MATHEMATICS

CLASS- 10

BOARD OF SECONDARY EDUCATION, RAJASTHAN
AJMER

Text Book Translation Committee

MATHEMATICS

Class - X

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MATHEMATICS

Class - X

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PREFACE

This book has been written in accordance with the new syllabus for class X prescribed by the Board of Secondary Education, Rajasthan, Ajmer. In presenting the book the basic object of the syllabus has been fully kept in mind and an attempt has been made to acquaint the students with the contribution of Indian Mathematician towards the development of scientific traditions. The contribution of Indian Mathematician have been mentioned at appropriate places. Every effort has been made to present the subject in simple and lucid manner Important principal have been explained in detail.

In the interest of the students sufficient number the illustrative examples have been given. At the end of each chapter a summary of the chapter is given in the form of important points, which will help the students in revision. In each chapter objective, short and essay type questions have been given in sufficient number in the miscellaneous exercise.

We hope the book will be useful to students. Students, teachers and reviewers are requested to send their comments, suggestions and to point out any shortcoming in the book, so that the desired improvement in the book can be made in the subsequent edition.

Authors

SYLLABUS

MATHEMATICS

Class-X

Time- 3.15 hours **Subject code- 09**

Question paper	Marks for question paper	Sessional Marks	Max. Marks
One	80	20	100

S.N.	Name of Unit	Marks
1.	Vedic Mathematics	4
2.	Numbers System	3
3.	Algebra	12
4.	Trigonometry	11
5.	Coordinate Geometry	6
6.	Geometry	20
7.	Mensuration	10
8.	Statistics and Probability	10
9.	Road Safety Education	4

Details of the Syllabus

Unit 1. Vedic Mathematics 4

Fundamental Concepts of Vedic mathematics

Expansion and exercise on fundamental operations, the meaning and applications of sutra Urdhva triyagbhyam, study to find square and cube by the sutra Nikhilam base – subbase, division operation (sutra Nikhilam, sutra paravartya yojayet and dhvajanka method) solution of simple equations by Vedic system, Navanka and Ekadashanka methods of checking the result for division operation.

Unit 2. Number System 3

Real numbers

Euclid's division lemma, Fundamental theorem of Arithmetic- Statements, After reviewing work done earlier and after illustrating through examples, Proofs of results irrationality of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, decimal expansions of rational numbers in terms of terminating/non-terminating recurring decimals.

Unit 3. Algebra 12

(a) Polynomials 4

Zeroes of polynomials. Relationship between zeroes and Coefficients of quadratic polynomials, statement and simple problems on division algorithm for polynomials with real coefficients. Standard form of a quadratic equation and its solution. Discriminant and nature of roots. LCM and HCF of Algebraic expression.

(b) Linear equation and Inequalities in two variables 5

Pair of linear equations in two variables. The graphical solution of pair of linear equation and its different possibilities, linear inequalities in two variable.

(c) Arithmetic Progressions 3

Motivation for studying A.P. Derivation of Standard results of finding the n^{th} term and sum of first n terms.

Unit -4. Trigonometry 11

(a) Trigonometric ratios

Trigonometric ratios of an acute angle of right-angled triangle. Values of trigonometric ratios of 0°, 30°, 45°, 60°, 90°. Relationship between the ratios.

(b) Trignometric identities

The application of trigonometric identities Trigonometric ratios of complementary angles.

(c) Height and Distance

Angles of elevation/depression. Simple problems on height and distance. (based on 30°, 45°, 60°)

Unit 5. Coordinate Geometry 6

Coordinate Geometry

Cartesian plane, Coordinates, Distance between two points, section formulae. Area of a triangle.

Unit 6. Geometry 20

(a) Point and Concurrent lines

Locus, concurrent points of a triangle (Circumcentre, incentre, orthocentre)

(b) Similar Triangles

Similarity, similar triangle and its related theorms, Theorms related areas of similar triangles.

(c) Circle

Relation between arc and angle of congruent circles. Chord and its related theorm, arc and angles subtended by it, cyclic quadrilateral, Tangents of circle and its related theorm. Chord and angle of Alternate segment.

(d) Practical Geometry

Internal division of a line segment in a given ratio, Construction of tengents from the outer point of circle. Construction of common tengents of two circles, construction of circumcircle and in circle of triangles.

Unit- 7. Mensuration **10**

(a) Area of plane figures 4

Circumference of a circle and its area, area of sectors and segments of a circle.

