted that the terms inside the box should not be seperated by comma.

13.29 Geometrical Interpretation of Scalar Triple Product

Let $\overline{OA} = \vec{a}, \overline{OB} = \vec{b}$ and $\overline{OC} = \vec{c}$. Draw a rectangular parallelopiped with concurrent edges $\vec{a}, \vec{b}, \vec{c}$

Now the vector area of parallelogram OBDC $= \vec{b}\times\vec{c}$

$\therefore\ \vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = |\vec{a}||\vec{b}\times\vec{c}|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}\times\vec{c}$

$= |\vec{b}\times\vec{c}|(|a|\cos\theta)$

= (area of parallelogram OBDC)

(height of rectangular parallelopiped)

= (area of base × height)

$\Rightarrow \vec{a}\cdot(\vec{b}\times\vec{c})$ = volume of rectangular parallelopiped whose concurrent edges are $\vec{a}, \vec{b}$ and $\vec{c}$

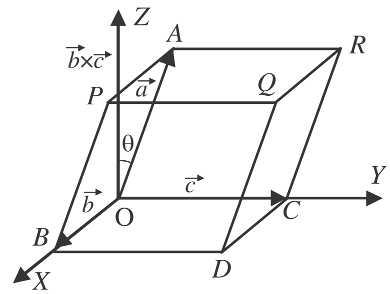


Fig. 13.23

similarly we can show $\vec{b}.\left(\vec{c}\times\vec{a}\right) = \vec{c}.\left(\vec{a}\times\vec{b}\right)$ the concurrent edges of rectangular parallelopipped

$$\vec{a}.\left(\vec{b}\times\vec{c}\right) = \vec{b}.\left(\vec{c}\times\vec{a}\right) = \vec{c}.\left(\vec{a}\times\vec{b}\right)$$

or $$[\vec{a}\ \ \vec{b}\ \ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$$

is equal to volume of rectangular parallelopiped whose concurrent edges are given.

13.30 Properties of Scalar Triple Product

(i) $$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \vec{b}\cdot\left(\vec{c}\times\vec{a}\right) = \vec{c}\cdot\left(\vec{a}\times\vec{b}\right) \quad (1)$$

again $$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \left(\vec{b}\times\vec{c}\right)\cdot\vec{a} \quad (2)$$

similarly $$\vec{b}\cdot\left(\vec{c}\times\vec{a}\right) = \left(\vec{c}\times\vec{a}\right)\cdot\vec{b} \quad (3)$$

and $$\vec{c}\cdot\left(\vec{a}\times\vec{b}\right) = \left(\vec{a}\times\vec{b}\right)\cdot\vec{c} \quad (4)$$

from equation (1) and (4) $$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \vec{c}\cdot\left(\vec{a}\times\vec{b}\right) = \left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$$

i.e. $$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$$

If the cyclic order remains unchanged then dot and cross signs can be changed.

(ii) If the cyclic order changes then the sign of scalar triple product changes.

$\because$ $$(\vec{b}\times\vec{c}) = -(\vec{c}\times\vec{b})$$

$\therefore \qquad \vec{a}\cdot(\vec{b}\times\vec{c}) = -\vec{a}\cdot(\vec{c}\times\vec{b})$

$\therefore \qquad [\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}]$

(iii) In scalar triple product if two vectors are parallel then the product is zero.

Let $\vec{a},\vec{b},\vec{c}$ are three vectors and $\vec{b}$ and $\vec{c}$ are parallel then $\vec{b}=\lambda\vec{c}$, where λ is a scalar,

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \vec{a}\cdot\left(\lambda\vec{c}\times\vec{c}\right) = \lambda\left(\vec{a}\cdot\vec{0}\right) = 0 \qquad \because \left[\vec{c}\times\vec{c}=\vec{0}\right]$$

Note: If two vectors are same then also the result is zero.

13.31 Volume of a Tetrahedron

Let in tetrahedron OABC, O be the origin and $A(\vec{a}), B(\vec{b})$ and $C(\vec{c})$ are other vertices.

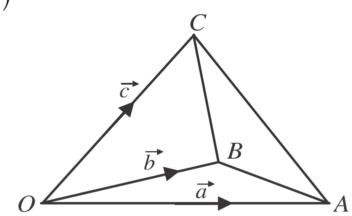


Fig. 13.24

Volume of Tetrahedron $(V) = \frac{1}{3}$ (area of base) × (height)

$$= \frac{1}{3}\left[\frac{1}{2}\left(\vec{a}\times\vec{b}\right)\right]\cdot\vec{c} = \frac{1}{6}\left[\vec{a}\ \vec{b}\ \vec{c}\right]$$

Thus Volume fo Tetrahedron = (1 / 6) (Volume of rectangular parallelopipped whose three concurrent edges are $\vec{a},\vec{b},\vec{c}$)

Note: If the four vertices of a tetrahedron are $A(\vec{a}), B(\vec{b}), C(\vec{c})$ and $D(\vec{d})$ then the volume is

$$= \frac{1}{6}[\vec{a}-\vec{b} \quad \vec{a}-\vec{c} \quad \vec{a}-\vec{d}]$$

13.32 Necessary and sufficient condition for the three non-parallel and non-zero vector $\vec{a},\vec{b},\vec{c}$ to be coplanar is $\left[\vec{a}\ \vec{b}\ \vec{c}\right]=0$

Necessary Condition : Let $\vec{a},\vec{b}$ and $\vec{c}$ are three non-zero non-parallel coplaner vectors then $\vec{b}\times\vec{c}$ is a vector perpendicular to the plane i.e. $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=0$

($\because \vec{a}$ is in a plane and $\vec{b}\times\vec{c}$ N is perpendicular to the plane and scalar product of two vectors is always zero)

$\Rightarrow \quad \left[\vec{a}\ \vec{b}\ \vec{c}\right]=0$

Sufficient condition : Let

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right]=0 \quad \Rightarrow \quad \vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=0$$

$\Rightarrow \quad \vec{a}\perp\left(\vec{b}\times\vec{c}\right)$, But $\vec{b}\times\vec{c}$, is perpendicular to vectors $\vec{b}$ and $\vec{c}$ i.e. vector $\vec{a}$ lies in the plane of vector $\vec{b}$ and $\vec{c}$ therefore $\vec{a},\vec{b}$ and $\vec{c}$ are coplaner.

Illustrative Examples

Example 22. Prove that $\left[\hat{i}\ \hat{j}\ \hat{k}\right]+\left[\hat{j}\ \hat{k}\ \hat{i}\right]+\left[\hat{k}\ \hat{i}\ \hat{j}\right]=3.$

Solution : $\left[\hat{i}\ \hat{j}\ \hat{k}\right]=\hat{i}\cdot\left(\hat{j}\times\hat{k}\right)=\hat{i}\cdot\hat{i}=1$

$\because \quad \left[\hat{i}\ \hat{j}\ \hat{k}\right]=\left[\hat{j}\ \hat{k}\ \hat{i}\right]=\left[\hat{k}\ \hat{i}\ \hat{j}\right]$

$\therefore \quad \left[\hat{i}\ \hat{j}\ \hat{k}\right]+\left[\hat{j}\ \hat{k}\ \hat{i}\right]+\left[\hat{k}\ \hat{i}\ \hat{j}\right]=1+1+1=3$

Example 23. If $\vec{a}=\hat{i}-2\hat{j}+\hat{k}, \vec{b}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}+2\hat{j}+\hat{k}$ then find the value of $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)$ and $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$, also show that $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$

Solution :

$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\begin{vmatrix}1 & -2 & 1\\ 1 & 1 & 1\\ 1 & 2 & 1\end{vmatrix}=0$$ ($\because$ first and third columns are same)

$$\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=\vec{c}\cdot\left(\vec{a}\times\vec{b}\right)=\begin{vmatrix}1 & 2 & 1\\ 1 & -2 & 1\\ 1 & 1 & 1\end{vmatrix}=0$$ ($\because$ first and third columns are same)

$\therefore \quad \vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$

Example 24. Prove that $\left[\vec{a}+\vec{b}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}\right]=2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$

Solution : since $\left(\vec{b}+\vec{c}\right)\times\left(\vec{c}+\vec{a}\right)=\vec{b}\times\left(\vec{c}+\vec{a}\right)+\vec{c}\times\left(\vec{c}+\vec{a}\right)$ (distributive law)

$=\left(\vec{b}\times\vec{c}\right)+\left(\vec{b}\times\vec{a}\right)+\left(\vec{c}\times\vec{c}\right)+\left(\vec{c}\times\vec{a}\right)$ (distributive law)

$=\left(\vec{b}\times\vec{c}\right)+\left(\vec{b}\times\vec{a}\right)+\left(\vec{c}\times\vec{a}\right)$ (1)

$\therefore \quad \left[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}\right]=\left(\vec{a}+\vec{b}\right).\left\{\left(\vec{b}+\vec{c}\right)\times\left(\vec{c}+\vec{a}\right)\right\}$

$=\left(\vec{a}+\vec{b}\right).\left\{\left(\vec{b}\times\vec{c}\right)+\left(\vec{b}\times\vec{a}\right)+\left(\vec{c}\times\vec{a}\right)\right\}$ (from (1))

$=\left(\vec{a}+\vec{b}\right).\left(\vec{b}\times\vec{c}\right)+\left(\vec{a}+\vec{b}\right).\left(\vec{b}\times\vec{a}\right)+\left(\vec{a}+\vec{b}\right).\left(\vec{c}\times\vec{a}\right)$ (distributive law)

$=\vec{a}.\left(\vec{b}\times\vec{c}\right)+\vec{b}.\left(\vec{b}\times\vec{c}\right)+\vec{a}.\left(\vec{b}\times\vec{a}\right)+\vec{b}.\left(\vec{b}\times\vec{a}\right)+\vec{a}.\left(\vec{c}\times\vec{a}\right)+\vec{b}.\left(\vec{c}\times\vec{a}\right)$

$=\left[\vec{a}\ \vec{b}\ \vec{c}\right]+0+0+0+0+\left[\vec{b}\ \vec{c}\ \vec{a}\right]$ ($\because$ property of triple product)

$$=2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$$

Example 25. For what value of λ are the vectors $\vec{a}=2\hat{i}-\hat{j}+\hat{k}, \vec{b}=\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{c}=3\hat{i}+\lambda\hat{j}+5\hat{k}$ coplaner.

Solution : Condition of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ to be coplaner is $\left[\vec{a}\,\vec{b}\,\vec{c}\right]=0$

i.e. $\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & \lambda & 5 \end{vmatrix}=0$ or $\begin{vmatrix} 3 & \lambda & 5 \\ 2 & -1 & 1 \\ 1 & 2 & -3 \end{vmatrix}=0$

$\Rightarrow \quad 3(3-2)+\lambda(1+6)+5(4+1)=0 \qquad \Rightarrow 3+7\lambda+25=0$

$\Rightarrow \quad \lambda=-4$

thus for $\lambda=-4$ the three vectolrs $\vec{a}, \vec{b}$ and $\vec{c}$ are coplaner.

Example 26. Prove that the points $A(4,8,12), B(2,4,6), C(3,5,4), D(5,8,5)$ are coplaner.

Solution : If the points $\overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{BD}$ are coplaner, again by the codnition $\left[\overrightarrow{BA}\ \ \overrightarrow{BC}\ \ \overrightarrow{BD}\right]=0$

now $\overrightarrow{BA}=\left(4\hat{i}+8\hat{j}+12\hat{k}\right)-\left(2\hat{i}+4\hat{j}+6\hat{k}\right)=2\hat{i}+4\hat{j}+6\hat{k}$

$\overrightarrow{BC}=\left(3\hat{i}+5\hat{j}+4\hat{k}\right)-\left(2\hat{i}+4\hat{j}+6\hat{k}\right)=\hat{i}+\hat{j}-2\hat{k}$

$\overrightarrow{BD}=\left(5\hat{i}+8\hat{j}+5\hat{k}\right)-\left(2\hat{i}+4\hat{j}+6\hat{k}\right)=3\hat{i}+4\hat{j}-\hat{k}$

$$\therefore \qquad \left[\overrightarrow{BA}\,\overrightarrow{BC}\,\overrightarrow{BD}\right]=\begin{vmatrix} 2 & 4 & 6 \\ 1 & 1 & -2 \\ 3 & 4 & -1 \end{vmatrix}=2(7)+4(-5)+6(1)=0$$

Thus the four points are coplaner.

Example 27. If four points $A(\vec{a}), B(\vec{b}), C(\vec{c})$ and $D(\vec{d})$ are copaner, then prove that

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right]=\left[\vec{b}\ \vec{c}\ \vec{d}\right]+\left[\vec{c}\ \vec{a}\ \vec{d}\right]+\left[\vec{a}\ \vec{b}\ \vec{d}\right]$$

Solution : Four points are coplaner thus vectors $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ are are coplaner.

$$\Rightarrow \qquad \left[\overrightarrow{AB}\ \ \overrightarrow{AC}\ \ \overrightarrow{AD}\right]=0$$

$$\Rightarrow \qquad \left[\left(\vec{b}-\vec{a}\right)\ \left(\vec{c}-\vec{a}\right)\ \left(\vec{d}-\vec{a}\right)\right]=0$$

$$\Rightarrow \qquad \left(\vec{b}-\vec{a}\right)\cdot\left\{\left(\vec{c}-\vec{a}\right)\times\left(\vec{d}-\vec{a}\right)\right\}=0$$

$$\Rightarrow \qquad \left(\vec{b}-\vec{a}\right)\cdot\left\{\vec{c}\times\vec{d}-\vec{c}\times\vec{a}-\vec{a}\times\vec{d}+\vec{a}\times\vec{a}\right\}=0$$

$$\Rightarrow \vec{b}\cdot(\vec{c}\times\vec{d})-\vec{b}\cdot(\vec{c}\times\vec{a})-\vec{b}\cdot(\vec{a}\times\vec{d})-\vec{a}\cdot(\vec{c}\times\vec{d})=0$$

$$\therefore \quad [\vec{a}\ \vec{b}\ \vec{c}]=[\vec{b}\ \vec{c}\ \vec{d}]+[\vec{c}\ \vec{a}\ \vec{d}]+[\vec{a}\ \vec{b}\ \vec{d}]$$

Example 28. Find the volume of the rectangular parallelopipped whose concurrent edges are $2\hat{i}-3\hat{j}+4\hat{k}, \hat{i}+2\hat{j}-\hat{k}$ P and $2\hat{i}-\hat{j}+2\hat{k}$.

Solution : Let $\vec{a}=2\hat{i}-3\hat{j}+4\hat{k}, \vec{b}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+2\hat{k}$, volume of parallelopipped $=[\vec{a}\,\vec{b}\,\vec{c}]$

$$=\begin{vmatrix}2 & -3 & 4\\ 1 & 2 & -1\\ 2 & -1 & 2\end{vmatrix}=2(3)+3(-4)+4(-5)=6-12-20=-26 \text{ unit}$$

Since Volume is positive, hence the result is 26 units.

Example 29. Find the volume of tetrahedron if the vertices are $O(0,0,0)$, $A(1,2,1), B(2,1,3)$ and $C(-1,1,2)$.

Solution : Here $O(0,0,0)$ is the origin and the position vector are $\vec{a}=\hat{i}+2\hat{j}+\hat{k}, \vec{b}=2\hat{i}+\hat{j}+3\hat{k}$ and $\vec{c}=-\hat{i}+\hat{j}+2\hat{k}$.

$$\text{volume of tetrahedron} \quad =\frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]=\frac{1}{6}\begin{vmatrix}1 & 2 & 1\\ 2 & 1 & 3\\ -1 & 1 & 2\end{vmatrix}$$

$$=\frac{1}{6}[1(-1)+2(-7)+1(3)]=-2 \text{ unit}$$

Since the volume is positive thus the result is 2 units.

Exercise 13.4

1. Prove that

   (i) $[\hat{i}\ \hat{j}\ \hat{k}]+[\hat{i}\ \hat{k}\ \hat{j}]=0$ (ii) $[2\hat{i}\ \hat{j}\ \hat{k}]+[\hat{i}\ \hat{k}\ \hat{j}]+[\hat{k}\ \hat{j}\ 2\hat{i}]=-1$

2. If $\vec{a}=2\hat{i}-3\hat{j}+4\hat{k}, \vec{b}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{c}=3\hat{i}-\hat{j}+2\hat{k}$ then find $[\vec{a}\,\vec{b}\,\vec{c}]$.

3. Prove that the vectors $-2\hat{i}-2\hat{j}+4\hat{k}, -2\hat{i}+4\hat{j}-2\hat{k}$ and $4\hat{i}-2\hat{j}-2\hat{k}$ are coplaner.

4. For what value of λ are the vectors copalner

   (i) $\vec{a}=2\hat{i}-\hat{j}+\hat{k}, \vec{b}=\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{c}=3\hat{i}+\lambda\hat{j}+5\hat{k}$

   (ii) $\vec{a}=\hat{i}-\hat{j}+\hat{k}, \vec{b}=2\hat{i}+\hat{j}-\hat{k}$ and $\vec{c}=\lambda\hat{i}-\hat{j}+\lambda\hat{k}$

5. Prove that the following four points are coplaner

   (i) $A(-1,4,-3), B(3,2,-5), C(-3,8,-5), D(-3,2,1)$

(ii) $A(0,-1,0), B(2,1,-1), C(1,1,1), D(3,3,0)$

6. Prove that $\vec{a}=2\hat{i}-\hat{j}+\hat{k}, \vec{b}=\hat{i}-3\hat{j}-5\hat{k}$ and $\vec{c}=3\hat{i}-4\hat{j}-4\hat{k}$ are the vector sides of a right angle triangle.

7. Find the volume of the rectangular parallelopipped whose three concurrent edges are given by the vectors:

(i) $\vec{a}=4\hat{i}-3\hat{j}+\hat{k}, \vec{b}=3\hat{i}+2\hat{j}-\hat{k}$ and $\vec{c}=3\hat{i}-\hat{j}+2\hat{k}$

(ii) $\vec{a}=2\hat{i}-3\hat{j}+\hat{k}, \vec{b}=\hat{i}-\hat{j}+2\hat{k}$ and $\vec{c}=2\hat{i}+\hat{j}-\hat{k}$

13.33 Vector Triple Product

Definition : The product of vector with the vector product of two vectors is known as vector triple product.

If $\vec{a}, \vec{b}, \vec{c}$ are three vectors then their vector product will be $\vec{a}\times(\vec{b}\times\vec{c}), (\vec{b}\times\vec{c})\times\vec{a}, (\vec{a}\times\vec{b})\times\vec{c}$ etc.

Geometrical Proof:

Here $\vec{a}\times(\vec{b}\times\vec{c})$, is perpendicular to vector $\vec{a}$ and vector $(\vec{b}\times\vec{c})$

$\therefore \quad \vec{a}\times(\vec{b}\times\vec{c})=\lambda\vec{b}+\mu\vec{c}$ where λ and μ are scalar

Note: It is clear from the vector triple product $\vec{a}\times(\vec{b}\times\vec{c})\neq(\vec{a}\times\vec{b})\times\vec{c}$, it is not associative.

13.34 For vectors $\vec{a},\vec{b},\vec{c}$ Prove that

$$\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$$

Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}, \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$

now $\vec{a}\times(\vec{b}\times\vec{c})=(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times\begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$=(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times\left\{(b_2c_3-b_3c_2)\hat{i}+(b_3c_1-b_1c_3)\hat{j}+(b_1c_2-b_2c_1)\hat{k}\right\}$$

$$=\sum\left\{a_2(b_1c_2-b_2c_1)-a_3(b_3c_1-b_1c_3)\right\}\hat{i}$$

$$=\sum\left\{b_1(a_2c_2+a_3c_3)-c_1(a_2b_2+a_3b_3)\right\}\hat{i}$$

$$=\sum\left\{(a_1c_1+a_2c_2+a_3c_3)b_1-(a_1b_1+a_2b_2+a_3b_3)c_1\right\}\hat{i}$$ (adding and subtracting *a, b, c*)

$$=\sum\left\{(\vec{a}\cdot\vec{c})b_1-(\vec{a}\cdot\vec{b})c_1\right\}\hat{i}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$$

$\therefore \quad \vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

similarly $\left(\vec{a}\times\vec{b}\right)\times\vec{c}=-\vec{c}\times\left(\vec{a}\times\vec{b}\right)=-\left\{\left(\vec{c}\cdot\vec{b}\right)\vec{a}-\left(\vec{c}\cdot\vec{a}\right)\vec{b}\right\}=\left(\vec{c}\cdot\vec{a}\right)\vec{b}-\left(\vec{c}\cdot\vec{b}\right)\vec{a}$

Illustrative Examples

Example 30. If $\vec{a}=3\hat{i}+2\hat{j}+\hat{k}, \vec{b}=\hat{i}-2\hat{j}+2\hat{k}$ and $\vec{c}=2\hat{i}+\hat{j}-\hat{k}$ then find the value of $\vec{a}\times(\vec{b}\times\vec{c})$

Solution :

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$$

$$\vec{a}\cdot\vec{c}=\left(3\hat{i}+2\hat{j}+\hat{k}\right)\cdot\left(2\hat{i}+\hat{j}-\hat{k}\right)$$

$$=(3)(2)+(2)(1)+(1)(-1)=7$$

$$\vec{a}\cdot\vec{b}=\left(3\hat{i}+2\hat{j}+\hat{k}\right)\cdot\left(\hat{i}-2\hat{j}+2\hat{k}\right)$$

$$=(3)(1)+(2)(-2)+(1)(2)=1$$

$\therefore$ $$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$$

$$=7\left(\hat{i}-2\hat{j}+2\hat{k}\right)-1\left(2\hat{i}+\hat{j}-\hat{k}\right)=5\hat{i}-15\hat{j}+15\hat{k}$$

Example 31. Prove that $\left(\vec{a}\times\vec{b}\right)\times\vec{c}=\vec{a}\times\left(\vec{b}\times\vec{c}\right)$, if and only if $\left(\vec{c}\times\vec{a}\right)\times\vec{b}=\vec{O}$

Solution : Let $\left(\vec{a}\times\vec{b}\right)\times\vec{c}=\vec{a}\times\left(\vec{b}\times\vec{c}\right)$

$\Rightarrow$ $\left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{b}\cdot\vec{c}\right)\vec{a}=\left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$

$\Rightarrow$ $-\left(\vec{b}\cdot\vec{c}\right)\vec{a}=-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$

$\Rightarrow$ $\left(\vec{b}\cdot\vec{c}\right)\vec{a}-\left(\vec{b}\cdot\vec{a}\right)\vec{c}=\vec{O}$

$\therefore$ $\left(\vec{c}\times\vec{a}\right)\times\vec{b}=\vec{O}$

Example 32. Prove that the vectors $\vec{a}\times\left(\vec{b}\times\vec{c}\right), \vec{b}\times\left(\vec{c}\times\vec{a}\right)$ and $\vec{c}\times\left(\vec{a}\times\vec{b}\right)$ are coplaner.

Solution : Let $\vec{P}=\vec{a}\times\left(\vec{b}\times\vec{c}\right), \vec{Q}=\vec{b}\times\left(\vec{c}\times\vec{a}\right)$ and $\vec{R}=\vec{c}\times\left(\vec{a}\times\vec{b}\right)$, then

$$\vec{P}+\vec{Q}+\vec{R}=\left\{\left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}\right\}+\left\{\left(\vec{b}\cdot\vec{a}\right)\vec{c}-\left(\vec{b}\cdot\vec{c}\right)\vec{a}\right\}+\left\{\left(\vec{c}\cdot\vec{b}\right)\vec{a}-\left(\vec{c}\cdot\vec{a}\right)\vec{b}\right\}=\vec{O}$$

$\Rightarrow$ $\vec{P}=(-1)\vec{Q}+(-1)\vec{R}$

$\Rightarrow$ $\vec{P}, \vec{Q}$ and $\vec{R}$ are in one plane

$\Rightarrow$ $\vec{P}, \vec{Q}, \vec{R}$ are coplaner

Example 33. Prove that $\left[\left(\vec{a}\times\vec{b}\right)\left(\vec{b}\times\vec{c}\right)\left(\vec{c}\times\vec{a}\right)\right]=\left[\vec{a}\,\vec{b}\,\vec{c}\right]^2$

Solution :

$$\left[\left(\vec{a}\times\vec{b}\right)\left(\vec{b}\times\vec{c}\right)\left(\vec{c}\times\vec{a}\right)\right]=\left\{\left(\vec{a}\times\vec{b}\right)\times\left(\vec{b}\times\vec{c}\right)\right\}\cdot\left(\vec{c}\times\vec{a}\right)$$

$$=\left\{\vec{d}\times\left(\vec{b}\times\vec{c}\right)\right\}\cdot\left(\vec{c}\times\vec{a}\right), \qquad (\text{Let } \vec{d}=\vec{a}\times\vec{b})$$

$$=\left\{\left(\vec{d}\cdot\vec{c}\right)\vec{b}-\left(\vec{d}\cdot\vec{b}\right)\vec{c}\right\}\cdot\left(\vec{c}\times\vec{a}\right)$$

$$=\left\{\left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{b}-\left[\vec{a}\ \vec{b}\ \vec{b}\right]\vec{c}\right\}\cdot\left(\vec{c}\times\vec{a}\right)$$

$$\left[\because \vec{d}\cdot\vec{c}=\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=\left[\vec{a}\ \vec{b}\ \vec{c}\right] \text{ and } \vec{d}\cdot\vec{b}=\left(\vec{a}\times\vec{b}\right)\cdot\vec{b}=\left[\vec{a}\ \vec{b}\ \vec{b}\right]=0\right]$$

$$=\left[\vec{a}\ \vec{b}\ \vec{c}\right]\left\{b\cdot\left(\vec{c}\times\vec{a}\right)\right\} \quad \left\{\because \left[\vec{c}\ \vec{c}\ \vec{a}\right]=0\right\}$$

$$=\left[\vec{a}\ \vec{b}\ \vec{c}\right]\left[\vec{b}\ \vec{c}\ \vec{a}\right]=\left[\vec{a}\ \vec{b}\ \vec{c}\right]^2 \qquad \left[\because \left[\vec{b}\ \vec{c}\ \vec{a}\right]=\left[\vec{a}\ \vec{b}\ \vec{c}\right]\right]$$

Exercise 13.5

1. Find the value of $\vec{a}\times\left(\vec{b}\times\vec{c}\right)$ it

   (i) $\vec{a}=3\hat{i}-\hat{j}+\hat{k}, \vec{b}=\hat{i}+3\hat{j}-\hat{k}$ and $\vec{c}=-\hat{i}+\hat{j}+3\hat{k}$

   (ii) $\vec{a}=2\hat{i}+\hat{j}-3\hat{k}, \vec{b}=\hat{i}-2\hat{j}+\hat{k}$ and $\vec{c}=-\hat{i}+\hat{j}-4\hat{k}$

2. Prove that $\vec{a}\times\left(\vec{b}\times\vec{c}\right)\neq\left(\vec{a}\times\vec{b}\right)\times\vec{c}$ it

   (i) $\vec{a}=2\hat{i}+5\hat{j}-7\hat{k}, \vec{b}=-3\hat{i}+4\hat{j}+\hat{k}, \vec{c}=-\hat{i}-2\hat{j}-3\hat{k}$

   (ii) $\vec{a}=2\hat{i}+3\hat{j}-5\hat{k}, \vec{b}=-\hat{i}+\hat{j}+\sqrt{2}\hat{k}, \vec{c}=4\hat{i}-2\hat{j}+\sqrt{3}\hat{k}$

3. Verify the formula $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$ where

   (i) $\vec{a}=\hat{i}+\hat{j}-2\hat{k}, \vec{b}=2\hat{i}-\hat{j}+\hat{k}, \vec{c}=\hat{i}+3\hat{j}-\hat{k}$

   (ii) $\vec{a}=\hat{i}-2\hat{j}+\hat{k}, \vec{b}=2\hat{i}+\hat{j}-\hat{k}, \vec{c}=3\hat{i}+5\hat{j}+2\hat{k}$

4. For any vector $\vec{a}$ prove that

   $$\hat{i}\times\left(\vec{a}\times\hat{i}\right)+\hat{j}\times\left(\vec{a}\times\hat{j}\right)+\hat{k}\times\left(\vec{a}\times\hat{k}\right)=2\vec{a}$$

5. Prove that

   $$\vec{a}\times\left(\vec{b}\times\vec{c}\right)+\vec{b}\times\left(\vec{c}\times\vec{a}\right)+\vec{c}\times\left(\vec{a}\times\vec{b}\right)=\vec{0}$$

6. Prove that $\vec{a},\vec{b},\vec{c}$ are coplaner if and only if $\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}$ are coplaner

7. Prove that

$$\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)=\left[\vec{a}\,\vec{b}\,\vec{c}\right]\vec{c}-\left[\vec{a}\,\vec{c}\,\vec{d}\right]\vec{d}$$

8. If the magnitude of two vectors $\vec{a}$ and $\vec{b}$ are $\sqrt{3}$ and 2 and $\vec{a}\cdot\vec{b}=\sqrt{6}$ then find the angle between vector $\vec{a}$ and $\vec{b}$.

9. Find the angle between the vectors $\hat{i}-2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$.

10. Find the projection of vector $\hat{i}+\hat{j}$ on $\hat{i}-\hat{j}$.

11. Fidn the projection of vector $\hat{i}+3\hat{j}+7\hat{k}$ on $7\hat{i}-\hat{j}+8\hat{k}$.

12. Find the valueof $(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})$.

13. Find the magnitude of the two vectors $\vec{a}$ and $\vec{b}$ if there magnitude is same and the angle between them is 60° and their scalar products is $\frac{1}{2}$.

14. For a unit vector $\vec{a}$, if $\left(\vec{x}-\vec{a}\right)\cdot\left(\vec{x}+\vec{a}\right)=12$ then find the valeu of $|\vec{x}|$.

15. If $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k}, \vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+3\hat{j}$ are such that $\vec{a}+\lambda\vec{b}$ is perpendicular to vector $\vec{c}$ then find the valeu of λ.

16. If $\vec{a},\vec{b},\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ then find the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$.

17. If the vertices of triangle ABC are $(1,2,3)(-1,0,0)(0,1,2)$ then find $\angle ABC$.

Important Points

1. $\vec{a}\cdot\vec{b}=ab\cos\theta,\ \therefore\ \vec{a}\cdot\vec{b}=0\Leftrightarrow\vec{a}\perp\vec{b}\left(\vec{a}\neq 0\neq\vec{b}\right)$

$\cos\theta=\dfrac{\vec{a}\cdot\vec{b}}{ab}$

.	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{i}$	1	0	0
$\hat{j}$	0	1	0
$\hat{k}$	0	0	1

2. If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then $\vec{a}.\vec{b}=a_1b_1+a_2b_2+a_3b_3$

3. $\vec{a}\times\vec{b}=\left(ab\sin\theta\right)\hat{n}$

$\sin\theta=\dfrac{\left|\vec{a}\times\vec{b}\right|}{ab}$ and $\hat{n}=\dfrac{\vec{a}\times\vec{b}}{\left|\vec{a}\times\vec{b}\right|}$

$\hat{i}\times\hat{j}=\hat{k},\ \hat{j}\times\hat{k}=\hat{i},\ \hat{k}\times\hat{i}=\hat{j}$

X	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{i}$	0	$\hat{k}$	$\hat{j}$
$\hat{j}$	$-\hat{k}$	0	$\hat{i}$
$\hat{k}$	$\hat{j}$	$-\hat{i}$	0

$\hat{j}\times\hat{i}=-\hat{k}, \hat{k}\times\hat{j}=-\hat{i}, \hat{i}\times\hat{k}=-\hat{j}$

$\hat{i}\times\hat{i}=\vec{O}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}$

$\vec{a}\times\vec{b}=\vec{O}\Leftrightarrow\vec{a}\parallel\vec{b} \qquad \left(\vec{a}\neq\vec{O}\neq\vec{b}\right)$

4. $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{a}\times\vec{b}=\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

5. Area of Parallelogram of two vectors is $=\vec{a}\times\vec{b}$, where $\vec{a}$ and $\vec{b}$ are the adjacent sides of the parallelogram.

6. Area of ΔABC =Qy$=\frac{1}{2}\left|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right|$, where $\vec{a},\vec{b},\vec{c}$, are position vectors of vertices of triangle.

7. The collinearity of three vectors $\vec{a},\vec{b}$ and $\vec{c}$ is given by $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=\vec{O}$

8. Area of parallelogram whose diagonals are $\vec{a}$ and $\vec{b}$ $=\frac{1}{2}\left|\vec{a}\times\vec{b}\right|$

9. We represent the scalar or dot product of three vectors $\vec{a},\vec{b},\vec{c}$ is $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)$ and $\left[\vec{a}\ \vec{b}\ \vec{c}\right]$.

10. If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$,

$\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$, then $\left[\vec{a}\ \vec{b}\ \vec{c}\right]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

11. Volume of rectangular parallelopipped $=\left[\vec{a}\ \vec{b}\ \vec{c}\right]$, (where $\vec{a},\vec{b},\vec{c}$ denoted its concurrent edges).

12. Volume of Tetrahedron$=\frac{1}{6}\left[\vec{a}\ \vec{b}\ \vec{c}\right]$ where $\vec{a},\vec{b},\vec{c}$ are its concurrent edges.

13. The triangular product of three vectors $\vec{a},\vec{b},\vec{c}$ is $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$.

14. In vectors, vector product does not follows associative property i.e. $\vec{a}\times\left(\vec{b}\times\vec{c}\right)\neq\left(\vec{a}\times\vec{b}\right)\times\vec{c}$

Answers

Exericse 13.1

(1) $|\vec{a}| = \sqrt{3}; |\vec{b}| = \sqrt{62}; |\vec{c}| = 1$ (2) any two vectors (3) any two vectors (4) $x = 2,\ y = 3$

(5) $-7,\ 6$ rFkk $-7i,\ 6j$ (6) $-4\hat{j} - \hat{k}$ (7) $\dfrac{\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{6}}$ (8) $\dfrac{(\hat{i} + \hat{j} + \hat{k})}{\sqrt{3}}$

(9) $\dfrac{\hat{i} + \hat{k}}{\sqrt{2}}$ (10) $\dfrac{8(5\hat{i} - \hat{j} + 2\hat{k})}{\sqrt{30}}$ (11) $-4\hat{i} + 6\hat{j} - 8\hat{k} = -2(2\hat{i} - 3\hat{j} + 4\hat{k})$

(12) (i) $\dfrac{1}{3}\hat{i},\ \dfrac{4}{3}\hat{j},\ \dfrac{1}{3}\hat{k}$ (ii) $-3\hat{i} + 3\hat{k}$ (13) $3\hat{i} + 2\hat{j} + \hat{k}$ (14) $\dfrac{-\hat{i} + 4\hat{j} + \hat{k}}{3},\ -3\hat{i} + 3\hat{k}$

(15) (3, 2, 1)

Exercise 13.2

(1) (i) 10 ; (ii) 0 ; (iii) $10\sqrt{3}$ (2) (i) –4 ; (ii) 7 ; (iii) 7 (4) $\theta = \cos^{-1}\left(\dfrac{72}{75}\right)$

(5) (i) 3 ; (ii) 3 (6) $\dfrac{2}{7}$ (7) $5\hat{i} - 15\hat{j} + 7\hat{k}$

Exercise 13.3

(1) $4\hat{i} - 5\hat{j} + 7\hat{k}$ (2) $\dfrac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$ (6) 16 (7) $-3\hat{i} + 6\hat{j} + 6\hat{k}$ (10) $\dfrac{5\sqrt{5}}{2}$

Exercise 13.4

(2) –7 (5) (i) –4 ; (ii) 1 (8) (i) 30 ; (ii) 14

Exercise 13.5

(1) (i) $-2\hat{i} - 2\hat{j} + 4\hat{k}$; (ii) $8\hat{i} - 19\hat{j} - \hat{k}$

(8) $\dfrac{\pi}{4}$ (9) $\cos^{-1}\left(\dfrac{5}{7}\right)$ (10) 0 (11) $\dfrac{60}{\sqrt{114}}$ (12) $6|\vec{a}|^2 + 11\vec{a}\cdot\vec{b} - 35|\vec{b}|^2$

(13) $|\vec{a}| = 1, |\vec{b}| = 1$ (14) $\sqrt{13}$ (15) – 4 (16) $-\dfrac{3}{2}$ (17) $\cos^{-1}\left(\dfrac{10}{\sqrt{102}}\right)$

14

Three Dimensional Geometry

14.01 Introduction

The objects we come acrose around us are only three dimenstional. So, the study of such objects is of utmost importance for our better understanding of this world. In the previous chapter, we have studied vectors in the 3-dimensional space. Vectors are very useful tools to study the 3-dimensional analytic geometry, which is also called the solid geometry. Most of the results are obtained in vector form, which look very simple, and then translate these results to the cartesian form. In solving the problems, we may use either of these two forms.

14.02 Direction Cosines of a Line

Directions cosines of any line L are defined as direction cosines of any vector $\overrightarrow{AB}$ whose support is given line. Let $\overrightarrow{OP} \parallel \overrightarrow{AB}$. If $\overrightarrow{OP}$ makes angles α, β and γ with positive directions of axes OX, OY and OZ then $\cos\alpha, \cos\beta, \cos\gamma$ are direction cosines of $\overrightarrow{OP}$. Direction cosines of $\overrightarrow{OP}$ and $\overrightarrow{AB}$ are similar, because they are parallel and make same angles with axes. In general, direction cosines are represented by ℓ, m, n respectively

$$\therefore \qquad \ell = \cos\alpha, m = \cos\beta, n = \cos\gamma .$$

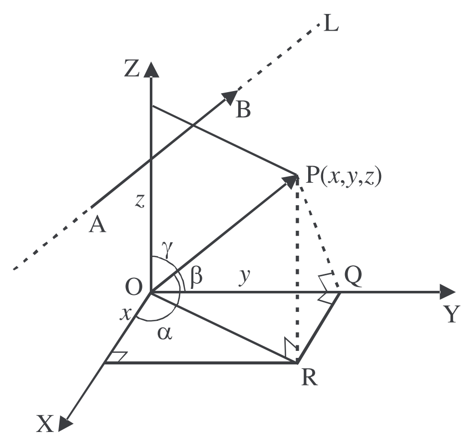


Fig. 14.01

Note:

1. Direction cosines never be written in bracket.
2. $\overrightarrow{BA}$ makes angle $\pi-\alpha$, $\pi-\beta$ and $\pi-\gamma$ with co-ordinate axes OX, OY and OZ respectively. Therefore, directions cosines of $\overrightarrow{BA}$ will be $\cos(\pi-\alpha), \cos(\pi-\beta), \cos(\pi-\gamma)$ i.e. $-\ell, -m, -n$.
   So, if ℓ, m, n are direction cosines of any line, then $-\ell, -m, -n$ are also its direction cosines just because $\overrightarrow{AB}$ and $\overrightarrow{BA}$ have a common support line L.
3. Direction cosines of X-axis : 1, 0, 0
   Direction cosines of Y-axis : 0, 1, 0
   Direction cosines fo Z-axis : 0, 0, 1

14.03 Relation among the Direction Cosines of a Line

Consider a vector $\overrightarrow{AB}$ with direction cosines ℓ, m, n with base line L. Through the origin, draw a line parallel to the given line and take a point P (x, y, z) on this line, such that $\overrightarrow{OP} \parallel \overrightarrow{AB}$. From P, draw a perpendicular PQ on the Y-axis (Fig. 14.01)

If $OP = r$, then $\cos\beta = \dfrac{y}{r}$

$\Rightarrow \qquad y = r\cos\beta = mr$. Similarly, $z = nr$ and $x = \ell r$

Again, $\qquad OP = r$

$\Rightarrow \qquad (OP)^2 = r^2$

$\Rightarrow \qquad x^2 + y^2 + z^2 = r^2$

$\Rightarrow \qquad r^2\left(\ell^2 + m^2 + n^2\right) = r^2$

$\Rightarrow \qquad \ell^2 + m^2 + n^2 = 1$

14.04 Direction ratios of a line

Definition : The direction ratios of a line are proportional to the direction cosines of the vector whose support is the line.