(b) Surface areas and volumes 6

Surface areas and volumes of cube, cuboids, sphere, hemisphere, right circular cylinder, cone . Converting one type of metallic solid into another

Unit- 8 Statistics and probability **10**

(a) Statics 6

Mean, Median and mode of ungrouped and grouped data.

(b) Probability 4

Random experiment, classical definition of probability. Simple problems on single events.

Unit- 9 Road Safety Education 4

Arithmetic Progression (Objective, content, exercise) Data collection (objective) Applications of trigonometry (Objective, content, exercise) Problems on two variables (Objective).

Prescribed book : **Mathematics**
Board of Secondary Education, Rajasthan, Ajmer

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1

Vedic Mathematics

1.01 Introduction :

In the previous class we have studied that Swami Bharati Krishna ji Tirth meditated hard for eight years while residing at Sringeri.During his meditation in a highly enlightened state, he realized through introspection the Sutras or Aphorisms of Vedic Mathematics. According toVedic scholars Vedic classics is neither man made nor the knowledge of Vedas it is the result of man's mental thinking or mental labour. It is the knowledge in the form of mantras received after his realization of the Divine power. In this context the Sutras reconstructed by Swami ji are Vedic Mathematical Sutras.

1.02 Importance of Vedic Mathematics :

Regular mental practice of Vedic Mathematcal sutras imporves both the concentration and memory of the human being and enhances analytical and synthetical acumen of thought. The easy, interesting and attractive approach of Vedic Mathematics arouses an attitude of curiosity in the individual. This curiosity makes the man sentient.

The consciousness of the man is awakened which develops the neocortex and personality of a man.

1.03 Revision and Extension of Basic Operations :

(i) Addition :

In the previous class we studied the addition operation of whole numbers and measure unit distance (Km.-m.) by the Sutra Ekadhikena Purvena. In fact every question of addition of measurments like – money, weight, capacity, distance, time and whole numbers, decimal numberts etc. can be performed by this Sutra.

Precautions before Addition :

1. Keep the number of coloumns of sub unit according to the rule i.e. to write 05 paisa for 5 paisa in the sub unit of money, or 084 metres for 84 metres in the sub unit of distance.
2. After writing the figures columnwise add accordingly by the Sutra.
3. In the time unit, while adding keep the base (sum) = 10 in first column of second and minute. Keep base (sum) = 6 in the second column of minute and second. In hour's column base (sum) = 10 is always chosen.

   Let us see the following example.

Example 1 : Adding the following :

kg	gm
112	065 ↓
360	$\dot{0}\dot{8}5$
$\dot{2}\dot{8}\dot{9}$	$\dot{8}72$
156	345
918	367

Steps :

(i) Write 065 gms. for 65 gms and 085 gms for 85 gms.

(ii) Start adding from unit column.

(iii) $5+5=10$,

hence Ekadhika dot on digit 8 prior to 5,

Remainder $=10-10=0$

(iv) Further $0+2+5=7$, write at answer's place.

(v) Proceed accordingly.

(ii) Addition by Mental Process :

(Sutra Ekadhikena Purvena + Shunyant Number Method)

By a little pratice of above process addition of big numbers can be preformed very fast. Concept of a Shunyat number process is a speciality of Ancient Indian Mathematics which is very easy and effective in addition operation. In this method two digits at unit and ten's place of the numbers can also be added, if needed.

Method :- Out of two numbers make one a shunyat number. Complement its deficiency by the second number. Now add both the new numbers. The sum if so obtained is greater than 100, mark an Ekadhika dot on the previous number. Add the remainder to the next number. In the end write the last remainder on the ansewer's place. For the remaining columns repeat the process as carried before. The method will be clear from the follwing examples.

Example 2. Add 35 and 58.

Solution : To make 58 a shuyant number 60, 2 is needed. This deficiency of two is covered form the number 35. Hence

$35 + 58 = 33 + 2 + 58 = 33 + 60 = 93$

Example 3. Add 19 and 65.

Solution : $19 + 65 = 19 + 1 + 64 = 20 + 64 = 84$

Note: Any numbers can be added by this method.

Example 4. Add the following :

$$\begin{array}{r} 4\,9\,9\,8 \\ \dot{0}\,6\,\dot{7}\,8\,9 \\ 5\,\dot{7}\,1\,5 \\ \dot{0}\,4\,8\,3\,7 \\ \dot{0}\,8\,\dot{9}\,7\,6 \\ \hline 3\,1\,3\,1\,5 \\ \hline \end{array}$$

Steps :

(i) $98 + 89 = 98 + 2 + 87 = 100 + 87 = 187$
Hence mark an Ekadhika dot on 7 before 89.