Let a, b, c be direction ratios of a line and let ℓ, m, n be the direction cosines of the vector whose support is give line. Then

$$\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$$

$\therefore$ Direction ratios of any line be the direction ratio of that vector whose support is the given line.

Notes:

1. If a, b, c are direction ratios of a line, then $ka, kb, kc,$ where $k \neq 0$ are also a set of direction ratios. So, any two sets of direction ratios of a line are also proportional. Also, for any line there are infinitely many sets of direction ratios.

2. For direction cosine ℓ, m, n, we have $\ell^2 + m^2 + n^2 = 1$ but for direction ratios a, b, c, we have $a^2 + b^2 + c^2 \neq 1$ till a, b, c become direction cosines.

3. $\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k$ (let)

   $\therefore \qquad \ell = ak, m = bk, n = ck$

   but $\qquad \ell^2 + m^2 + n^2 = 1$

   $\Rightarrow \qquad k^2\left(a^2 + b^2 + c^2\right) = 1$

   $\Rightarrow \qquad k = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}}$

   $\therefore \qquad \ell = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}; m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}; \quad n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$

4. Let $\qquad \vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$

   $\therefore \qquad \hat{r} = \frac{\vec{r}}{|\vec{r}|} = \left(\frac{a}{\sqrt{a^2 + b^2 + c^2}}\right)i + \left(\frac{b}{\sqrt{a^2 + b^2 + c^2}}\right)j + \left(\frac{c}{\sqrt{a^2 + b^2 + c^2}}\right)k$

   $\qquad = \ell\hat{i} + m\hat{j} + n\hat{k}$

   where $\qquad \ell = \frac{a}{\sqrt{a^2 + b^2 + c^2}}; m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}; \quad n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$

   Thus in vector $\vec{r}$, coefficient of $\hat{i}, \hat{j}, \hat{k}$ are the direction ratios of that vector.

14.05 Direction cosines of a line passing through Two Points

Let L be the line passing through the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

$$\overrightarrow{PQ} = \text{(position vector of } Q) - \text{(position vector of } P)$$

$$= \left(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}\right) - \left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right)$$

$$= (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$$

$\therefore$ d.r's (direction ratios) of $\overrightarrow{PQ}$ are $x_2 - x_1, y_2 - y_1, z_2 - z_1$ and its d.c's (direction cosines) are

$$\frac{x_2 - x_1}{|\overrightarrow{PQ}|}, \frac{y_2 - y_1}{|\overrightarrow{PQ}|}, \frac{z_2 - z_1}{|\overrightarrow{PQ}|},$$

where, $\left|\overrightarrow{PQ}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Illustrative Examples

Example 1. A line makes an angle of 30° and 60° with the positive direction of X and Y-axis. Find the angle formed by the line with the positive direction of Z–axis.

Solution : Let the line makes an angle γ with the positive direction of Z-axis. Thus it makes angle 30°, 60° and γ with the three axes.

$\therefore$ the d.c's of line are $\cos 30^\circ$, $\cos 60^\circ$, $\cos\gamma$ i.e. $\frac{\sqrt{3}}{2}, \frac{1}{2}$, $\cos\gamma$

We know that, $\ell^2 + m^2 + n^2 = 1$

$\therefore$ $\left(\sqrt{3}/2\right)^2 + (1/2)^2 + (\cos\gamma)^2 = 1$

or $\cos^2\gamma = 1 - 1$

$\Rightarrow$ $\cos^2\gamma = 0$

$\Rightarrow$ $\cos\gamma = 0$

or, $\gamma = 90^\circ$

Thus the line makes an angle of 90° with the Z-axis.

Example 2. If the vector makes an angle of α, β and γ with OX, OY and OZ axes respectively, then Prove that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

Solution : Let the d.c's of the given vector be ℓ, m, n

then, $\cos\alpha = \ell$, $\cos\beta = m$, $\cos\gamma = n$

we know that $\ell^2 + m^2 + n^2 = 1$

$\therefore$ $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow$ $\left(1 - \sin^2\alpha\right) + \left(1 - \sin^2\beta\right) + \left(1 - \sin^2\gamma\right) = 1$

$\Rightarrow$ $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

Example 3. Find the direction cosines of a line joining the points (1, 0, 0) and (0, 1, 1).

Solution : The direction ratios of the line joining (1, 0, 0) and (0, 1, 1) are

$$0-1,\ 1-0,\ 1-0 = -1,\ 1,\ 1$$

Thus, the direction cosine will be

$$\mp\frac{1}{\sqrt{3}},\ \pm\frac{1}{\sqrt{3}},\ \pm\frac{1}{\sqrt{3}}$$

Example 4. Show that the points A(2, 3, 4), B(–1, 2, –3) and C(–4, 1, –10) are collinear.

Solution : The direction ratios of the line joining the points A and B thus it is clear that direction ratios of AB and BC are proportional therefore

AB || BC

But in AB and BC B is common

$\therefore$ A, B and C are colinear.

Example 5. If a line makes an angle 90°, 135° and 45° with the *X*, *Y* and *Z*-axes respectively then find the direction cosine of the line.

Solution : Direction angles are $90^\circ,\ 135^\circ,\ 45^\circ$

$\therefore$ direction cosines are

$$\ell = \cos 90^o = 0,\ m = \cos 135^o = -\frac{1}{\sqrt{2}},\ n = \cos 45^o = \frac{1}{\sqrt{2}}$$

thus, the d.c's of the given line are

$$0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}.$$

Exercise 14.1

1. Find the direction cosines of a line whcih makes equal angles with the coordinate axes.
2. Find the direction cosines of the line passing through two points (4, 2, 3) and (4, 5, 7).
3. If the direction ratios of the line are 2, –1, –2, then find the direction cosines.
4. A vector $\vec{r}$, makes angle of 45°, 60°, 120° with the *X*, *Y* and *Z*-axes respectively and the magnitude of $\vec{r}$ is 2 units, then find $\vec{r}$.

14.6 Equation of a line in Space

We shall now study the vector and cartesian equations of a line in space. A line is uniquely determined if

(i) it passes through a given point and has given direction, or

(ii) it passes through two given points.

(i) Equation of a line through a given point A $(\vec{a})$ and parallel to a given vector $\vec{m}$

Let the line be *L* whose equation is to be determined. Let the line is parallel to the vector $\vec{m}$ and passes through the point A whose position vector is $\vec{a}$. Let *O* be the origin, therefore $\overrightarrow{OA} = \vec{a}$.

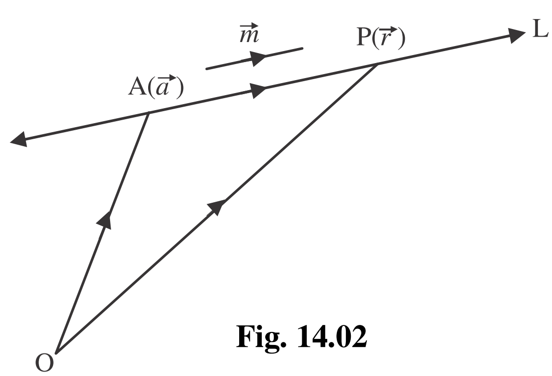


Fig. 14.02

Let P be any point on the line L whose position vector is $\vec{r}$,

then $\overrightarrow{OP} = \vec{r}$

clearly $\overrightarrow{AP} \parallel \overrightarrow{m}$

$\Rightarrow \quad \overrightarrow{AP} = \lambda \overrightarrow{m}$

$\Rightarrow \quad$ (position vector P) $-$ (position vector of A) $= \lambda \vec{m}$

$\Rightarrow \quad \overrightarrow{OP} - \overrightarrow{OA} = \lambda \vec{m}$

$\Rightarrow \quad \vec{r} - \vec{a} = \lambda \vec{m}$

$\Rightarrow \quad \vec{r} = \vec{a} + \lambda \vec{m}$

for each value of the parameter λ , this equation gives the position vector of a point P on the line. Hence, the vector equation of the line is given by

$$\vec{r} = \vec{a} + \lambda \vec{m} \qquad (1)$$

Cartesian Form

Let $A(x_1, y_1, z_1)$ be the given point and the direction ratios of the line be a, b, c. Consider the coordinates of any point P be (x, y, z) then,

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$

Since, the direction ratios of the given line be a, b, c therefore, it is parallel to $\vec{m}$

$$\therefore \quad \vec{m} = a\hat{i} + b\hat{j} + c\hat{k}$$

Now, vector equation of the line is

$$\vec{r} = \vec{a} + \lambda \vec{m}$$

$\Rightarrow \quad x\hat{i} + y\hat{j} + z\hat{k} = \left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right) + \lambda\left(a\hat{i} + b\hat{j} + c\hat{k}\right)$

$\Rightarrow \quad x\hat{i} + y\hat{j} + z\hat{k} = (x_1 + \lambda a)i + (y_1 + \lambda b)\, j + (z_1 + \lambda c)$

$\Rightarrow \quad x = x_1 + \lambda a; \quad y = y_1 + \lambda b; \quad z = z_1 + \lambda c$

$\Rightarrow \quad \dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c} = \lambda$

$\therefore$ equation of line passing through $A(x_1, y_1, z_1)$ with direction ratios a, b, c is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

(ii) Equation of a line passing through two given points

Vector form

Let a line L passes through the two points A and B whose position vectors are $\vec{a}_1$ and $\vec{a}_2$. If O is the origin, then $\overrightarrow{OA} = \vec{a}_1$ and $\overrightarrow{OB} = \vec{a}_2$

$\therefore \qquad \overrightarrow{AB} =$ (position vector of B) $-$ (position vector of A)

$= \vec{a}_2 - \vec{a}_1$

Let there be point P on the line L whose position vector is $\vec{r}$, then $\overrightarrow{OP} = \vec{r}$

$\therefore \qquad \overrightarrow{AP} = \vec{r} - \vec{a}_1$

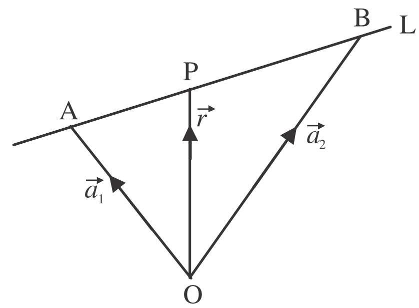


Fig. 14.03

since $\overrightarrow{AP}$ and $\overrightarrow{AB}$ are collinear vectors, then

$\Rightarrow \qquad \overrightarrow{AP} = \lambda\left(\overrightarrow{AB}\right), \lambda \in R$

$\Rightarrow \qquad \vec{r} - \vec{a}_1 = \lambda\left(\vec{a}_2 - \vec{a}_1\right)$

$\Rightarrow \qquad \vec{r} = \vec{a}_1 + \lambda\left(\vec{a}_2 - \vec{a}_1\right)$

$\therefore$ The vector equation of line L passing through the points $A(\vec{a}_1)$ and $B(\vec{a}_2)$ is

$$\vec{r} = \vec{a}_1 + \lambda\left(\vec{a}_2 - \vec{a}_1\right) \qquad (2)$$

Cartesian Form:

Let the line L, passes through the points $A\left(x_1, y_1, z_1\right)$ and $B\left(x_2, y_2, z_2\right)$. Let the coordinates of any point P on the line be $\left(x, y, z\right)$.

Since $\overrightarrow{AP}$ and $\overrightarrow{AB}$ are collinear, then

$\Rightarrow \qquad \left(x\hat{i} + y\hat{j} + z\hat{k}\right) - \left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right) = \lambda\left\{\left(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}\right) - \left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right)\right\}$

$\Rightarrow \qquad \left(x - x_1\right)\hat{i} + \left(y - y_1\right)\hat{j} + \left(z - z_1\right)\hat{k} = \lambda\left(x_2 - x_1\right)\hat{i} + \lambda\left(y_2 - y_1\right)\hat{j} + \lambda\left(z_2 - z_1\right)\hat{k}$

$\Rightarrow \qquad x - x_1 = \lambda\left(x_2 - x_1\right);\ y - y_1 = \lambda\left(y_2 - y_1\right);\ z - z_1 = \lambda\left(z_2 - z_1\right)$

$\Rightarrow \qquad \dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$

which is the required equation of line.

Illustrative Examples

Example 6. Find the vector and cartesian equation of the line passing through the point (5, 2, –4) and parallel to the vector $3\hat{i} + 2\hat{j} - 8\hat{k}$.

Solution :

Let $\qquad \vec{a} = 5\hat{i} + 2\hat{j} - 4\hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{j} - 8\hat{k}$

The vector equation of the line is $\qquad \vec{r} = \vec{a} + \lambda\left(\vec{m}\right)$

$\therefore \qquad x\hat{i} + y\hat{j} + z\hat{k} = 5\hat{i} + 2\hat{j} - 4k + \lambda\left(3\hat{i} + 2\hat{j} - 8\hat{k}\right)$

or, $\qquad x\hat{i} + y\hat{j} + z\hat{k} = (5 + 3\lambda)\hat{i} + (2 + 2\lambda)\hat{j} + (-4 - 8\lambda)\hat{k}$

or, $\qquad x - 5 = 3x,\ y - 2 = 2x,\ z + 4 = -8x$

or, $$\frac{x-5}{3}=\frac{y-2}{2}=\frac{z+4}{-8}=\lambda$$

Thus, equation in cartesian form will be $\frac{x-5}{3}=\frac{y-2}{2}=\frac{z+4}{-8}$

Example 7. Find the vector equation of the line passign through the points (–1, 0, 2) and (3, 4, 6).

Solution : Let the position vector of points A(–1, 0, 2) and B (3, 4, 6) be $\vec{a}$ and $\vec{b}$ respectively.

then, $$\vec{a}=-\hat{i}+2\hat{k}$$

and $$\vec{b}=3\hat{i}+4\hat{j}+6\hat{k}$$

$\therefore$ $$\vec{b}-\vec{a}=4\hat{i}+4\hat{j}+4\hat{k}$$

Let the position vector of any point P be $\vec{r}$, then the vector equation of the line is-

$$\vec{r}=-\hat{i}+2\hat{k}+\lambda\left(4\hat{i}+4\hat{j}+4\hat{k}\right)$$

Example 8. Find the vector equation of a line passing through point A (2, –1, 1) and parallel to the line joining the points B (–1, 4, 1) and C (1, 2, 2). Also find its the cartesian equation.

Solution : For the vector equation

position vector of $B=-\hat{i}+4\hat{j}+\hat{k}$

and position vector of $\text{C}=\hat{i}+2\hat{j}+2\hat{k}$

$\therefore$ $\vec{BC}=$ position vector of C– position vector of B

$$=\left(\hat{i}+2\hat{j}+2\hat{k}\right)-\left(-\hat{i}+4\hat{j}+\hat{k}\right)=2\hat{i}-2\hat{j}+\hat{k}$$

position vector of A, $\vec{r}_1=2\hat{i}-\hat{j}+\hat{k}$

$\therefore$ Vector equation of the line

$$\vec{r}=\vec{r}_1+\lambda(\overrightarrow{BC})$$

$\Rightarrow$ $$\vec{r}=\left(2\hat{i}-\hat{j}+\hat{k}\right)+\lambda\left(2\hat{i}-2\hat{j}+\hat{k}\right) \qquad (1)$$

Certesian equation of the line,

$$\left(x\hat{i}+y\hat{j}+z\hat{k}\right)=\left(2\hat{i}-\hat{j}+\hat{k}\right)+\lambda\left(2\hat{i}-2\hat{j}+\hat{k}\right),\ \text{when } \vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

$\Rightarrow$ $$\left(x\hat{i}+y\hat{j}+z\hat{k}\right)=(2+2\lambda)\hat{i}+(-1-2\lambda)\hat{j}+(1+\lambda)\hat{k}$$

On comparing,

$\Rightarrow$ $$\frac{x-2}{2}=\frac{y+1}{-2}=\frac{z-1}{1}=\lambda$$

Thus, the cartesian equation of line is $\frac{x-2}{2}=\frac{y+1}{-2}=\frac{z-1}{1}$

Example 9. The cartesian equation of a line $6x-2=3y+1=2z-2$. Find

(a) direction ratios of the line.

(b) the vector and cartesian equation of a line passing through (2, –1, –1) and parallel to the given line.

Solution : Equation of a line

$$6x-2=3y+1=2z-2$$

$$\Leftrightarrow \quad \frac{x-(1/3)}{1/6}=\frac{y+(1/3)}{1/3}=\frac{z-1}{1/2}$$

$$\Leftrightarrow \quad \frac{x-(1/3)}{1}=\frac{y+(1/3)}{2}=\frac{z-1}{3}$$

(a) Therefore, the d.r's of the given line are 1, 2, 3.

(b) Equation of a line passing through (2, –1, –1) and parallel to the given line.

$$\frac{x-2}{1}=\frac{y+1}{2}=\frac{z+1}{3}$$

New vector equation of a line passing through A (2, –1, –1) and i.e. $\vec{a}=2\hat{i}-\hat{j}-\hat{k}$ parallel to $\vec{m}=\hat{i}+2\hat{j}+3\hat{k}$ is

$$\vec{r}=\vec{a}+\lambda\vec{m}$$

or, $$\vec{r}=\left(2\hat{i}-\hat{j}-\hat{k}\right)+\lambda\left(\hat{i}+2\hat{j}+3\hat{k}\right)$$

Exercise 14.2

1. Find the equation of the line passing through the point (5, 7, 9) and parallel to the following given axis:
   (i) *X*–axis (ii) *Y*–axis (iii) *Z*–axis
2. Find the equation of the line in vector and in cartesian form that passes through the point with position vector $2\hat{i}-3\hat{j}+4\hat{k}$ and is parallel to the vector $3\hat{i}+4\hat{j}-5\hat{k}$.
3. Find the equation of the line which passes through the point (5, –2, 4) and is parallel to the vector $2\hat{i}-\hat{j}+3\hat{k}$
4. Find the eqation of the line whcih passes through the point (2, –1, 1) and is parallel to the line $\frac{x-3}{2}=\frac{y+1}{7}=\frac{z-2}{-3}$.
5. Find the vector equation of the line whose cartesian equation is
   $$\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$$
6. Find the cartesian equation of the line whcih passes through the point (1, 2, 3) and is parallel to the line $\frac{x-2}{1}=\frac{y+3}{7}=\frac{2z-6}{3}$.

7. The coordinates of the three vertices of a parallelogram ABCD are A (4, 5, 10), B (2, 3, 4) and C (1, 2, –1). Find the vector and cartesian equation of AB and BC. Also find the coordinates of D.

8. The cartesian equation of a line is $3x+1=6y-2=1-z$. Find the point through which it passes and also find the direction ratios and vector equation.

9. Find the equation of the line whcih passes through the point (1, 2, 3) and is parallel to the vector $3\hat{i}+2\hat{j}-2\hat{k}$.

10. Find the vector and cartesian equation of a line passing through the point whose position vector is $2\hat{i}-\hat{j}+4\hat{k}$ and in the direction of the vector $\hat{i}+2\hat{j}-\hat{k}$.

11. Find the cartesian equation of the line which passes through the point (–2, 4, –5) and is parallel to the line $\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$.

12. The cartesian equation of a line is $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$. Find its vector equation.

13. Find the vector and cartesian equation of a line passing through the origin and the point (5, –2, 3).

14. Find the vector and cartesian equation of a line passing through the point (3, –2, –5) and (3, –2, 6).

14.07 Angle between Two Lines

Vector form:

Let the vector equation of two lines be

$$\vec{r}=\vec{a}_1+\lambda\vec{m}_1,\ \lambda\in R \quad \text{and} \quad \vec{r}=\vec{a}_2+\mu\vec{m}_2,\ \mu\in R$$

If the angle between them is θ, then it is clear from figure 14.04 that, the angle between vector $\vec{m}_1$ and vector $\vec{m}_2$ is also θ. Thus $\cos\theta=\frac{\vec{m}_1\cdot\vec{m}_2}{|\vec{m}_1||\vec{m}_2|}$.

Cartesian form:

Let the cartesian equation of two lines be

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1} \quad \text{and} \quad \frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$$

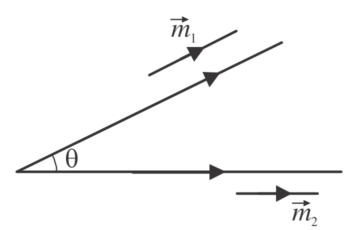


Fig. 14.04

$\therefore$ $\vec{m}_1=a_1\hat{i}+b_1\hat{j}+c_1\hat{k}$ and $\vec{m}_2=a_2\hat{i}+b_2\hat{j}+c_2\hat{k}$

but $$\cos\theta=\frac{\vec{m}_1\cdot\vec{m}_2}{|\vec{m}_1||\vec{m}_2|}$$

$\Rightarrow$ $$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Note:

1. If the direction cosines of two lines are ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 and the angle between them is θ, then $\cos\theta=\ell_1\ell_2+m_1m_2+n_1n_2$.

2. If the two lines are perpendicualr, then $a_1a_2 + b_1b_2 + c_1c_2 = 0$ or $\ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$.

3. If the two lines are parallel, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ or $\frac{\ell_1}{\ell_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$.

Illustrative Examples

Example 10. Find the angle between the lines $\frac{5-x}{3} = \frac{y+3}{-4} = \frac{z-7}{0}$ and $\frac{x}{1} = \frac{1-y}{2} = \frac{z-6}{2}$.

Solution : Given lines are

$$\frac{x-5}{-3} = \frac{y+3}{-4} = \frac{z-7}{0} \qquad (1)$$

$$\frac{x}{1} = \frac{y-1}{-2} = \frac{z-6}{2} \qquad (2)$$

Let the vectors parallel to line (1) and (2) be $\vec{m}_1$ and $\vec{m}_2$ respectively, then $\vec{m}_1 = -3\hat{i} - 4\hat{j} + 0k$ and $\vec{m}_2 = \hat{i} - 2\hat{j} + 2\hat{k}$. Let the angle between $\vec{m}_1$ and $\vec{m}_2$ be θ, then

$$\cos\theta = \frac{\vec{m}_1 \cdot \vec{m}_2}{|\vec{m}_1||\vec{m}_2|}$$

$$\Rightarrow \qquad \cos\theta = \frac{\{(-3)\times 1 + (-4)\times(-2) + 0\times 2\}}{\left\{\sqrt{(-3)^2 + (-4)^2 + 0^2}\right\}\left\{\sqrt{1^2 + (-2)^2 + 2^2}\right\}} = \frac{1}{3}$$

$$\Rightarrow \qquad \theta = \cos^{-1}(1/3).$$

Example 11. Find the angle betwene the lines

$$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda\left(\hat{i} + 2\hat{j} + 2\hat{k}\right) \text{ and } \vec{r} = 5\hat{i} - 2\hat{j} + \mu\left(3\hat{i} + 2\hat{j} + 6\hat{k}\right).$$

Solution : Let the angle between the lines which are parallel to $\vec{b}_1 = \hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b}_2 = 3\hat{i} + 2\hat{j} + 6\hat{k}$ respectively be θ, therefore

$$\cos\theta = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right| = \left|\frac{\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)\cdot\left(3\hat{i} + 2\hat{j} + 6\hat{k}\right)}{\sqrt{1+4+4}\sqrt{9+4+36}}\right|$$

$$= \left|\frac{3+4+12}{3\times 7}\right| = \frac{19}{21}$$

$$\therefore \qquad \theta = \cos^{-1}(19/21)$$

Example 12. Find the equation of line passing through (–1, 3, –2) and perpendicular to the line

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \text{ and } \frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5}$$

Solution : Let < a, b, c > be the d.r's of the required line. Since this required line is are perpendicular to the given lines, then

$$a + 2b + 3c = 0 \quad (1)$$

and $$-3a + 2b + 5c = 0 \quad (2)$$

By cross-multiplication method in (1) and (2), we get

$$\frac{a}{4} = \frac{b}{-14} = \frac{c}{8}$$

or $$\frac{a}{2} = \frac{b}{-7} = \frac{c}{4} = k \text{ (Let)}$$

∴ The line passes through (–1, 3, –2), with d.r's < 2, –7, 4 > be given by

$$\frac{x+1}{2} = \frac{y-3}{-7} = \frac{z+2}{4}$$

Exercise 14.3

1. Find the angle between the lines-

$$\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda\left(3\hat{i} + 2\hat{j} + 6\hat{k}\right) \text{ and } \vec{r} = 7\hat{i} - 6\hat{j} + \mu\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$$

2. Find the angle between the lines-

$$\frac{x}{2} = \frac{y}{2} = \frac{z}{1} \text{ and } \frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$$

3. Show that the line passing through the points (1, –1, 2) and (3, 4, –2) is perpendicular to the line passing through the points (0, 3, 2) and (3, 5, 6).

4. If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ are mutually perpendicular, then find the value of k.

5. Find the vector equation of the line passing through the point (1, 2, –4) and perpendicular to the lines

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \text{ and } \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}.$$

6. Find the cartesian equation of the line passing through (–2, 4, –5) and parallel to the line $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$.

14.08 Intersection of Two Lines

If two lines intersect in a plane, then there is one common point between them so that the distance between them is zero. The following methods are used to find the point of intersection of two lines.

(1) Equation of lines in vector form:

Let two lines be $$\vec{r} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) + \lambda(m_1\hat{i} + m_2\hat{j} + m_3\hat{k}) \quad (1)$$

and $$\vec{r} = (a'_1\hat{i} + a'_2\hat{j} + a'_3\hat{k}) + \mu(m'_1\hat{i} + m'_2\hat{j} + m'_3\hat{k}) \quad (2)$$

(i) ∵ Lines intersect, therefore

$$(a_1 i + a_2 j + a_3 k) + \lambda(m_1\hat{i} + m_2\hat{j} + m_3\hat{k}) = \vec{r} = (a'_1\hat{i} + a'_2\hat{j} + a'_3\hat{k}) + \mu(m'_1\hat{i} + m'_2\hat{j} + m'_3\hat{k})$$

On comparing, we get

$$a_1+\lambda m_1=a'_1+\mu m'_1;\quad a_2+\lambda m_2=a'_2+\mu m'_2;\quad a_3+\lambda m_3=a'_3+\mu m'_3$$

(ii) On solving the two equations, we get the value of λ and μ. If these values satisfy the third equation, then the lines are intersecting otherwise not.

(iii) To get the position vector of intersecting point, put the value of λ, μ in (1) and (2).

(2) Equation of lines in cartesian form

lines $$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}=r_1 \text{ (let)} \quad (1)$$

and $$\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}=r_2 \text{ (let)} \quad (2)$$

(i) Point on line (1) and (2) are

$$(a_1r_1+x_1,\ b_1r_1+y_1,\ c_1r_1+z_1) \text{ and } (a_2r_2+x_2,\ b_2r_2+y_2,\ c_2r_2+z_2)$$

∵ Lines intersect, therefore

$$a_1r_1+x_1=a_2r_2+x_2;\quad b_1r_1+y_1=b_2r_2+y_2 \text{ and } c_1r_1+z_1=c_2r_2+z_2$$

(ii) Find the value of r_1 and r_2 by solving any two of the equations. If the values of r_1 and r_2 satisfy the third equation, then the lines intersect otherwise not.

(iii) Substituting the values of r_1 and r_2 in the general point, we get the point of intersection.

Illustrative Examples

Example 13. Prove that the lines

$$\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7} \text{ and } \frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$$

intersect each other. Find the coordinates of their intersecting points.

Solution : Let the coordinates of any point on $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}=r_1$ (let)

be $(r_1+4,\ -4r_1-3,\ 7r_1-1)$. Similarly,

$\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}=r_2$ the coordinates of the point be $(2r_2+1,\ -3r_2-1,\ 8r_2-10)$ on the line.

These lines will intersect each other, if they have a common point between them i.e.

$$r_1+4=2r_2+1 \quad (1)$$

$$-4r_1-3=-3r_2-1 \quad (2)$$

$$7r_1-1=8r_2-10. \quad (3)$$

Solving equation (1) and (2), we have $r_1=1,\ r_2=2,$ which satisfies equation (3) also. Thus, the two lines intersect each other at the point (5, –7, 6).

Example 14. Prove that the lines

$$\vec{r} = (i + j - k) + \lambda(3i - j) \text{ and } \vec{r} = (4i - k) + \mu(2i + 3k)$$

intersect each other and find the point of intrsection.

Solution : Let the position vector of the points of intersection be $\vec{r}$.

$$\therefore \qquad (i + j - k) + \lambda(3i - j) = (4i - k) + \mu(2i + 3k)$$

$$1 + 3\lambda = 4 + 2\mu \quad \Rightarrow \quad 3\lambda - 2\mu = 3 \qquad (1)$$

$$1 - \lambda = 0 \quad \Rightarrow \quad \lambda = 1 \qquad (2)$$

$$-1 = -1 + 3\mu \quad \Rightarrow \quad \mu = 0 \qquad (3)$$

(on comparing the coefficients of i, j, k)

From (2) and (3), $\lambda = 1, \mu = 0$, which satisfie (1). Also putting $\lambda = 1$ in the equation $\vec{r} = (i + j - k) + \lambda(3i - j)$, we have

$$\vec{r} = 4i + 0j - k$$

thus, the coordinates of point of intersection are (4, 0, –1).

Example 15. Show that the lines,

$$\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-1}{5} \text{ and } \frac{x-2}{4} = \frac{y-1}{3} = \frac{z+1}{-2}$$

do not intersect each other.

Solution : Given lines are

$$\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-1}{5} = \lambda \qquad (1)$$

$$\frac{x-2}{4} = \frac{y-1}{3} = \frac{z+1}{-2} = \mu \qquad (2)$$

Let $P(3\lambda + 1, 2\lambda - 1, 5\lambda + 1)$ be any point on (1) and $Q(4\mu + 2, 3\mu + 1, -2\mu - 1)$ be any point on (2). If the lines (1) and (2) intersect, then

$$3\lambda + 1 = 4\mu + 2 \quad ; \quad 2\lambda - 1 = 3\mu + 1 \quad ; \quad 5\lambda + 1 = -2\mu - 1$$

$$\Leftrightarrow \qquad 3\lambda - 4\mu = 1 \qquad (3)$$

$$2\lambda - 3\mu = 2 \qquad (4)$$

$$5\lambda + 2\mu = -2 \qquad (5)$$

Solving (3) and (4), we have $\lambda = -5$ and $\mu = -4$.

But the value of λ and μ , do not satisfy (5) Therefore, these two lines do not intersect each other.

14.09 Perpendicular distance of a point from a line

Vector form:

Let the foot of perpendicular drawn from point $P(\vec{\alpha})$ on the line be L

$\because$ $\vec{r}$ is any arbitrary point on the line. Therefore, the position vector of point L will be $\vec{a} + \lambda\vec{b}$

$\therefore \quad \overrightarrow{PL} =$ Position vector of L – position vector of P

$$= \vec{a} + \lambda\vec{b} - \vec{\alpha}$$

$$= (\vec{a} - \vec{\alpha}) + \lambda\vec{b}$$

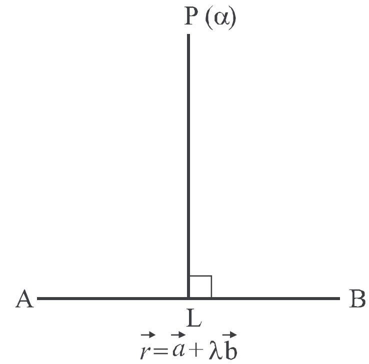


Fig. 14.05

$\because \quad$ vector $\overrightarrow{PL}$ is perpendicular to the line parallel to $\vec{b}$ therefore

$$\overrightarrow{PL} \cdot \vec{b} = 0$$

$$\left\{(\vec{a} - \vec{\alpha}) + \lambda\vec{b}\right\} \cdot \vec{b} = 0$$

$$(\vec{a} - \vec{\alpha}) \cdot \vec{b} + \lambda\,|\vec{b}\,|^2 = 0$$

$$\lambda = -\frac{(\vec{a} - \vec{\alpha}) \cdot \vec{b}}{|\vec{b}\,|^2}$$

Now position vector of L $= \vec{a} + \lambda\,\vec{b}$

$$= \vec{a} - \left(\frac{(\vec{a} - \vec{\alpha}) \cdot \vec{b}}{|\vec{b}\,|^2}\right)\vec{b}$$

$\therefore \quad$ Equation of $\overrightarrow{PL}$

$$\vec{r} = \vec{\alpha} + \mu\left[\left\{\vec{a} - \left(\frac{(\vec{a} - \vec{\alpha}) \cdot \vec{b}}{|\vec{b}\,|^2}\right)\vec{b}\right\} - \alpha\right]$$

$$= \vec{\alpha} + \mu\left[(a - \vec{\alpha}) - \left\{\frac{(\vec{a} - \vec{\alpha}) \cdot \vec{b}}{|\vec{b}\,|^2}\right\}\vec{b}\right]$$

$\therefore \quad$ Magnitude of $\overrightarrow{PL}$ is length of PL

Cartesian Form : To find the length of perpendicular drawn from $P(\alpha, \beta, \gamma)$ on the line $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$.

Let the foot of perpendicular drawn from point $P(\alpha, \beta, \gamma)$ to the line $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c} = \lambda$

Let the coordinates of L be $(x_1 + a\lambda,\ y_1 + b\lambda,\ z_1 + c\lambda)$

$\therefore \quad$ direction ratios of PL be $x_1 + a\lambda - \alpha,\ y_1 + b\lambda - \beta$ and $z_1 + c\lambda - \gamma$

d.r's of line AB be a, b, c

$\because \quad$ PL and AB are mutually perpendicular. Therefore,

$$(x_1 + a\lambda - \alpha)a + (y_1 + b\lambda - \beta)b + (z_1 + c\lambda - \gamma)c = 0$$

$$\Rightarrow \quad \lambda = \frac{a(\alpha - x_1) + b(\beta - y_1) + c(\gamma - z_1)}{a^2 + b^2 + c^2}$$

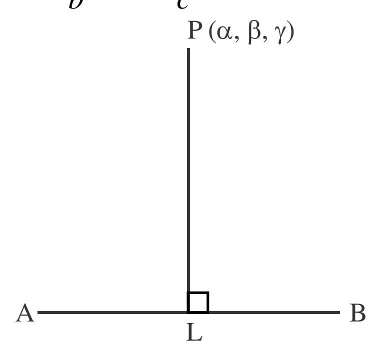


Fig. 14.06

By putting the value of λ in the coordinates of L, we get the actual coordinates of L. We can find the distance of PL by using distance formula.

Illustrative Examples

Example 16. Find the length of perpendicular drawn from point the (1, 2, 3) on the line $\frac{x-6}{3}=\frac{y-7}{2}=\frac{z-7}{-2}$.

Solution : Let the foot of perpendicular from point P (1, 2, 3) on the line be L.

$\therefore$ Coordinates of L are $(3\lambda+6,\ 2\lambda+7,\ -2\lambda+7)$ (1)

$\therefore$ d.r's of PL

$3\lambda+6-1,\ 2\lambda+7-2,\ -2\lambda+7-3$

i.e. $3\lambda+5,\ 2\lambda+5,\ -2\lambda+4$

d.r's of line are 3, 2, –2. Since PL is perpendicular to the given line. Therefore,

$$3(3\lambda+5)+2(2\lambda+5)+(-2)(-2\lambda+4)=0$$

$$\Rightarrow \qquad \lambda=-1$$

Putting the value of $\lambda=-1$ in (1), the coordiantes of L are (3, 5, 9)

$$PL=\sqrt{(3-1)^2+(5-2)^2-(9-3)^2}$$

$$= 7 \text{ units}$$

Required length of perpendicular is 7 units.

Exercise 14.4

1. Show that the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-4}{5}=\frac{y-1}{2}=z$ are mutually intersecting. Find the point of intersection.
2. Examine that the lines $\vec{r}=(\hat{i}-\hat{j})+\lambda(2\hat{i}+\hat{k})$ and $\vec{r}=(2\hat{i}-\hat{j})+\mu(\hat{i}+\hat{j}-\hat{k})$ are intersecting or not.
3. Find the foot of perpendicular from the point (2, 3, 4) to the line $\frac{4-x}{2}=\frac{y}{6}=\frac{1-z}{3}$. Also find the perpendicular distance of the line from the point.
4. Find the vector equation of the line passing through the point (2, 3, 2) and parallel to the line $\vec{r}=(-2\hat{i}+3\hat{j})+\mu(2\hat{i}-3\hat{j}+6\hat{k})$. Also find the distance between them.

14.10 Skew Lines and Shortest Distance between Two Skew Lines

If two lines in space intersect at a point, then the shortest distance between them is zero. Also, if two lines in space are parallel, then the shortest distance between them will be the perpendicular distance, i.e. the length of the perpendicular drawn from a point on one line onto the other line. Further, in a space, there are lines whcih are neither intersecting nor parallel. In fact, such pair of lines are *non coplanar* and are called *skew lines.*

By the shortest distance between two lines, we mean the join of a point in one line with one point on the other line so that the length of the segment so obtained is the smallest. For skew lines, the line of the shortest distance will be perpendicular to both the lines.

Note: If two lines intersect at a point, then the shortest distance between them is zero.

14.11 To find the Shortest Distance between Two Skew Lines

Vector form

We now determine the shortest distance between two skew lines in the following way:

Let L_1 and L_2 be two skew lines with equations

$$L_1 : \vec{r} = \vec{a}_1 + \lambda \vec{b}_1$$

$$L_2 : \vec{r} = \vec{a}_2 + \lambda \vec{b}_2$$

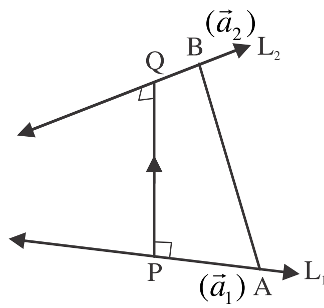


Fig. 14.07

Take any point A on L_1 with position vector $A(\vec{a}_1)$ and B on L_2, with position vector $B(\vec{a}_2)$. Then, the magnitude of the shortest distance vector will be equal to that of the projection of AB along the direction of the line of shortest distance. If $\overrightarrow{PQ}$ is the shortest distance vector between L_1 and L_2, then it is being perpendicular to both $\vec{b}_1$ and $\vec{b}_2$. Thus, unit vector $\hat{n}$ along $\overrightarrow{PQ}$ will be

$$\hat{n} = \frac{\vec{b}_1 \times \vec{b}_2}{\left|\vec{b}_1 \times \vec{b}_2\right|}$$

$$\therefore \qquad \overrightarrow{PQ} = (PQ)\hat{n} = d\hat{n}, \text{ where } PQ = d \text{ (Shortest Distance)}$$

Let θ be the angle between $\overrightarrow{AB}$ and $\overrightarrow{PQ}$, then

$$PQ = AB\cos\theta \qquad (1)$$

But, $$\cos\theta = \frac{\overrightarrow{AB}.\overrightarrow{PQ}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{PQ}\right|} \qquad (2)$$

$$\therefore \qquad \cos\theta = \frac{(\vec{a}_2 - \vec{a}_1)\cdot(d\hat{n})}{(AB)(d)}, \qquad \overrightarrow{AB} = \vec{a}_2 - \vec{a}_1$$

$$= \frac{(\vec{a}_2 - \vec{a}_1)\cdot\hat{n}}{(AB)}$$

From (1), $$PQ = (AB)\frac{(\vec{a}_2 - \vec{a}_1)\cdot\hat{n}}{(AB)}$$

$$= (\vec{a}_2 - \vec{a}_1)\cdot\hat{n}$$

$$= \frac{(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}$$

$\therefore$ Required shortest distance $= d = PQ = \left|\dfrac{(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right|$

Note: If two lines mutually intersect, then the shortest distance distance between them is zero.

i.e. $\left|\dfrac{(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right| = 0$

$\Rightarrow \quad (\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2) = 0$

$\Rightarrow \quad \left[(\vec{a}_2 - \vec{a}_1)\ \vec{b}_1\ \ \vec{b}_2\right] = 0$

Cartesian Form:

The shortest distance between lines $L_1 : \dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1}$ and $L_2 : \dfrac{x - x_2}{a_2} = \dfrac{y - y_2}{b_2} = \dfrac{z - z_2}{c_2}$

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

14.12 Distance between two parallel lines

If two lines L_1 and L_2 are parallel, then they are coplanar. Let the lines be given by

$\vec{r} = a_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$

where, $\vec{a}_1$ is the position vector of a point A to L_1 and $\vec{a}_2$ is the position vector of a point B to L_2.

As L_1, L_2 are coplanar. Therefore, according to fig. 14.08 the foot of the perpendicualr from B on the line L_1 is C, then the distance between the lines L_1 and $L_2 = BC$

Let θ be the angle between $\overrightarrow{AB}$ and $\vec{b}$

$\therefore \quad \vec{b} \times \overrightarrow{AB} = \left(|\vec{b}||\overrightarrow{AB}|\sin\theta\right)\hat{n}$

where $\hat{n}$, is the unit vector perpendicular to the plane of the lines L_1 and L_2

$\Rightarrow \quad \vec{b} \times (\vec{a}_2 - \vec{a}_1) = |\vec{b}|(BC)\hat{n}$, where $BC = (AB)\sin\theta$

$\Rightarrow \quad |\vec{b} \times (\vec{a}_2 - \vec{a}_1)| = |\vec{b}|(BC)$, where $|\hat{n}| = 1$

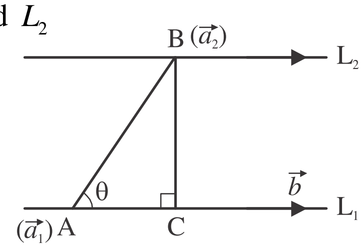


Fig. 14.08

$$\Rightarrow \qquad BC = \frac{\left|\vec{b} \times (\vec{a}_2 - \vec{a}_1)\right|}{|\vec{b}|}$$

Thus, the distance between the two given parallel lines,

$$d = BC = \left|\frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$$

Illustrative Examples

Example 17. Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k}) \text{ and } \vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k})$$

Solution : $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$

We see that $\vec{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k},\ \vec{a}_2 = 4\hat{i} + 5\hat{j} + 6\hat{k}$

$\vec{b}_1 = \hat{i} - 3\hat{j} + 2\hat{k}$ and $\vec{b}_2 = 2\hat{i} + 3\hat{j} + \hat{k}$

$$\therefore \qquad (\vec{a}_2 - \vec{a}_1) = (4\hat{i} + 5\hat{j} + 6\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = (3\hat{i} + 3\hat{j} + 3\hat{k})$$

and
$$(\vec{b}_1 \times \vec{b}_2) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= \hat{i}(-3-6) + \hat{j}(4-1) + \hat{k}(3+6) = -9\hat{i} + 3\hat{j} + 9\hat{k}$$

$$\therefore \qquad |\vec{b}_1 \times \vec{b}_2| = \sqrt{81+9+81} = \sqrt{171}$$

$$\text{S.D.} = \frac{|(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}$$

$$\therefore \qquad \text{S.D.} = \frac{|(3\hat{i} + 3\hat{j} + 3\hat{k})\cdot(-9\hat{i} + 3\hat{j} + 9\hat{k})|}{\sqrt{171}}$$

$$= \frac{|-27+9+27|}{\sqrt{171}} = \frac{9}{\sqrt{171}} = \frac{9}{3\sqrt{19}} = \frac{3}{\sqrt{19}}$$

Example 18. Find the shortest distance between two lines whose equations are

$$\frac{x-3}{2} = \frac{y-4}{1} = \frac{z+1}{-3} \text{ and } \frac{x-1}{-1} = \frac{y-3}{3} = \frac{z-1}{2}.$$

Solution : The given equations are

$$\frac{x-3}{2} = \frac{y-4}{1} = \frac{z+1}{-3} \qquad (1)$$

$$\frac{x-1}{-1}=\frac{y-3}{3}=\frac{z-1}{2} \qquad (2)$$

From (1) line passes through (3, 4, –1) and its d.r's are 2, 1, –3.

∴ Its the vector equation is $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$, where $\vec{a}_1=3\hat{i}+4\hat{j}-\hat{k},\ \vec{b}_1=2\hat{i}+\hat{j}-3\hat{k}$

Similarly from line (2),

$$\vec{a}_2=\hat{i}+3\hat{j}+\hat{k},\ \vec{b}_2=-\hat{i}+3\hat{j}+2\hat{k}$$

Now, $$\vec{a}_2-\vec{a}_1=\left(\hat{i}+3\hat{j}+\hat{k}\right)-\left(3\hat{i}+4\hat{j}-\hat{k}\right)=-2\hat{i}-\hat{j}+2\hat{k}$$

and $$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & -3\\ -1 & 3 & 2\end{vmatrix}=11\hat{i}-\hat{j}+7\hat{k}$$

∴ $$|\vec{b}_1\times\vec{b}_2|=|11\hat{i}-\hat{j}+7\hat{k}|=\sqrt{121+1+49}=\sqrt{171}=3\sqrt{19}$$

Shortest distance $$=\left|\frac{(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$$

$$=\left|\frac{(-2\hat{i}-\hat{j}+2\hat{k})\cdot(11\hat{i}-\hat{j}+7\hat{k})}{3\sqrt{19}}\right|=\left|\frac{-22+1+14}{3\sqrt{19}}\right|=\frac{7}{3\sqrt{19}}$$

Example 19. Find the shortest distance between the lines L_1 and L_2 whose vector equations are

$$\vec{r}=\hat{i}+2\hat{j}-4\hat{k}+\lambda\left(2\hat{i}+3\hat{j}+6\hat{k}\right) \text{ and } \vec{r}=3\hat{i}+3\hat{j}-5\hat{k}+\mu\left(2\hat{i}+3\hat{j}+6\hat{k}\right)$$

Solution : The given lines are parallel. Comparing with $\vec{r}=\vec{a}_1+\lambda\vec{b}$ and $\vec{r}=\vec{a}_2+\mu\vec{b}$, we have

$$\vec{a}_1=\hat{i}+2\hat{j}-4\hat{k},\ \vec{a}_2=3\hat{i}+3\hat{j}-5\hat{k}$$

and $\vec{b}=2\hat{i}+3\hat{j}+6\hat{k}$

Hence, the distance between the lines

$$d=\left|\frac{\vec{b}\times\left(\vec{a}_2-\vec{a}_1\right)}{|\vec{b}|}\right|=\left|\frac{\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 6\\ 2 & 1 & -1\end{vmatrix}}{\sqrt{4+9+36}}\right|=\frac{\left|-9\hat{i}+14\hat{j}-4\hat{k}\right|}{\sqrt{49}}=\frac{\sqrt{293}}{\sqrt{49}}=\frac{\sqrt{293}}{7}$$

Exercise 14.5

1. Find the shortest distance between the lines whose vector equations are $\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k})$.

2. Find the shortest distance between the lines whose equation are $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$.

3. Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k}) \quad \text{and} \quad \vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k})$$

4. Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k} \quad \text{and} \quad \vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$$

5. Find the shortest distance between the lines

$$\frac{x-1}{2} = \frac{y+1}{3} = z \quad \text{and} \quad \frac{x+1}{3} = \frac{y-2}{1}, z = 2$$

Also find the equations of line of shortest distance.

14.13 Plane

Definition : A plane is a surface such that if any two points are taken on it, the line segment joining them, lies completely on the surface.

A plane is determined uniquely if any one of the following be known:

(i) the normal to the plane and its distance from the origin be given, i.e., equation of a plane in normal form.

(ii) it passes through a point and it is perpendicualr to a given direction.

(iii) it passes through three given non collinear points.

Now we shall find vector and Cartesian equations of the planes.

14.14 General Equation of a Plane

To prove that, every first degree equation in x, y and z represents a plane.

Let the equation be

$$ax + by + cz + d = 0, \qquad (1)$$

where a, b, c and d are the constants and a, b, c are non-zero.

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ satisfy the equation (1)

$$ax_1 + by_1 + cz_1 + d = 0 \qquad (2)$$

and

$$ax_2 + by_2 + cz_2 + d = 0 \qquad (3)$$

multiplying (2) with m_2 and (3) with m_1 (where $m_1 + m_2 \neq 0$) and adding

$$a(m_2x_1 + m_1x_2) + b(m_2y_1 + m_1y_2) + c(m_2z_1 + m_1z_2) + d(m_1 + m_2) = 0$$

or

$$a\left(\frac{m_2x_1 + m_1x_2}{m_1 + m_2}\right) + b\left(\frac{m_2y_1 + m_1y_2}{m_1 + m_2}\right) + c\left(\frac{m_2z_1 + m_1z_2}{m_1 + m_2}\right) + d = 0$$

The point dividing the points P and Q in the ratio $m_1 : m_2$ is given by

$$R\left(\frac{m_2x_1+m_1x_2}{m_1+m_2},\ \frac{m_2y_1+m_1y_2}{m_1+m_2},\ \frac{m_2z_1+m_1z_2}{m_1+m_2}\right)$$

for every value of m_1, m_2 (except $m_1=-m_2$), point R satisfies equation (1)

Here, we have shown that $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ lie on (1) and the point R joining the point P and Q also lies on (1) i.e. line lies in the plane given by (1).

Thus, equation (1) denotes a plane in General form. Therefore, a linear equation with variables x, y, z always denotes an equation of plane.

Corollary : One Point Form:

To prove that, the equation of a plane passing through the point (x_1, y_1, z_1) is

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0,$$

Let equation of plane be

$$ax+by+cz+d=0, \tag{1}$$

since, it passes through (x_1, y_1, z_1)

$$\therefore \quad ax_1+by_1+cz_1+d=0. \tag{2}$$

subtracting (2) from (1),

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0, \tag{3}$$

whcih is the required equation of plane.

Special cases: In the general equation of plane $ax+by+cz+d=0$,

	If		**Form of Plane**		**Conclusion**
1.	$d=0$	$\Rightarrow$	$ax+by+cz=0$	$\Rightarrow$	Plane passes through the origin
2. (i)	$a=0$	$\Rightarrow$	$by+cz+d=0$	$\Rightarrow$	Plane parallel to X - axis
(ii)	$b=0$	$\Rightarrow$	$ax+cz+d=0$	$\Rightarrow$	Plane parallel to Y - axis
(iii)	$c=0$	$\Rightarrow$	$ax+by+d=0$	$\Rightarrow$	Plane parallel to Z - axis
3. (i)	$a=0, d=0$	$\Rightarrow$	$by+cz=0$	$\Rightarrow$	Plane passes through X - axis
(ii)	$b=0, d=0$	$\Rightarrow$	$ax+cz=0$	$\Rightarrow$	Plane passes through Y - axis
(iii)	$c=0, d=0$	$\Rightarrow$	$ax+by=0$	$\Rightarrow$	Plane passes through Z - axis
4. (i)	$b=0, c=0$	$\Rightarrow$	$ax+d=0$	$\Rightarrow$	Plane perpendicular to X - aixs
(ii)	$a=0, c=0$	$\Rightarrow$	$by+d=0$	$\Rightarrow$	Plane perpendicular to Y - axis
(iii)	$a=0, b=0$	$\Rightarrow$	$cz+d=0$	$\Rightarrow$	Plane perpendicular to Z - axis
5- (i)	$a=b=d=0$	$\Rightarrow$	$cz=0$	$\Rightarrow$	Plane coincides to with XY - plane
(ii)	$b=c=d=0$	$\Rightarrow$	$ax=0$	$\Rightarrow$	Plane coincides to with YZ - plane
(iii)	$a=c=d=0$	$\Rightarrow$	$by=0$	$\Rightarrow$	Plane coincides to with ZX - plane

Note: Since there are three independent constants in the equation of plane, hence to get the complete equation of plane, we must find the vlaues of the three constants.

Illustrative Examples

Example 20. Find the ratio in which the line joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is divided by the plane $ax + by + cz + d = 0$.

Solution : Let the line joining the points P and Q is divided by the plane $ax + by + cz + d = 0$ in the ratio $\lambda : 1$.

Let the intersecting point of the line and Plane be R. Thus, R lies on PQ whcih divides PQ in the ratio $\lambda : 1$. Therefore, the coordinates of R will be $\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}, \frac{\lambda z_2 + z_1}{\lambda + 1}\right)$.

Since the point R lies on the plane, therefore it will satisfy the equation of the plane

$$\therefore \quad a\left(\frac{\lambda x_2 + x_1}{\lambda + 1}\right) + b\left(\frac{\lambda y_2 + y_1}{\lambda + 1}\right) + c\left(\frac{\lambda z_2 + z_1}{\lambda + 1}\right) + d = 0$$

or, $$a(\lambda x_2 + x_1) + b(\lambda y_2 + y_1) + c(\lambda z_2 + z_1) + d(\lambda + 1) = 0$$

or, $$\lambda(ax_2 + by_2 + cz_2 + d) = -(ax_1 + by_1 + cz_1 + d)$$

or, $$\lambda = -\frac{(ax_1 + by_1 + cz_1 + d)}{(ax_2 + by_2 + cz_2 + d)}$$

This is the required ratio.

Example 21. Find the ratio in whcih the line joining the points P (–2, 4, 7) and Q (3, –5, 8) is cut by the co-ordinate planes.

Solution : The coordinates of point R on the line joining the points P (–2, 4, 7) and Q (3, –5, 8) and dividing in the ratio $\lambda : 1$ be $\left(\frac{3\lambda - 2}{\lambda + 1}, \frac{-5\lambda + 4}{\lambda + 1}, \frac{8\lambda + 7}{\lambda + 1}\right)$.

(i) If R, lies on YZ plane i.e. at $x = 0$, then $\frac{3\lambda - 2}{\lambda + 1} = 0$ or $\lambda = \frac{2}{3}$ i..e the required ratio is 2 : 3.

(ii) If R, lies on ZX plane i.e. at $y = 0$, then $\frac{-5\lambda + 4}{\lambda + 1} = 0$ or $\lambda = \frac{4}{5}$ i.e. the required ratio is 4 : 5.

(iii) If R, lies on XY plane i.e. at $z = 0$, then $\frac{8\lambda + 7}{\lambda + 1} = 0$ or $\lambda = -\frac{7}{8}$ i.e. the required ratio is –7 : 5.

14.15 Intercept Form of a Plane

In this section, we shall deduce the equation of a plane in terms of the intercepts a, b and made by the plane on the coordinate axes. i.e. on X, Y and Z-axes resectively as -

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Let the equation of the plane be

$$Ax + By + Cz + D = 0 \qquad (1)$$

Let the plane makes intercepts a, b, c on X Y and Z axes respectively, such that

$$OP = a,\ OQ = b \text{ and } OR = c$$

Thus, the coordinates P, Q and R be $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$.

Since, point P $(a, 0, 0)$ lies on plane (1),

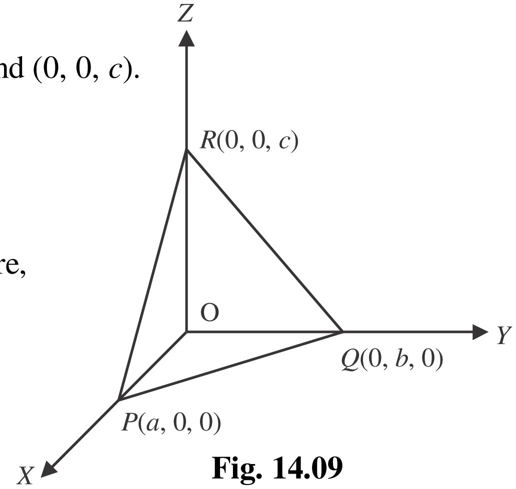


Fig. 14.09

$$\therefore \quad A \cdot a + B \cdot 0 + C \cdot 0 + D = 0 \quad \Rightarrow \quad A = -\frac{D}{a}$$

Similarly, plane (1) passes through point Q and R. Therefore,

$$B = -D/b \text{ and } C = -D/c$$

Substituting the vlaues of A, B, C in (1), we have

$$-\frac{D}{a}x - \frac{D}{b}y - \frac{D}{c}z + D = 0 \quad \text{or} \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

This is the required equation of plane in intercept form.

Note: By converting general equation of plane in intercept from, we obtain the intercepts made by plane on axes.

Illustrative Examples

Example 22. Convert the equation of plane $3x - 4y + 2z = 12$ in the intercept form and find the intercepts made on the coordinate axes.

Solution : Given equation is $3x - 4y + 2z = 12$,

$$\Rightarrow \quad \frac{3x}{12} - \frac{4y}{12} + \frac{2z}{12} = 1$$

$$\Rightarrow \quad \frac{x}{4} + \frac{y}{(-3)} + \frac{z}{6} = 1$$

On comparing with the intercept form $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, we have the intercepts made on X, Y and Z axes are 4, –3 and 6 respectively.

Example 23. A plane meets the coordinate axes at points A, B and C such that the coordinates of the centroid of the triangle ABC so formed is $K(p, q, r)$. Show that the required equation of the plane is $\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 3$.

Solution : Let the equation of plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. Thus, the coordinates of A, B and C are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$. Therefore, the coordinates of the centroid will be $K(a/3, b/3, c/3)$. But it is given that the centroid is $K(p, q, r)$,

$$\therefore \quad \frac{a}{3} = p, \qquad \frac{b}{3} = q, \qquad \frac{c}{3} = r$$

$$\Rightarrow \quad a = 3p, \qquad b = 3q, \qquad c = 3r.$$

Substituting the values of a, b and c in, we get the required equation

$$\frac{x}{3p} + \frac{y}{3q} + \frac{z}{3r} = 1, \quad \text{i.e.} \quad \frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 3,$$

Example 24. A variable plane moves in a space in such a way that the sum of reciprocals of the intercepts made by it on the coordinate axes is a constant. Prove that the plane passes through the fixed point.

Solution : Let the equation of the plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$ (1)

$\therefore$ The intercpets made by the plane on the coordinate axes are a, b and c respectively.

According to the question, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\text{constant}=\frac{1}{\lambda}$ (let)

or, $$\frac{\lambda}{a}+\frac{\lambda}{b}+\frac{\lambda}{c}=1 \qquad (2)$$

$\therefore$ Equation (2) shows that, $(\lambda, \lambda, \lambda)$ satisfies equation (1). Thats means, plane (1) passes through the fixed point $(\lambda, \lambda, \lambda)$

14.16 Equation of a Plane in Normal Form

Vector form : Consider a plane whose perpendicualr distance from the origin is p and $\hat{n}$ is the unit normal vector. Now, we have to find the equation of this plane.

Let O be the origin.

Let $ON = p$, length of perpendicular from the origin to the plane

Let $\hat{n}$ is the unit normal vector along ON whose direction from O to N is positive

$\therefore$ $$\overrightarrow{ON} = p\hat{n} \qquad (1)$$

Let the position vector of any point P be $\vec{r}$ then $\overrightarrow{NP} \perp \overrightarrow{ON}$.

$\therefore$ $$\overrightarrow{NP}\cdot\overrightarrow{ON}=0 \qquad (2)$$

but $$\overrightarrow{NP}=\vec{r}-p\hat{n} \qquad (3)$$

From (1), (2) and (3),

$$\left(\vec{r}-p\hat{n}\right)\cdot p\hat{n}=0$$

or $$\left(\vec{r}-p\hat{n}\right)\cdot\hat{n}=0 \qquad [\because p\neq 0]$$

or $$\vec{r}\cdot\hat{n}-p\hat{n}\cdot\hat{n}=0$$

or $$\vec{r}\cdot\hat{n}=p \quad [\because \hat{n}\cdot\hat{n}=1]$$

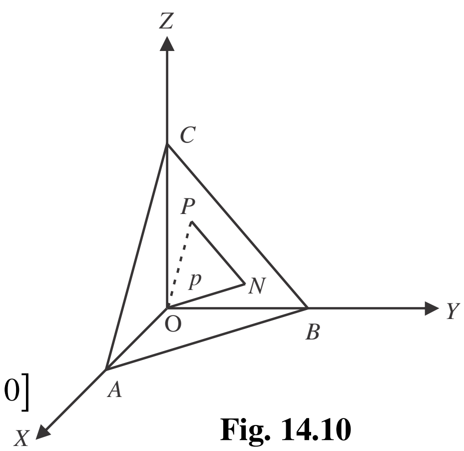


Fig. 14.10

Cartesian Form : Let ABC is any plane and ON is perpendicular from the origin, where N is the foot of perpendicular. If the length of perpendicular from the origin to ON is p and direction cosines are l, m, n, then equation of plane will be in terms of l, m, n and p.

Clearly the coordinates of point N are (lp, mp, np). Let $P(x, y, z)$ be any point on the line lying in a plane. Then, the d.c's of PN are $\frac{x-lp}{PN}, \frac{y-mp}{PN}, \frac{z-np}{PN}$. Now since ON is perpendicualr on the plane, thus it is perpendicular to every line lying in the plane. Therefore, ON and PN are mutually perpendicular.

$$l\left(\frac{x-lp}{PN}\right)+m\left(\frac{y-mp}{PN}\right)+\left(\frac{z-np}{PN}\right)=0$$

$\Rightarrow$ $$lx+my+nz=p\left(l^2+m^2+n^2\right)$$

$\Rightarrow$ $$lx+my+nz=p \qquad \left[\because\ l^2+m^2+n^2=1\right]$$

This is the required eqation of a plane in Normal form.

Note: Let $\vec{n}$ be any vector in the direction of n, then $\vec{n}=n\hat{n}$.

From (4), $\vec{r}\cdot\left(\vec{n}/n\right)=p \Rightarrow \quad \vec{r}\cdot\vec{n}=np$

or, $\vec{r}\cdot\vec{n}=q$, where (5)

$q=np$ (6)

$\therefore$ This is the vector equation of the plane.

2. When origin lies on the plane, then $p=0$, therefore, equation of plane passing through the origin and perpendicular to the vector $\vec{n}$ is $\vec{r}\cdot\vec{n}=0$.

3. In the normal form of a plane, the direction of vector $\vec{n}$ is from origine to the plane and p is positive.

4. If the intercepts made by the plane $\vec{r}\cdot\vec{n}=q$ are $x_1,\ y_1,\ z_1$, respectively such that OA=x_1, OB=y_1 and OC=z_1 then the position vector of these points are $x_1 i,\ y_1 j$ and $z_1 k$. Since the point lies on the plane, therefore.

$$x_1 i\cdot\vec{n}=q \qquad x_1 j\cdot\vec{n}=q, \qquad z_1 k\cdot\vec{n}=q$$

$$\Rightarrow \qquad x_1=\frac{q}{i\cdot\vec{n}}, \qquad y_1=\frac{q}{j\cdot\vec{n}}, \qquad z_1=\frac{q}{k\cdot\vec{n}}.$$

5. Vector equation of a plane is an equation which have the position vector of any arbitrary point lying in the plane.

Illustrative Examples

Example 25. Find the equation of plane whcih is at a distance of 4 units from the origin and perpendicular to the vector $i-2j+2k$.

Solution : Vector form : Here $p=4$ and $\vec{n}=i-2j+2k$

$$\therefore \qquad \hat{n}=\frac{\vec{n}}{|\vec{n}|}=\frac{i-2j+2k}{\sqrt{(1+4+4)}}=\frac{1}{3}i-\frac{2}{3}j+\frac{2}{3}k$$

thus, the required equation of plane is $\vec{r}\cdot\left(\frac{1}{3}i-\frac{2}{3}+\frac{2}{3}k\right)=4$

or, $$\vec{r}\cdot\left(i-2j+2k\right)=12$$

This is the required equation of plane.

Cartesian form : Substituting $\vec{r}=xi+yj+zk$ in the above equation, we have

equation $$\left(xi+yj+zk\right)\cdot\left(i-2j+2k\right)=12$$

i.e. $$x-2y+2z=12,$$

Example 26. Reduce the equation of plane $\vec{r}\cdot(i-2j+2k)=12$ into the Normal form and find the perpendicular distance from the origin.

Solution : Vector form : Given equation of plane is $\vec{r}\cdot(i-2j+2k)=12$

i.e. $\vec{r}\cdot\vec{n}=12,$

where, $\vec{n}=i-2j+2k.$ $\therefore$ $|\vec{n}|=\sqrt{(1+4+4)}=3\neq 1$

Thus, the given equation is not in the nromal form.

$\therefore$ Dividing both the sides by $|\vec{n}|=3$

$$(\vec{r}\cdot\vec{n})/3=12/3 \qquad \Rightarrow \qquad \vec{r}\cdot\left(\frac{1}{3}i-\frac{2}{3}j+\frac{2}{3}k\right)=4$$

This equation represents the equation of plane in normal form and the distance from the origin is 4 units.

Cartesian form: Cartesian form of the equation is

$$x-2y+2z=12$$

Here R.H.S. is positive, now dividing the equation by $\sqrt{(1+4+4)}=3\neq 1$ we have

$$\frac{1}{3}x-\frac{2}{3}y+\frac{2}{3}z=4,$$

The given equation represents the normal form, with d.r's $\frac{1}{3},-\frac{2}{3},\frac{2}{3}$

Example 27. Find the equation of plane whose distance from the origin is 2 units and the d.r's of its normal be 12, –3, 4.

Solution : Given $p=2$ and the d.r's of its normal be $12, -3, 4$

Thus, the direction cosines of the normal be $12/13, -3/13, 4/13$ $\{\because \sqrt{(12)^2+(-3)^2+(4)^2}=13\}$

$\therefore$ Thus, the equation of plane is,

$$\frac{12}{13}x-\frac{3}{13}y+\frac{4}{13}z=2, \qquad [\text{From } \ell x+my+nz=p]$$

or, $12x-3y+4z=26,$

which is the required equation of the plane.

Vector form : Let $\vec{n}$ be the vector perpendicular to the plane and the d.r's of $\vec{n}$ are 12, –3, 4

$$\therefore \quad \vec{n}=12i-3j+4k \qquad \Rightarrow \qquad |\vec{n}|=\sqrt{\{(12)^2+(-3)^2+(4)^2\}}=13\neq 1$$

$$\therefore \quad \hat{n}=\frac{\vec{n}}{|\vec{n}|}=\frac{12}{13}i-\frac{3}{13}j+\frac{4}{13}k$$

$\because$ The required plane is at a distance of 2 units from the origin. Therefore, the equation will be

$$\vec{r}\cdot\hat{n}=2$$

or, $$\vec{r}\cdot\left(\frac{12}{13}i-\frac{3}{13}j+\frac{4}{13}k\right)=2$$

This is the required equation of the plane in vector form.

Example 28. Find the direction cosines of the perpendicular dropped from the origin to the plane $\vec{r}\cdot(6i+2j-3k)+7=0$.

Solution : Cartesian form : The given equation of the plane can be written as

$$(xi+yj+zk)\cdot(6i+2j-3k)+7=0$$

or, $$6x+2y-3z+7=0$$

or, $$-6x-2y+3z=7 \qquad (1)$$

On dividing by 7, we have

$$-\frac{6}{7}x-\frac{2}{7}y+\frac{3}{7}z=1, \qquad (2)$$

Comparing the equation (2) with $\ell x+my+nz=p$, we get the required direction cosines as $-6/7,\ -2/7,\ 3/7$

Vector form : To find the direction cosines of the perpendicular we need to convert the given plane into normal form

Equation of plane $$\vec{r}\cdot(6i+2j-3k)+7=0,$$

i.e. $$\vec{r}\cdot(6i+2j-3k)=-7$$

$\Rightarrow$ $$\vec{r}\cdot(-6i-2j+3k)=7$$

$\Rightarrow$ $$\vec{r}\cdot\vec{n}=7, \qquad \text{where } \vec{n}=(-6i-2j+3k)$$

now $$|\vec{n}|=\sqrt{\{(-6)^2+(-2)^2+(3)^2\}}=7\neq 1.$$

On dividing by $|\vec{n}|=7$ in (1), we have

$$\frac{\vec{r}\cdot\vec{n}}{7}=\frac{7}{7}$$

or $$\vec{r}\cdot\left(-\frac{6}{7}i+\frac{2}{7}j+\frac{3}{7}k\right)=1$$

Therefore the d.c's of the perpendicular dropped from origin to the plaen are $-\frac{6}{7},-\frac{2}{7},\frac{3}{7}$.

Exercise 14.6

1. Find the equation of plane passing through the point (2, –1, 3) and perpendicular to the X-axis.
2. Find the equation of plane passing through the point (3, 2, 4) and X-axis.
3. A variable plane passes through the point (p, q, r) and meets the coordinate axes in point A, B and C respectively. Show that the locus of the common points of the planes parallel to the coordinate axes and passing through A, B and C is

$$\frac{p}{x}+\frac{q}{y}+\frac{r}{z}=1$$

4. Find the vector equation of the plane which is at a distance of 7 units from the origin and $\hat{i}$ is the unit normal vector to it.
5. Find the vector equation of the plane whcih is at a distance of 7 units from the origin and normal to the vector $6i+3j-2k$.
6. Write the equation of plane $\vec{r}\cdot(3i-4j+12k)=5$ in normal form and find the perpendicular distance from the origin, Also find the d.c's of the normal so obtained.

or

Write the equation of plane $3x-4y+12z=5$ in normal form and find the perpendicular distance from the origin. Also find the d.c's of the normal so obtained.

7. Find the vector equation of the plane whcih is at a distance fo 4 units from the origin and the direction ratios of the normal are 2, –1, 2.
8. Find the normal form of the equation of the plane $2x-3y+6z+14=0$.
9. Find the equation of plane, if the length of perpendicular drawn from origin is 13 units and the direction ratios of the perpendicular are 4, –3, 12.
10. Find the unit normal vector of the plane $x+y+z-3=0$.

14.17 Angle Between Two Planes

The angle between two planes is defined as the angle between their normals

Vector form: Let the equation of the plane be

$$\vec{r}\cdot\vec{n}_1=d_1 \qquad \text{and} \qquad \vec{r}\cdot\vec{n}_2=d_2$$

where $\vec{n}_1$ and $\vec{n}_2$ are the perpendicular vectors. Observe that if θ is an angle between the two planes, then angle between their normals is also θ

$$\cos\theta=\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|} \qquad \text{or} \qquad \theta=\cos^{-1}\left(\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right)$$

Note: (i) Two planes are perpendicular if $\vec{n}_1\cdot\vec{n}_2=0$.

(ii) Two planes are parallel if $\vec{n}_1=\lambda\vec{n}_2$, where λ is a constant.

Cartesian form: Let the angle between the two planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ be θ . Let $\vec{n}_1$ and $\vec{n}_2$ are normal vectors to the plane.

$\therefore \qquad \vec{n}_1 = a_1 i + b_1 j + c_1 k$

and $\qquad \vec{n}_2 = a_2 i + b_2 j + c_2 k$

$$\therefore \qquad \cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$

Note: (i) Two planes are mutually perpendicular, if $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$.

(ii) Two planes are parallel, if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.

14.18 Angle Between a Plane and a Line

The angle between a plane and a line is the complement of the angle between the line and normal to the plane

Vector form: Let the equation of the line is $\vec{r} = \vec{a} + \lambda\vec{b}$ and the equation of the plane is $\vec{r} \cdot \vec{n} = d$, where $\vec{n}$ is normal vector of plane. If θ is angle between plane and lien, then angle between line and normal to the plane will be $\left(\dfrac{\pi}{2} - \theta\right)$.

$$\cos\left(\frac{\pi}{2} - \theta\right) = \frac{\vec{b} \cdot \vec{n}}{|\vec{b}||\vec{n}|} \qquad \text{or} \qquad \sin\theta = \frac{\vec{b} \cdot \vec{n}}{|\vec{b}||\vec{n}|}$$

Note: (i) line is perpendicualr to the plane, if $\vec{b} \times \vec{n} = \vec{O}$ or $\vec{b} = \lambda\vec{n}$.

(ii) line is parallel to the plane, if $\vec{b} \cdot \vec{n} = 0$.

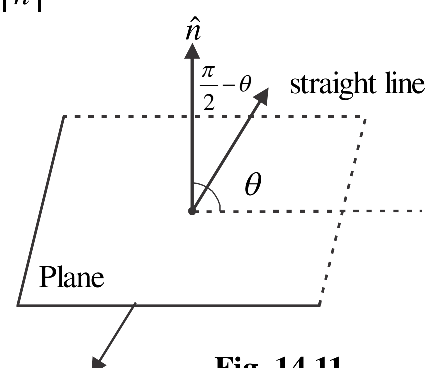


Fig. 14.11

Cartesian form: Let the equation of the plane be

$$ax + by + cz + d = 0 \qquad (1)$$

and equation of line be

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \qquad (2)$$

The d.c's of (1) are *a*, *b*, *c* and the d.r's of the line (2) are *l*, *m*, *n*. If the angle between the line and the plane is θ , then the angle between the normal and the line will be $\left(\dfrac{\pi}{2} - \theta\right)$.

$$\therefore \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \frac{al + bm + cn}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{(l^2 + m^2 + n^2)}}$$

or

$$\sin\theta = \frac{al + bm + cn}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{(l^2 + m^2 + n^2)}}$$

Note: (i) line is perpendicualr to the plane, if $\frac{a}{\ell}=\frac{b}{m}=\frac{c}{n}$.

(ii) line is parallel to the plane, if $a\ell+bm+cn=0$.

Illustrative Examples

Example 29. Find the angle between the planes $\vec{r}\cdot(2i-3j+4k)=1$ and $\vec{r}\cdot(-i+j)=4$.

Solution : We know that the angle between the planes $\vec{r}\cdot\vec{n}_1=d_1$ and $\vec{r}\cdot\vec{n}_2=d_2$ is

$$\cos\theta=\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

Here, $\vec{n}_1=2i-3j+4k$ and $\vec{n}_2=-i+j+0k$

$$\therefore \qquad \cos\theta=\frac{-2-3+0}{\sqrt{4+9+16}\sqrt{1+1}}=\frac{-5}{\sqrt{29}\sqrt{2}}$$

$$\Rightarrow \qquad \theta=\cos^{-1}\left(\frac{-5}{\sqrt{58}}\right)$$

Example 30. Prove that the planes $2x+6y+6z=7$ and $3x+4y-5z=8$ are mutually perpendicular.

Solution : We know that the planes

$$2x+6y+6z=7$$

and $$3x+4y-5z=8$$

are mutually perpendicular, if their normals are mutually perpendicular

i.e., $2(3)+6(4)+6(-5)=0$

or, $6+24-30=0$, which is true.

Hence, the planes are mutually perpendicular.

Example 31. If the planes $\vec{r}\cdot(i+2j+3k)=7$ and $\vec{r}\cdot(\lambda i+2j-7k)=26$ are mutually perpendicular then find the vlaue of λ.

Solution : The planes $\vec{r}\cdot\vec{n}_1=d_1$ and $\vec{r}\cdot\vec{n}_2=d_2$ are mutually perpendicualr if

$$\vec{n}_1\cdot\vec{n}_2=0$$

Here, $\vec{n}_1=(i+2j+3k)$ and $\vec{n}_2=(\lambda i+2j-7k)$, therefore

$$(i+2j+3k)\cdot(\lambda i+2j-7k)=0$$

$$\Rightarrow \qquad \lambda+4-21=0 \quad \text{or} \quad \lambda=17$$

Example 32. Find the angle between the line $\vec{r}=(2i+2j+9k)+\lambda(2i+3j+4k)$ and plane $\vec{r}\cdot(i+j+k)=5$.

Solution : If the angle between the line $\vec{r}=\vec{a}+\lambda\vec{b}$ and plane $\vec{r}\cdot\vec{n}=d$ is θ, then

$$\sin\theta=\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}$$

On comparing with the standard equation, we have

$$\vec{b} = 2i + 3j + 4k \qquad \text{and} \qquad \vec{n} = i + j + k$$

$$\therefore \qquad \sin\theta = \frac{(2i+3j+4k)\cdot(i+j+k)}{\sqrt{4+9+16}\sqrt{1+1+1}} = \frac{9}{\sqrt{87}}$$

$$\Rightarrow \qquad \theta = \sin^{-1}\left(\frac{9}{\sqrt{87}}\right) \qquad \text{or} \qquad \theta = \sin^{-1}\left(\frac{3\sqrt{3}}{\sqrt{29}}\right)$$

Example 33. Find the angle between the line $\frac{x-1}{1} = \frac{y-2}{-1} = \frac{z+1}{1}$ and the plane $2x + y - z = 4$.

Solution : The perpendicular vector to the plane $\qquad 2x + y - z = 4 \qquad$ (1)

is $\vec{n} = 2i + j - k$ and the parallel vector to the line $\frac{x-1}{1} = \frac{y-2}{-1} = \frac{z+1}{1}$ is $\vec{b} = i - j + k$

If the angle between the line adn the plane is θ then

$$\sin\theta = \frac{(i-j+k)\cdot(2i+j-k)}{\sqrt{1+1+1}\sqrt{4+1+1}} = \frac{2-1-1}{\sqrt{3}\sqrt{6}} = 0 \qquad \Rightarrow \qquad \theta = 0$$

Example 34. If the line $\vec{r} = (i - 2j + k) + \lambda(2i + j + 2k)$, is parallel to the plane $\vec{r}\cdot(3i - 2j + mk) = 4$, then find the vlaue of *m*.