(ii) Rest $87 + 15 = 87 + 3 + 12 = 90 + 12 = 102$
Hence mark an Ekadhika dot on 7 before 15.

(iii) Rest $02 + 37 = 39$. Now

$$39 + 76 = 35 + 4 + 76 = 35 + 80$$
$$= 15 + 20 + 80 = 115$$

(iv) Hence mark an Ekadhika dot on 9 before 76 write 15 at answer's place.

(v) The remaining addition process will be completed accordingly.

Example 5. Add the following :

$$\begin{array}{r} 7\,5\,3\,4 \\ 2\,4\,5\,9 \\ \dot{0}\,1\,\dot{9}\,3\,2 \\ 6\,5\,4\,7 \\ \hline 1\,8\,4\,7\,2 \\ \hline \end{array}$$

Steps :

(i) $34 + 59 = 33 + 1 + 59 = 33 + 60 = 93$

(ii) $93 + 32 = 93 + 7 + 25 = 100 + 25 = 125$
hence Ekadhika dot on 9.

(iii) Remaining $25 + 47 = 22 + 3 + 47 = 22 + 50 = 72$ write at answer's place.

(iv) The remaining addition process can be completed as above.

(iii) Subtraction

In the previous class we studied two Vedic methods of substraction as given below :

1. Sutra Ekadhikena Purvena + Param mitra Aunka method
2. Sutra Ekanyunea Purvena + Param mitra Aunka method

By the first method every question of substraction of an measure unit or whole numbers can be performed. Let us revised this same method again. We know that the two numbers are called best friends fo each other if their sum is equal to ten. Vyojya and Viyojak are also known to us. This method will be clear by the following illustrations.

Examples 6. By Vedic method subtract the follwing :

$$\begin{array}{r} 8\,0\,0 \\ -\dot{2}\,\dot{6}\,3 \\ \hline 5\,3\,7 \\ \hline \end{array}$$

Steps :

(i) 3 can not be subtracted form 0, hence add param mitra aunka i.e. best friend of 3 to 0 $\therefore$ $0 + 7 = 7$. Write it at answer's place. Also mark an Ekadhika dot on 6 prior to 3.

(ii) $\dot{6} = 7$ can not be subtracted form 0 hence add param mitra aunka of 7 to 0 i.e. $0 + 3 = 3$. Write it at answer's place. Also mark an Ekadhika dot on 2 prior to 6.

(iii) $8 - \dot{2} = 5$, write it at answer's place. Hence Remainder = 537

Example 7. Subtract the following by Vedic method.

km.	m.	cm.
3 7	4 6 7	3 5
$\dot{2}$ 8	$\dot{3}$ 7 $\dot{5}$	$\dot{4}$ 6
0 9	0 9 1	8 9

Steps :

(i) Columns of cm and metre are arranged accordingly.

(ii) In cm column 6 is subtracted form 5. Hence add param mitra aunka of 6 to 5.

(iii) Sum = 4 + 5 = 9, write it at anwer's place and make an Ekadhika dot on 4 prior 6.

(iv) $\dot{4}$ = 5 can not be subtracted from 3 hence add best friend of 5 to 3i.e. 5 + 3 = 8.

(v) Write 8 at answer's place. Also mark an Ekadhika dot on 5of metre's unit place.

(vi) 7 - $\dot{5}$ = 1, write it at answer's place.

(vii) 7 can not be subtractedfor 6, hence add best friend of 7 to 6, i.e, 3 + 6 = 9 write 9 at the answer's place and mark an Ekahdika dot on 3.

(viii) $4 - \dot{3} = 0$, write it at answer's place.

(ix) Proceed further in the same way.

Finally Remainder = 9 km. 91 m. 89 cm.

(iv) Multiplication :

In the previous class we studied in details the multiplication methods based on three important sutras. We should practice these methods extensively so that just by throwing a glance on any question, we may able to choose the best method for a quick solution.

Sutra Urdhva Triyagbhyam

Any question of multiplication can be done orally by Sutra Urdha Triyagbhyam. Only a line space is required for writing the answer. Sutra can be applied from both sides i.e. left or right.

(a) Meaning :

Sutra is composed of two words 'Urdhva' and 'Triyank' Its meaning is 'Vertically and crosswise'. Urdha means vertically or straight. Its symbol is ↑ or ↓ or ↕ and its function is to multiply the digits written vertically. The word Triyak means crosswise. Its symbol is ↗ or ↖ or ⤧ and its function is to multiply the digits written at cross.

(b) Applications:

(i) Multiplication :

Method : Firstly the digits of numbers in question are arranged in columns. The deficiency of the digits is fulfilled by introducing zeroes at those places. Groups are formed by these columns.

No. of Groups = No. of columns × 2 – 1 = an odd number always

Now in groups symbols are marked. According to these symbols multiplication is carried out. Lastly the products are written in a special arrangement. Finally their sum gives the product of two numbers.

Note: (1) In any groups the total number of symbols is always equal to the number of columns of that group.

(2) All these symbols of a group pass through a common point.

(3) A group formed on even place consists of cross symbols in pairs.

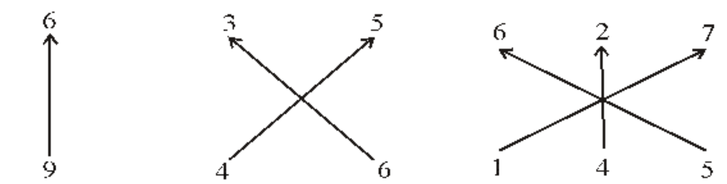
(4) A group formed at odd place consists of one vertical symbol only which is marked in the first or last group. The remaining symbols of this group are cross symbols in pairs.

(5) The middle group is always the largest one and is equal to the question itself. The method is explained by the following examples.

(6) Multiply by Sutra vertically and crosswise:

Example 8 : Calculate the products by marking symbols of the following groups.

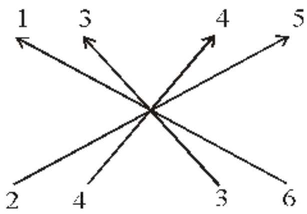
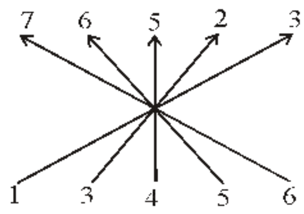
(1) (2) (3)



(1) $= 6 \times 9$ $= 54$ Vetically

(2) $= 6 \times 3 + 4 \times 5$ $= 38$ Cross in pair

(3) $= 5 \times 6 + 1 \times 7 + 4 \times 2$ $= 45$ Two cross pairs + Vertically

(4)	(5)
	
$=(6\times1+2\times5)$ $+(4\times4+3\times3)$ $=41$	$=(6\times7+1\times3)$ $+(5\times6+3\times2)$ $+4\times5$ $=101$
Two cross pairs	Two cross pairs + vertically

Find : 147 x 28

			1	4	7
		×	0	2	8
0	2	6	6	6	
	1	4	5		
=	4	1	1	6	

Five groups formed

V	IV	III	II	I
1	14	147	47	7
↑	×	×	×	↑
0	02	028	28	8
$=0$	$=2$	$=16$	$=46$	$=56$

Addition of all above five products are as follows :

(1) In 56 write 6 as unit place and 5 in second line of ten's place.

(2) From 46, write 6 in first line of ten's place and 4 write in second line of hundred's place.

(3) Similary 16, 2, 0 and 1 adjusted in I and II line as shown above. Sum all terms.

Example 9. Choose the best sutra based method in finding the product of 588×512 .

I Solution : In this multiplication let us examine first method based on the Sutra Ekadhikena Purvena.

, Sum of the digits of unit and ten's place $=88+12=100$

Rest Nikhilam digit is equal to each other $=5$, hence Sutra Ekadhikena Purvena is effective.

According to Sutra

$$588\times512=5\times6/88\times12 \quad \text{(Four digits in R.H.S.)}$$

$$=301056$$

II Solution : Sutra Nikhilam (Sub-base) method is applicable in this question

$$588\times512$$

$$=\begin{array}{ll} 588 & +88 \\ \times512 & +12 \end{array}$$

$$=5(588+12)/88\times12$$

$$=5\times600/_{10}56$$

$$=3000/_{10}56=301056$$

Steps :

(i) Base $=100$

(ii) Sub Base $=100\times5$

(iii) Upadhar aunka $= 5$

(iv) Deviations are +12 and +88.

(v) Two digits in R.H.S.

III Solution : Sutra Eka nyunena Purvena is not applicable in this question as out if these two numbers one must be composed of digit 9 only.

IV Solution : Product of 512 and 588 can be found out by Sutra Urdhva Tiriyak method. There are three columns in the question hence five groups are to be evaluated. Find groupwise products and write them in a special way. In the end add them to get the final product.

$$\begin{array}{r} 5\,8\,8 \\ \times\,5\,1\,2 \\ \hline 2\,5\,5\,8\,4\,6 \\ 4\,5\,2\,1 \\ \hline =3\,0\,1\,0\,5\,6 \end{array}$$

Results

1. Final product = 301056.
2. In the firtst solution, the product is quickly recieved by Sutra Ekadhikena Purvena and this is the best sutra out of all.