Solution : Given line is parallel to the vector $\vec{b} = 2i + j + 2k$ and normal vector to the plane is $\vec{n} = 3i - 2j + mk$. Since the given line is parallel to the plane,

$$\therefore \qquad \vec{b} \perp \vec{n}.$$

$$\Rightarrow \qquad \vec{b}\cdot\vec{n} = 0$$

$$\Rightarrow \qquad (2i + j + 2k)\cdot(3i - 2j + mk) = 0$$

$$\Rightarrow \qquad 6 - 2 + 2m = 0 \qquad \Rightarrow \qquad m = -2$$

14.19 Distance of a Point From a Plane

Consider a point P with position vector $\vec{a}$ and a plane whose equation is $\vec{r}\cdot\vec{n} = q$ We have to find the length of perpendicular from a point to the given plane.

Let π be the given plane and the position vector of point *P* is $\vec{a}$. Let the length of perpendicualr drawn from point *P* on plane π be *PM*.

$\because$ line *PM*, passes through $P(\vec{a})$ and the unit normal vector $\vec{n}$ is parallel to the plane π

$\therefore$ the vector equation of the line *PM* is $\vec{r} = \vec{a} + \lambda\vec{n}$, where λ is a sacalr. (1)

Again point *M*, is the intersecting point of line *PM* and plane π, therefore point M will satisfy the equation of plane

$$\therefore \qquad (\vec{a} + \lambda\vec{n})\cdot\vec{n} = q$$

$$\Rightarrow \qquad \vec{a}\cdot\vec{n} + \lambda\vec{n}\cdot\vec{n} = q$$

$$\Rightarrow \qquad \vec{a}\cdot\vec{n}+\lambda|\vec{n}|^2=q$$

$$\Rightarrow \qquad \lambda=\frac{q-\vec{a}\cdot\vec{n}}{|\vec{n}|^2}$$

substituting the value of λ in (1), the position vector of M will be

$$\vec{r}=\vec{a}+\frac{q-\vec{a}\cdot\vec{n}}{|\vec{n}|^2}\vec{n}$$

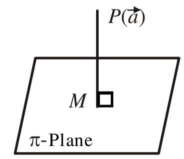


Fig. 14.12

$$\therefore \qquad \overrightarrow{PM} = \text{(position vector M)} - \text{(position vector of P)}$$

$$=\vec{a}+\frac{q-\vec{a}\cdot\vec{n}}{|\vec{n}|^2}\vec{n}-\vec{a}=\frac{(q-\vec{a}\cdot\vec{n})\vec{n}}{|\vec{n}|^2}$$

$$\therefore \qquad PM=|\overrightarrow{PM}|=\frac{|(q-\vec{a}\cdot\vec{n})\vec{n}|}{|\vec{n}|^2}=\frac{|(q-\vec{a}\cdot\vec{n})||\vec{n}|}{|\vec{n}|^2}=\frac{|(q-\vec{a}\cdot\vec{n})|}{|\vec{n}|}$$

Thus, the required length is $\frac{|q-\vec{a}\cdot\vec{n}|}{|\vec{n}|}$ or $\frac{|\vec{a}\cdot\vec{n}-q|}{|\vec{n}|}$

Note: (i) $$\overrightarrow{PM}=(PM)\hat{n}=\frac{|\vec{a}\cdot\vec{n}-q|}{|\vec{n}|}\hat{n}$$

$$=\frac{|\vec{a}\cdot\vec{n}-q|}{|\vec{n}|}\times\frac{\vec{n}}{|\vec{n}|}=\frac{|\vec{a}\cdot\vec{n}-q|\vec{n}}{|\vec{n}|^2}$$

(ii) length of perpendicular drawn from origin to the plane $\vec{r}\cdot\vec{n}=q$ is

$$=\frac{q}{|\vec{n}|} \qquad \text{[here } \vec{a}=\vec{0}\text{]}$$

Cartesian form: To find the length of perpendicualr drawn from point $P(x_1, y_1, z_1)$ to the plane $ax+by+cz+d=0$.

Let the foot of perpendicualr drawn from point $P(x_1, y_1, z_1)$ to the plane $ax+by+cz+d=0$ is M.

Therefore, the equation of the line *PM* is

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c} \qquad (1)$$

($\because$ The direction ratios *a*, *b*, c of normal to the plane will also be the direction ratios of the line *PM)*

Now, the coordinates of any point on the line are $(x_1+ar,\ y_1+br,\ z_1+cr)$, where *r* is real number. If these are the coordinates of point *M,* then they will satisfy the equation of plane

$$\therefore \qquad a(x_1+ar)+b(y_1+br)+c(z_1+cr)+d=0$$

or, $$r=-\frac{ax_1+by_1+cz_1+d}{a^2+b^2+c^2} \qquad (2)$$

Now, $PM = \sqrt{\left\{(x_1 + ar - x_1)^2 + (y_1 + br - y_1)^2 + (z_1 + cr - z_1)^2\right\}}$

$= |r| \sqrt{(a^2 + b^2 + c^2)}$

Now, $PM = \left| -\dfrac{(ax_1 + by_1 + cz_1 + d)}{(a^2 + b^2 + c^2)} \right| \sqrt{\left(a^2 + b^2 + c^2\right)}$ [using (2)]

Therefore, the required length is $\left| \dfrac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$

Illustrative Examples

Example 35. Find the perpendicualr distance of the point with position vector $2i - j - 4k$ from the plane $\vec{r} \cdot (3i - 4j + 12k) - 9 = 0$.

Solution : We know that the perpendicular distance of the point, whose position vector is $\vec{a}$, from the plane $\vec{r} \cdot \vec{n} = q$ is $\left| \dfrac{\vec{a} \cdot \vec{n} - q}{|\vec{n}|} \right|$.

Here $\vec{a} = 2i - j - 4k$, $\vec{n} = 3i - 4j + 12k$ and $q = 9$.

$\therefore$ Required distance $= \dfrac{|(2i - j - 4k) \cdot (3i - 4j + 12k) - 9|}{\sqrt{(9 + 16 + 144)}} = \dfrac{47}{13}$

Example 36. Show that the points $A(1, -1, 3)$ and $B(3, 3, 3)$ are at equal distance from the plane $\vec{r} \cdot (5i + 2j - 7k) + 9 = 0$.

Solution : Position vector of point A is $i - j + 3k$.

$\therefore$ perpendicular distance of point A from the plane is

$$= \frac{|(i - j + 3k) \cdot (5i + 2j - 7k) + 9|}{\sqrt{(25 + 4 + 49)}} = \frac{9}{\sqrt{78}} \quad (1)$$

Position vector of point B is $3i + 3j + 3k$.

$\therefore$ Perpendicular distance of point B from the plane is

$$= \frac{|(3i + 3j + 3k) \cdot (5i + 2j - 7k) + 9|}{\sqrt{(25 + 4 + 49)}} = \frac{9}{\sqrt{78}} \quad (2)$$

Therefore, from (1) and (2), we conclude that the point is at equal distance from the given plane.

Exercise 14.7

1. Find the angle between the planes:

   (i) $\vec{r} \cdot (2i - j + 2k) = 6$ and $\vec{r} \cdot (3i + 6j - 2k) = 9$

   (ii) $\vec{r} \cdot (2i + 3j - 6k) = 5$ and $\vec{r} \cdot (i - 2j + 2k) = 9$

   (iii) $\vec{r} \cdot (i + j + 2k) = 5$ and $\vec{r} \cdot (2i - j + 2k) = 6$

2. Find the angle between the planes:

(i) $x+y+2z=9$ and $2x-y+z=15$

(ii) $2x-y+z=4$ and $x+y+2z=3$

(iii) $x+y-2z=3$ and $2x-2y+z=5$

3. Prove that the following planes are mutually perpendicular:

(i) $x-2y+4z=10$ and $18x+17y+4z=49$

(ii) $\vec{r}\cdot(2i-j+k)=4$ and $\vec{r}\cdot(-i-j+k)=3$

4. If the following planes are mutually perpendicualr, then find the vlaue of λ:

(i) $\vec{r}\cdot(2i-j+\lambda k)=5$ and $\vec{r}\cdot(3i+2j+2k)=4$

(ii) $2x-4y+3z=5$ and $x+2y+\lambda z=5$

5. Find the angle between the line $\frac{x+1}{3}=\frac{y-1}{2}=\frac{z-2}{4}$ and plane $2x+y-3z+4=0$.

6. Find the angle between the line $\frac{x-2}{3}=\frac{y+1}{-1}=\frac{z-3}{2}$ and plane $3x+4y+z+5=0$.

7. Find the angle between the line $\vec{r}=(\hat{i}+2\hat{j}-\hat{k})+\lambda(\hat{i}-\hat{j}+\hat{k})$ and plane $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})=4$.

8. Find the angle between the line $\vec{r}=(2i+3j+k)+\lambda(i+2j-k)$ and plane $\vec{r}\cdot(2i-j+k)=4$.

9. If the line $\vec{r}=(\hat{i}-2\hat{j}+\hat{k})+\lambda(2\hat{i}+\hat{j}+2\hat{k})$ is parallel to the plane $\vec{r}\cdot(3\hat{i}-2\hat{j}+m\hat{k})=3$, then find the vlaue of *m*.

10. If the line $\vec{r}=i+\lambda(2i-mj-3k)$ is parallel to the plane $\vec{r}\cdot(mi+3j+k)=4$, then find the vlaue of *m*.

Miscellaneous Exercise 14

1. Whcih of the following group is not the direction cosines of a line:

(A) 1, 1, 1 (B) 0, 0, –1 (C) –1, 0, 0 (D) 0, –1, 0

2. Point *P* is such that *OP* = 6 and vector $\overrightarrow{OP}$ makes an angle 45° and 60° with *OX*-axis and *OY*-axis respectively, then the position vector of *P* will be

(A) $3i+3j\pm3\sqrt{2}k$ (B) $6i+6\sqrt{2}j\pm6k$ (C) $3\sqrt{2}i+3j\pm3k$ (D) $3i+3\sqrt{2}j\pm3k$

3. The angle between the two diagonals of the cube will be

(A) 30° (B) 45° (C) $\cos^{-1}(1/\sqrt{3})$ (D) $\cos^{-1}(1/3)$

4. The direction cosines of vector 3*i* are:

(A) 3, 0, 0 (B) 1, 0, 0 (C) –1, 0, 0 (D) –3, 0, 0

5. The vector form of the line $\frac{x-3}{-2}=\frac{y-4}{-5}=\frac{x+7}{13}$ is

(A) $(3i+4j-7k)+\lambda(-2i-5j+13k)$ (B) $(-2i-5j+13k)+\lambda(3i+4j-7k)$

(C) $(-3i-4j+7k)+\lambda(-2i-5j+13k)$ (D) none of these

6. If lines $\frac{x+1}{1}=\frac{y+2}{\lambda}=\frac{z-1}{-1}$ and $\frac{x-1}{-\lambda}=\frac{y+1}{-2}=\frac{z+1}{1}$ are mutually perpendicular, then the value of λ is

(A) 0 (B) 1 (C) –1 (d) 2

7. The shortest distance between the lines $\vec{r}=(5i+7j+3k)+\lambda(5i-16j+7k)$ and $\vec{r}=(9i+13j+15k)+\mu(3i+8j-5k)$ is

(A) 10 units (B) 12 units (C) 14 units (D) 7 units

8. The angle between the line $\vec{r}=(2i-j+k)+\lambda(-i+j+k)$ and the plane $\vec{r}\cdot(3i+2j-k)=4$ is

(A) $\sin^{-1}(-2/\sqrt{42})$ (B) $\sin^{-1}(2/\sqrt{42})$ (C) $\cos^{-1}(-2/\sqrt{42})$ (D) $\cos^{-1}(2/\sqrt{42})$

9. If the equation $lx+my+nz=p$ is the normal form of the plane then which of the following is true or false

(A) *l*, *m*, *n* are the d.c's of the normal to the plane

(B) *p* is the perpendicualr distance from the origin to the plane

(C) for every vlaue of *p,* the plane passes through the origin

(D) $\ell^2+m^2+n^2=1$

10. A plane meets the coordinate axes at the points A, B and C respectively such that the centroid of the triangle *ABC* is (1, 2, 3), then the equation of the plane is

(A) $\frac{x}{1}+\frac{y}{2}+\frac{z}{3}=1$ (B) $\frac{x}{1}+\frac{y}{2}+\frac{z}{3}=\frac{1}{6}$ (C) $\frac{x-1}{1}+\frac{y-2}{2}+\frac{z-3}{3}=1$ (D) $\frac{x}{3}+\frac{y}{6}+\frac{z}{9}=1$

11. If two points are $P(2i+j+3k)$ and $Q(-4i-2j+k)$, then the equation of the plane passing through point Q and perpendicualr to PQ is

(A) $\vec{r}\cdot(6i+3j+2k)=28$ (B) $\vec{r}\cdot(6i+3j+2k)=32$

(C) $\vec{r}\cdot(6i+3j+2k)+28=0$ (D) $\vec{r}\cdot(6i+3j+2k)+32=0$

12. The direction cosines of two lines are expressed with the following given relations, find them

$$l-5m+3n=0 \quad \text{and} \quad 7l^2+5m^2-3n^2=0$$

13. The projection of the line segment on the axes are –3, 4, –12 respectively. Find the length and direction cosines of the line segment.

14. Prove that the line joining the points (a, b, c) and (a', b', c') passes through the origin, If $aa'+bb'+cc'=pp'$, p and p' are the distances from the origin.

15. Find the equation of plane passing through $P(-2, 1, 2)$ and parallel to the vectors $\vec{a}=-i+2j-3k$ and $\vec{b}=5i-j+k$.

IMPORTANT POINTS

1. Any line OP (Vector $\overrightarrow{OP}$) makes angle α, β, γ with positive direction of co-ordinate axes, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are direction cosines of line OP (Vector $\overrightarrow{OP}$), whcih are generally denoted by l, m, n . Hence, $0 \le \alpha, \beta, \gamma \le \pi$.

   (i) Vector $\overrightarrow{PO}$ makes angle $\pi-\alpha,\ \pi-\beta,\ \pi-\gamma$ with axes OX, OY, OZ respectively, then direction cosines of $\overrightarrow{PO}$ are $\cos(\pi-\alpha), \cos(\pi-\beta),\ \cos(\pi-\gamma)$ i.e. $-l, -m, -n$.

   Therefore, if l, m, n are direction cosines of any line, then $-l, -m, -n$ are also direction cosines of the same line.

   (ii) Direction cosines of X, Y and Z axes are respectively 1, 0, 0; 0, 1, 0 and 0, 0, 1.

2. **Projection of any vector on co-ordinate axes :** If $\vec{r}$ is given position vector and l, m, n are its direction cosines, then its projection on X, Y, Z axes are lr, mr, nr respectively.

3. **Co-ordinates of a point in the form of direction cosines:** If $P(x, y, z)$ is a point, then its co-ordinates will be (lr, mr, nr), where l, m, n are direction cosines of $\overrightarrow{OP}$ and OP = r.

4. **To represent a unit vector $\hat{r}$ in the form of direction cosines:**

   $\hat{r}$ (unit vector in direction of $\vec{r}$) $= l\hat{i} + m\hat{j} + n\hat{k}$,

   where l, m, n are direction cosines of $\vec{r}$.

5. $l^2 + m^2 + n^2 = 1$, where l, m, n are direction cosines.

6. **Direction ratios of a line :** A set of three numbers for $\vec{r}$, which are proportional to the direction cosines l, m, n are called direction ratios.

7. **Conversion of direction ratios into direction cosines :** Let $\vec{r} = ai + bj + ck$ is a vector having direction ratios a, b, c, then its direction cosines l, m, n are given as follows:

$$l = \frac{a}{\sqrt{(a^2+b^2+c^2)}},\quad m = \frac{b}{\sqrt{(a^2+b^2+c^2)}},\quad n = \frac{c}{\sqrt{(a^2+b^2+c^2)}}$$

8. **Direction ratio and direction cosines of a line joining two points:** Let two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, then $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$ are direction ratio of line PQ and direction cosines are $\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$,

   where $$PQ = \sqrt{\{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2\}}$$

9. Equation of a line whcih passes through point $P(x_1, y_1, z_1)$ and parallel to line having direction cosines l, m, n is $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$.

10. Co-ordinates of a point lying on line, which is at a distance *r* from a point $P(x_1, y_1, z_1)$ on same line are $(lr + x_1, mr + y_1, nr + z_1)$, where *r* is a parameter.

11. If direction ratios *a*, *b*, *c* are given then the equation of line is

$$\frac{x-x_1}{a/\sqrt{a^2+b^2+c^2}} = \frac{y-y_1}{b/\sqrt{a^2+b^2+c^2}} = \frac{z-z_1}{c/\sqrt{a^2+b^2+c^2}} = k(\text{let})$$

or, $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = R$, where $R = \frac{k}{\sqrt{a^2+b^2+c^2}}$

12. A point has co-ordinates $(ar + x_1, br + y_1, cr + z_1)$, then at this position, it is not at a distance *r* from point $P(x_1, y_1, z_1)$.

13. Equation of line passes through a point having position vector $\vec{a}$ and parallel to a vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$, where λ is a real number.

14. If above line passes through origin, then $\vec{r} = \lambda\vec{b}$.

15. **Non coplanar lines (skew lines) :** Non parallel and non-intersecting lines which doesn't lie on same plane are called 'Non coplanar or Skew lines'.

16. **Shortest distance :** Distance between two skew lines, which is perpendicular to both, is called "Shortest Distance".

17. **Shortest distance :** Shortest distance between two skew lines

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \text{ and } \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \text{ is}$$

$$= \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \div \sqrt{\left\{\sum\left(m_1n_2-m_2n_1\right)^2\right\}}$$

18. If shortest distance becomes zero, then lines are coplanar with the following condition.

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

19. **Shortet Distance :** Shortest distance between two skew lines

$$\vec{r} = \vec{a}_1 + \lambda\vec{b}_1 \quad \text{and} \quad \vec{r} = \vec{a}_2 + \lambda\vec{b}_2$$

$$d = \left|\frac{\left(\vec{b}_1 \times \vec{b}_2\right)\cdot\left(\vec{a}_2 - \vec{a}_1\right)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

20. If θ is an angle between two planes $\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$, Then

$$\cos\theta = \frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|} \quad \text{or} \quad \theta = \cos^{-1}\left(\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right)$$

(i) If planes are mutually perpendicular, then $\vec{n}_1\cdot\vec{n}_2 = 0$.

(ii) If planes are parallel, then $\vec{n}_1 = \lambda\vec{n}_2$, where λ is constant

21. If θ is an angle between two planes

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

(i) Planes are mutually perpendicular, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

(ii) Planes are parallel, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

22. If θ is an angle between two lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$, then

$$\cos\theta = \frac{\vec{b}_1\cdot\vec{b}_2}{|\vec{b}_1||\vec{b}_2|} \quad \text{or} \quad \theta = \cos^{-1}\left(\frac{\vec{b}_1\cdot\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right)$$

(i) Lines are perpendicular, if $\vec{b}_1\cdot\vec{b}_2 = 0$.

(ii) Lines are parallel, if $\vec{b}_1 = \lambda\vec{b}_2$, where λ is cosntant.

23. If θ is an angle between two lines

$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$, then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

(i) Lines are perpendicualr, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

(ii) Lines are parallel, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

24. Angle between a line and a plane is complement of angle between normal of plane and given line. Let equation of plane is $\vec{r}\cdot\vec{n} = d$ and equation of line is $\vec{r} = \vec{a} + \lambda\vec{b}$ and θ is angle between them, then

$$\sin\theta = \frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}$$

(i) Line is perpendicular to plane, if $\vec{b} \times \vec{n} = \vec{0}$, or $\vec{b} = \lambda \vec{n}$.

(ii) Line is parallel to plane, if $\vec{b} \cdot \vec{n} = 0$.

25. **General equation of plane :**

$$ax + by + cz + d = 0,$$

where a, b, c, d are scalar quantity or constant and all a, b, c are not zero.

(a) Every first degree equation in x, y, z represetns a plane.

(b) There is only three independent constant in plane.

26. Equation of plane passing through a point (x_1, y_1, z_1)

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0,$$

where a, b, c are constant.

27. Equation of plane in intercpet form:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

where a, b, c are respectively intercepts on X, Y, Z axes respectively.

28. **Equation of plane in normal form:**

$$\vec{r} \cdot \hat{n} = p,$$

Here p is perpendicualr distance from origin to the plane and $\hat{n}$ is unti vector of normal of plane.

Note : Equation of plane in normal form may also be written as

$$\vec{r} \cdot \vec{n} = q$$

Here $q = |\vec{n}|\, p$.

29. Distance of a point from plane :

$$d = \frac{|\vec{a} \cdot \vec{n} - q|}{|\vec{n}|},$$

where $\vec{a}$ is position vector of point and $\vec{r} \cdot \vec{n} = q$ is equation of plane.

Answers

Exercise 14.1

1. $\pm\frac{1}{\sqrt{3}},\ \pm\frac{1}{\sqrt{3}},\ \pm\frac{1}{\sqrt{3}}$ 2. $0,\ \frac{1}{\sqrt{5}},\ \frac{4}{5}$ 3. $\frac{2}{3},\ -\frac{1}{3},\ -\frac{2}{3}$ 4. $\sqrt{2}\hat{i}+\hat{j}-\hat{k}$

Exercise 14.2

1. (i) $\frac{x-5}{0}=\frac{y-7}{0}=\frac{z-9}{0}$; (ii) $\frac{x-5}{0}=\frac{y-7}{1}=\frac{z-9}{0}$; (iii) $\frac{x-5}{0}=\frac{y-7}{0}=\frac{z-9}{1}$

2. $\vec{r}=\left(2\hat{i}-3\hat{j}+4\hat{k}\right)+\lambda\left(3\hat{i}+4\hat{j}-5\hat{k}\right);\ \frac{x-2}{3}=\frac{y+3}{4}=\frac{z-4}{-5}$ 3. $\vec{r}=5\hat{i}-2\hat{j}+4\hat{k}+\lambda\left(2\hat{i}-\hat{j}+3\hat{k}\right)$

4. $\vec{r}=\left(2\hat{i}-\hat{j}+\hat{k}\right)+\lambda\left(2\hat{i}+7\hat{j}-3\hat{k}\right)$ 5. $\vec{r}=\left(5\hat{i}-4\hat{j}+6\hat{k}\right)+\lambda\left(3\hat{i}+7\hat{j}+2\hat{k}\right)$

6. $\frac{x-1}{-2}=\frac{y-2}{14}=\frac{z-3}{3}$

7. (i) Equation of AB : $\vec{r}=\left(4\hat{i}+5\hat{j}+10\hat{k}\right)+\mu\left(\hat{i}+\hat{j}+3\hat{k}\right);\ \frac{x-4}{1}=\frac{z-5}{1}=\frac{z-10}{3}$

(ii) Equation of BC : $\vec{r}=\left(2\hat{i}+3\hat{j}+4\hat{k}\right)+\lambda\left(\hat{i}+\hat{j}+5\hat{k}\right);\ \frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{5}$;

(iii) Co-ordinates of D are (3, 4, 5)

8. $\left(-\frac{1}{3},\frac{1}{3},1\right)$; 2, 1, −6 ; $\vec{r}=-\frac{1}{3}\hat{i}+\frac{1}{3}\hat{j}+\hat{k}+\lambda\left(2\hat{i}+\hat{j}-6\hat{k}\right)$

9. $\vec{r}=\left(2\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(3\hat{i}+2\hat{j}-2\hat{k}\right);\ \frac{x-1}{3}=\frac{y-2}{2}=\frac{z-3}{-2}$

10. $\vec{r}=\left(2\hat{i}-\hat{j}+4\hat{k}\right)+\lambda\left(\hat{i}+2\hat{j}-\hat{k}\right);\ \frac{x-2}{1}=\frac{y+1}{2}=\frac{z-4}{-1}$ 11. $\frac{x+2}{3}=\frac{y-4}{5}=\frac{z+5}{6}$

12. $\vec{r}=\left(5\hat{i}-4\hat{j}+6\hat{k}\right)+\lambda\left(3\hat{i}+7\hat{j}+2\hat{k}\right)$ 13. $\frac{x}{5}=\frac{y}{-2}=\frac{z}{3};\ \vec{r}=\lambda\left(5\hat{i}-2\hat{j}+3\hat{k}\right)$

14. $\vec{r}=\left(3\hat{i}-2\hat{j}-5\hat{k}\right)+\lambda\left(11\hat{k}\right);\ \frac{x-3}{0}=\frac{z+2}{0}=\frac{z+5}{11}$

Exercise 14.3

1. $\theta=\cos^{-1}\left(19/21\right)$ 2. $\theta=\cos^{-1}\left(2/3\right)$ 4. $k=-10/7$

5. $\vec{r}=\hat{i}+2\hat{j}-4\hat{k}+\lambda\left(2\hat{i}+3\hat{j}+6\hat{k}\right);\ \frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6}$ 6. $\frac{x+}{3}=\frac{y-4}{5}=\frac{z+5}{6}$

Exercise 14.4

1. (–1, –1, –1) 2. No 3. $\left(\frac{170}{49}, \frac{78}{49}, \frac{10}{49}\right); \frac{3}{7}\sqrt{101}$

4. $\vec{r} = \left(2\hat{i} + 3\hat{j} + 2\hat{k}\right) + \lambda\left(2\hat{i} - 3\hat{j} + 6\hat{k}\right); \ \frac{\sqrt{580}}{7}$

Exercise 14.5

1. $\frac{3\sqrt{2}}{2}$ 2. $2\sqrt{29}$ 3. $\frac{3}{\sqrt{19}}$ 4. $\frac{8}{\sqrt{29}}$ 5. $\frac{3}{\sqrt{59}}; \ \frac{59x-253}{1} = \frac{59y-232}{-3} = \frac{592-97}{7}$

Exercise 14.6

1. $x - 2 = 0$ 2. $2y - z = 0$ 4. $\vec{r} \cdot i = 7$

5. $\vec{r} \cdot \left(\frac{6}{7}i + \frac{3}{7}j - \frac{2}{7}k\right) = 7$;k $\vec{r} \cdot (6i + 3j - 2k) = 49$

6. $\vec{r} \cdot \left(\frac{3}{13}i - \frac{4}{13}j + \frac{12}{13}k\right) = \frac{5}{13}; \ \frac{5}{13}; \ \frac{3}{13}, -\frac{4}{13}, \frac{12}{13}$ or $\frac{3}{13}x - \frac{4}{13}y + \frac{12}{13}z = \frac{5}{13}; \ \frac{5}{13}; \ \frac{3}{13}, -\frac{4}{13}, \frac{12}{13}$

7. $\vec{r} \cdot \left(\frac{2}{3}i - \frac{1}{3}j + \frac{2}{3}k\right) = 4$ 8. $-\frac{2}{7}x + \frac{3}{7}y - \frac{6}{7}z = 2$ 9. $\frac{4}{13}x - \frac{3}{13}y + \frac{12}{13}z = 13$ 10. $\frac{1}{\sqrt{3}}(i + j + k)$

Exercise 14.7

1. (i) $\cos^{-1}\left(-\frac{4}{21}\right)$; (ii) $\cos^{-1}\left(-\frac{16}{21}\right)$; (iii) $\cos^{-1}\left(\frac{5}{3\sqrt{6}}\right)$

2. (i) $\theta = \frac{\pi}{3}$; (ii) $\theta = \frac{\pi}{3}$; (iii) $\cos^{-1}\left(-\frac{2}{3\sqrt{6}}\right)$

4. (i) $\lambda = -2$; (ii) $\lambda = 2$ 5. $\sin^{-1}\left(\frac{-4}{\sqrt{406}}\right)$ 6. $\sin^{-1}\left(\sqrt{\frac{7}{52}}\right)$

7. $\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$ 8. $\sin^{-1}\left(-\frac{1}{6}\right)$ 9. $m = -2$ 10. $m = -3$

Miscellaneous Exercise 14

1. (A) 2. (C) 3. (D) 4. (B) 5. (C) 6. (B) 7. (A)
8. (A) 9. (C) 10. (D) 11. (C)

12. $-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}; \ \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$ 13. 13; $-\frac{3}{13}, \frac{4}{13}, -\frac{12}{13}$

14. $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right); \ x - 2y + z = 0$ 15. $x + 14y + 9z = 30$

15

Linear Programming

15.01 Introduction

A large number of decision problems faced by a business manager involve allocations to various activities, with the objective of increasing profits or decreasing costs, or both. The manager has to take a decision as to how best to allocate the resources among the various activities. The decision problems can be formulated, and solved, as mathematical programming problems. Mathematical programming involves optimisation of a certain function, called objective function, subject to certain constraints.

Definition : Linear programming deal with the optimization of a linear function of a number of variables subject to a number of conditions on the variables in the form of linear inequation or equations in variables involved.

15.02 Linear programming problem and its mathematical formulation

Let us understand the linear programming and its mathematical formulation with the help of following example:

Example : A developer produced two product P_1 and P_2 with the help of two machines M_1 and M_2. To make a unit of P_1, M_1 takes one hour and M_2 takes 3 hours and to make a unit of P_2, each take two hours. Profit on per unit of P_1 and P_2 be ₹ 60 and ₹ 50 respectively and M_1 and M_2 can work for 40 hrs. and 60 hrs. respectively in a week, then how much unit it can produce for maximum proft. It is clear from this example that,

(i) Developer can produce only P_1 or P_2 or both. Thus he gains maximum profit from different additive incorporate.

(ii) There are certain over riding conditions or constraints like M_1 and M_2 can work only 40 and 60 hrs respectively in a week.

Let developer only wants to produce P_1, then only 10 unit can produce and net profit $=60 \times 20 =$ ₹ 1200

Let developer only wants to produce P_2, then only 20 unit can produce and net profit $=50 \times 20 =$ ₹ 1000

There are too many possibilities. So, we have to know that how developer gain maximum profit from different method. Now, there is a problem, how developer can gain maximum profit from different method of Production. To find the answer, we have to formulate it mathematically.

15.03 Mathematical formulation of the problem

Let, x and y is number of desirable units of product P_1 and P_2 for favourable solution. Now, represent the problem in form of following table :

Machine	**Product**		**Availability (per week)**
	P_1	P_2	
M_1	1 hr.	2 hrs.	40 hrs
M_2	3 hrs.	2 hrs.	60 hrs.
Profit (per unit)	₹ 60	₹ 50	

Per unit profit on product P_1 and P_2 are ₹ 60 and ₹ 50 respectively. So, total profit on x unit of P_1 and y unit of P_2.

$$Z = 60x + 50y$$

So, we can relate the total profit with variable x and y linearly. Developer try to maximize that profit.

$$Z = 60x + 50y$$

Constraint for machines M_1 and M_2 : We know that, for production of P_1 and P_2, M_1 occupy for 1 and 2 hours.

So, occupation of M, for production of x unit of P_1 and y unit of P_2 will be $x + 2y$ but availability of M, is 40 hrs. per week then

$$x + 2y \le 40$$

Similarly for M_2 $\quad 3x + 2y \le 60$

Non-negative constraint : Since x and y is number of developing unit whcih never be negative.

So, $\quad x \ge 0, \quad y \ge 0$

Maximize : $\quad Z = 60x + 50y$

Constraint : $\quad x + 2y \le 40$

$$3x + 2y \le 60$$

and $\quad x \ge 0, y \ge 0$

Now, we have define some terms whcih is used in linear programming problems.

Objective Function :

If $c_1, c_2, ..., c_n$ are constants and $x_1, x_2, ..., x_n$ are variables then linear functions $Z = c_1x_1 + c_2x_2 + ... + c_nx_n$ which has to be maximize or minimise, is called objective function.

Constraints : Restriction on the variables of a linear programming problem are called constraints. These are represented in form of linear equation or inequalities.

In above example $x + 2y \le 40$ and $3x + 2y \le 60$ are constraints $x \ge 0$ and $y \ge 0$ are non-negative constraints.

Solution : The set of all values which satisfy the constraints of linear programming problems is called 'Solution'.

Feasible solution : Set of values of variables which satisfy the all constraints with non-negative constraints also, called feasible solution.

Optimal solution : Optimal solutions of linear programming problem is a feasible solution for which objective fucntion has maximum or minimum value.

Note : Optimal solution is actual solution of linear programming problem.

15.04 Graphical method to solve linear programming problems :

Graphical method is easiest method to solve linear programming problem. Graphical method is possible only if there is only two variable in linear programming problem.

Corner point method :

This method is based on 'Fundamental Extreme point Theorem', which states that, "If any linear programming problems attains an optimal solution, then one of the corner points (vertices) of the convex polygon at all feasible solution gives the optimal solution",.

Following algorithm can be used to solve a linear programming problem in two variables graphically by using corner-point method:

1. Formulate the given linear programming problem in mathematical form if it is not given in mathematical form.
2. Convert all inequalities (constraints) into equations and drawn their graphs. To draw graph of a linear equation, but $y = 0$ in it and obtain a point on x-axis similarly by putting $x = 0$. Obtain a point on y-axis. Join these points to obtaine graph of the equation.
3. Determine the region represented by each inequation. To determine the region represented by an inequation replace x and y both by zero, if the inequation reduces to as valid statement, then the region containing the origin is the region represented by the given inequation. Otherwise, the region not containing the origin is the region represented by the given inequation.
4. Obtain the region in xy-plane containing all points that simultaneously satisfy all constraints including non-negativity restrictions. The polygonal region so obtained is the feasible region and is known as the convex polygon of the set of all feasible solutions of linear programming problem.
5. Determine the co-ordinates of vertices (corner points) of the convex polygon obtained in setp 2.
6. Obtain the values of the objective function at each vertices of convex polygon. The point where objective function attains its optimum (maximum or minimum) value is the optimal solution of the given linear programming problem.

Now we have to solve the example of 15.03 by graphical method when problem is given following:

Maximize $Z = 60x + 50y$

Constraints $x + 2y \le 40$

$3x + 2y \le 60$

and $x \ge 0, y \ge 0$

Firstly we have to convert the constraints into equations;

$$x + 2y = 40 \qquad (1)$$

$$3x + 2y = 60 \qquad (2)$$

So, there are two points A (40, 0) and B (0, 20). Just like that putting $x = 0$ in equation (2) then $y = 30$ and for $y = 0$, $x = 20$, then we have two point C (0, 30) and D (20, 0). After joining A, B, C and D we have obtained the graph of line (1) and (2).

$x + 2y = 40$

x	40	0
y	0	20

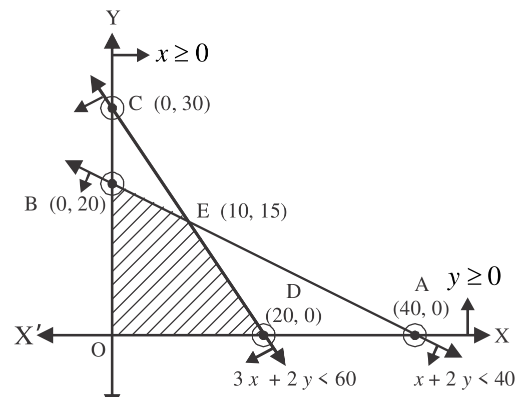
$A(40, 0)$; $B(0, 20)$

$3x + 2y = 60$

x	0	20
y	30	0

$C(0, 30)$; $D(20, 0)$

To determine the region of enequality $x+2y \leq 40$, we have to put value of x and y equal to zero, inequality $(0)+2(0) \leq 40$ satisfied. So, feasible region of inequality is toward origin. Just like that we have investigate the inequality $3x+2y \leq 60$ by putting $x = 0$, $y = 0$ which satisfied the inequality. So, feasible region of given inequality is also towards the origin.



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Shaded region ODEB is set of all possible values which satisfy the all constraints including non-negative constraints. There is no any solution beyond this region. next step is to find a solution from feasible solution of region ODEB by which we can obtain the optimal solution.

After inspecting the feasible solutions we have find that optimal solution will be on border line of ODEP. Now we have to tabulate the objective function on corner points O, D, E, B of feasible region ODEB.

Corner points	***x*-coordinate**	***y*-coordinate**	**Objective function** $Z = 60x + 50y$
O	0	0	$Z_0 = 0$
D	20	0	$Z_D = 1200$
E	10	15	$Z_E = 1350$
B	0	20	$Z_B = 1000$

From the above table, it is clear that objective function has its maximum value at E(10, 15), so, solution given by E is optional solution.

Note :

(1) If feasible solution of any linear programming problem gives a convex polygon then any corner point of polygon attain maximum value of objective fucntion and any other corner point attain minimum value of objective function.

(2) Sometimes the feasible region of linear programming problem is not a bounded convex polygon. That is, it extends indefenetely in any direction. In such case, we say that the feasible region is unbounded. Above algorithm is applicable when the feasible region is bounded. If the feasible region is unbounded, then we find the values of the objective function $Z = ax + by = M$ by at each corner point of the feasible region. Let M and m respectively denote the largest and smallest values of Z at there points. In order to check whether Z has maximum and minimum values at M and m respectively, we proceed as follows:

(i) Draw the line $ax+by > M$ and find the open half plane $ax+by > M$. If the open half plane represented by $ax+by > M$ has no point common with the unbounded feasible region, then M is the maximum value of Z otherwise Z has no maximum value.

(ii) Draw the line $Z = ax+by = m$ and find the half plane $ax+by < m$. If the half plane $ax+by < m$ has no point common with the unbounded feasible region, then m is the minimum value of z. Otherwise, Z has no minimum value.

Illustrative Examples

Example 1. Solve the following LPP graphically

Maximize $Z = 5x + 3y$

Constraints $3x + 5y \le 15$

$5x + 2y \le 10$

and $x \ge 0,\ y \ge 0$

Solution : Converting the given inequalities into equations, we obtain the following equation :

$$3x + 5y = 15 \quad (1)$$

$$5x + 2y = 10 \quad (2)$$

Region represented by $3x + 5y \le 15$: The line $3x + 5y = 15$ meets the coordinate axes at A (5, 0) and B (0, 3). Join these points to obtain the line $3x + 5y = 15$. Clearly (0, 0) satisfies the inequality $3x + 5y \le 15$. So the region containing the origin represents the solution set of inequation $3x + 5y \le 15$.

3x + 5y = 15

x	5	0
y	0	3

A(5, 0), B (0, 3)

Region represented by $5x + 2y \le 10$. The line $5x + 2y = 10$ meets the coordinate axes at C (2,0) and D (0,5) respectively.

$5x + 2y = 10$

x	2	0
y	0	5

Join these points to obtain line $5x + 2y \le 10$. Clearly (0, 0) satisfies the inequation 5 (0) + 2 (0) = 0 ≤ 10. So, the region containing the origin represents the solution set of this inequation.

The shaded region OCEB in figure represents the common region of the inequations. This region is feasible region of given LPP.

The coordinates of the vertices (conrer points) of the shaded feasible region are 0(0, 0), C(2, 0), $E(20/19,\ 45/19)$ and B(0, 3). These points have been obtained by solving the equations at the corresponding intersecting lines, simultaneously.