Example 10. Choose the best sutra based method in finding the product of 842×858 .

I Solution : Sutra Ekadhikena Purvena is not effective in this case as in the R.H.S. the product 42×58 cannot be obtained easily.

II Solution : Sutra Nikhilam (Base) method is also not effective as deviations will be - 158 and - 142 w.r.t. base 1000. Nikhilam Sub base method is also not effective as the deviations w.r.t. sub base = 800 will be 42 and 58.

III Solution : Sutra Ekanyunena Purvena is not applicable in this case.

IV Solution : Sutra Urdhva Tiriyak is very effective in all types of multiplication. The calculations can be hard as the numbers are composed of big digits. Hence new alternative is to be searched out.

V Solution : In finding out the product of 842×858 start with the Sutra Ekadhikena Purvena and finish it up by Sutra Urdhva Triyak.

$$\begin{aligned} & 842 \times 858 \\ &= 8 \times 9 / 42 \times 58 \\ &= 72/2436 \\ &= 722436 \end{aligned}$$

Steps :

(i) By sutra Urdhva Triyak get the product mentally

$$\begin{array}{r} 42 \\ \times 58 \\ \hline 2436 \\ \hline \end{array}$$

1.04 Squaring (Sutra Nikhilam – Base, Sub-Base)

We have studied the multiplication of two numbers by Sutra Nikhilam - Base, Sub-Base methods. When both the numbers are equal to each other, it becomes a square operation.The squaring of the numbers is explained by the following examples.

Base method : Formula : $(\text{Number})^2$ = Number + deviation / $(\text{deviation})^2$

1. $17^2 = 17 \times 17$, Base $= 10$, deviation $= +7$

$= 17 + 7$
$ 17 + 7$

$17 + 7 / 49$

$= 24/_4 9 = 289$

or

$17^2 = 17 + 7/7^2$
$= 24/_4 9$

$= 289$

2. $98^2 = 98 \times 98$, Base $= 100$, deviation $= -02$

$= 98 - 02/(-02)^2$
$= 9604$

3. $104^2 = 104 + 04/(04)^2$

$= 10816$

4. $115^2 = 115 + 15/15^2$
$= 130/_2 25$
$= 13225$

Sub-Base Method :

Formula :– $(\text{Number})^2 = \text{Sub base digit } (\text{number} + \text{deviation})/(\text{deviation})^2$

5. $23^2 = 23 \times 23$, Base $= 10$, Sub-Base $= 10 \times 2$, Deviation $= +3$

Nikhilam sub base method or Squaring in one line-

number deviation

$= 23 \quad +3$
$ 23 \quad +3$

$= 2(23 + 3)/3^2$

$= 529$

$23^2 = 2(23 + 3)/3^2 = 529$

6. 64^2, Base $= 10$, Sub-base $= 10 \times 6$, Deviation $= +04$

$= 6(64 + 4)/4^2$
$= 408/_1 6$
$= 4096$

7. 308^2, Base $= 100$, Sub-base $= 100 \times 3$, Deviation $= +08$

$= 3(308 + 08)/(08)^2$
$= 94864$

1.05 Cube (Sutra Nikhilam - Base, Sub-Base)

We have studied the methods of multiplication of three numbers by Sutra Nikhilam Base, Sub-Base When all the three numbers are equal then cube can be obtained by the also method.

Base method :

Formula : $(\text{Number})^3$ = Number + 2 x Deviation / 3 x $(\text{Deviation})^2$ / $(\text{Deviation})^3$

1. 12^3, Base $=10$, Deviation $=+2$

$$=12+2\times(2)\Big/3\times(2)^2\Big/(2)^3$$

$$=16\Big/12\Big/8=1728$$

2. 105^3, Base $=100$, Deviation $=+05$

$$=105+2\times(05)\Big/3\times(05)^2\Big/(05)^3$$

$$=115\Big/75\Big/_{1}25=1157625$$

3. 98^3, Base $=100$, Deviation $=-02$ (Here deviation is negative)

$$=98+2\times(-02)\Big/3\times(-02)^2\Big/(-02)^3$$

$$=94\Big/12\Big/-08$$ (taking 1 from 12 in third section)

$$=94\,11\,92$$ (in third section value of 1 is 100, ∵ base = 100)

Sub-Base Method Formula of cubing a number

Formula : $(\text{Number})^3$ = $(\text{Sub base digit})^2$ (number + 2 x deviation)
sub-base x 3 x $(\text{deviation})^2$ / $(\text{deviaton})^3$

4. 35^3, Base $=10$, Upadhar Unka $=3$, deviaton $=5$

$$=3^2(35+2\times5)\Big/3\times3\times(5)^2\Big/5^3$$

$$=9\times45\Big/9\times25\Big/125$$

$$=405\Big/_{22}5\Big/_{12}5=42875$$

5. 497^3

$$=5^2\{497+2\times(-03)\}\Big/5\times3\times(-03)^2\Big/(-03)^3$$

$$=25\times491\Big/5\times27\Big/-27$$

$= 12275 \,/_1\, 35 \,/\, (-27)$

$= 12276 \,/\, 34 \,/\, 100 - 27$

$= 122763473$

1.06 Division Operation :

In this class we will studied the division operation based on the following three sutras :

1. Sutra Nikhilam
2. Sutra Paravartya Yojeyeta
3. Sutra Urdhva Tiriyak

Sutra Nikhilam based division method is effective only when the digits of the divisor are greater than 5 and with respect to base 10 or any power of 10 the complement of the divisor can be found out. In this method the main operation is done by the complement of the divisor and digit 1 on its extreme left.

If the divisor contains digits lower than five or it can be reduced into a number of digits lower than five with digit 1 on its left side and deviations can be calculated w.r.t. base 10 or power of 10, the division based on Sutra paravartya Yojeyeta can be performed. This is the only method which can be applied in Algebra.

Dhwajanka Method of Division : (Sutra Urdhva Tiriyak) Every question of division can be performed by this method. Selection of the Mukhyanka and Dhwajanka of the divisor is important. There can be any number of digits in Dhwajanka as well as in Mukhyanka but division by Mukhyanka is a necessary condition. So many digits form unity are written in third part as there are digits in the Dhwajanka. The method is explained here with by the following examples.

Division (Sutra Nikhilam) :

Sutra Nikhilam based method is always suitable in case the each digit of divisor is greater than 5.

Method to write the question

Divide the required place in three parts by drawing two vertical lines. From the left in the first place write the divisor and its complementary number beneath it. Write as many digits of dividend from unit place in the third part as there are zero's in the base. Write the remainder digits of dividend in the middle part.

Nikhilam method:

From left hand side write the first digit of dividend down at the sum's place below the horizontal line. Multiply this number by the complementary number and write the product in the middle part below the second digit of the dividend. If there are two digits in the complementary number, write the product below the third digit of the dividend also. Add only second place digits upper and lower and write it down at the sum's place don't add the digits of third place. Multiply the second digit of sum's by the complementary number and write the product in the middle part below the third digit of the divident and add. Repeat this process until the product may cover the last digit i.e. unit digit of the dividend in the third part. Add, again lastly the sum so achieved in the middle part is equal to the quotient and sum achieved in the third part is equal to the remainder.

If the remaidner is greater than the divisor than subtract the divisior from it and get the final quoteint and final remainder.

The method is explained by the following examples.

Example 11 : (i) $311 \div 8$, Base $= 10$

First	Second		Third
8	3	1	1
2		6	–
			14
	3	7	15
		+1	–8
	3	8	7

Steps

(i) Quotient = 37, Remainder = 15

(ii) Remainder 15 > divisor 8

(iii) Adjustment is necessary

(iv) New Quotient = 38
Remainder = 7

(ii) $10025 \div 88$, Base $= 100$

8	8	1	0	0	2	5
1	2		1	2	–	–
				1	2	–
					3	6
		1	1	3	8	1

Steps

(i) Complementary number = 1000 – 88 = 12

(ii) In the middle part write 1 beneath 1 as shown. Now 1 x 12 = 12 is written in middle part beneath the next digits as shown.

(iii) 0 + 1 = 1 is written in middle part as shown.

(iv) 1 x 12 = 12 is written in middle and third part beneath the digits as shown.

(v) 0 + 2 + 1 = 3 is written in middle part.

(vi) 3 x 12 = 36 is written in the third part digits as shown on adding.

(vii) Quotient = 113

(vii) Remainder = 81

Sutra Paravartya Yojayet

Sutra is widely employed in several fields like solution of equations, formation of magic squares and also in division specially in Algebra.

(a) Meaning:

Meaning of Sutra Paravartya yojayet is "Transpose and adjust", e.g. in case of transposition the sign are changed into opposite one i.e. $(+)$ into $(-)$, $(-)$ into $(+)$, $(\times)$ into $(\div)$, and $(\div)$ into $(\times)$. Similalry in case of magic squares after every last line or last column, there starts the formation of a new number in a coming line or column.