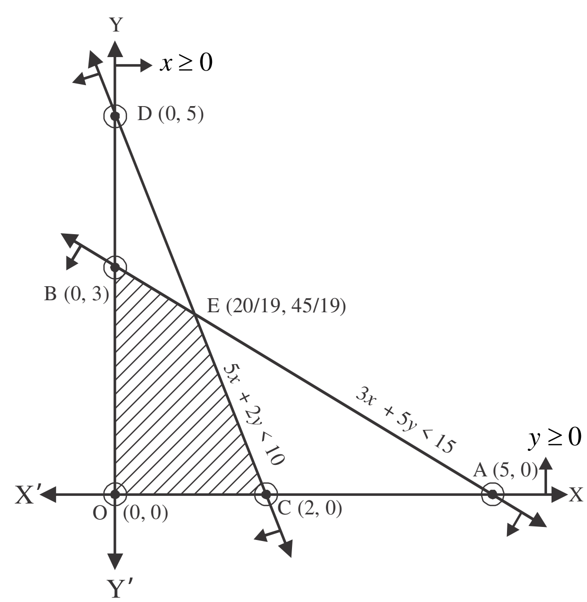

Fig. 15.02

The values of objective function at these points are given in the following table.

Points	*x*-co-ordiante	*y*-co-ordinate	Objective function Z = 5*x* + 3*y*
O	0	0	$Z_0 = 5(0) + 3(0) = 0$
C	2	0	$Z_C = 5(2) + 3(0) = 10$
E	20 / 19	45 / 19	$Z_E = 5(20/19) + 3(45/19) = 235/19$
B	0	3	$Z_B = 5(0) + 3(3) = 9$

Clearly Z is maximum at $E(20/19,\ 45/19)$. Hence $x = 20/19,\ y = 45/19$ is the optimal solution of the given LPP. The optimal value of Z is 235 / 19.

Example 2. Solve the following linear programming problem graphically.

Minimize $\quad Z = 200x + 500y$

Subject to the constraints $\quad x + 2y \geq 10$

$\quad 3x + 4y \leq 24$

and $\quad x \geq 0,\ y \geq 0$

Solution : The inequalities in the form of equations are

$$x + 2y = 10 \qquad (1)$$

$$3x + 4y = 24 \qquad (2)$$

Area shown by the inequality $x + 2y \geq 10$

Line $x + 2y = 10$ meets the coordinate axes at points A (10, 0) and B (0,5).

x + 2y = 10

x	10	0
y	0	5

A (10, 0) ; B (0, 5)

Area shown by the inequality $3x + 4y \leq 24$

Line $3x + 4y = 24$ meets the coordinate axes at points C(8, 0) and D (0, 6).

3x + 4y = 2y

x	8	0
y	0	6

C (8, 0) ; D (0, 6)

The shaded region in figure is the feasible region determined by the system of constraints. We observe that the feasible region BED is bounded. So, we now use corner points method to determine the maximum value of Z.

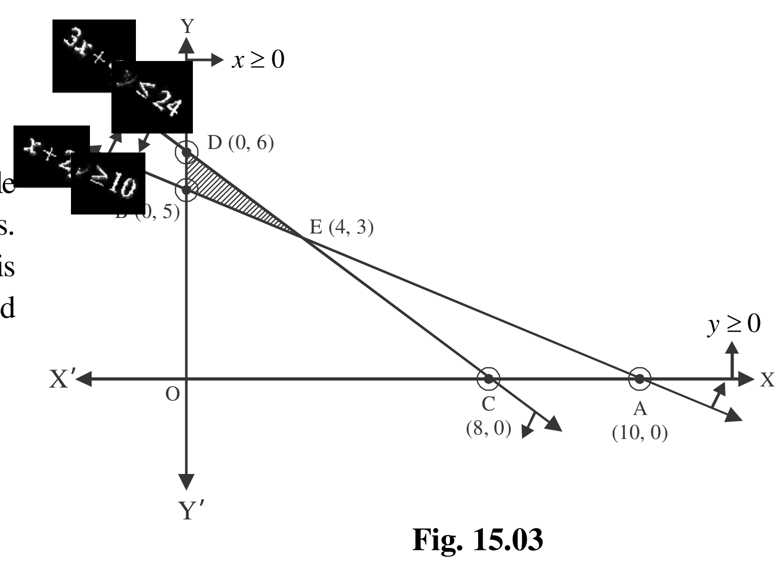

Fig. 15.03

Now we evaluate Z at each corner point.

Points	*x*-co-ordinate	*y* co-ordinate	Objective function Z = 200*x* + 500*y*
B	0	5	$Z_B = 200(0) + 500\ (5) = 2500$
E	4	3	$Z_E = 200\ (4) + 500\ (3) = 2300$
D	0	6	$Z_D = 200\ (0) + 500\ (6) = 3000$

Hence the minimum value at point E (4, 3) is 2300.

Example 3. Solve the following linear programming problem graphically.

Maximize $$Z = y + \frac{3}{4}x$$

subject to the constraints $$x - y \geq 0$$

$$-x/2 + y \leq 1$$

and $$x \geq 0,\ y \geq 0$$

Solution : The inequalities in the form of equations are

$$x - y = 0 \qquad (1)$$

$$-x/2 + y = 1 \qquad (2)$$

Area shown by the inequality $x - y \geq 0$

Line $x - y = 0 \Rightarrow x = y$ meets at points O (0, 0) ; A (1, 1).

x = y

x	0	1
y	0	1

Area shown by the inequality $-x/2 + y \leq 1$

Line $-x/2 + y = 1$ meets the coordinate axes at points B(–2,0) and C(0,1).

$-x/2 + y = 1$

x	–2	0
y	0	1

B(–2, 0) ; C(0, 1)

We draw the graph of the equations. The shaded region in fig 15.04 is the feasible region determined by the system of constraints. We observe that the feasible region is unbounded. So, We can see that there is no point satisfying all the constraints simultaneosuly. Thus the problem is having no feasible region and hence no feasible solution.

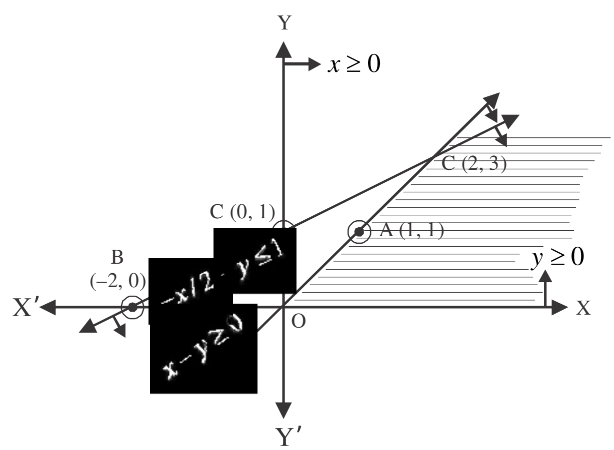

Fig. 15.04

Example 4. Solve the following linear programming problem graphically:

Maximize $Z = 3x + 4y$

Subject to constraint $x + y \le 3$

$2x + 2y \le 12$

and $x \ge 0,\ y \ge 0$

Solution : The inequalities in the form of equations are

$$x + y = 3 \quad (1)$$

$$2x + 2y = 12 \quad (2)$$

Area shown by the inequality $x + y \le 3$:

Line $x + y = 3$ meets the coordinate axes at points A (3, 0) and B (0, 3).

x + y = 3

x	3	0
y	0	3

A (3, 0) ; B (0, 3)

Area shown by the inequality $2x + 2y \ge 12$:

Line $2x + 2y = 12$ meets the coordinate axes at points C(6, 0) and D(0, 6)

2x + 2y = 12

x	6	0
y	0	6

C (6, 0) ; D (0, 6)

We draw the graph of the equations. The shaded region in fig. 15.05 is the feasible region determined by the system of constraints. We observe that the feasible region is unbounded. So, we can see that there is no point satisfying all the constraints simultanesouly. Thus, the problem is having no feasible region and hence no feasible solution.

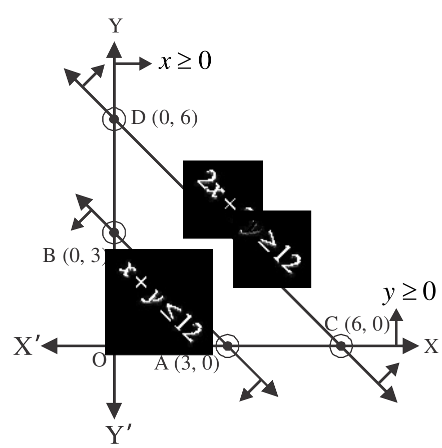

Fig. 15.05

Example 5. Solve the following linear programming problem graphically:

Maximize $Z = 2x + 3y$

Subject to constraints $4x + 6y \le 60$

$2x + y \le 20$

and $x \ge 0,\ \ y \ge 0$

Solution : The inequalities in the form of equations are

$$4x + 6y = 60 \quad (1)$$

$$2x + y = 20 \quad (2)$$

Area shown by the inequality $4x + 6y \le 60$:

Line $4x + 6y = 60$ meets the coordinate axes at points A (15, 0) and B(0,10).

$4x + 6y = 60$

x	15	0
y	0	10

A(15, 0) ; B (0, 10)

Area shown by the inequality $2x + y \le 20$:

Line $2x + y = 20$ meets the coordiante axes at points C(10, 0) and D(0, 20).

$2x + y = 20$

x	10	0
y	0	20

C (10, 0) ; D (0, 20)

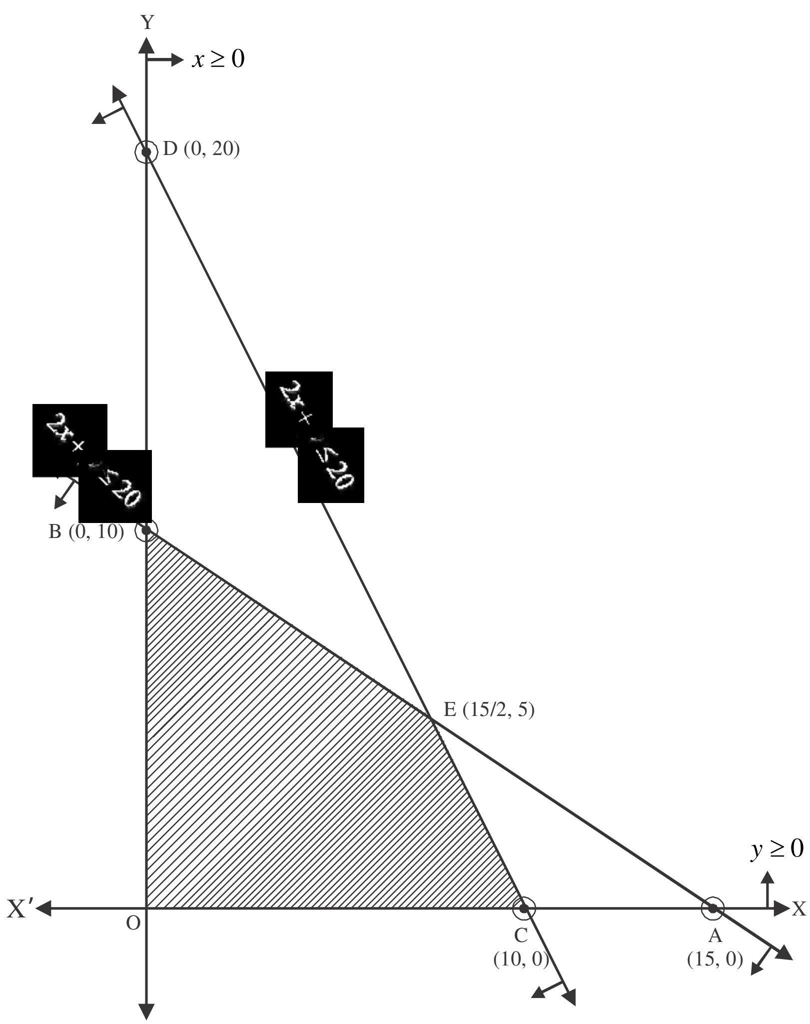

Fig. 15.06

The shaded region in Fig. 15.06 is the feasible region determined by the system of constraints. We observe that the feasible region OCEB is bounded. So, we now use corner point method to determine the maximum value of Z. The coordinates of the corner points O, C, E and B are O(0, 0), C (10, 0), $E(15/2,\ 5)$ and B (0, 10). Now we evaluate Z at each corner point.

Points	***x*-coordinate**	***y*-coordiante**	**Objective function Z = 2*x* + 3*y***
O	0	0	$Z_O = 2(0) + 3(0) = 0$
C	10	0	$Z_C = 2(10) + 3(0) = 20$
E	15 / 2	5	$Z_E = 2(15/2) + 3(5) = 30$
B	0	10	$Z_B = 2\ (0) + 3(10) = 30$

Hence the maximum value at point $E(15/2,\ 5)$ and $B(0,\ 10)$ is the maximum value is obtained at points E and B.

Note: The reason for the inifinte solution is the objective function $Z = 2x + 3y$ which is parallel to the line $4x + 6y = 60$.

Exercise 15.1

Solve the following linear Programming problems graphically:-

1. Minimize $Z = -3x + 4y$

   Subject to the constraints $x + 2y \le 8$

   $3x + 2y \le 12$

   and $x \ge 0,\ \ y \ge 0$

2. Maximize $Z = 3x + 4y$

   Subject to the constraints $x + y \le 4$

   and $x \ge 0,\ \ y \ge 0$

3. Minimize $Z = -50x + 20y$

   Subject to the constraints $2x - y \ge -5$

   $3x + y \ge 3$

   $2x - 3y \le 12$

   and $x \ge 0,\ \ y \ge 0$

4. Minimize $Z = 3x + 5y$

   Subject to the constraints $x + 3y \ge 3$

   $x + y \ge 2$

   and $x \ge 0,\ \ y \ge 0$

5. Find the maximum and minimum value of $Z = 3x + 9y$

Subject to the constraints $x + 3y \le 60$

$x + y \ge 10$

and $x \ge 0, \quad y \ge 0$

6. Minimize $Z = x + 2y$

Subject to the constraints $2x + y \ge 3$

$x + 2y \ge 6$

and $x \ge 0, \quad y \ge 0$

7. Find the maximum and minimum value of $Z = 5x + 10y$

Subject to the constraints $x + 2y \le 120$

$x + y \ge 60$

$x - 2y \ge 0$

and $x \ge 0, \quad y \ge 0$

8. Maximize $Z = x + y$

Subject to the constraints $x - y \le -1$

$-x + y \le 0$

and $x \ge 0, \quad y \ge 0$

9. Maximize $Z = 3x + 2y$

Subject to the constraints $x + y \ge 8$

$3x + 5y \le 15$

and $x \ge 0, \quad y \ge 0$

10. Maximize $Z = -x + 2y$

Subject to the constraints $x \ge 3$

$x + y \ge 5$

$x + 2y \ge 6$

and $x \ge 0, \quad y \ge 0$

15.05 Different types of linear programming problems

In this section, we will discuss about some important linear programming problem like diet related problem, manufacturing related problem and transportation related problem.

Diet related problem:

In these problems, we determine the amount of different kind of constituents / nutrients which should be included in a diet so as to minimize the cost of the desired diet such that it contains a certain minimum amount of each constituent / nutrients.

Illustrative Examples

Example 6. A human requires definite amount of two type of vitamin (Vitamin A and Vitamin B) for balanced food. These vitamins find in two different food product (F_1 and F_2). Vitamin contained in one unit of each food product, minimum requirement for balanced food and prices of per unit food product is given in table.

Vitamin	Food product		Daily requirement
	F_1	F_2	
A	2	4	40
B	3	2	50
Price per unit (in ₹)	3	2.5	

How much unit of both produce its used os that the minimum requirement for balanced food is fullfilled?

Solution : Let x unit of F_1 and y unit of F_2 is required for minimum necesstiy. Then price of x unit of F_1 will be $3x$ and y unit of F_2 will be 2.54. Total price will be $3x + 2.5y$, we have to calculate minimum value.

Objective function is $Z = 3x + 2.5y$

Subject to constraint for vitamin A :

$$2x + 4y \geq 40$$

Subject to constraint for vitamin B :

$$3x + 2y \geq 50$$

Since units of required food product may not be negative, so, non-negative constraint

$$x \geq 0, \quad y \geq 0$$

So, mathematical formulation of given LPP

Minimize $Z = 3x + 2.5y$

Constraint $2x + 4y \geq 40$

$3x + 2y \geq 50$

and $x \geq 0, \quad y \geq 0$

Region represented by indequation $2x + 4y \geq 40$:

Line $2x + 4y = 40$ meets the coordiante axes at A(20, 0) and B(0, 10) respectively.

2x + 4y = 40

x	20	0
y	0	10

Join these points to obtain line $2x \;\; 4y = 40$, But (0, 0) doesn't satisfy the inequation $2(0) + 4(0) = 0 \geq 40$, So, the region opposite to the origin represents the solution set of this inequation.

Region represented by $3x + 2y \geq 50$

Line $3x + 2y \geq 50$ meets the coordiante axes at point $C(50/3, 0)$ and D (0, 25).

3x + 2y = 50

x	50/3	0
y	0	25

Join these points to obtain line $3x + 2y = 50$. But (0, 0) doesn't satisfy the in equation $3x + 2y \geq 50$, so, the region opposite the origin represents the solution set of inequations.

Region represented by $x \geq 0$ and $y \geq 0$.

Since every point in first quadrant satisfies, the both inequation. So, the region represented by inequations $x \geq 0$ and $y \geq 0$ in first quadrant.

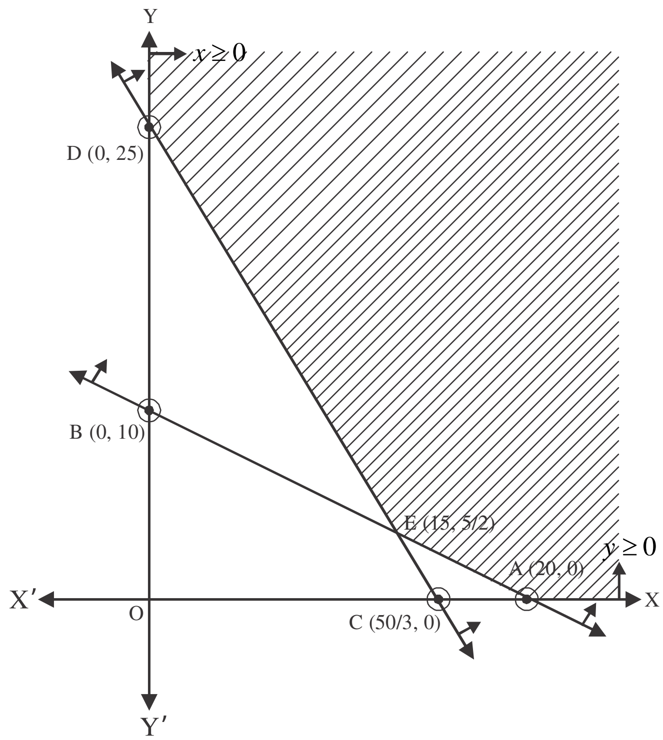

Fig. 15.07

The coordinates of vertices (corner points) of shaded region are A (20, 0) ; $E(15, 5/2)$ and D (0, 25). Where E is intersection point of line $2x + 4y = 40$ and $3x + 2y = 50$.

The values of objective function at these points are given in following table:

Points	*x* co-ordinate	*y* co-ordinate	Objective function Z = 3*x* + 2.5*y*
A	20	0	$Z_A = 3(20) + 2.5(0) = 60$
E	15	5 / 2	$Z_E = 3(15) + 2.5\ (5/2) = 51.25$
D	0	25	$Z_D = 3(0) + 2.5\ (25) = 62.5$

Clearly Z is minimum at point $E(15, 5/2)$. Since feasible region is unbounded. So, we have to draw graph of $3x + 2.5y < 51.25$. Resultant open half plane represented by in equation $3x + 2.5y < 51.25$ doesn't have any common point with feasible region. So, minimum value of LPP is 51.25, Rs. 50, for optimal solution we have 15 unit of F_1 and 5/2 unit of F_2.

Manufacturing problems:

In these prloblems, we determine the number of units of different products which should be produced and sold by a firm when each product requires a fixed manpower, maching hours, labour hour per unit of product, warehouse space per unit of output etc., in order to make maximum profit.

Illustrative Examples

Example 7. A firm manufacturing two types of electric items, A and B. Can make a profit of ₹ 20 per unit of A and ₹ 30 per unit of B. Each unit of A requires 3 motors and 2 transformers and each unit of B requires 12 motors and 4 transformers. The total supply of these per month is restricted to 210 motors and 300 transformers. Type B is an expert model requiring a voltage stabilzer which has a supply restricted to 65 units per month. Formulate the LPP for maximum profit and solve it graphically.

Solution : Left firm manufactures x and y unit respectively of A and B to get maximum profit. Profit per unit of A and B is ₹ 20 and ₹ 30 respectively. So, profit from x and y unit of A and B is,

Z is objective function. $Z = 20x + 30y$

Constraint for motor

For manufacturing x unit of A and y unit of B we have need of $3x$ and $2y$ motors and total supply of motor per month is 210 only. So,

$$3x + 2y \leq 210$$

Constraint for transformer.

For manufacturing of x unit of A and y unit of B we have need of $2x$ and $4y$ transformers and total supply at transformer per month is 300 only, So,

$$2x + 4y \leq 300$$

Voltage stabilizer is used in only B and its supply per month is only 65, 50

$$y \leq 65$$

Manufactured unit may not be negative. So,

$$x \geq 0, \quad y \geq 0$$

So, mathematical formulation of LPP is given below,

Maximize $Z = 20x + 30y$

constraint $3x + 2y \leq 210$

$2x + 4y \leq 300$

$y \leq 65$

and $x \geq 0, \quad y \geq 0$

Convert all the inequations into equation,

$$3x + 2y = 210 \qquad (1)$$

$$2x + 4y = 300 \qquad (2)$$

$$y = 65 \qquad (3)$$

Region represented by $3x + 2y \leq 210$:

Line $3x + 2y = 210$ meets the coordinate axes at point A(70, 0) and B(0, 105).

3x + 2y = 210

x	70	0
y	0	105

Join A and B to obtain the line $3(0) + 2(0) = 0 \leq 210$. (0, 0) satisifes the inequation, So the region containing the origin represents the solution set of inequation.

Region represented by 2x + 4y $\leq$ 300:

Line $2x + 4y = 300$ meets the co-ordinate axes at C(150, 0) and D(0, 75) respectively.

2x + 4y = 300

x	150	0
y	0	75

Join C and D to obtain the line $2(0) + 4(0) = 0 \leq 300$. (0, 0) satisifes the inequation, so, region containing the origin represents the solution set of inequation.

Region represented y $\leq$ 65 :

Line $0x + y = 65$ meets at point E (5, 65) an F(10, 65).

0x + y = 65

x	5	10
y	65	65

Join E and F to obtain line 0x + y = 65. (0, 0) satisifes the inequation, so, region containing the origin represent the solution set of inequation.

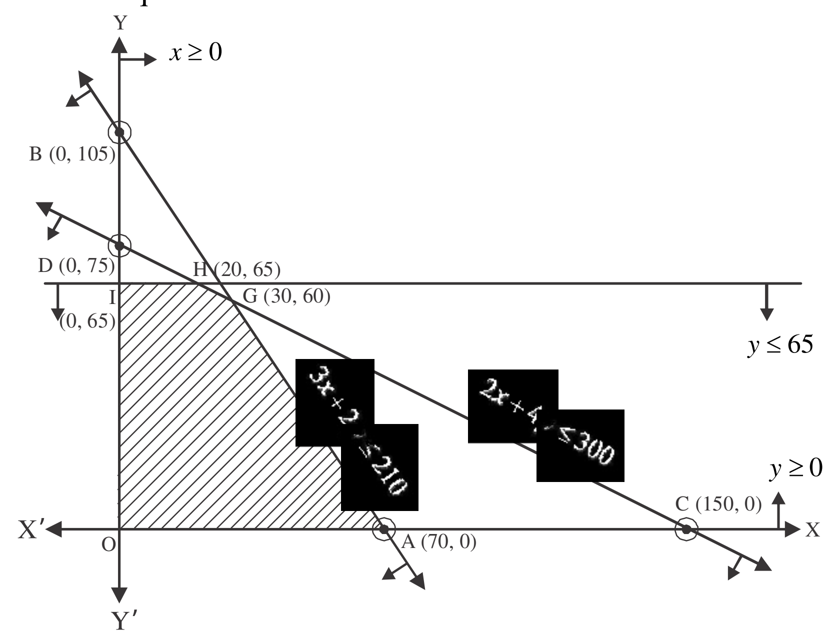

Fig. 15.08

Region represented by x $\geq$ 0 and y $\geq$ 0 :

Since points on first quadrant satisfy the both inequation. So, region represented by $x \geq 0$ and $y \geq 0$ is first quadrant.

Shaded region OAGHI represents the common region of above inequations. This region is feasible region of given LPP. Vertices of shaded feasible region are O(0,0), A (70,0), G(30, 60), H (20, 65) and I(0, 65). Where G and H are intersection points of $2x + 4y = 300$ and $3x + 2y = 210$ and $y = 65$. Values of objective function is given in following table at these points.

Points	x-co-ordinate	y-co-ordinate	Objective function $Z = 20x + 30y$
O	0	0	$Z_O = 20(0) + 30(0) = 0$
A	70	0	$Z_A = 20(70) + 30(0) = 1400$
G	30	60	$Z_G = 20(30) + 30(60) = 2400$
H	20	65	$Z_H = 20(20) + 30(65) = 2350$
I	0	65	$Z_I = 20(0) + 30(65) = 1950$

Clearly, it is clear from table that objective function has its maximum value at point G(30, 60). So, for maximum profit, firm will manufacture 30 unit of A and 60 unit of B from which it gain maximum profit of ₹2400.

Transportation problems :

In this type of problems, we have transport diffrent objects from diefferent factories and different-different places according to demands on market. This type of transport according to supply from factories to the market so that cost of transportation is minimum.

Illustrative Examples

Example 8. There are two factories located one at place P and the other at place Q. From these locations, a certain commodity is to be delivered to each of the three depots situated at A, B and C. The weekly requirements of the depots are respectively 5, 5 and 4 units of the commodity while the production capacity of the factories at P and Q are respectively 8 and 6 units. The cost of transportation per unit is given below:

From / To	Cost (In ₹)		
	A	**B**	**C**
P	16	10	15
Q	10	12	10

How many units should be transported from each factory to each depot in order that the transportation cost is minimum. What will be the minimum transportation cost?

Solution : The problem can be explained diagrammatically as follows : Let x units and y units of the commodity be transported from the factory at P to the depots at A and B respectively. Then $(8 - x - y)$ units will be transported to depot at C.

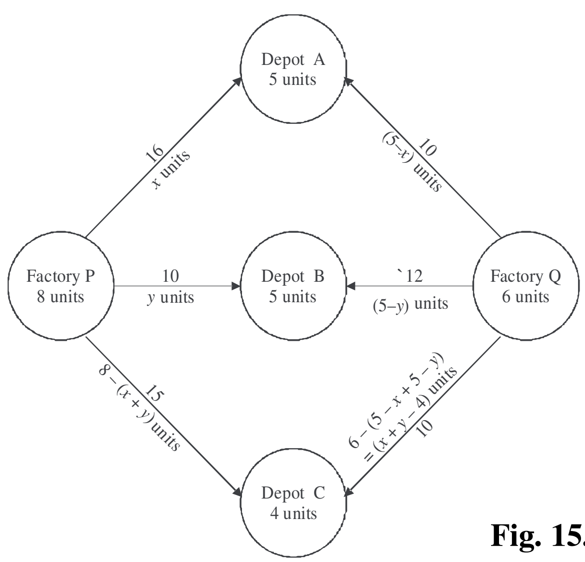

Fig. 15.09

Hence, we have $x \geq 0, \quad y \geq 0$ and $\quad 8 - x - y \geq 0$

$\Rightarrow \quad x \geq 0, \quad y \geq 0$ and $\quad x + y \leq 8$

Now, the weekly requirement of the depot at A is 5 units of the commodity. Since x units are transported from the factory at P, the remaining $(5 - x)$ units need to be transported from the factory at Q. Obviously $x \leq 5$

Similarly, $\quad y \leq 5$ and $x + y \geq 4$

Total transportation cost Z is given by

$$Z = 16x + 10y + 15(8 - x - y) + 10(5 - x) + 12(5 - y) + 10(x + y - 4)$$

$$Z = x - 7y + 190$$

Therefore, the problem reduces to

Minimize $\quad Z = (x - 7y + 190)$

subject to the constraints $\quad x \geq 0, \; y \geq 0$

$$y \leq 5$$

$$x \leq 5$$

$$x + y \geq 4$$

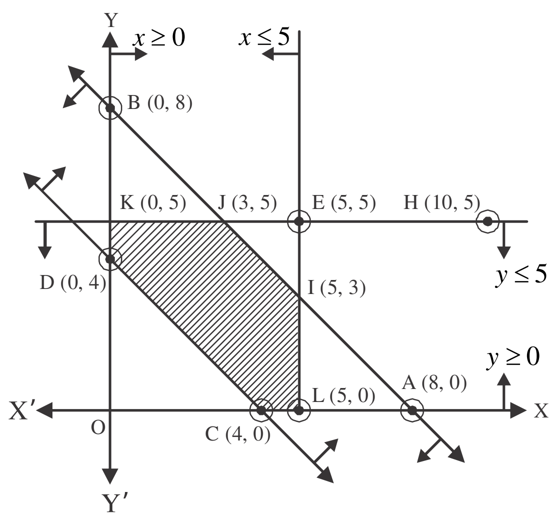

Fig. 15.10

The shaded region CLIJKD represented by the constraints above

Observe that the feasible region is bounded. The coordinates of the corner points of the feasible region are (0, 4), (0, 5), (3, 5), (5, 3), (5, 0) and (4, 0). Let us evaluate Z at these points.

Corner Point	Z = 10 (x – 7y + 190)
(4, 0)	162
(5, 0)	155
(5, 3)	158
(3, 5)	174
(0, 5)	195
(0, 4)	194

From the table, we see that the minimum vlaue of Z is 155 at the point (5, 0)

Hence, the optimal transportation strategy will be to deliver 0, 5 and 3 units from the factory at P and 5, 0 and 1 units from the factory at Q to the depots at A, B and C respectively. Corresponding to this strategy, the transportation cost would be minimum, i.e., ₹ 155.

Exercise 15.2

1. A dietician wishes to mix two type sof foods in such a way that the vitamin content of the mixture contains atleast 8 units of vitamin A and 10 units of vitamin C. Food 1 contains 2 units / kg of vitamin A and 1 unit / kg of vitamin C. Food II contains 1 unit / kg of vitamin A and 2 units / kg of vitamin C. It costs Rs. 50 / kg to purchase Food I and Rs. 70 / kg to purchase Food II. Formulate a linear programming problem to minimise the cost of the mixture.
2. A dietician wishes to mix together two kinds of food X and Y in such a way that the mixture cotnains atleast 10 uns if vitamin A, 12 units of vitamin B and 8 units of vitamin C the vitmain contents of one kg food is given below:

	Vitamin A	**Vitamin B**	**Vitamin C**
Food X	1	2	3
Food Y	2	2	1

   One kg of food X costs ₹ 16 and one kg of food Y costs ₹ 20. Find the least cost of the mixture which will produce the required diet?
3. One kind of cake requires 300 grams of flour and 15 grams of fat and another kind of cake requires 150 grams of flour and 30 grams of fat. Find the maximum number of cake which can be made from 7.5 kg of flour and 600 grams of fat assuming that their is no shortage of other ingredients in making the cake.
4. A manufacturer produces nuts and bolts. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts. It takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of ₹ 17.50 per package nuts and ₹ 7 pr package on bolts. How many packages of each should be produced each day so as to maximize his profit, if he operates his machines for at the most 12 hours a day?
5. A furniture dealer deals in tables and chairs. He has ₹ 5760 to invest and has a storage space of atmost 20 pieces. A table cost ₹ 360 and a chair cost ₹ 240. He estimates that from the sale of one table he can make a profit of ₹ 22 and by selling one chair. He makes a profit of ₹ 18. He wants to know how many tables and chairs he should buy from the available money, so as to maximize his profit, assuming that he can sell all the items which he buys. Solve the following optimising problem graphically.
6. A factory manufactures two types of screws A and B. Each type of screw requires the use of two machines automatic and a hand operated. It takes 4 minutes on automatic and 6 minutes on hand operated machines to manufacture a package of screws A while it takes 6 minutes on automatic and 3 minutes on hand operated machine to manufacture a package of screws B. Each machine is available for at the most 4 hours on any day. The manufactuer can sell a package of screws A at a profit to 70 paise and screw B at a profit of ₹ 1. Assuming that he sells all the screws he manufactures, how many packages of each type should the factory owner produce in a day in order to maximize his profit? Determine the maximum profit.
7. A company manufactures two types of novelty sourvenirs made of plywood. Sourvenirs of type A require 5 minutes each for cutting and 10 minutes each for assembling. Souvenirs of type B require 8 minutes each for cutting and 8 minutes for assembling. There are 3 hours and 20 minutes available for cuttign and 4 hours for assembling. The profit is Rs 5 each for type A and ₹ 6 each for type B sourvenirs. How many sourvenirs of each type should the company manufacture in order to maximize the profit?

8. There are two types of fertilisers F_1 and F_2. F_1 consists of 10% nitrogen and 6% phosphoric acid and F_2 consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, a farmer finds that he needs atleast 14 kg of nitrogen and 14 kg of phosphoric acid for his corporation. If F_1 costs ₹ 6 / kg and F_2 costs ₹ 5 / kg, determine how much of each type of fertiliser should be used so that nutrient requirements are met at a minimum cost. What is the minimum cost?

9. A merchant plans to sell two types of personal computers - desktop model and a portable model that will cost ₹ 25,000 and ₹ 40,000 respectively he estimates that total monthly demand of computer will not exceed 250 units. Determine the number of units of each type of computers which the merchant should stock to get maximum profit if he does not want to invest more than ₹ 70 lakhs.

10. Two godowns A and B have grain capacity of 100 quintals and 50 quintals resp. They supply to three ration shops D, E and F whose requirements are 60, 50 and 40 quintals respectively. The cost of transportantion per quintal from the godwon to the shops are given in the following table:-

Transportation cost per quintal (in ₹)

From / To	**A**	**B**
D	6	4
E	3	2
F	2.50	3

How should the supplies be transported in order that the transportation cost is minimum? What is the minimum cost?

Miscellaneous Examples

Example 9. A company produces two types of leather belts, say type A and B. Belt A is a superior quality and belt B is of lower quality. Profits on each type of belt are ₹ 2 and ₹ 1.50 per belt respectively. Each belt of type A requires twice as much time as required by a blet of type B. If all belts were of type B, the company could produce 1000 belts per day,. But the supply of leather is sufficient only for 800 belts per day (Both A and B comined). Belt A requires a fancy buckle and only 400 fancy buckles are available for this per day. For belt B, only 700 buckles are available per day.

Solution : Let company produces x unit of A and y unit of B. Profit from A and B are ₹2 and ₹ 1.50 respectively. So, objective function is.

Maximize $$Z = 2x + 1.50y$$

If all belts be of type B then company produces 1000 belts per day. Time taken to produce y unit of B type belt $= \frac{y}{1000}$

Since, time taken to produce A-type belt is twice with respect to B. So, time taken to produce A type belt $= \frac{x}{500}$

$$\frac{x}{500} + \frac{y}{1000} \le 1$$

$$\Rightarrow \quad 2x + y \le 1000$$

Supply of leather is limited produce only 800 belt. So,

$$x + y \le 800$$

Since 400 buckles are available for A-type belt and 700 buckle for B-type belt.

$$x \le 400, \qquad y \le 700$$

No. of belt never be negative. So,

$$x \ge 0, \qquad y \ge 0$$

Mathematical formulation of given LPP is

Maximize $\quad Z = 2x + 1.50y$

Constraint $\quad 2x + y \le 1000$

$$x + y \le 800$$

$$x \le 400$$

$$y \le 700$$

and $\quad x,\ y \ge 0$

Convert the inequations in equations,

$$2x + y = 1000 \qquad (1)$$

$$x + y = 800 \qquad (2)$$

$$x = 400 \qquad (3)$$

$$y = 700 \qquad (4)$$

Region represented by 2*x* + *y* $\le$ 1000:

Line $2x + y = 1000$ meets the co-ordinate axes at A (500, 0) and B(0, 1000).

2x + y = 1000

x	500	0
y	0	1000

Join A and B to obtain line $2(0) + (0) = 0 \le 1000$. (0, 0) satisfies the inequation. So, region containing the origin represents the solution set of inequation.

Region represented by *x* + *y* $\le$ 800

Line $x + y = 800$ meets the coordinate axes at point C(800, 0) and D (0, 800).

x + y = 800

x	800	0
y	0	800

Join C and D to obtain line $x + y = 800$. (0, 0) satisfies the inequlaity. So, region containg the origin represents the solution set of inequation.

Region represented by *x* $\le$ 400:

Line $x + 0y = 400$ meets at the point E(400, 10) and F(400, 20).

x + 0y = 400

x	400	400
y	10	20

Join E and F to obtain the line $x + 0y = 400$. (0, 0) satisfies the inequation $x \le 400$. So, region containing the origin represents the solution set of inequation.

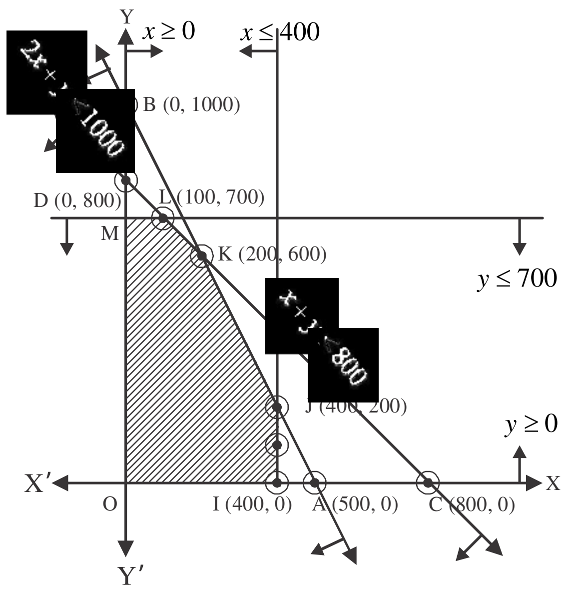

Fig. 15.11

Region represented by $x \geq 0$ and $y \geq 0$

Since, every point in first quadrant satisfy the both inequalities, So, region represented by $x \ge 0$, $y \ge 0$ is in first quadrant.

Shaded region of IKLM is common region of all inequation. That is feasible region of given LPP. Vertices of this region are O (0, 0), I (400, 0) J (400, 200), K (200, 600), L (100, 700), M (0, 700). Where J, K, L are intersection points of lines $x = 400$ and $2x + y = 1000$; $2x + y = 1000$ and $x + y = 800$; $y = 700$ and $x + y = 800$.

Values of objective function at these points are–

Points	x Co-ordinate	y Co-ordinate	Objective functions Z = 2x + 1.50y
O	0	0	$Z_O = (2)(0) + (1.50)(0) = 0$
I	400	0	$Z_I = (2)(400) + (1.50)(0) = 800$
J	400	200	$Z_J = (2)(400) + (1.50)(200) = 1100$
K	200	600	$Z_K = (2)(200) + (1.50)(600) = 1300$
L	100	700	$Z_L = 2(100) + (1.50)(700) = 1250$
M	0	700	$Z_M = (2)(0) + (1.50)(700) = 1050$

It is clear from table, objective function is maximum at K(200, 600). So, company produces 200 unit of A and 600 unit of B for maximum profit.