(b) Application :

Division : Divisor based on Sutra Paravartya Yojayet is convenient only when the divisor is very close to the base = 10 or power of 10 and its first digit from left is one. When the first digit is not one but it can be adjusted into one even then this method is applicable.

Method:

(1) Subtract the nearest base from divisor and get the deviation. If the digit of deviation is greater than 5, convert it into small digit by Vinculum operation. Now transpose the sign of every digit of deviation.

(2) (i) Divide the place of division in three parts.

(ii) In the first part from the L.H.S. write the divisor, below it the deviation, and finally the transposed digits below the deviation. With a slight practice the transposed digits can be directly written below the divisor.

(iii) According to the number of deviation digits or the number of zeroes in the base the third part is fulfilled by the dividend digits. Lastly the second part is fulfilled by the remaining dividend digits.

(iv) The further steps are like that of Sutra Nikhilam.

The method is explained by the following examples:

Example 12 : $1358 \div 113$, Base $= 100$

1 1 3	1 3	5 8
1 3	–1	–3 –
–1 –3		–2 –6
	1 2	0 2

Steps

(i) Write dividend digit 1 of the second part at the answer's place.

(ii) Multiply transposed digits –1–3 by this dividend digit 1.
Write the product –1–3 below 3 and 5.

(iii) 3 – 1 = 2, Again write the product –2 – 6 below 5 and 8

∵ $2[-1-3] = -2-6$

By adding Quotient = 12, Remainder = 02

(2) $395166 \div 1321$, Base $= 1000$

1 3 2 1	3 9 5	1 6 6
$\overline{3}$ $\overline{2}$ $\overline{1}$	$\overline{9}$ $\overline{6}$	$\overline{3}$ – –
	0	0 0 –
		3 2 1
	3 0 $\overline{1}$	1 8 7

Quotient $= 30\overline{1} = 299$

Remainder $= 187$

(Dhvajanka Sutra)

Any big question of division can be solved with minimum calculations by this universal method based on Dhvajanka Sutra. In this method divisor is divided into two parts (i) main divisor i.e. Mukhyanka (ii) Top flag digit i.e. Dhvajanka. The unit digit of divisor or unit digit with more digits with more digits of divisor are written on the top of the flag and known as Dhvajanka. The remainder part of the divisor is known as main divisor (Mukhyanka) and is written on the base place. It conducts the whole divisior process.

Method :

(1) Divide the place of division in three parts.

(2) In the first part from L.H.S., write Mukhyanka at base and Dhvajanka at the power place.

(3) Write so many digits of dividend as that of Dhvajanka in the third part starting from the unit place.

(4) Write now the remaining digits of dividend in the middle or second part.

More details of the method are given in the following illustrations.

Example 13 :

(1) $23754 \div 74$ (Dhvajanka method)

^{4}	2 3 7 5	4
7	2 1	0
	3 2 1	$4 - 1 \times 4 = 0$

Steps

(i) $23 \div 7$, Quotient $\text{digit}_1 = 3$, write it below the horizontal line.
Write Remainder $= 2$ below and before dividend digit 7.

(ii) New Remainder $= 27$, Adjust it by the formula,
Dividend adjusted $=$ New Dividend $-$ Quotient digit $\times$ Dhvajanka
$= 27 - 3 \times 4 = 15$

(iii) $15 \div 7$, Quotient $\text{digit}_2 = 2$, Remainder $= 1$
Write it according to step no. (i)

(iv) New Dividend $= 15$, Dividend adjusted $= 15 - 2 \times 4 = 7$

(v) $7 \div 7$, Quotient $\text{digit}_3 = 1$, Remainder $= 0$

(vi) Last Remainder $= 04 - 1 \times 4 = 0$

$\therefore$ Quotient $= 321$, Remainder $= 0$

(2) $21112 \div 812$ (Dhvajanka method)

^{12}	2 1 1	1 2
8	5	1
	2 6	112-100-12=0

Steps

(i) $21 \div 8$, Quotient digit $_1 = 2$, Remainder $= 5$

(ii) New Dividend $= 51$, Dividend adjusted $= 51 - 2 \times 1 = 49$

(iii) $49 \div 8$, Quotient digit $_2 = 6$, Remainder $= 1$

(iv) Dividend adjusted or Last Remainder

$$= 112 - (6 \times 1 + 2 \times 2)10 - 6 \times 2$$
$$= 112 - 100 - 12 = 0$$

Final Quotient $= 26$, Remainder $= 0$

Note: (1) Dhvajanka 1 2

Quotient 2 6

Three groups are formed by Dhvajanka and Quotient.