Example 10. The old hen can be purchased at ₹ 2 per hen whereas the price of new hen is 5 Rs. per hen. Old hense give 3 eggs and new hens give 5 eggs per week. Price of one egg is 30 paise. Investment on food of a hen per week is ₹ 1. How many hens of both type a man buy if he has only ₹ 80 and he earned profit more than ₹ 6. If than person can not keep more than 20 hens the solve the LPP by graphical method.

Solution: Let he purchases x new hens and y old hens. Since, a new hen gives 5 eggs per week, So, he earns ₹ 1.50 earn per week. After deducting food investment, gross profit is 50 paise.

Similarly, profit from old hen = ₹ (0.30 × 3 – 1) = ₹ (–0.10). So, objective function is

$Z = (.50)x - (.10)y$. Price of old hen is ₹ 2 per hen and price of new hen is ₹ 5 per hen. Also, the person has only ₹ 80. So, $5x + 2y \le 80$.

Again, that person can not keep more 20 hens in his house.

So, $x + y \le 20$. Person wants to get profit more than ₹ 6, $0.5x - 0.1y \ge 6$.

Purchased hens never be negative.

So, $x \ge 0,\ y \ge 0$

Mathematical formulation of given LPP is,

Maximize $Z = (\cdot 50)x - (\cdot 10)y$

Constraints $5x + 2y \le 80$

$x + y \le 20$

$0\cdot 5x - 0\cdot 1y \ge 6$

and $x \ge 0,\ y \ge 0$

Since the person wants to get profit more than ₹6. Therefore, it is not necessary to consider $0.5x - 0.1y \ge 6$.

∴ The LPP is maximize $Z = (\cdot 50)x - (\cdot 10)y$

Such that $5x + 2y \le 80$, $x + y \le 20$ and $x \ge 0,\ y \ge 0$

On converting the inequation into the equation, we get

$$5x + 2y = 80 \qquad (1)$$

$$x + y = 20 \qquad (2)$$

Region represented by 5x + 2y ≤ 80:

Line $5x + 2y = 80$ meets the coordinate axes at A (16, 0) and B (0, 40).

5x + 2y = 80

x	16	0
y	0	40

Join A and B to obtain line $5(0) + 2(0) = 0 \le 80$. (0, 0). Satisfy the inequation. So, region containing the origin gives the solution set of inequation.

Region represented by $x + y \le 20$:

Line $x + y = 20$ meets the coordinate axes at C(20, 0) and D(0, 20).

x + y = 20

x	20	0
y	0	20

Join C and D to obtain line $x + y = 20$, (0, 0) satisfy the inequation. So, region containing the origin represent the solution set of inequation.

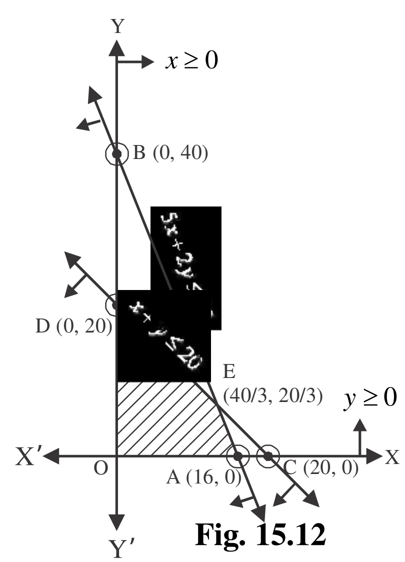

Fig. 15.12

Region represented by $x \geq 0$ and $y \geq 0$:

Since every point of first quadrant satisfy both the inequation. So, region represented by $x \geq 0$ and $y \geq 0$ is in first quadrant.

Shaded region OAED represents the common region of inequations. That is the feasible region. Vertices (corner points) of this region are 0(0, 0), A (16, 0), $E(40/3,\ 20/3)$ and D (0, 20) and E is intersection point of $x + y = 20$ and $5x + 2y = 80$.

So, value of objective function on these point given in table.

Points	*x*-co-ordinate	*y*-co-ordinate	Objective function Z = (.50) *x* – (.10) *y*
0	0	0	$Z_O = (\cdot 50)(0) - (\cdot 10)(0) = 0$
A	16	0	$Z_A (\cdot 50)(16) - (\cdot 10)(0) = 8$
E	40 / 3	20 / 3	$Z_E = (\cdot 50)(40/3) - (\cdot 10)(20/3) = 6$
D	0	20	$Z_D = (\cdot 50)(0) = (\cdot 10)(20) = -2$

It is clear from above table that objective function is maximum at corner point (16, 0). So, for maximum profit the purchase 16 new hens to get profit of ₹ 8.

Miscellaneous Exercise – 15

Solve the following Linear Programming Problems graphically:

1. Maximize $Z = 4x + y$

   constraints $x + y \leq 50$

   $3x + y \leq 90$

   and $x \geq 0,\ y \geq 0$

2. Maximize $Z = 3x + 2y$

   constraints $x + y \geq 8$

   $3x + 5y \leq 15$

   and $x \geq 0,\ y \leq 15$

3. Maximize and Minimize $Z = x + 2y$

   constraints $x + 2y \geq 100$

   $2x - y \leq 0$

   $2x + y \leq 200$

   and $x \geq 0,\ y \geq 0$

4. Maximize $Z = 3x + 2y$

   constraints $x + 2y \leq 10$

   $3x + y \leq 15$

   and $x \geq 0,\ y \geq 0$

5. Food for pateint must include a mixture of atleast 4000 units of vitamin 50 units mineral and 1400 units

calories. Two food products A and B are available at the cost of ₹ 3 and ₹ 4 per unit. Food product A constains 200 units of vitmain, 1 unit of mineral and 40 calories and food B contains 100 units of vitamin, 2 units mineral and 40 calories. What should be the mixture of food so that the cost is minimum.

6. A diet is to contain atleast 80 units of vitamin A and 100 units of minerals. Two foods F_1 and F_2 are available. Food F_1 costs ₹ 4 per unit food and F_2 costs ₹ 6 per unit. One unit of food F_1 contains 3 units of minerals. Formulate this as a linear programming problem. Find the minimum cost for diet that consist of mixture of these two foods and also meets the minimal nutritional requriements.

7. A furniture manufacturer makes table and chairs. These are made on two machines A and B. Machine A takes 2 hours and machine B takes 6 hours to make a chair, whereas machine A taskes 4 hours and machine B takes 2 hours to make a table. Machine A and B are used for 16 hours and 30 hours respectively. The manufacturer earns a profit of ₹ 3 and ₹ 5 on selling one chair and one table. Find the number of chairs and tables to be manufactured per day so as to get the maximum profit.

8. A firm manufactures two types of pills for headache size A and size B. Size A pill contains 2 grams aspirin, 5 grams bicorbonate and 1 gram sulphur whereas size B pill contains 1 gram aspirin, 8 grams bicorbonate and 66 grams sulphur. It is been found that for quick relief atleast 12 grams aspirin, 7.4 grams bicorbonate and 24 grams sulphur is required. For quick relief from pain what should be the minimum number of pill a patient should take.

9. A brick manufacturer has tow depots A and B with a storage capacity of 30,000 and 20,000 bricks. He takes the order from three builder P, Q and R of 15,000, 20,000 and 15,000 number of bricks. The cost of transportation to deliver 1000 bricks is given below in the table.

From/To	**P**	**Q**	**R**
A	40	20	30
B	20	60	40

keeping the transportation cost minimum how would the manufacturer send the bricks.

10. Constraints $x + y \le 3$

$y \le 6$

and $x,\ y \le 0$

The area bounded by the above inequalities

(A) unbounded in first quadrant

(B) unbounded in first and second quadrant

(C) bounded in first quadrant

(D) None of these

IMPORTANT POINTS

1. Linear programming is mathematical method which is used to distribute the limited resources in optimized manner in competitive activities, while all used variables have linear relationship.
2. Set of values of variable which satisfied the all constraint of LPP is called a solution LPP.
3. Solution of LPP which satisifed the non-negative constraint is feasible solution and set of all feasible solution is called feasible region.
4. A feasible solution which gives optimal solution of LPP is called optimal solution.
5. Graphical method is applicable in LPP when there is only two variable in problem.
6. Graphical method maninly depends upont he extreme point theorem which states that 'An optimal solution

of a LPP, if it exists, occurs at one of the extrem (corner) points of the convex polygon of the set of all feasible solutions'.

7. Following algorithm can be used to solve a LPP in two variables graphically by using corner point method:
   (i) Formulate the given LPP in mathematical form it is not given in mathematical form.
   (ii) Convert all inequations into equations and draw their graphs. To draw the graph of a linear equation, puty y =0 and obtain the point on x-axis. Simiarly by putting $x = 0$ obtain a point on y-axis. Join these points to obtain the graph of the equation.
   (iii) Determine the region represented by each inequation. To determine the region represented by an inequation replace x and y both by zero, if the inequation reduces to a valid statement, then the region containing the origin is the region represented by the given inequation. Otherwise, the region not containing the origin is the region represented by the given inequation.
   (iv) Obtain the region in xy- plane containing all points that simultaneosuly satisfy all constraint including non-negative restrictions. The polygonal region is so obtained is the feasible region and is known as the convex polygon of the set of all feasible solutions of LPP.
   (v) Determine the coordinates of the vertices (corner points) of the convex polygon obtained in step II. There vertices are known as extreme points of the set of all feasible solutions of LPP.
   (vi) Obtain the values of the objective functions to each of vertices of the convex polygon. The point where the objective function attains its optimum (maximum or minimum) value is the optimal solutions of given LPP.
8. If the feasible region of LPP is bounded, i.e., it is a convex polygon. Then, the objective function $Z = ax + by$ has both maximum value M and minimum value m and each of these values is the optimal solution of given LPP.
9. Sometimes the feasible region of a LPP is not a bounded convex polygon. That is, it extends indefinitely in any direction. In such cases, we say that the feasible region is unbounded. The above algorithm is applicable when the feasible region is bounded. If the feasible region is unbounded, then we find values of objective function $Z = ax + by$ at each corner points of feasible region. Let M and m respectively largest and smallest values of Z at these points. In order to check whether Z has maximum and minimum values at M and m respectively.

We proced as follows:

(i) Draw the line $ax + by > M$ and find the open half plane $ax + by > M$. if the open half plane represented by $ax + by > M$ has no point common with the unbounded feasible region, then M is maximum value of Z has no maximum value.

(ii) Draw the line $ax + by < m$ and find the open half plane represented by $ax + by < m$. If the half-plane $ax + by < m$ has no point common with the unbounded feasible region, then m is the minimum value of Z, otherwise Z has no minimum value.

Answers

Exericse 15.1

1. point (4, 0), minimum $Z = -12$
2. point (0, 4), maximum $Z = 16$
3. For the given constraints no minimum value exists.
4. point (3 / 2, 1 / 2), minimum $Z = 7$
5. point (5, 5) minimum $Z = 60$ and points (15, 15) and (0, 20), maximum z = 120
6. points (6, 0) and (0, 3), minimum $Z = 6$
7. point (60, 0), minimum $Z = 300$ points (120, 0) and (60, 30) maximum $Z = 600$

8. For the given constraints no maximum valeu exists.
9. For the given constraints no feasible solution exists.
10. For the given constraints no maximum value of objective function exists.

Exercise 15.2

1. Minimum $Z = 5x + 7y$

   constraints $2x + y \ge 8$

   $x + 2y \ge 10$

   $x \ge 0,\ y \ge 0$

   For food I, 2 kg and for food II. 4 kg mixture is requried whose minimum value is ₹ 38

2. Minimum $Z = 6x + 10y$

   constraints $x + 2y \ge 10$

   $2x + 2y \ge 12$

   $3x + y \ge 8$

   $x \ge 0,\ y \ge 0$

   For food I, 2 kg and for food II, 4 kg mixture is requried whose minimum valeu is ₹ 52.

3. 20, 10

4. Maximize $Z = 2.50x + y$

   constraints $x + 3y \le 12$

   $3x + y \le 12$

   $x \ge 0,\ y \ge 0$

   3 and 4 packets of nuts and bolts everday with a profit of ₹ 10.50

5. Maximize $Z = 22x + 18y$

   constraints $x + y \le 20$

   $360x + 240y \le 5760$

   $x \ge 0,\ y \ge 0$

   the dealer would buy 8 fans and 12 sweing macnine sto get the profit of ₹ 392

6. Maximize $Z = 0.7x + y$

   constraints $4x + 6y \le 240$

   $6x + 3y \le 240$

   and $x \ge 0,\ y \ge 0$

   the dealer would make 30 packets of bolts A and 20 packets of bolt B to get the maximum profit of ₹ 41.

7- Maximize $Z = 5x + 6y$

   constraints $5x + 8y \le 200$

   $10x + 8y \le 240$

   and $x \ge 0,\ y \ge 0$

   Firm should make 8 mementos of type A and s 20 memntos of type B to get the maximum profit of ₹ 160

8. Minimize $Z = (.60)x + (.40)y$

constraints $\frac{10x}{100} + \frac{5y}{100} \le 14$

$\frac{6x}{100} + \frac{10y}{100} \le 14$

and $x \ge 0,\ y \ge 0$

1000 kg of fertiliser F_1 and 80 kg of fertiliser F_2 should be used

minimum vlaue = ₹ 92

9. Maximize $Z = 4500x + 5000y$

Constraints $25000x + 40000y \le 7000000$

$x + y \le 250$

and $x \ge 0,\ y \ge 0$

The dealer will store 200 desktop and 50 portable computers so that to get the maximum profit of ₹ 1150000

10. Let depot A, provide x and y quintals of food to D and E

Minimize $Z = 6x + 3y + 5/2(100 - x - y) + 4(60 - x) + 2(50 - y) + 3(x + y - 60)$

Constraints $x + y \le 100$

$x \le 60$

$y \le 50$

$x + y \ge 60$

and $x \ge 0,\ y \ge 0$

10, 50 and 40 Quintals for depot A to D, E and F

50, 0 and 0 Quintals for depot B to D, E and F

Miscellaneous Exercise - 15

1. At point (30, 0) Maximum is $Z = 120$
2. There is no feasible solution
3. At points (0, 50) and (20, 40) Minimum $Z = 100$ at (0, 200) and maximum Z = 400
4. At point (4, 3), Maximum $Z = 18$
5. 5 units of food material A and 30 units of food material B
6. Let x and y denotes the units of food material F_1 and F_2

Minimize $Z = 4x + 6y$

constraints $3x + 6y \ge 80$

$4x + 3y \ge 100$

and $x \ge 0,\ y \ge 0$

Minimum value = ₹ 104

7. 22 / 5 Chairs and 9 / 5 tables, maximum profit = ₹ 22.2
8. 2 tablest of size A and 8 tablets of size B
9. From depot A, the builders P, Q and R should be supplied 0, 20000, 10000 number of bricks and from depot B, the builders P, Q and R should be supplied, 15000, 0, 5000 number of bricks

10- (C)

16

Probability and Probability Distribution

16.01 Introduction

We often make statements about probability. For example, a weather forecaster may predict that there is an 80% chance of rain tomorrow. A health news reporter may state that a smoker has a much greater chance of getting cancer than a non smoker does.

In earlier classes, we have studied the probability as a measure of uncertainty of an event in a random experiment. We have also established a relationship between the exiomatic theory and the classical theory of probability in case of equally likely outcomes. On the basis of this relationship, we obtain probabilities of events associated with discrete sample space. In this chapter, we shall discuss the important concept of conditional probability, multiplication rule of probality and independence of events, the Baye's theorem, random variable and its probability distribution, the mean and variance of a probability distribution.

16.02 Conditional Probability

If we have two events form the same sample space, Does the information about the occurance of one of the events affect the probability of the other event ? Let us try to answer this question by taking up a random experiment in which the outcomes are equally likely to occur. Consider the experiment of tossing two fair coins. The sample space of the experiment is

$$S=\{HH,\ HT,\ TH,\ TT\},\ \text{H = Head, T = Tail}$$

Since the coins are fair, we can assign the probability 1/4 to each sample point. Let A be the event at least one head appears and B be the event "first coin shows tail". Then

$$A=\{HT,\ TH,\ HH\},\quad B=\{TH,\ TT\}$$

$\therefore$ $$P(A)=P(\{HT\})+P(\{TH\})+P(\{HH\})$$

$$=(1/4)+(1/4)+(1/4)=3/4$$

and $$P(B)=P(\{TH\})+P(\{TT\})$$

$$=(1/4)+(1/4)=1/2$$

also $$A\cap B=\{TH\}$$

$\therefore$ $$P(A\cap B)=P(\{TH\})=1/4$$

Now, we have to find the probability of A, when event B has already occured with the information of occurence of B, we are sure that the case in which first coin does not result into a tail, should not be considered while finding the probability of A. This information reduces our sample space form the set S to its subset B for the event A

Thus sample point of event A which is favourable to event B is {TH}

Thus , Probability of A considering B as the sample space = 1/2

or, Probability of A given that the event B has occured = 1/2

This probability of the event A is called the *conditional probability of A given that B has already occured,* and is denoted by *P(A/B)*

i.e. $$P\left(\frac{A}{B}\right)=\frac{1}{2}$$

Thus, we can also write the conditional probability of A given that B has occured as $P(A/B)$

$$P\left(\frac{A}{B}\right)=\frac{\text{Number of elementary events favourable to } (A\cap B)}{\text{Number of elementary events which are favourable to } B}$$

$$=\frac{n(A\cap B)}{n(B)}$$

Dividing the numerator and the denominator by total number of elementary events of the sample space, we see that

$P(A/B)$ can also be written as

$$P\left(\frac{A}{B}\right)=\frac{\dfrac{n(A\cap B)}{n(S)}}{\dfrac{n(B)}{n(S)}}=\frac{P(A\cap B)}{P(B)}$$

note that it is valid only when $P(B)\neq 0$

Definition : If A and B are two events associated with the same sample space of a random experiment, the conditional probability of the event A given that B has occured is given by

$$P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)};\ P(B)\neq 0$$

Similarly the conditional probability of the event B given that A has occured is given by

$$P\left(\frac{B}{A}\right)=\frac{P(A\cap B)}{P(A)};\ P(A)\neq 0$$

16.03 Properties of conditional probability

Let A and B be events of a sample space S of an experiment, then we have

(i) $$P\left(\frac{S}{B}\right)=P\left(\frac{B}{B}\right)=1$$

We know that, $$P\left(\frac{S}{B}\right)=\frac{P(S\cap B)}{P(B)}=\frac{P(B)}{P(B)}=1$$

again $$P\left(\frac{B}{B}\right)=\frac{P(B\cap B)}{P(B)}=\frac{P(B)}{P(B)}=1$$

thus $$P\left(\frac{S}{B}\right)=P\left(\frac{B}{B}\right)=1$$

(ii) $$P\left(\frac{\bar{A}}{B}\right)=1-P\left(\frac{A}{B}\right)$$

using property (i) $$P\left(\frac{S}{B}\right)=1$$

$\Rightarrow$ $$P\left(\frac{A\cup\bar{A}}{B}\right)=1 \qquad \left[\because S=A\cup\bar{A}\right]$$

$\Rightarrow$ $$P\left(\frac{A}{B}\right)+P\left(\frac{\bar{A}}{B}\right)=1 \qquad [\because A \text{ and } \bar{A} \text{ are disjoint events}]$$

$\therefore$ $$P\left(\frac{\bar{A}}{B}\right)=1-P\left(\frac{A}{B}\right).$$

(iii) *If* A *and* B *are any two events of a sample space* S *and* F *is an event of* S *such that* $P(F)\neq 0$ then

(a) $$P\left(\frac{A\cup B}{F}\right)=P\left(\frac{A}{F}\right)+P\left(\frac{B}{F}\right)-P\left(\frac{A\cap B}{F}\right)$$

In particular, if A *and* B *are disjoint events, then*

(b) $$P\left(\frac{A\cup B}{F}\right)=P\left(\frac{A}{F}\right)+P\left(\frac{B}{F}\right)$$

We have

$$P\left(\frac{A\cup B}{F}\right)=\frac{P\left[(A\cup B)\cap F\right]}{P(F)}$$

$$=\frac{P\left[(A\cap F)\cup(B\cap F)\right]}{P(F)}$$

$$=\frac{P(A\cap F)+P(B\cap F)-P(A\cap B\cap F)}{P(F)}$$

$$=\frac{P(A\cap F)}{P(F)}+\frac{P(B\cap F)}{P(F)}-\frac{P\left[(A\cap B)\cap F\right]}{P(F)}$$

$$=P\left(\frac{A}{F}\right)+P\left(\frac{B}{F}\right)-P\left(\frac{A\cap B}{F}\right).$$

Special Condition : When A and B are disjoint events, then $P\left(\frac{A\cap B}{F}\right)=0$

$\therefore$ $$P\left(\frac{A\cup B}{F}\right)=P\left(\frac{A}{F}\right)+P\left(\frac{B}{F}\right).$$

Illustrative Examples

Example 1. If $P(A)=6/11,\ P(B)=5/11$ and $P(A\cup B)=7/11$ then find

(i) $P(A\cap B)$ (ii) $P(A/B)$ (iii) $P(B/A)$

Solution : (i) We know that $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$\Rightarrow$ $$P(A\cap B)=P(A)+P(B)-P(A\cup B)$$

$$=\frac{6}{11}+\frac{5}{11}-\frac{7}{11}=\frac{4}{11}$$

(ii) $$P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)}=\frac{4/11}{5/11}=\frac{4}{5}$$

(iii) $$P\left(\frac{B}{A}\right)=\frac{P(A\cap B)}{P(A)}=\frac{4/11}{6/11}=\frac{2}{3}$$

Example 2. An instructor has a question bank consisting of 300 easy true / false questions, 200 difficult true/ false questions, 500 easy multiple choice questions. If a question is selected at random from the question bank, what is the probabiliy that it will be an easy question given that it is a multiple choice question ?

Solution : Let event A 'it is an easy question' and event B 'It is a multiple choice question' and we have to find $P(A/B)$

$$n(A)=300+500=800,\ n(B)=500+400=900$$

Here set $A\cap B$ denotes 'it is an easy multiple choice question'

$\therefore$ $$n(A\cap B)=500$$

required probability $$=P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)}$$

$$=\frac{n(A\cap B)}{n(B)}=\frac{500}{900}=\frac{5}{9}.$$

Example 3. Determine $P(A/B)$ in each case when a coin is tossed three times, where

(i) A : head on third toss B : heads on first two tosses

(ii) A : at least two heads, B : at most two heads

(iii) A : at most two tails, B : at least one tail

Solution : The sample space when a coin is tossed three times is as follows

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

(i) $A = \{HHH, HTH, THH, TTH\}$, $B = \{HHH, HHT\}$

then $\qquad A \cap B = \{HHH\}$

$\Rightarrow \qquad n(A) = 4,\ n(B) = 2,\ n(A \cap B) = 1$

$\therefore \qquad P\left(\frac{A}{B}\right) = \frac{n(A \cap B)}{n(B)} = \frac{1}{2}.$

(ii) $A = \{HHH, HHT, HTH, THH\}$, $B = \{HHT, HTH, THH, HTT, THT, TTH, TTT\}$

$\therefore \qquad A \cap B = \{HHT, HTH, THH\}$

$\Rightarrow \qquad n(A) = 4,\ n(B) = 7,\ n(A \cap B) = 3$

$\therefore \qquad P\left(\frac{A}{B}\right) = \frac{n(A \cap B)}{n(B)} = \frac{3}{7}.$

(iii) $A = \{HHH, HHT, HTH, THH, HTT, THT, TTH\}$,

$B = \{HHT, HTH, THH, HTT, THT, TTH, TTT\}$

$\therefore \qquad A \cap B = \{HHT, HTH, THH, HTT, THT, TTH\}$

$\Rightarrow \qquad n(A) = 7,\ n(B) = 7,\ n(A \cap B) = 6$

$\therefore \qquad P\left(\frac{A}{B}\right) = \frac{n(A \cap B)}{n(B)} = \frac{6}{7}.$

Example 4. A black and a red die are thrwon, then

(a) Find the conditional probability of obtaining a sum greater than 9, giving that the black die resulted in a 5.

(b) Find the conditional probabilility of obtaining the sum 8, given that the red die resulted in a number less than 4.

Solution : (*i*) Let event A denotes 'sum greater then 9' and event B denotes 'black dice resulted in a 5' now we have to find $P(A/B)$

$$A = \{(5, 5), (6, 4), (4, 6), (6, 5), (5, 6), (6, 6)\}, \quad B = \{(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\}$$

$$A \cap B = \{(5, 5), (5, 6)\}$$

$\Rightarrow \qquad n(A) = 6,\ n(B) = 6,\ n(A \cap B) = 2$

thus required probability $\qquad = P\left(\frac{A}{B}\right) = \frac{n(A \cap B)}{n(B)} = \frac{2}{6} = \frac{1}{3}.$

(ii) Let event A denotes 'sum greater than 9' and event B denotes' red die resulted in a number less than 4' now we have to find $P(A/B)$

then $A=\{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$

$$B=\{(6, 1), (6, 2), (6, 3), (5, 1), (5, 2), (5, 3), (4, 1), (4, 2), (4, 3), (3, 1), (3, 2), (3, 3), (2, 1), (2, 2), (2, 3), (1, 1), (1, 2), (1, 3)\}$$

$$A \cap B=\{(6, 2), (5, 3)\}$$

$$\Rightarrow \quad n(A)=5,\ n(B)=18,\ n(A \cap B)=2$$

thus required probability $=P\left(\frac{A}{B}\right)=\frac{n(A \cap B)}{n(B)}=\frac{2}{18}=\frac{1}{9}.$

Example 5. A die is thrown three times, then event A and B defined as follows :

A : 4 appears on the third throw,

B : 6 and 5 appears respectively on first two tosses

determine $P(A/B)$.

Solution : When a coin is tossed three times the sample space S contains $=6 \times 6 \times 6=216$ equally likely outcomes.

then $A=\{(1, 1, 4), (1, 2, 4), (1, 3, 4), (1, 4, 4), (1, 5, 4), (1, 6, 4)$

$(2, 1, 4), (2, 2, 4), (2, 3, 4), (2, 4, 4), (2, 5, 4), (2, 6, 4)$

$(3, 1, 4), (3, 2, 4), (3, 3, 4), (3, 4, 4), (3, 5, 4), (3, 6, 4)$

$(4, 1, 4), (4, 2, 4), (4, 3, 4), (4, 4, 4), (4, 5, 4), (4, 6, 4)$

$(5, 1, 4), (5, 2, 4), (5, 3, 4), (5, 4, 4), (5, 5, 4), (5, 6, 4)$

$(6, 1, 4), (6, 2, 4), (6, 3, 4), (6, 4, 4), (6, 5, 4), (6, 6, 4)\}$

$$B=\{(6, 5, 1), (6, 5, 2), (6, 5, 3), (6, 5, 4), (6, 5, 5), (6, 5, 6)\}$$

$$A \cap B=\{(6, 5, 4)\}$$

$$\Rightarrow \quad n(A)=36,\ n(B)=6,\ n(A \cap B)=1$$

Thus required probability $=P\left(\frac{A}{B}\right)=\frac{n(A \cap B)}{n(B)}=\frac{1}{6}.$

Example 6. Consider the experiment of throwing a die, if a multiple of 3 or 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event' the coin shows a tail', given that' at least one die shows a 3'.

Solution : The results of the experiments can be shown as

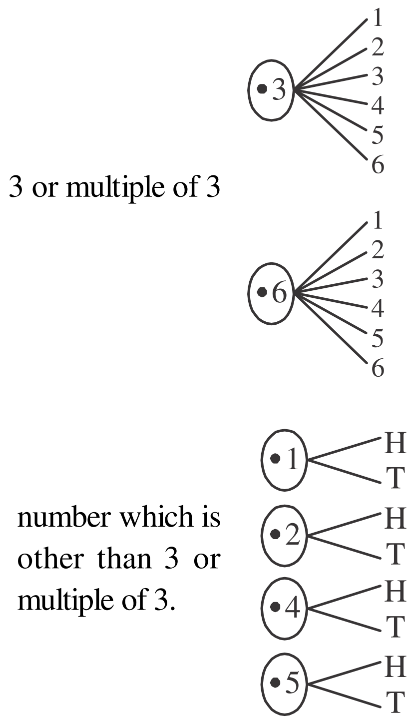

Fig. 16.01

The sample space is as follows

$$S = \{(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (6, 1), (6, 2), (6, 3), (6, 4)$$
$$(6, 5), (6, 6), (1, H), (1, T), (2, H), (2, T), (4, H), (4, T), (5, H), (5, T)\}$$

Let event A denotes ' tail on the coin' and event B denotes 'at least one die show a 3'.

then $A = \{(1, T), (2, T), (4, T), (5, T)\}$; $B = \{(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (6, 3)\}$

$$A \cap B = \phi$$

$\Rightarrow$ $n(A) = 4,\ n(B) = 7,\ n(A \cap B) = \phi$

Required probability $= P\left(\dfrac{A}{B}\right) = \dfrac{n(A \cap B)}{n(B)} = \dfrac{0}{7} = 0$

Exercise 16.1

1. If $P(A) = 7/13$, $P(B) = 9/13$ and $P(A \cap B) = 4/13$ then find $P(A/B)$.
2. If $P(B) = 0.5$ and $P(A \cap B) = 0.32$ then find $P(A/B)$.
3. If $2P(A) = P(B) = 5/13$ and $P\left(\dfrac{A}{B}\right) = \dfrac{2}{5}$ then find $P(A \cup B)$.

4. If $P(A)=0.6,\ P(B)=0.3$ and $P(A\cap B)=0.2$ then find $P\left(\frac{A}{B}\right)$ and $P\left(\frac{B}{A}\right)$.

5. If $P(A)=0.8,\ P(B)=0.5$ and $P\left(\frac{B}{A}\right)=0.4$ then find that

(i) $P(A\cap B)$ (ii) $P\left(\frac{A}{B}\right)$ (iii) $P(A\cup B)$

6. Assune that each born child equally likely to be a boy or a girl. If a family has two children, it is given that if atleast one of them is a boy than find the probability that both the children to be a boy.

7. Two coins are tossed once then find $P(A/B)$

(i) A : tail appear on one coin, B : one coin shows head

(ii) A : no tail appears, B : no head appears

8. Mother, father and son line up at random for a family picture. If A and B are two event as follows then find $P(A/B)$

A : son on one end

B : father in middle

9. A fair die is rolled. Consider events $A=\{1, 3, 5\}$] $B=\{2, 3\}$ and $C=\{2, 3, 4, 5\}$ then find

(i) $P\left(\frac{A}{B}\right)$ and $P\left(\frac{B}{A}\right)$ (ii) $P\left(\frac{A}{C}\right)$ and $P\left(\frac{C}{A}\right)$ (iii) $P\left(\frac{A\cup B}{C}\right)$ and $P\left(\frac{A\cap B}{C}\right)$

10. Given that the two numbers apperaring on throwing two dice are different. Find the probability of the event 'the sum of numbers on the dice is 4'.

11. Ten cards numbered 1 to 10 are placed in box, mixed up throughly and then one card is drawn randomly. If it is known that the number on the drawn card is more than 3, what is the probability that it is an even number ?

12. In a school, there are 1000 students, out ot which 430 are girls. It is known that 10% girls out of 430 study in class XII. What is the probability that a student chosen randomly studies in class XII if given that the chosen student is a girl ?

13. A die is thrown twice and the sum of the numbers appearing is observed to be 6. What is the conditional probability that the number 4 has appeared at least once ?

14. Consider the experiment of tossing a coin. If the coin shows head, toss it again but if it shows tail, then throw a die. Find the conditional probability of the event that 'the die shows a number greater than 4' if given that 'there is at least one tail'.

16.04 Multiplication theorem on probability

Let A and B be two events associated with a sample space S. Clearly, the set A denotes the event that both A and B have occured. In other words $A\cap B$ denotes the simultaneous occurence of the events A and B. The event $A\cap B$ is also written as AB

We know that the conditional probability of event A given that B has occurred is

$$P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)};\ P(B)\neq 0$$

$\Rightarrow$ $$P(A\cap B)=P(B)P\left(\frac{A}{B}\right) \quad \text{(i)}$$

again $$P\left(\frac{B}{A}\right)=\frac{P(B\cap A)}{P(A)};\ P(A)\neq 0$$

or $$P\left(\frac{B}{A}\right)=\frac{P(A\cap B)}{P(A)} \quad [\because\ B\cap A=A\cap B]$$

$\therefore$ $$P(A\cap B)=P(A)P\left(\frac{B}{A}\right) \quad \text{(ii)}$$

from (i) and (ii) $$P(A\cap B)=P(A)P\left(\frac{B}{A}\right)=P(B)P\left(\frac{A}{B}\right), \text{where } P(A)\neq 0 \text{ and } P(B)\neq 0$$

The above result is known as the *multiplication rule of probability.*

Note : Let A, B and C be any events of sample space then

$$P(A\cap B\cap C)=P(A)P\frac{B}{A}P\frac{C}{A\cap B}$$

$$=P(A)P\left(\frac{B}{A}\right)P\left(\frac{C}{AB}\right)$$

Thus the above expression denotes the multiplication rule of probability for more than two events

Illustrative Examples

Example 7. An urn contains 10 white and 15 black balls. Two balls are drawn from the urn one after the other without replacement. What is the probability that first ball is white and second is black.

Solution : Let A and B denote respectively the events that ball drawn is white and second ball drawn is black then we have to find $(A\cap B)$

Now $P(A)=P$ (white ball in first draw) $=\frac{^{10}C_1}{^{25}C_1}=\frac{10}{25}$

Also given that the first ball drawn is white, i.e. event A has occured, now there are 9 white balls and fifteen black balls left in the urn. Therefore, the probability that the second ball drawn is black, given that the ball in the first draw is white, is nothing but the conditional probability of B given that A has occured. i.e.

$\therefore$ $$P\left(\frac{B}{A}\right)=\frac{^{15}C_1}{^{24}C_1}=\frac{15}{24}$$

By multiplication rule of probability, we have

$$P(A\cap B)=P(A)P\left(\frac{B}{A}\right)=\frac{10}{25}\times\frac{15}{24}=\frac{1}{4}.$$

Example 8. Three cards are drawn successively, without replacement form a pack of 52 well shuffled cards. What is the probability that first two cards are kings and the third card drawn is a queen ?

Solution : Let K denotes the event that the card drawn is king and Q be the event that the card drawn is a queen.

Clearly, we have to find $P(KKQ)$

Now $P(K) = P$ (card drawn is a king) $= 4/52$

Now there are three kings in (52-1) = 51 cards.

$\therefore \quad P\left(\frac{K}{K}\right) = P$ (the probability of second king with the condition that one king has already been drawn) $= \frac{3}{51}$

Now there are four queens left in 50 cards.

$\therefore \quad P\left(\frac{Q}{KK}\right) = P$ (the probability of third drawn card to be a queen, with the condition that two kings have already been drawn $= \frac{4}{50}$

By multiplication rule of probability, we have

$$P(KKQ) = P(K)P\left(\frac{K}{K}\right)P\left(\frac{Q}{KK}\right)$$

$$= \frac{4}{52} \times \frac{3}{51} \times \frac{4}{50} = \frac{2}{5525}.$$

16.05 Independent Events

If A and B are two events such that the probability of occurence of one of them is not affected by occurence of the other. Such events are called *independent events*

Two events A and B are said to be independent, if

$$P\left(\frac{A}{B}\right) = P(A) \quad \text{when} \quad P(B) \neq 0$$

and

$$P\left(\frac{B}{A}\right) = P(B) \quad \text{when} \quad P(A) \neq 0$$

Now, by the multiplication rule of probability, we have

$$P(A \cap B) = P(A)P\left(\frac{B}{A}\right)$$

If A and B are independent, then

$$P(A \cap B) = P(A)P(B)$$

Note : Three events A, B and C are said to be mutually independent, if

$$P(A \cap B) = P(A)P(B)$$

$$P(B \cap C) = P(B)P(C)$$

$$P(A \cap C) = P(A)P(C)$$

and $$P(A \cap B \cap C) = P(A)P(B)P(C)$$

If at least one of the above is not true for three given events, we say that the events are not independent

Example : An unbaised die is thrown twice. Let the event A be 'odd number on the first throw' and B the event' odd number on the second throw'. Check the independence of the events A and B.

The sample space in tossing two coins is

$$S = \begin{Bmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{Bmatrix}$$

$\Rightarrow$ $$n(S) = 36$$

Also getting an odd number on the first throw we have

$$n(A) = 18$$

$$P(A) = \frac{18}{36} = \frac{1}{2}$$

similarly $$P(B) = \frac{18}{36} = \frac{1}{2}$$

and $$P(A \cap B) = P(\text{getting odd number on both the throws}) = \frac{9}{36} = \frac{1}{4}$$

{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5)} [be the sample points w.r. to event A and event B.]

clearly $$P(A \cap B) = 1/4 = 1/2 \times 1/2 = P(A)P(B)$$

Thus A and B are independent events.

Illustrative Examples

Example 9. Events A and B are such that $P(A) = 1/2,\ P(B) = 7/12$ and P(A – not or B – not) $= 1/4$ then are A and B independent events ?

Solution : Given $$P(A) = \frac{1}{2},\ P(B) = \frac{7}{12},\ P(\bar{A} \cup \bar{B}) = \frac{1}{4}$$

$$P(\bar{A} \cup \bar{B}) = \frac{1}{4}$$

$\Rightarrow$ $$P(\overline{A \cap B}) = \frac{1}{4} \qquad \left[\because P(\bar{A} \cup \bar{B}) = P(\overline{A \cap B})\right]$$

$\Rightarrow$ $$1 - P(A \cap B) = \frac{1}{4} \qquad \left[\because P(\overline{A \cap B}) = 1 - P(A \cap B)\right]$$

$\Rightarrow$ $$P(A \cap B) = 1 - 1/4 = 3/4$$

also $$P(A)P(B) = 1/2 \times 7/12 = 7/24$$

$\because$ $$P(A \cap B) \neq P(A)P(B)$$

therefore A and B are not independent events.

Example 10. A fair coin and an unbaised die are tossed. Let A be the event 'head appears on the coin' and B be the event '3 on the die'. Check whether A and B are independent events or not.

Solution : The sample space related to the experiment is -

$$S = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$$

and $A = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6)\}$, $B = \{(H, 3), (T, 3)\}$

$\therefore$ $$A \cap B = \{(H, 3)\}$$

$$P(A) = \frac{6}{12} = \frac{1}{2}, \quad P(B) = \frac{2}{12} = \frac{1}{6}, \quad P(A \cap B) = \frac{1}{12}$$

clearly $$P(A \cap B) = P(A)P(B) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

Therefore A and B are independent events.

Example 11. A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event, 'the numeber is even,' and B be the event,'the number is red'. Are A and B independent ?

Solution : The sample space in rolling a die once is $= \{1, 2, 3, 4, 5, 6\}$

then $A = \{2, 4, 6\}$, $B = \{1, 2, 3\}$ also $A \cap B = \{2\}$

$$P(A) = \frac{3}{6} = \frac{1}{2}, \qquad P(B) = \frac{3}{6} = \frac{1}{2}, \qquad P(A \cap B) = \frac{1}{6}$$

clearly $$P(A \cap B) = \frac{1}{6} \neq P(A).P(B).$$

Therefore A and B are not independent events.

Example 12. A die is thrown. If A is the event 'the number apperaring is a multiple of 3' and B be the event ' the number appearing is even' then find whether A and B are independent ?

Solution : The sample space in rolling a die once is $= \{1, 2, 3, 4, 5, 6\}$

then $A = \{3, 6\}$, $B = \{2, 4, 6\}$ and $A \cap B = \{6\}$

$$P(A) = \frac{2}{6} = \frac{1}{3}, \quad P(B) = \frac{3}{6} = \frac{1}{2}, \quad P(A \cap B) = \frac{1}{6}$$

clearly $$P(A \cap B) = \frac{1}{6} = \frac{1}{2} \times \frac{1}{3} = P(A)P(B)$$

Thus events A and B are independent events.

Example 13. Events A and B are such that $P(A)=1/2,\ P(A\cup B)=3/5$ and $P(B)=r$ then find r if ,

(i) the events are mutually exculsively

(ii) the events are independent

Solution : (i) If events A and B are mutually exclusively then

$$P(A\cup B)=P(A)+P(B)$$

$$3/5=1/2+r \quad\Rightarrow\quad r=1/10$$

(ii) If events A and B are independent events

$$P(A\cap B)=P(A)P(B)=(1/2)r$$

Given $\qquad P(A\cup B)=3/5$

$$\Rightarrow \qquad P(A)+P(B)-P(A\cap B)=3/5$$

$$\Rightarrow \qquad 1/2+r-P(A\cap B)=3/5$$

$$\Rightarrow \qquad 1/2+r-(1/2)r=3/5$$

$$\Rightarrow \qquad 1/2+(1/2)r=3/5$$

$$\Rightarrow \qquad r/2=3/5-1/2$$

$$\Rightarrow \qquad r=1/5$$

Example 14. Three coins are tossed simultaneously. Consider the event A 'three heads or three tails', B' at least two heads' and C' at most two heads'. Of the pairs (A,B), (A,C) and (B,C), which are independent ? which are dependent ?

Solution : The sample space of tossing three coins is

$$S=\{HHH,\ HHT,\ HTH,\ THH,\ HTT,\ THT,\ TTH,\ TTT\}$$

then $\qquad A=\{HHH,\ TTT\},\quad B=\{HHT,\ HTH,\ THH,\ HHH\}$

and $\qquad C=\{TTT,\ TTH,\ THT,\ HTT,\ THH,\ HTH,\ HHT\}$

Also

$$A\cap B=\{HHH\},\quad A\cap C=\{TTT\}\ \text{ and }\ B\cap C=\{HHT,\ HTH,\ THH\}$$

$$P(A)=2/8=1/4,\qquad P(B)=4/8=1/2,\qquad P(C)=7/8$$

$$P(A\cap B)=1/8,\qquad P(A\cap C)=1/8,\quad P(B\cap C)=3/8$$

Clearly $P(A\cap B)=P(A)P(B)=1/4\times 1/2=1/8$

similarly $\qquad P(A\cap C)\neq P(A)P(C)$

and $\qquad P(B\cap C)\neq P(B)P(C)$

Thus A and B are independent events and A and C, and B and C are dependent.

Example 15. If in a random experiment A and B are independent events then prove that

(i) $\overline{A}$ and B are dependent events.

(ii) A and $\overline{B}$ are independent events

(iii) $\overline{A}$ and $\overline{B}$ are also independent events

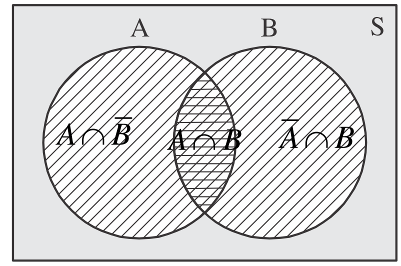

Fig. 16.02

Solution : Given A and B are independent events

$$P(A\cap B)=P(A)P(B)$$

It is clear form the venn diagram that $A\cap B$ and $\overline{A}\cap B$ are mutually exclusive such that

$$(A\cap B)\cup(\overline{A}\cap B)=B$$

By addition theroem of Probability

$$P(B)=(A\cap B)+P(\overline{A}\cap B)$$

$$\Rightarrow \quad P(\overline{A}\cap B)=P(B)-P(A\cap B)$$

$$=P(B)-P(A)P(B) \qquad [\because P(A\cap B)=P(A)P(B)]$$

$$=P(B)[1-P(A)]$$

$$=P(B)P(\overline{A})$$

$$=P(\overline{A})P(B)$$

Therefore $\overline{A}$ and B are independent events.

(ii) It is clear form the venn diagram that $A\cap B$ and $A\cap\overline{B}$ are mutually exclusive events such that

$$(A\cap B)\cup(A\cap\overline{B})=A$$

By addition theroem of probability

$$P(A)=P(A\cap B)+P(A\cap\overline{B})$$

$$\Rightarrow \quad P(A\cap\overline{B})=P(A)-P(A\cap B)$$

$$=P(A)-P(A)P(B)$$

$$=P(A)[1-P(B)]$$

$$=P(A)P(\overline{B})$$

Therefore A and $\overline{B}$ are independent events

(iii) $$P(\overline{A}\cap\overline{B})=P(\overline{A\cup B})$$

$$=1-P(A\cup B)$$

$$P(\overline{A}\cap\overline{B})=1-[P(A)+P(B)-P(A\cap B)]$$
$$=1-[P(A)+P(B)-P(A)P(B)]$$
$$=1-P(A)-P(B)+P(A)P(B)$$
$$=[1-P(A)]-P(B)[1-P(A)]$$
$$=[1-P(A)][1-P(B)]$$
$$=P(\overline{A})P(\overline{B})$$

Therefoere $\overline{A}$ and $\overline{B}$ are independent events

Example 16. If A and B are two independent events, then find the probability of occurence of at least one of A and B.

Solution : P (at least one of A and B) $=P(A\cup B)$

$$=P(A)+P(B)-P(A\cap B)$$
$$=P(A)+P(B)-P(A)P(B)$$ [∵ Events A and B are indepndent]
$$=P(A)+P(B)[1-P(A)]$$
$$=P(A)+P(B)P(\overline{A}) \qquad [\because P(A)+P(\overline{A})=1]$$
$$=1-P(\overline{A})+P(B)P(\overline{A})$$
$$=1-P(\overline{A})[1-P(B)]$$
$$=1-P(\overline{A})P(\overline{B})$$

Exercise 16.2

1. If A and B are two events such that $P(A)=1/4$, $P(B)=1/2$ and $P(A\cap B)=1/8$, then find $P(\overline{A}\cap\overline{B})$

2. If $P(A)=0.4$, $P(B)=p$ and $P(A\cup B)=0.6$ and A and B are independent events then find the value of p.

3. If A and B are independent events and $P(A)=0.3$ and $P(B)=0.4$ then find

   (i) $P(A\cap B)$ (ii) $P(A\cup B)$ (iii) $P\left(\frac{A}{B}\right)$ (iv) $P\left(\frac{B}{A}\right)$

4. If A and B are independent events and $P(A)=0.3,\ P(B)=0.6$ then find

(i) $P(A\cap B)$ (ii) $P(A\cap\overline{B})$ (iii) $P(A\cup B)$ (iv) $P(\overline{A}\cap\overline{B})$

5. A bag contains 5 white, 7 Red and 8 black balls. If four balls are drawn without replacement then find the probability that all are white.
6. If a coin is tossed thrice then find the probability of getting an odd number atleast once.
7. Two cards are drawn without replacement form a well shuffled pack of 52 cards Find the probability that both are black.
8. Two coins are tossed. Find the probability of getting two heads when it is known that one Head has already occured.
9. In a hostel, 60% of the students read Hindi newspaper, 40% read English newspaper and 20% read both Hindi and English newspapers. A student is selected at random.
   (i) Find the probability that she reads neither Hindi nor English newspapers.
   (ii) If she reads Hindi newspaper, find the probability that she reads English newspaper.
   (iii) If she reads English newspaper, find the probability that she reads Hindi newspaper.
10. A, solves 90% of the problems of the book and B, solves 70 % of the problems of the same book. If a question is taken at random then find the probability that at least one of them solve the question.
11. Three students are given the mathematical question to solve. Probability of solving the problem by the three are 1/2, 1/3 and 1/4. What is the probability that the question will solved ?
12. A bag contains 5 white and 3 black balls. Four balls are drawn one by one without replacements. Find the probability that the balls are of different colors.
13. Probability of solving specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that
   (i) the problem is solved
   (ii) exacty one of them solves the problem

16.06 Partition of a Sample Space

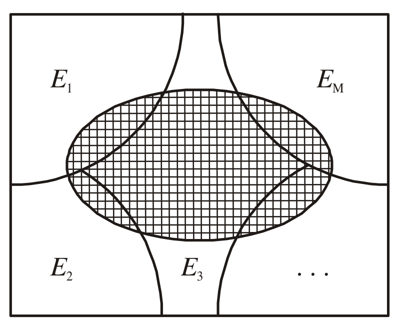

Fig. 16.03

A set of events $E_1, E_2, ..., E_n$ is said to represent a partition of the sample space S if

(i) $E_i \cap E_j = \phi,\ i = j,\ i,\ j = 1, 2, 3, ..., n$

(ii) $E_1 \cup E_2 \cup E_3 \cup ... \cup E_n = S$ and

(iii) $P(E_i) > 0,$ for all $i = 1, 2, ..., n$

In other words , the events $E_1, E_2, ..., E_n$ represent a partition of the sample space S if they are pairwise disjoint, exhaustive and have non zero probabilities.

Example : As an example, we see that any non empty event E and its complement E form a partition of the sample space S since they satisfy

$$E \cap E' = \phi \quad \text{and} \quad E \cup E' = S.$$

16.07 Theorem on Total Probability

Statement : Let $\{E_1, E_2, ..., E_n\}$ be a partition of the sample space S, and suppose that each of the events $E_1, E_2, ..., E_n$ has non zero probability of occurence. Let A be any event associated with S, then

$$P(A) = P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right) + ... + P(E_n)P\left(\frac{A}{E_n}\right)$$

$$= \sum_{j=1}^{n} P(E_j)P\left(\frac{A}{E_j}\right)$$

Statement : Given that $E_1, E_2, ..., E_n$ is a partition of the sample space S. Therefore

$\therefore$ $\quad S = E_1 \cup E_2 \cup ... \cup E_n$ $\quad$ (1)

and $\quad E_i \cap E_j = \phi \;\; \forall\; i \neq j,\; i,\; j = 1, 2, ..., n$

for any event A

$$A = A \cap S$$

$$= A \cap (E_1 \cup E_2 \cup ... \cup E_n)$$

$$= (A \cap E_1) \cup (A \cap E_2) \cup ... \cup (A \cap E_n)$$

$\because$ $\quad A \cap E_i$ and $A \cap E_j$ are the subsets of set E_i and E_j which are also disjoin for $i \neq j$

$\therefore$ $\quad$ for $i \neq j,\; i,\; j = 1,\; 2, ..., n$, $A \cap E_i$ and $A \cap E_j$ are also disjoint.

$\therefore$ $\quad P(A) = P\left[(A \cap E_1) \cup (A \cap E_2) \cup ... \cup (A \cap E_n)\right]$

$$= P(A \cap E_1) + P(A \cap E_2) + ... + P(A \cap E_n)$$

now $\quad P(A \cap E_i) = P(E_i)P\left(\frac{A}{E_i}\right), \qquad \left[\because\; P(E_i) \neq 0 \;\; \forall\;\; i = 1,\; 2, ..., n\right]$

Now, by multiplication rule of probability, we have

$$P(A) = P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right) + ... + P(E_n)P\left(\frac{A}{E_n}\right)$$

$\Rightarrow$ $$P(A) = \sum_{j=1}^{n} P(E_j)P\left(\frac{A}{E_j}\right).$$

Illustrative Examples

Example 17. In a class two- third of the students are boys and remainig are girls. Probability of a girl securing first division is 0.25 whereas probability of a boy securing first division is 0.28. A student is selected at random, find the probability that he or she gets first division.

Solution : Let event E_1 denotes ' a boy is selected' and event E_2 deontes ' a girl is selected' and let event A represent ' a student gets first divisions '.

then $$P(E_1)=2/3, P(E_2)=1/3$$

and $$P\left(\frac{A}{E_1}\right)=0.28, \quad P\left(\frac{A}{E_2}\right)=0.25$$

using theroem of total probability

$$P(A)=P(E_1)P\left(\frac{A}{E_1}\right)+P(E_2)P\left(\frac{A}{E_2}\right)=\frac{2}{3}\times 0.28+\frac{1}{3}\times 0.25=0.27$$

16.08 Baye's Theorem

Famous mathematician, John Baye's solved the problem of finding inverse probability by using conditional probability. The formula developed by him is known as '*Baye's theroem'* which was published posthumously in 1763

Statement : If $E_1, E_2, ..., E_n$ are n non empty events which constitute a partition of sample space S, ie., $E_1, E_2, ..., E_n$, are pairwise disjoint and $E_1 \cup E_2 \cup, ..., \cup E_n = S$ and A is any event of non zero probability, then

$$P\left(\frac{E_i}{A}\right)=\frac{P(E_i)P\left(\frac{A}{E_i}\right)}{\sum_{j=1}^{n} P(E_j)P\left(\frac{A}{E_j}\right)}, \; i=1, 2, 3, ..., n$$

Proof : By formula of conditional probability, we know that

$$P\left(\frac{E_i}{A}\right)=\frac{P(A\cap E_j)}{P(A)}=\frac{P(E_i)P\left(\frac{A}{E_i}\right)}{P(A)}$$ (by multiplication rule of probability)

$$=\frac{P(E_i)P\left(\frac{A}{E_i}\right)}{\sum_{j=1}^{n} P(E_j)P\left(\frac{A}{E_j}\right)}$$ (by the result of theorem of total probability)

Illustrative Examples

Example 18. In a factory which manufactures bolts, machines A, B and C maufacture respectively 25%, 35%, and 40% of the bolts. Of their outputs 5, 4, and 2 percent are respectively defective bolts. A bolt is drawn at random form the product and is found to be defective. What is the probability that it is manufactured by the machine B ?

Solution : Let events B_1, B_2 and B_3 be the following :

B_1 : the bolt is manufactured by machine A

B_2 : the bolt is manufactured by machine B

B_3 : the bolt is manufactured by machine C

Clearly B_1, B_2, B_3 are mutually exclusive and exhaustive events and hence, they represent a partition of the sample space. Let the event E be 'the bolt is defective'.

The event E occurs with B_1 or with B_2 or with B_3. Given that

$$P(B_1) = 25\% = \frac{25}{100} = 0\cdot25$$

$$P(B_2) = 35\% = \frac{35}{100} = 0\cdot35$$

and $$P(B_3) = 4\% = \frac{40}{100} = 0\cdot40$$

Again $P\left(\frac{E}{B_1}\right) =$ Probability that the bolt drawn is defective given that it is manufactured by machine A

= 5% = 0.05

Similarly,

$$P\left(\frac{E}{B_2}\right) = 0\cdot04,\ P\left(\frac{E}{B_3}\right) = 0\cdot02$$

Hence, by Baye's Theorem, we have

$$P\left(\frac{B_2}{E}\right) = \frac{P(B_2)P\left(\frac{E}{B_2}\right)}{P(B_1)P\left(\frac{E}{B_1}\right) + P(B_2)P\left(\frac{E}{B_2}\right) + P(E_3)P\left(\frac{E}{B_3}\right)}$$

$$= \frac{0\cdot35\times0\cdot04}{0\cdot25\times0\cdot05 + 0\cdot35\times0\cdot04 + 0\cdot40\times0\cdot02}$$

$$= \frac{0\cdot0140}{0\cdot0345} = \frac{28}{69}$$

Example 19. Given three identical boxex I, II, and III each containing two coins. In box I, both coins are gold coins, in box II, both are silver coins and in the box III, there is one gold and one silver coin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold.

Solution : Let E_1, E_2, E_3 be the events that boxes I, II, and III are chosen, respectively

Then $$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

Also, let A be the event that ' the coin drawn is of gold'

P (a gold coin form bag I) = $P\left(\frac{A}{E_1}\right) = \frac{2}{2} = 1$

P (a gold coin form bag II) $P\left(\frac{A}{E_2}\right) = 0$

P(a gold coin from bag III) $P\left(\frac{A}{E_3}\right) = \frac{1}{2}$

Now, the probability that the other coin in the box is of gold = the probability that gold coin is drawn form the box I.

$$P\left(\frac{E_1}{A}\right)$$

By Bayes' theroem, we know that

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1)P\left(\frac{A}{E_1}\right)}{P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right) + P(E_3)P\left(\frac{A}{E_3}\right)}$$

$$= \frac{1/3 \times 1}{1/3 \times 1 + 1/3 \times 1/2}$$

$$= \frac{1/3}{1/3 + 1/6} = \frac{1/3}{2+1/6} = \frac{1/3}{3/6} = \frac{1}{3} \times \frac{6}{3} = \frac{2}{3}$$

Example 20. A man is know to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

Solution : Let E be the event that the man reports that six occurs in the throwing of the die and let S_1 be the event that six occurs and S_2 be the event that six does not occur Then

Probability that six occurs $= P(S_1) = \frac{1}{6}$

Probability that six does not occur $= P(S_2) = \frac{5}{6}$

Probability that the man reports that six occurs when six has actually occured on the die = Probability that the man speaks the truth =

$$= P\left(\frac{E}{S_1}\right) = \frac{3}{4}$$

Probability that the man reports that six occurs when six has not actually occured on the die = Probability that the man does not speak the truth

$$= P\left(\frac{E}{S_2}\right) = 1 - \frac{3}{4} = \frac{1}{4}$$

Thus, by Baye's theroem, we get

Probability that the report of the man that six has occured is actually a six

$$= P\left(\frac{S_1}{E}\right) = \frac{P(S_1)P\left(\frac{E}{S_1}\right)}{P(S_1)P\left(\frac{E}{S_1}\right) + P(S_2)P\left(\frac{E}{S_2}\right)}$$

$$= \frac{1/6 \times 3/4}{1/6 \times 3/4 + 5/6 \times 1/4} = \frac{3/24}{3/24 + 5/24} = \frac{3/24}{8/24}$$

$$= \frac{3}{24} \times \frac{24}{8} = \frac{3}{8}$$

Hence, the required probability is 3/8

Example 21. Suppose that the reliability of a HIV test is specified as follows: Of perople having HIV, 90 % of the test detect the disease but 10% go undetected. Of people free of HIV, 99% of the test are judged HIV -ive but 1% are diagnosed as showing HIV +ive. From a large population of which only 0.1% have HIV, one person is selected at random, given the HIV test, and the pathologist reports him/her as HIV +ive. What is the probability that the person actually has HIV ?

Solution : Let E denotes the event that the person selected is actually having HIV and A the event that the person's HIV test is diagnosed as +ive. We need to find $P\left(\frac{E}{A}\right)$

Also E' denotes the event that the person selected is actually not having HIV. Clearly , $\{E, E'\}$ is a partition of the sample space of all people in the population. We are given that

$$P(E) = 0 \cdot 1\% = \frac{0 \cdot 1}{100} = 0 \cdot 001$$

$$P(E') = 1 - P(E) = 1 - 0 \cdot 001 = 0 \cdot 999$$

P(Person tested as HIV +ive given that he/she is actually having HIV)

$$P\left(\frac{A}{E}\right) = 90\% = \frac{9}{10} = 0 \cdot 9$$

P(Person tested as HIV+ive given that he/she is actually not having HIV)

$$P\left(\frac{A}{E'}\right)=1\%=\frac{1}{100}=0\cdot 01$$

Now, by Baye's theorem

$$P\left(\frac{E}{A}\right)=\frac{P(E)P\left(\frac{A}{E}\right)}{P(E)P\left(\frac{A}{E}\right)+P(E')P\left(\frac{A}{E'}\right)}$$

$\therefore$ $$P\left(\frac{E}{A}\right)=\frac{0\cdot 001\times 0\cdot 9}{0\cdot 001\times 0\cdot 9+0\cdot 999\times 0\cdot 01}$$

$$=\frac{90}{1089}=0\cdot 083 \text{ approx.}$$

Thus, the probability that a person selected at random is actually having HIV given that he/she is tested HIV +ive is 0.083.

Exercise 16.3

1. Bag I contain 3 red and 4 black balls while another Bag II contains 5 red and 6 black balls. One ball is drawn at random form one of the bags and it is found to be red. Find the probability that it was drawn form Bag II.
2. A doctor is to visit a patient. From the past experience, it is known that the probabilites that he will come by train, bus, scooter or by other means of transport are respectively $\frac{3}{10},\frac{1}{5},\frac{1}{10}$ and $\frac{2}{5}$ The probabilities that he will be late are $\frac{1}{4},\frac{1}{3}$ and $\frac{1}{12}$, if he comes by train, bus and scooter respectively, but if he comes by he comes by train ?
3. Bag I contains 3 Red and 4 black balls while Bag II contains 4 Red and 5 black balls. One ball is transfered from Bag I to Bag II and then a ball is drawn form Bag II and it was found to be Red. Find the probability that the transfered ball is black.
4. A bag contains 3 Red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.
5. There are three coins. One is a two headed coin (having head on both faces), another is a baised coin that comes up heads 75% of the time and third is an unbaised coin. One of the three coins is chosen at random and tossed, it shows heads, what is the probability that it was the two headed coin ?
6. A laboratory blood test is 99% effective in detecting a certain disease when it is in fact, present. However, the test also yields a false positive result for 0.5% of the healthy person tested (i.e. if a healthy person is tested, then, with probability 0.005, the test will imply he has the disease). If 0.1 percent of the population actually has the disease, what is the probability that a person has the disease given that his test result is positive ?

7. Students in a college, it is known that 60% reside in hostel and 40% are day scholar (not residing in hostel). Previous year results report that 30% of all students who reside in hostel attain A grade and 20% if day scholar attain A grade in their annual examination. At the end of the year, one student is chosen at random from the college and he has an A grade, what is the probabilioty that the student is a hostlier ?
8. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probabilioty of accidents are 0.01, 0.03 and 0.15 respectively. One of the insured persons meet with an accident, What is the probability that he is a scooter driver ?
9. In answering a question on a multiple choice test, a student either knows the answer or gusses. Let 3/4 be the probability that he knows the answer and 1/4 be the probability that he gusses. Assuming that a student who guesses at the answer will be correct with probability 1/4 What is the probability that the student know the answer given that he answered it correctly ?
10. Suppose that 5% of men and 0.25% of women have grey hair. A grey haired person is selected at random. What is the probability of this person being male ? Assume that there are equal number of males and females.
11. Two groups are competing for the position on the Board of directors of corporation. The probabilities that the first and the second groups will win are 0.6 and 0.4 respectively. Further, if the first group wins, the probability of introducing a new product is 0.7 and the corresponding probability is 0.3 if the second group wins . Find the probability that the new product introduced was by the second group.
12. Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin three times and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3, or 4 with the die ?
13. A card form a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be both diamonds. Find the probability of the lost card being a diamond.
14. A bag contains 3 Red and 7 Black balls. Two balls are selected at random without replacement. If the second drawn ball is Red then what is the probability that the first ball drawn is also Red ?

16.09 Random variable and its Probability Distribution

We have already learnt about random experiments and formation of sample spaces. Sample spaces are set of all possible results of any random experiment. The result of any random experiments may be numerical or non-numerical. In most of these experiments, we were not only interested in the particular outcome that occurs but rather in some number associated with that outcomes as shown in following example / experiments.

(*i*) In tossing two dice, we may be interested in the sum of the number on the two dice.

(*ii*) In tossing a coin 50 times, we may be interested in the sum of the number of heads obtained.

(*iii*) In the experiment of taking out four articles (one after the other) at random from a lot of 20 articles in which 6 are defective, we want to know the number of defective in the sample of four and not in the particular sequence of defective and non defective articles. In all of the above experiments, we have a rule which assigns to each outcome of the experiment a single real number. This single real number may vary with different outcome of a random experiment and hence, is called random variable. A random variable is usually denoted by X. If you recall the defination of a function, you will realise that the random variable X is really speaking a function whose domain is the set of outcomes(or sample space) of a random experiment. A random variable can take any real value, therefore, its co-domain is the set of real numbers. Hence, a random variable can be defined as follows :

Definition : A random variable is a real valued function whose domain is the sample space of a random experiment

Random variables are generally expresses as X, Y, Z

For example, let us consider the experiment of tossing a coin three times in succession.

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

If X denotes the number of heads obtained, then X is a random variable and for each outcome, its value is as given below :

$$X(HHH) = 3, \ X(HHT) = 2 = X(HTH) = X(THH),$$

$$X(HTT) = 1 = X(THT) = X(TTH), \ X(TTT) = 0$$

NOTE : more than one random variables can be defined on the same sample space

Random variables are of two types :

(i) Discrete Random variable

(ii) Continuous Random variable

(i) Discrete Random variable : If a random variable takes a finite or infinite value then that variable is called as discrete random variable. For example -

(a) number of students in a class.

(b) the number of printed errors in a book

(c) the number of complaints received in an office

(ii) Continuous Random variable : If a random variable takes all the values in a fixed interval then it is called as continuous random variable for example -

(a) height of a person

(b) $X = \{x \in R \ : \ 0 < x < 1\}$ etc.

NOTE : In this chapter random variable means discrete random variable only.

16.10 Probability distribution of a Random Variable

Probability distribution of a random variable is discription of collection of all possible results and probability related to them. The probability distribution of a random variable X is the system of numbers

$$X = x \ : \ x_1 \quad x_2 \quad x_3 \ldots x_n$$

$$P(X) \ : \ p_1 \quad p_2 \quad p_3 \ldots p_n$$

$$\text{where} \quad p_i > 0 \quad \text{and} \quad \sum_{i=1}^{n} p_i = 1; \quad i = 1, 2, \ldots, n$$

The real numbers $x_1, x_2, x_3, \ldots, x_n$ are the possible values of the random variable X with possible probabilities $p_1, p_2, p_3, \ldots, p_n$ etc.

For example, let us consider the experiment of tossing a coin two times in succession. The sample space of the experiment is

$$S = \{HH, HT, TH, TT\}$$

If X denotes the number of heads obtained, then X is a random variable and for each outcome, its value is as given below :

$$X(HH) = 2, \ X(HT) = 1 = X(TH), \ \ X(TT) = 0$$

Here X tales the values 0, 1 and 2 whose corresponding probabilities are $1/4$, $2/4$ and $1/4$, thus the proabbility distribution is

$X = x$:	0	1	2
$P(X)$:	$1/4$	$2/4$	$1/4$

where $p_i > 0$ and $\sum p_i = \frac{1}{4} + \frac{2}{4} + \frac{1}{4} = 1,$

Illustrative Examples

Example 22. The probability distribution of a random variable X is given below :

X	0	1	2	3	4	5	6	7
$P(X)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Find

(i) k (ii) $P(X<6)$ (iii) $P(X \geq 6)$ (iv) $P(0<X<5)$

Solution : (i) The sum of probabilities in a probability distribution is always 1. Therefore

$$P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)+P(X=6)+P(X=7)=1$$

$\Rightarrow$ $0+k+2k+2k+3k+k^2+2k^2+7k^2+k=1$

$\Rightarrow$ $10k^2+9k-1=0$

or $(10k-1)(k+1)=0$

or $10k-1=0$ $[\because k \geq 0]$

$\Rightarrow$ $k = \frac{1}{10}$

(ii) $P(X<6)=P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)$

$\Rightarrow$ $0+k+2k+2k+3k+k^2$

$\Rightarrow$ k^2+8k

$\Rightarrow$ $(1/10)^2+8(1/10)=\frac{81}{100}$

(iii) $P(X \geq 6)=P(X=6)+P(X=7)$

$\Rightarrow$ $2k^2+7k^2+k$

$\Rightarrow$ $9k^2+k$

$\Rightarrow$ $9(1/10)^2+1/10=\frac{19}{100}$

(iv) $P(0<X<5)=P(X=1)+P(X=2)+P(X=3)+P(X=4)$

$\Rightarrow$ $k+2k+2k+3k=8k$

$\Rightarrow$ $8/10=4/5$

Example 23. Three cards are drawn succesively with replacement form a well-shuffled deck of 52 cards. Find the probability distribution of the number of aces.

Solution : The number of aces is a random variable. Let it be denoted by X. Clearly, X can take the values 0, 1, 2 or 3.

$$P(X=0) = \text{P (non-ace and non-ace)} = \frac{{}^{48}c_3}{{}^{52}c_3} = \frac{4324}{5525}$$

$$P(X=1) = \text{P (ace and two non-ace)} = \frac{{}^{4}c_1 \times {}^{48}c_2}{{}^{52}c_3} = \frac{1128}{5525}$$

$$P(X=2) = \text{P (two ace and one non-ace)} = \frac{{}^{4}c_2 \times {}^{48}c_1}{{}^{52}c_3} = \frac{72}{5525}$$

$$P(X=3) = \text{P (ace and ace and ace)} = \frac{{}^{4}c_3}{{}^{52}c_3} = \frac{1}{5525}$$

Thus, the required probability distribution is

X	0	1	2	3
$P(X)$	$\frac{4324}{5525}$	$\frac{1128}{5525}$	$\frac{72}{5525}$	$\frac{1}{5525}$

Example 24. Let X denote the number of hours you study during a randonly selected school day. The probability that X can take the values *x*, has the following form, where *k* is some unkonwn constant.

$$P(X=x) = \begin{cases} 0\cdot 1 & ; \quad If\ \ x=0 \\ kx & ; \quad If\ \ x=1 or\ 2 \\ k(5-x) & ; \quad If\ \ x=3 or\ 4 \\ 0 & ; \quad otherwise \end{cases}$$

(i) Find the value of *k*.

(ii) What is the probability that

(a) you study at least two hours ?

(b) Exactly two hours ?

(c) At most two hours ?

Solution : The probability disrtibution of X is

$$\begin{array}{lccccccc} X & : & 0 & 1 & 2 & 3 & 4 \\ P(X) & : & 0\cdot 1 & k & 2k & 2k & k \end{array}$$

(*i*) We know that

$$P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)=1$$

$$0\cdot 1+k+2k+2k+k=1$$

$\Rightarrow$ $$6k=0\cdot 9$$

or $$k=0\cdot 15$$

(*ii*) (a) required probability

when $P(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4)$

$$= 2k + 2k + k = 5k = 5 \times 0{\cdot}15 = 0{\cdot}75.$$

(b) required probability

when $P(X = 2) = 2k = 2 \times 0{\cdot}15 = 0{\cdot}30.$

(c) required probability

when $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$$= 0{\cdot}1 + k + 2k = 3k + 0{\cdot}1$$

$$= 0{\cdot}1 + 3 \times 0{\cdot}15 = 0{\cdot}55$$

Example 25. A coin is baised so that the head is 3 times as likely to occur as tail. If the coin is tosses twice, find the probability distribution of number of tails.

Solution : Let p denotes the probability of getting tail in tossing a coin once then probability of getting head will $3p$

Thus getting " number of head " and " number of tails " are mutually exclussive and exhaustive events

$$P(H) + P(T) = 1$$

$\Rightarrow$ $3p + p = 1$

or $p = 1/4$

$\therefore$ $P(H) = \dfrac{3}{4}$ and $P(T) = \dfrac{1}{4}$

Let X denote the number of tails in tossing a coin twice then X will take values 0, 1 and 2

$P(X = 0) =$ P (not getting Tail)

= P (getting both Heads)

= P(HH)

= P(H) P(H) {$\because$ both are independent}

$$= \frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$$

$P(X = 1) =$ P (getting one Tail and one Head) = P(HT) + P(TH)

= P(H) P(T) + P(T) P(H)

$$= \frac{3}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{3}{4} = \frac{3}{8}$$

$P(X = 2) =$ P(getting both Tails)

$$= P(TT) = P(T)\,P(T) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

Now the probability distribution of X

X	:	0	1	2
$P(X)$	:	$\frac{9}{16}$	$\frac{3}{8}$	$\frac{1}{16}$

16.11 Mean of a Random Variable

In many problems, it is desirable to describe some feature of the random variable by means of a single number that can be computed form its probability distribution. Few such numbers are mean, median and mode. In this section, we shall discuss mean only. Mean is a measure of location or central tendency in the sense that it roughly locates a *middle* or *average value* of the random varible

Let X be a random variable whose possible values $x_1, x_2, x_3, ..., x_n$ occur with probabilities $p_1, p_2, ..., p_n$ respectively.

The mean of a random variable X is also called the expection of X, denoted by E(X).

$$E(X) = \mu = \sum_{i=1}^{n} x_i p_i$$

$$= x_1 p_1 + x_2 p_2 + ... + x_n p_n$$

The mean of X, denoted by μ is the number $\sum_{i=1}^{n} x_i p_i$ i.e. the mean of X is the weighted average of the possible values of X, each being weighted by its probability with which it occurs.

Let a dice be thrown and the random variable X be the number that appears on the dice. Find the mean or expectation of X.

The sample space is $= \{1, 2, 3, 4, 5, 6\}$

Now the probability distribution with random variable X–

$X = x$	:	1	2	3	4	5	6
$P(x)$	:	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

$\therefore$ $$\mu = E(X) = \sum x_i p_i$$

$$= x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4 + x_5 p_5 + x_6 p_6$$

$$= 1 \times 1/6 + 2 \times 1/6 + 3 \times 1/6 + 4 \times 1/6 + 5 \times 1/6 + 6 \times 1/6$$

$$= 21/6 = 7/2$$

NOTE : This does not mean at all that in the experiment of tossing a coin we get the number 7/2. This number indicates that if the coin is tossed for longer period then the number we get in average tossing is 7/2

Illustrative Examples

Example 26. Three coins are tossed, If X denotes the number of Heads then find the mean or expectations of X

Solution : Here X takes the values 0, 1, 2, and 3

$$P(X = 0) = P(TTT) = \frac{1}{8}$$

$$P(X=1)=P\text{ (HTT या TTH या THT)}=\frac{3}{8}$$

$$P(X=2)=P\text{(HHT या THH या HTH)}=\frac{3}{8}$$

and $$P(X=3)=P(HHH)=\frac{1}{8}$$

Probability distribution of variable X is–

$X=x$	0	1	2	3
$P(x)$	$1/8$	$3/8$	$3/8$	$1/8$

Mean of $$X=\overline{X}=E(X)=\sum x_i p_i$$

$$=0\times 1/8+1\times 3/8+2\times 3/8+3\times 1/8=12/8=3/2$$

Example 27. Two cards are drawn simultaneously (or successively with replacement) from a well shuffled pack of 52 cards. Find the mean and probability of the number of aces.

Solution : Let X denote the number of aces.

Variable X take the values 0, 1 and 2

P(X = 0) = P (not getting an ace)

= P (no ace and no ace) =P (no ace) . P (no ace)

$=48/52\times 48/52=144/169$

P(X = 1) = P (getting an ace)

= P(ace and no ace or no ace and ace)

= P(ace) P(no ace) + P(no ace) P(ace)

$=4/52\times 48/52+48/52\times 4/52=24/169$

P(X = 2) = P(gettting both the aces ace)

=P(ace and ace)

=P(ace) P(ace)

$=4/52\times 4/52=1/169$

Probability distribution of variable X-

X	:	0	1	2
$P(X)$	:	$144/169$	$24/169$	$1/169$

Mean $$=\overline{X}=E(X)=\sum x_i p_i$$

$$=0\times 144/169+1\times 24/169+2\times 1/169=26/169.$$

16.12 Variance of a random variable

Let X be a random whose possible values $x_1, x_2, ..., x_n$ occur with probabilities $p_1, p_2, ..., p_n$ respectively then variance of X is given by var(X) or σ_X^2

$$\sigma_X^2=\text{var}(X)=E(X-\mu)^2=\sum_{i=1}^{n}(x_i-\mu)^2 p_i$$

The positive square root of variance as "$+\sqrt{\text{var}(X)}$" is called as standard deviation

$$\sigma_X = +\sqrt{\text{var}(X)} = +\sqrt{\sum_{i=1}^{n}(x_i-\mu)^2 p_i}$$

Alternative Formula to find Variance.

$$\text{var}(X) = \sum_{i=1}^{n}(x_i-\mu)^2 p_i$$

$$= \sum_{i=1}^{n}\left(x_i^2+\mu^2-2\mu x_i\right)p_i$$

$$= \sum_{i=1}^{n}x_i^2 p_i + \sum_{i=1}^{n}\mu^2 p_i - 2\sum_{i=1}^{n}\mu x_i p_i$$

$$= \sum_{i=1}^{n}x_i^2 p_i + \mu^2\sum_{i=1}^{n}p_i - 2\mu\sum_{i=1}^{n}x_i p_i$$

$$= \sum_{i=1}^{n}x_i^2 p_i + \mu^2(1) - 2\mu(\mu)$$

$$= \sum_{i=1}^{n}x_i^2 p_i - \mu^2$$

$$= \sum_{i=1}^{n}x_i^2 p_i - \left(\sum_{i=1}^{n}x_i p_i\right)^2$$

$$\text{var}(X) = E\left(X^2\right) - \left\{E(X)\right\}^2 \qquad \text{Where } E\left(X^2\right) = \sum_{i=1}^{n}x_i^2 p_i$$

For Example : Find the Variance of head in three tosses of a fair coin.

Solution : We have to find the variance of head in three tosses of a fair coin

The sample space S={HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Here X can take the values 0, 1, 2 and 3 whose probabilities are $1/8, 3/8, 3/8 \; ; 1/8$

The probability distribution of X is -

$X=x$	:	0	1	2	3
$P(X)$	:	1/8	3/8	3/8	1/8

Variance of X var (X) $= E\left(X^2\right) - \left[E(X)\right]^2$

where $$E(X) = \sum_{i=1}^{n}x_i p_i = x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4$$

$$= 0\times 1/8 + 1\times 3/8 + 2\times 3/8 + 3\times 1/8 = 3/2$$

and $$E(X^2)=\sum_{i=1}^{n} x_i^2 p_i = x_1^2 p_1 + x_2^2 p_2 + x_3^2 p_3 + x_4^2 p_4$$

$$=(0)^2\times 1/8+(1)^2\times 3/8+(2)^2\times 3/8+(3)^2\times 1/8$$

$$=0+3/8+12/8+9/8=3$$

$\therefore$ $$\text{var}(X)=E(X^2)-[E(X)]^2$$

$$=3-(3/2)^2=3-9/4=3/4.$$

Example 28. Two dice are thrown simultaneously. If X denotes the number of sixes, find the variance of X,

Solution : The Sample Space is tossing two coins is -

$$X=\begin{Bmatrix} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6) \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6) \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6) \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6) \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6) \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{Bmatrix}$$

X can take the values 0, 1 and 2

$$P(X=0)=\text{P(not getting six)}=25/36$$

$$P(X=1)=\text{P(getting six on one dice)}=10/36$$

$$P(X=2)=\text{P(getting six on both the die)}=1/36$$

The probability distribution of variable X -

X	:	0	1	2
$P(X)$	:	25/36	10/36	1/36

$$E(X)=\sum_{i=1}^{n} x_i p_i = 0\times 25/36+1\times 10/36+2\times 1/36=12/36=1/3$$

$$E(X^2)=\sum_{i=1}^{n} x_i^2 p_i=(0)^2\times 25/36+(1)^2\times 10/36+(2)^2\times 1/36=14/36=7/18$$

$\therefore$ $$\text{var}(X)=E(X^2)-\{E(X)\}^2=\frac{7}{18}-\left(\frac{1}{3}\right)^2=\frac{5}{18}.$$

Example 29. Two numbers are selected at random (without replacement) from the first six positive integers. Let X denote the larger of the two numbers obtained. Find the variance

Solution : X takes the value 2, 3, 4, 5, 6

$P(X=2)=$ P (getting a number greater than 2)

$= P$ (getting 1 and then 2) or (getting 2 and then 1)

$=1/6\times1/5+1/6\times1/5=2/30=1/15$

$P(X=3)=$ P (getting a number greater than 3)

$= P$ (getting a number lesser than 3 and than 3) (getting 3 or a number lesser than 3)

$=2/6\times1/5+1/6\times2/5=4/30=2/15$

similarly $P(X=4)=3/6\times1/5+1/6\times3/5=6/30=1/5$

and $P(X=5)=4/6\times1/5+1/6\times4/5=8/30=4/15$

also $P(X=6)=5/6\times1/5+1/6\times5/5=10/30=1/3$

Thus the probability distribution of X -

X	:	2	3	4	5	6
$P(X)$	:	$1/15$	$2/15$	$1/5$	$4/15$	$1/3$

$E(X)=\sum x_i p_i = 2\times1/15+3\times2/15+4\times1/5+5\times4/15+6\times1/3=70/15=14/3$

$E(X^2)=\sum x_i^2 p_i = (2)^2\times1/15+(3)^2\times2/15+(4)^2\times1/5+(5)^2\times4/15+(6)^2\times1/3$

$=4/15+18/15+16/5+100/15+36/3=350/15=70/3$

$\text{var}(X)=E(X^2)-\{E(X)\}^2$

$=70/3-(14/3)^2=70/3-196/9=14/9.$

Example 30. A class has 15 students whose ages are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19 and 20 years. One student in such a manner that each has the same chance of being chosen and the age X of the selected student is recorded. What is the proabability distribution of the random variable X? Find mean, variance and standard deviation of X.

Solution : X can take values 14, 15, 16, 17,18, 19, 20 and 21

$\therefore\ P(X=14)=2/15,\quad P(X=15)=1/15,\quad P(X=16)=2/15,\quad P(X=17)=3/15,$

$P(X=18)=1/15,\quad P(X=19)=2/15,\quad P(X=20)=3/15,\quad P(X=21)=1/15$

The probability distribution of X -

X	:	14	15	16	17	18	19	20	21
$P(X)$	:	$2/15$	$1/15$	$2/15$	$3/15$	$1/15$	$2/15$	$3/15$	$1/15$

Mean of X $=E(X)=\sum x_i p_i$

$=14\times2/15+15\times1/15+16\times2/15+17\times3/15+18\times1/15+19\times2/15+20\times3/15+21\times1/15$

$=263/15=17{\cdot}53$

$=E(X^2)=\sum x_i^2 p_i$

$$=(14)^2\times(2/15)+(15)^2\times(1/15)+(16)^2\times(2/15)+(17)^2\times(3/15)+(18)^2\times(1/15)+(19)^2\times(2/15)+(20)^2\times(3/15)+(21)^2\times(1/15)$$

$$=\frac{392}{15}+\frac{225}{15}+\frac{512}{15}+\frac{867}{15}+\frac{324}{15}+\frac{722}{15}+\frac{1200}{15}+\frac{441}{15}=\frac{4683}{15}$$

$$\text{var}(X)=E(X^2)-\{\text{E}(X)\}^2$$

$$=\frac{4683}{15}-\left(\frac{263}{15}\right)^2=\frac{70245-69169}{225}=\frac{1076}{225}$$

$$\text{Standard Deviation }=\sqrt{\frac{1076}{225}}=2{\cdot}186$$

Exercise 16.4

1. State which of the following are not the probability distribution of a random variable. Give reasons for your answer.

(i)

X	:	0	1	2
$P(X)$	:	0.4	0.4	0.2

(ii)

X	:	0	1	2
$P(X)$	:	0.6	0.1	0.2

(iii)

X	:	0	1	2	3	4
$P(X)$	:	0.1	0.5	0.2	–.01	0.3

2. Find the probability distribution of number of heads in two tosses of a coin.
3. Four rotten oranges by mistake are mixed with 16 good oranges. Two oranges are drawn and found to be rotten, find the probability distribution.
4. An urn contains 4 white and 3 red balls. Three balls are drawn at random and found to be red, find the probability distribution.
5. From a lot of 10 object which includes 6 defective, a sample of 4 objects is drawn at random. If the random variable of defective objects is denoted as X, then find-

   (i) Probability distribution of X (ii) $P(X\le 1)$ (iii) $P(X<1)$

   (iv) $P(0<X<2)$.
6. A die is rolled so that getting an even number is twice as likely to occur odd number. If a die is rolled twice then considering the random variable X as the square of the number , find the probability distribution.
7. An urn contains 4 white and 6 red balls. Four balls are drawn at random, find the probability distribution of number of white balls.
8. Find the probability distribution of getting a doublet in rolling two dice three times.
9. A pair of dice is rolled. Let X, the sum of the numbers on the dice. Find the mean of X.
10. Find the variance of the number obtained on a throw of an unbiased die.
11. In a meeting, 70% of the members favour and 30% oppose a certain proposal. A member is selected at random and we take X = 0 if he opposed, and X = 1 if he is in favour. Find E(X) and Var (X).
12. Two cards are drawn simultaneously (or successively without replacement) from a well shuffled pack of 52 cards. Find the mean, variance and standard deviation of the number of kings.

16.13 Bernoulli Trials

Each time we toss a coin or roll a die or perform any other experiments, we call it a trial. If a coin is tossed, say, 4 times, the number of trials is 4, each having exactly two outcomes, namely, success or faliure. The outcome of any trial is independent of the outcome of any other trial. In each of such trials, the probability of success or failure remains constant. Such independent trials which have only two outcomes usually referred as 'success' or 'failure' are called *Bernoulli trials*

(i) There should be a finite number of trials.

(ii) The trials should be independent.

(iii) Each trial has exactly two outcomes : success or failure.

(iv) The probability of success remains the same in each trial.

For example, throwing a die 50 times is a case of 50 Bernoulli trials, in which each trial results is success (say an even number) or failure (an odd number) and the probability of success (p) is same for all 50 throws. Obviously, the successive throws of the die are independent experiments.

16.14 Binomial Distribution

Let an experiments is repeated n times. Therefore it is an experiments of n-Bernoulli trials where every experiments is independent and let S and F denote respectively success and failure in each trial.

Let the probability of getting a success in an experiments is (p) and failure be $(q = 1 - p)$]

let in n- Bernoulli's trials experiment, the probability for r successes and (n - r) failure

$$P(X = r) = P(r\,\text{success}).P[(n-r)\, failure]$$

$$= P(\underbrace{SSS...S}_{r\,times}\ \underbrace{FFF...F}_{(n-r)\,times})$$

$$= P(S)P(S)P(S)...P(S)\quad P(F)P(F)P(F)...P(F)$$

$$= ppp....p\ \ qqq...q$$

$$P(X = r) = p^r q^{n-r}$$

This result shows r success and $(n - r)$ failure in an experiments but in n experiments r success can be found through ${}^nc_r$ procedures and probabilities of every procedure remains $p^r q^{n-r}$

Thus Probability of r *success in* n-Bernoulli's experiments is

$$P(X = r) = {}^nc_r p^r q^{n-r};\quad r = 0,1,2,....,n \text{ and } q = 1 - p$$

The distribution of number of successes X *in* n-Bernouli's experiments is given by-

X	0	1	2	...	r	...	n
$P(X)$	${}^nC_0 p^0 q^{n-0} = {}^nC_0 q^n$	${}^nc_1 p^1 q^{n-1}$	${}^nc_2 p^2 q^{n-2}$	...	${}^nc_r p^r q^{n-r}$	...	${}^nc_n p^n q^{n-n} = {}^nc_n p^n$

The above probability distribution is known as *binomial distribution with parameters n* and p, because for given values of n and p, we can find the complete probability distribution.

A binomial distribution with n-Bernoulli trial and probability of success in each trial as p, is denote by B(n,p).

NOTE : $$\sum_{r=0}^{n} P(X = r) = \sum_{r=o}^{n} {}^nC_r p^r q^{n-r}$$

$$= {}^nC_0 p^o q^n + {}^nC_1 p^1 q^{n-1} + ... + {}^nC_n p^n q^{n-n} = (q + p)^n = 1.$$

Illustrative Examples

Example 31. A die is thrown 7 times. If 'getting a sum 7' is a success, what is the probability of (*i*) no success ? (*ii*) 6 successes? (*iii*) at least 6 successes? (*iv*) at most 6 successes?

Solution : Let the probability of getting a sum 7 be p then $p = 6/36 = 1/6$

[∵ there are six ways of getting a sum 7 on the dice]

$$(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)$$

$$q = 1 - p = 1 - 1/6 = 5/6$$

Let the number of successes be X then

Let the number of successes be X then

$$P(X = r) = {}^7c_r (1/6)^r (5/6)^{7-r}; \quad r = 0,1,2,3,4,5,6,7$$

(i) P (no success) $= P(X = 0)$

$$= {}^7C_o \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^{7-0} = \left(\frac{5}{6}\right)^7$$

(ii) P (6 successes) $= P(X = 6)$

$$= {}^7C_6 \left(\frac{1}{6}\right)^6 \left(\frac{5}{6}\right)^{7-6} = \frac{35}{6^7}$$

(iii) P (atleast 6 successes) $= P(X \geq 6)$

$$= P(X = 6) + P(X = 7)$$

$$= {}^7C_6 \left(\frac{1}{6}\right)^6 \left(\frac{5}{6}\right)^{7-6} + {}^7C_7 \left(\frac{1}{6}\right)^7 \left(\frac{5}{6}\right)^{7-7}$$

$$= 7\left(\frac{1}{6}\right)^6 \left(\frac{5}{6}\right) + \left(\frac{1}{6}\right)^7 = \frac{1}{6^5}$$

(iv) P (at most 6 successes) $= P(X \leq 6)$

$$= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6)$$

$$= 1 - P(X > 6)$$

$$= 1 - P(X = 7)$$

$$= 1 - {}^7C_7 \left(\frac{1}{6}\right)^7 \left(\frac{5}{7}\right)^{7-7} = 1 - \left(\frac{1}{6}\right)^7.$$

Example 32. A die is thrown again and again until three sixes are obtained. Find the probability of obtaining the third six in the sixth throw of the die.

Solution : Let the probability of getting a number 6 is p then $p = \frac{1}{6}, \quad q = 1 - \frac{1}{6} = \frac{5}{6}$

Required probability = P (getting two 6 in first 5 throws). P (getting 6 in the sixth throw)

$$=\left({}^{5}C_{2}p^{2}q^{5-2}\right)(p)={}^{5}C_{2}\left(\frac{1}{6}\right)^{2}\left(\frac{5}{6}\right)^{3}\times\frac{1}{6}$$

$$=\frac{10\times125}{6^{6}}=\frac{625}{23328}.$$

Example 33. A fair coin is tossed 5 times. Find the probability of getting atleast 3 Heads.

Solution : Let the probability be p then $p=1/2,\ q=1/2$ Let X denote getting a number 5 then $n = 5$ and $p=1/2$ such that

$$P(X=r)={}^{5}C_{r}\left(\frac{1}{2}\right)^{5-r}\left(\frac{1}{2}\right)^{r}={}^{5}C_{r}\left(\frac{1}{2}\right)^{5};\ \text{where } r=0,1,2,3,4,5$$

Required probability $= P$ (atleast 3 Heads)

$$=P(X\geq3)$$

$$=P(X=3)+(X=4)+(X=5)$$

$$={}^{5}C_{3}\left(\frac{1}{2}\right)^{5}+{}^{5}C_{4}\left(\frac{1}{2}\right)^{5}+{}^{5}C_{5}\left(\frac{1}{2}\right)^{5}$$

$$=\left({}^{5}C_{3}+{}^{5}C_{4}+{}^{5}C_{5}\right)\left(\frac{1}{2}\right)^{5}=\left(\frac{10+5+1}{32}\right)=\frac{1}{2}$$

Example 34. A die is rolled 6 times. If getting an even number is a success than find the following probabilities -

(i) exactly 5 successes (ii) atleast 5 successes (iii) almost 5 successes

Solution : Let the probability be p then $p=3/6=1/2$ and $q=1-p=1-1/2=1/2$

and let $n = 6$ and $p=1/2$ then -

$$P(X=r)={}^{6}C_{r}\left(\frac{1}{2}\right)^{6-r}\left(\frac{1}{2}\right)^{r}={}^{6}C_{r}\left(\frac{1}{2}\right)^{6};\ \text{where } r=0,1,2,3,4,5,6$$

(i) P (exactly 5 successes) $=P(X=5)={}^{6}C_{5}\left(\frac{1}{2}\right)^{6}=\frac{3}{32}.$

(ii) P (atleast 5 successes) $=P(X\geq5)=P(X=5)+P(X=6)$

$$={}^{6}C_{5}\left(\frac{1}{2}\right)^{6}+{}^{6}C_{6}\left(\frac{1}{2}\right)^{6}$$

$$=\frac{6}{64}+\frac{1}{64}=\frac{7}{64}.$$

(iii) P (atmost 5 successes) $= P(X \le 5)$

$$= P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)$$

$$= 1 - P(X > 5)$$

$$= 1 - P(X = 6)$$

$$= 1 - {}^6C_6\left(\frac{1}{2}\right)^6 = 1 - \frac{1}{64} = \frac{63}{64}.$$

Example 35. The probability of a shooter hitting a target is 1/4 How many minimum number of times must he/she fire so that the probability of hitting the target at least once is more than 2/3 ?

Solution : Let a person hit the target n times

as per the question $p = 1/4$ and $q = 1 - p = 1 - 1/4 = 3/4$ then

$$P(X=r) = {}^nC_r\left(\frac{1}{4}\right)^r\left(\frac{3}{4}\right)^{n-r} \text{; where } r = 0, 1, 2, \ldots, n$$

Given P (hitting the target atleast once) $> 2/3$

$$P(X \ge 1) > 2/3$$

$$\Rightarrow \quad 1 - P(X = 0) > 2/3$$

$$\Rightarrow \quad 1 - {}^nC_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{n-0} > \frac{2}{3}$$

$$\Rightarrow \quad 1 - \left(\frac{3}{4}\right)^n > \frac{2}{3}$$

$$\Rightarrow \quad \left(\frac{3}{4}\right)^n < \frac{1}{3}$$

$$\Rightarrow \quad n = 4, 5, 6, \ldots\left[\because \left(\frac{3}{4}\right)^1 > \frac{1}{3}, \left(\frac{3}{4}\right)^2 > \frac{1}{3}, \left(\frac{3}{4}\right)^3 > \frac{1}{3}\ but \left(\frac{3}{4}\right)^4 < \frac{1}{3}, \left(\frac{3}{4}\right)^5 < \frac{1}{3}, \ldots\right]$$

The person should hit the target atleast 4 times.

Example 36. A man takes a step forward with probability 0.4 and backwards with probability 0.6. Find the probability that at the end of eleven steps he is just one step away from the starting point.

Solution : Let p denote the probability that the man takes a step forward. Then $p = 0.4$,

$$\therefore \quad q = 1 - p = 1 - 0.4 = 0.6$$

Let X denote the number of steps taken in the forward direction. Since the steps are independent of each other, therefore X is a binomial variate with parameters $n = 11$ and $p = 0.4$ such that

$$P(X=r) = {}^{11}C_r (0.4)^r (0.6)^{11-r}; r = 0, 1, 2, \ldots\ldots, 11$$

Since the man is one step away from the initial point, he is either one step forward or one step backward from the initial point at the end of eleven steps. If he is one step forward, then he must have taken six steps forward and five steps backward and if he is one step backward, then he must have taken five steps forward and six steps backward. Thus, either $X = 6$ or $X = 5$

$\therefore$ required probability $= P[(X = 5)$ or $(X = 6)]$

$\Rightarrow$ required probability $= P(X = 5) + P(X = 6)$ (both the event are mutually exclusive)

$\Rightarrow$ required probability $= {}^{11}c_5 (0\cdot4)^5 (0\cdot6)^{11-5} + {}^{11}c_6 (0\cdot4)^6 (0\cdot6)^{11-6}$

$\Rightarrow$ required probability $= {}^{11}c_5 (0\cdot4)^5 (0\cdot6)^6 + {}^{11}c_6 (0\cdot4)^6 (0\cdot6)^5 = 462(0\cdot24)^5$.

Exercise 16.5

1. If a fair coin is tossed 10 times, find the probability of
   (*i*) exactly six heads (*ii*) at least six heads (*iii*) at most six heads
2. An urn contains 5 white, 7 red and 8 black balls. If four balls are drawn with replacement then what is the probability that
   (*i*) all balls are white (*ii*) only three balls are white
   (*iii*) none of the balls is white (*iv*) alleast three balls are white
3. In a hurdle race, a player has to cross 10 hurdless. The probability that he will clear each hurdle is 5 / 6. What is the probability that he will knock down fewer that 2 hurdle.
4. Five dice are thrown at once. If getting an even number is a success then find the probability of three successes.
5. Ten eggs are drawn successively with replacement form a lot conatining 10% defective eggs. Find the probability that there is at least one defective egg.
6. A person buys a lottery ticket in 50 lotteries, in each of which his chance of winning a prize is 1 / 100. What is the probability that he will win a prize.
   (i) at least once (ii) exactly once (iii) at least twice?
7. The probability that a bulb produced by a factory will fuse after 150 days of use is 0.05. Find the probability that out of 5 such bulbs
   (i) none (ii) not more than one (iii) more than one (iv) at least one
   will fuse after 150 days of use.
8. In a multiple choice examination with three possible answers for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing?
9. In an examination, 20 questions of true-false type are asked. Suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answers 'true'; if it falls tails, he answers 'false'. Find the probability that he answers at least 12 questions correctly.
10. A bag consists of 10 balls each marked with one of the digits 0 to 9. If four balls are drawn successively with replacement from the bag, what is the probability that none is marked with the digit 0?
11. Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards,. What is the probability that
    (i) all the five cards are spades ? (ii) only 3 cards are spades ?
    (iii) none is a spade ?
12. Suppose X has a binomial $B(6, 1/2)$ Show that X = 3 is the most likely outcome.
13. A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability of two successes.

Miscellaneous Examples

Example 37. A and B throw two dice alternatively. If A throws 6 before B throws 7 then A wins and if B throws 7 before A throws 6 then B wins. If A starts playing then find the probability that A wins.

Solution : We can get 6 in five ways

$\{(1,5)\ (2,4)\ (3,3)\ (4,2)\ (5,1)\} = 5/36$

and Probability of not getting $6 = 1 - 5/36 = 31/36$

similarly we can get 7 in six different ways

$\{(1,6)\ (2,5)\ (3,4)\ (4,3)\ (5,2)(6,1)\}$

$\therefore$ probability of getting $7 = 6/36 = 1/6$

and probability of not getting $7 = 1 - 1/6 = 5/6$

Let two events A and B are defined such that

A 'getting 6 in one throw'

B 'getting 7 in one throw'

then $P(A) = \frac{5}{36}, \quad P(\overline{A}) = \frac{31}{36}$

and $P(B) = \frac{1}{6}$ o $P(\overline{B}) = \frac{5}{6}$

A	A_W	$A_L B_L A_W$	$A_L B_L A_L B_L A_W$	...
B	$A_L B_W$	$A_L B_L A_L B_W$	$A_L B_L A_L B_L A_L B_W$	...

where A_W and A_L are of winning and losing of events A, Similarly B_W and B_L are winning are losing of events B

If A starts playing then the probability of winning A

$$P(A_W) + P(A_L B_L A_W) + (A_L B_L A_L B_L A_W) + \ldots$$

$$= P(A) + P(\overline{A})P(\overline{B})P(A) + \ldots$$

$$= \frac{5}{36} + \frac{31}{36} \times \frac{5}{6} \times \frac{5}{36} + \ldots$$

$$= \frac{5}{36}\left[1 + \left(\frac{31}{36} \times \frac{5}{6}\right) + \ldots\right]$$

$$= \frac{5}{36} \frac{1}{\left[1 - \left(\frac{31}{36} \times \frac{5}{6}\right)\right]} \qquad [S_\infty = \frac{a}{1-r}]$$

$$= \frac{5}{36} \frac{36 \times 6}{216 - 155} = \frac{30}{61}.$$

Example 38. If each element of a second order determinant is either zero or one, what is the probability that the value of the determinant is positive? (Assuming that the individual entries of the determinant are chosen independently, each value being assumed with probability 1/2).

Solution : Let the given determinant be $\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$,

where $a_{ij} = 0$ or 1; $i, j = 1, 2$

It is clear that $\Delta \le 0$ if $a_{11} = 0$ or $a_{22} = 0$ neither $a_{11} = 0$ nor $a_{22} = 0$ $\Rightarrow$ $a_{11} = 1 = a_{22}$ when $a_{11} = a_{22} = 1$ then $\Delta = 0$ if $a_{12} = a_{21} = 1$ so $a_{12} \ne 1$, $a_{21} \ne 1$ following are three posibility of values of Δ.

$$a_{11} = a_{22} = 1, a_{12} = 1, a_{21} = 0$$

$$a_{11} = a_{22} = 1, a_{12} = 0, a_{21} = 1$$

$$a_{11} = a_{22} = 1, a_{12} = 0, a_{21} = 0$$

Retired probability $= P(a_{11} = a_{22} = 1, a_{12} = 1, a_{21} = 0) + P(a_{11} = a_{22} = 1, a_{12} = 0, a_{21} = 1)$

$$+P(a_{11} = a_{22} = 1, a_{12} = 0, a_{21} = 0)$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{3}{16}.$$

Example 39. Find the mean of binomial distribution $B(4, 1/3)$.

Solution : Let X be a random variable whose probability distribution is $B(4, 1/3)$

here $n = 4,\ p = 1/3,\ q = 1 - p = 1 - 1/3 = 2/3$

and $P(X = x) = {}^4C_x \left(\frac{2}{3}\right)^{4-x} \left(\frac{1}{3}\right)^x$; $x = 0, 1, 2, 3, 4$

thus the probability distribution is

$X:$	0	1	2	3	4
$P(x_i):$	${}^4C_0\left(\frac{2}{3}\right)^{4-0}\left(\frac{1}{3}\right)^0 = \frac{16}{81}$	${}^4C_1\left(\frac{2}{3}\right)^{4-1}\left(\frac{1}{3}\right)^1 = \frac{32}{81}$	${}^4C_2\left(\frac{2}{3}\right)^{4-2}\left(\frac{1}{3}\right)^2 = \frac{24}{81}$	${}^4C_3\left(\frac{2}{3}\right)^{4-3}\left(\frac{1}{3}\right)^3 = \frac{8}{81}$	${}^4C_4\left(\frac{2}{3}\right)^{4-4}\left(\frac{1}{3}\right)^4 = \frac{1}{81}$

Mean $\mu = E(X) = \sum x_i p_i$

$$= 0 \times \frac{16}{81} + 1 \times \frac{32}{81} + 2 \times \frac{24}{81} + 3 \times \frac{8}{81} + 4 \times \frac{1}{81}$$

$$= \frac{32 + 48 + 24 + 4}{81} = \frac{108}{81} = \frac{4}{3}$$

Miscellaneous Exercise 16

1. Two events A and B are mutually independent if -

   (A) $P(A)=P(B)$ (B) $P(A)+P(B)=1$

   (C) $P(\bar{A}\bar{B})=[1-P(A)][1-P(B)]$ (D) A and B are mutually exclusive

2. What is the probability of getting even prime number on both the dice if pair of dice is rolled together ?

   (A) 1 / 3 (B) 0 (C) 1 / 36 (D) 1 / 12

3. If A and B ae events so that $A \subset B$ and $P(B) \neq 0$, then which of the following statement is true ?

   (A) $P\left(\frac{A}{B}\right)<P(A)$ (B) $P\left(\frac{A}{B}\right) \geq P(A)$ (C) $P\left(\frac{A}{B}\right)=P\frac{(B)}{(A)}$ (D) None of these

4. Two cards are drawn from the well shuffled pack of 52 cards Let X denote the number of aces, then find X -

   (A) 5 / 13 (B) 1 / 13 (C) 37 / 221 (D) 2 / 13

5. Let X takes the value 0, 1, 2, 3. The mean of X is 1.3. If $P(X=3)=2P(X=1)$ and $P(X=2)=0{\cdot}3$ then find $P(X=0)$.

   (A) 0.2 (B) 0.4 (C) 0.3 (D) 0.1

6. The probability of a girl being a racer is 4 / 5. Find the probability of 4 girls being a racer out of 5 girls.

   (A) $\left(\frac{4}{5}\right)^4\left(\frac{1}{5}\right)$ (B) ${}^5C_1\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)$ (C) ${}^5C_4\left(\frac{4}{5}\right)^4\left(\frac{1}{5}\right)^4$ (D) None of these

7. A box contains 100 objects out of which 10 are defective. The probability of the given 5 objects , find the probability that none of them are defective-

   (A) $\left(\frac{1}{2}\right)^5$ (B) 10^{-1} (C) $\frac{9}{10}$ (D) $\left(\frac{9}{10}\right)^5$

8. A couple has two children, find the probability -

   (*i*) that both are males if it is known that the elder one is a male

   (*ii*) that both children are female, if it is known that the elder child is a female

   (*iii*) that both children are males, if it is known that at least one of the children is male

9. Two integers are chosen form the numbers 1 to 11. Find the probability that both the numbers are odd if it is known that the sum of both is an even number.

10. An electronic assembly consists of two sub system, say, A and B. Form previous testing procedures, the following probabilities are assumed to be known :

    P (A fails) = 0.2

    P (B fails alone) = 0.15

    P (A and B fail) = 0.15

    Evaluate the following probabilities

    (i) P(A fails | B has failed)

    (ii) P(A fails alone)

11. Let A and B be two independent events. The probability that both occur together is 1 / 8 and probability that both do not occur is 3 / 8. Determine $P(A)$ and $P(B)$.

12. Anil speaks truth in 60 % of the cases and Anand speaks truth in 90 % of the cases. Find the probability that both of them contradicts on a statement.

13. Three people A, B and C toss a coin one by one. A person wins if he gets Heads first. Assuming that the game continues , if A starts the game, find the probability that A wins.

14. The probability of a person remains alive for the next 25 years is 4 / 5 and the probability that his wife remain alive for the same 25 yeas is 3 / 4, Find the probabilities that -

    (*i*) both are alive for the 25 years

    (*ii*) atleast one of them remain alive for the next 25 years.

    (*iii*) Only wife remain alive for the next 25 years.

15. In a group of children there are 3 girls and 1 boy, 2 girls and 2 boys and 1 girl and 3 boys. One child is selected at random form each group. Find the probability that the out of the three children selected there is 1 girl and 2 boys.

16. Bag 1 contains 3 black and 4 white balls and Bag II contains 4 black and 3 white balls. A die is thrown. If it shows 1 or 3 then a ball is drawn form Bag I and if some other number appear a ball is drawn from Bag II. Find the probability the drawn ball is black.

17. A person has under taken a construction job. The probabilities are 0.65 that there will be strike , 0.80 that the construction job will be completed on time if there is no strike and 0.32 that the construction job will be completed on time.

18. Bag I contains 8 white and 4 black balls and Bag II contains 5 white and 4 black balls. One ball is transferred from Bag I to Bag II and then a ball is drawn from Bag II. Find the probability that the drawn ball is white in colour.

19. On a multiple choice examination with four choices a student either guesses or knowns or cheat the answer. Find the probability of guessing or cheating the answer if it is known that he answers the question correctly.

20. A letter comes form the two cities *TATANAGAR* or *CALCUTTA*. Only alphabets TA is visible on the envelope. Find the probability that the letter comes from the city.

    (*i*) *CALCUTTA*

    (*ii*) *TATANAGAR*,

21. A manufacturer has three machine operators A, B and C. The first operator A produces 1 %, whereas the other two opeartors B and C produce 5% and 7% defective items resp. A is on the job for 50 % of the time, B is on the job for 30 % of the time and C is on the job for 20 % of the time. A defective item is produced, what is the probability that it was produced by A ?

22. A random variable X has a probability distribution $P(X)$ of the following form where K is some number:

$$P(X=x)=\begin{cases} k & \text{if } x=0 \\ 2k & \text{if } x=1 \\ 3k & \text{if } x=2 \\ 0, & \text{otherwise} \end{cases}$$

    (i) Find the value of K

    (ii) Find $P(X<2)$, $P(X\le 2)$ and $P(X\ge 2)$

23. A random variable takes all negative integral values and the value of X is 'r' whose probability is directly

proportional to α^r where $0<\alpha<1$ then find $P(X=0)$

24. Let X be the random variable with values x_1, x_2, x_3, x_4 such that

$$2P(X=x_1)=3P(X=x_2)=P(X=x_3)=5P(X=x_4)$$

Find the probability distribution of X

25. A fair coin is tossed to get one head or five tails. if X denotes the number of tosses then find the mean of X.

26. Three cards are drawn from the well shuffled deck of 52 cards. Find the probability distribution of number of red cards drawn. Also find the mean of the distribution.

IMPORTANT POINTS

1. If any random experiment if A and B are two events related to sample space then the conditional probability of event A, given the occuence of the event B is given by

$$P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)}; \qquad P(B)\neq 0.$$

similarly

$$P\left(\frac{B}{A}\right)=\frac{P(A\cap B)}{P(A)}; \qquad P(A)\neq 0$$

2. $0\le P\left(\frac{A}{B}\right)\le 1, \qquad P\left(\frac{\overline{A}}{B}\right)=1-P\left(\frac{A}{B}\right)$

3. If S is a sample space and A and B are two events then event F is such that $P(F)\neq 0$ then

$$P\left(\frac{A\cup B}{F}\right)=P\left(\frac{A}{F}\right)+P\left(\frac{B}{F}\right)-P\left(\frac{A\cap B}{F}\right)$$

4. Multiplication Rule of Probability

$$P(A\cap B)=P(A)P\left(\frac{B}{A}\right); \quad P(A)\neq 0 \text{ ;k } P(A\cap B)=P(B)P\left(\frac{A}{B}\right); \quad P(B)\neq 0$$

5. If A and B are independent, then

$$P\left(\frac{A}{B}\right)=P(A),\ P(B)\neq 0; \qquad P\left(\frac{B}{A}\right)=P(B),\ P(A)\neq 0$$

and $P(A\cap B)=P(A)P(B)$

6. Therorem of total probability

Let $A_1, A_2, A_3, ..., A_n$ be 'n' partition of a sample spaces let A be any event associated with S, i.e. then

$P(A_j)\neq 0;\ j=1,2,...,n$

$$P(E)=P(A_1)P\left(\frac{E}{A_1}\right)+P(A_2)P\left(\frac{E}{A_2}\right)+...+P(A_n)P\left(\frac{E}{A_n}\right)=\sum_{j=1}^{n}P(A_j)P\left(\frac{E}{A_j}\right)$$

7. Baye's Theorem -

$$P\left(\frac{A_i}{E}\right)=\frac{P(A_i)P\left(\frac{E}{A_i}\right)}{\sum_{j=1}^{n}P(A_j)P\left(\frac{E}{A_j}\right)}$$

8. A random variable is a real valued function whose domain is the sample space of a random experiment.
9. The probability distribution of a random variable X is the system of numbers

$$\begin{matrix} X=x & : & x_1 & x_2 & x_3 \ldots x_n \\ P(x) & : & p_1 & p_2 & p_3 \ldots p_n \end{matrix}\ ; \text{ where } p_i>0, \quad \sum_{i=1}^{n}p_i=1; \quad i=1,2,\ldots,n$$

10. Let X be a random variable whose possible values $x_1, x_2, \ldots, x_n$ occur with probabilities $p_1, p_2, \ldots, p_n$ respectively. The mean of X, denoted by μ is the number $\sum_{i=1}^{n}x_i p_i$
   (a) The mean of a random variable X is also called the expectation of X, denoted by E (X).
   (b) Varience of X

   $$=\text{var}(X)=\sigma_x^{\,2}=E(X-\mu)^2=\sum_{i=1}^{n}(x_i-\mu)^2 p_i$$

   (c) $\text{var}(X)=E(X^2)-\{E(X)\}^2$
   (d) Standard Deviation

   $$\sigma_x=+\sqrt{\text{var}(x)}=+\sqrt{\sum_{i=1}^{n}(x_i-\mu)^2 p_i}$$

11. Trials of a random experiments are called Bernoulli trials, if they satisfy the following conditions :
   (i) There should be a finite number of trials.
   (ii) The trials should be independent.
   (iii) Each trial has exactly two outcomes : success or failure.
   (iv) The probability of success remains the same in each trial.
12. Probability of r successes in binomial distribution is $B(n, p)$

$$P(X=r)={}^nC_r p^r q^{n-r}; \quad r=0,1,2,\ldots,n \text{ where } q=1-p.$$

ANSWERS

Exercise 16.1

1. 4 / 9 2. 16 / 25 3. 11 / 26 4. $P\left(\frac{A}{B}\right)=\frac{2}{3}$, $P\left(\frac{B}{A}\right)=\frac{1}{3}$ 5. (i) 0.32 ; (ii) 0.64 ; (iii) 0.98

6. 1/ 3 7. (i) $P\left(\frac{A}{B}\right)=1$ (ii) $P\left(\frac{A}{B}\right)=0$ 8. $P\left(\frac{A}{B}\right)=1$

9. (i) $P\left(\frac{A}{B}\right)=\frac{1}{2}$, $P\left(\frac{B}{A}\right)=\frac{1}{3}$; (ii) $P\left(\frac{A}{C}\right)=\frac{1}{2}$, $P\left(\frac{C}{A}\right)=\frac{2}{3}$; (iii) $P\left(\frac{A\cup B}{C}\right)=\frac{3}{4}$, $P\left(\frac{A\cap B}{C}\right)=\frac{1}{4}$

10. 1 / 15   11. 4 / 7   12. 0.1   13. 2 / 5   14. 2 / 9

Exercise 16.2

1. 3 / 8   2. 1 / 3   3. (i) 0.12 ; (ii) 0.58 ; (iii) 0.3; (iv) 0.4   4. (i) 0.18 ; (ii) 0.12 ; (iii) 0.72 ; (iv) 0.28

5. 1 / 969   6. 7 / 8   7. 25 / 102   8. 1 / 3   9. (i) 1 / 5; (ii) 1 / 3 ; (iii) 1 / 2

10. 0.97   11. 3 / 4   12. 1 / 7   13. (i) 2 / 3 ; (ii) 1 / 2

Exercise 16.3

1. 35 / 68   2. 1 / 2   3. 16 / 31   4. 2 / 3   5. 4 / 9   6. 22 / 133   7. 9 / 13

8. 1 / 52   9. 12 / 13   10. 20 / 21   11. 2 / 9   12. 8 / 11   13. 11 / 50   14. 2 / 9

Exercise 16.4

1. (i)

2.
$X = x$	0	1	2
$P(x)$	1/4	1/2	1/4

3.
$X = x$	0	1	2
$P(x)$	12/19	32/95	3/95

4.
$X = x$	0	1	2	3
$P(x)$	4/35	18/35	12/35	1/35

5. (i)
$X = x$	0	1	2	3
$P(x)$	1/6	1/2	3/10	1/30

(ii) 2 / 3   (iii) 1 / 6   (iv) 1 /2

6.
$X = x$	0	1	2
$P(x)$	4/9	4/9	1/9

7.
$X = x$	0	1	2	3	4
$P(x)$	1/14	8/21	6/14	4/35	1/210

8.
$X = x$	0	1	2	3
$P(x)$	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$

9. 7   10. 35 / 12   11. 7 / 10, 21 / 100

12. $\frac{34}{221}, \frac{6800}{(221)^2}, 0{\cdot}37$

Exercise 16.5

1. (i) 105 / 512 ; (ii) 193 / 512 ; (iii) 53 / 64   2. (i) $\left(\frac{1}{4}\right)^4$ (ii) $3\left(\frac{1}{4}\right)^3$ (iii) $\left(\frac{3}{4}\right)^4$ (iv) $\frac{13}{4^4}$

3. $\frac{5^{10}}{2\times 6^9}$ 4. $\frac{13}{16}$ 5. $1-\frac{9^{10}}{10^{10}}$ 6. (i) $1-\left(\frac{99}{100}\right)^{50}$ (ii) $\frac{1}{2}\left(\frac{99}{100}\right)^{49}$ (iiii) $1-\frac{149}{100}\left(\frac{99}{100}\right)^{49}$

7. (i) $\left(\frac{19}{20}\right)^5$ (ii) $\frac{6}{5}\left(\frac{19}{20}\right)^4$ (iii) $1-\frac{6}{5}\left(\frac{19}{20}\right)^4$ (iv) $1-\left(\frac{19}{20}\right)^5$ 8. $\frac{11}{243}$

9. $\dfrac{{}^{20}C_{12} + {}^{20}C_{13} + \ldots + {}^{20}C_{20}}{2^{20}}$ 10. $\left(\dfrac{9}{10}\right)^4$ 11. (i) $\dfrac{1}{1024}$; (ii) $\dfrac{45}{512}$; (iii) $\dfrac{243}{1024}$

13. $\dfrac{25}{216}$

Miscellaneous Exercise - 16

1. (C) 2. (C) 3. (B) 4. (D) 5. (B) 6. (C) 7. (D)

8. (i) 1 / 2 ; (ii) 1 / 2; (iii) 1 / 3 9. 3 / 5 10. (i) 1 / 2 ; (ii) 0.05

11. $P(A) = \dfrac{1}{2},\ P(B) = \dfrac{1}{4}$;k $P(A) = \dfrac{1}{4},\ P(B) = \dfrac{1}{2}$ 12. 0.42 13. 4 / 7, 2 / 7, 1 / 7

14. (i) $\dfrac{3}{5}$; (ii) $\dfrac{19}{20}$; (iii) $\dfrac{3}{20}$ 15. $\dfrac{13}{32}$ 16. $\dfrac{11}{21}$ 17. 0.488 18. $\dfrac{83}{150}$ 19. $\dfrac{24}{29}$

20. (i) $\dfrac{4}{11}$; (ii) $\dfrac{7}{11}$ 21. $\dfrac{5}{34}$ 22.(i) $\dfrac{1}{6}$; (ii) $P(X < 2) = \dfrac{1}{2},\ P(X \le 2) = 1,\ P(X \ge 2) = \dfrac{1}{2}$

23. $(1-\alpha)$ 24.

X	x_1	x_2	x_3	x_4
$P(X)$	$\dfrac{15}{61}$	$\dfrac{10}{61}$	$\dfrac{30}{61}$	$\dfrac{6}{61}$

25. 1.9