Groups	1	1 2	2
	↑	⤧	↑
	2	2 6	6
	1×2	$(6 \times 1 + 2 \times 2)$	6×2
	$= 2$	$= 10$	$= 12$

(2) (i) Product of I group = 2 subtracted from 51

(ii) Product of II group $= 10 \times 10 = 100$ subtracted from 112

(iii) Product of III group = 12 also subtracted from 112

(3) In the end as soon as the remainder enter into third part, division process is over.

Example 14. $98765 \div 87$ **(Dhwajanka method)**

7	9	8	7	6	5
8		1	3	6	5
	1	1	3	5	$55 - 5 \times 7 = 20$

Steps :

(i) Divisor $= 87$, Mukhyanka $= 8$, Dhwajanka $= 7$

(ii) In III part only one digit of dividend is kept i.e. $= 5$

(iii) $9 \div 8$, Quotient first digit $= 1$, Remainder $= 1$

(iv) New Dividend $= 18$, Modified dividend $= 18 - 1 \times 7 = 11$

(v) $11 \div 8$, Quotient II digit $= 1$, Remainder $= 3$

(vi) New Dividend $= 37$, Modified dividend $= 37 - 1 \times 7 = 30$

(vii) $30 \div 8$, Quotient III digit $= 3$, Remainder $= 6$

(viii) New Dividend 66, Modified dividend $= 66 - 3 \times 7 = 45$

(ix) $45 \div 8$, Quotient IV digit $= 5$, Remainder $= 5$

(x) New Dividend 55,

Modified dividend or last Remainder $= 55 - 5 \times 7 = 20$

$\therefore$ Quotient $= 1135$, Remainder $= 20$

Example 15. $13579 \div 975$ (Dhwajanka method)

75	13	5	79
9		4	11
	1	3	$1179 - 260 - 15 = 904$

Steps:

(i) Divisor = 975, Mukhyanka = 9 and Dhwajanka = 75, hence two digits 79 are written in III part.

(ii) $13 \div 9$, Quotient I digit = 1, Remainder = 4

(iii) New dividend = 45, Modified dividend $= 45 - 1 \times 7 = 38$

(iv) $38 \div 9$, Quotient II digit $= 4$, Remainder $= 2$

(v) New dividend = 27,

Modified dividend $= 27 - (4 \times 7 + 1 \times 5) = 27 - 33 = -6$

As the modified dividend is negative, hence quotient II digit 4 is rejected and 3 will be more suitable ($\neq 4$).

* Therefore steps nos. (iv) and (v) are liable to be rejected.

(vi) Again $38 \div 9$, Quotient II digit = 3, Remainder = 11

(vii) New dividend $= 1179$, Modified dividend or last Remainder

$$= 1179 - (3 \times 7 + 1 \times 5) \times 10 - 3 \times 5 = 1179 - 260 - 15 = 904$$

Quotient = 13, Remainder = 904

Exmple 16. $21015 \div 879$ (Dhwajanka method)

Solution : The big digits of divisor 879 are changed into smaller digit by Vincular method. $879 = \dot{8}\bar{2}\bar{1} = 9\bar{2}\bar{1}$

Hence Mukhyanka = 9 and Dhwajanka = $\bar{2}\bar{1}$

$\bar{2}\bar{1}$	21	0	15
9		3	7
	2	3	$715 + 80 + 3 = 798$

Steps :

(i) $21 \div 9$, Quotient first digit = 2, Remainder = 3

(ii) New dividend = 30, Modified dividend $= 30 - 2 \times \bar{2} = 34$

(iii) $34 \div 9$, Quotient II digit = 3, Remainder = 7

(iv) New divided = 715, Modified dividend or

last Remainder $= 715 - (3 \times \bar{2} + 2 \times \bar{1})10 - 3 \times \bar{1}$

$$= 715 + 80 + 3 = 798$$

$\therefore$ Quotient = 23, Remainder = 798

Example 17. $7453 \div 79$

$\bar{1}$	74	5	3
8		2	2
	9	4	$23 + 4 = 27$

Steps :

(i) Divisor $79 = 8\bar{1}$, Mukhyanka = 8, Dhwajanka = $\bar{1}$

(ii) $74 \div 8$, Quotient first digit = 9, Remainder = 2

(iii) New dividend = 25, Modified dividend $= 25 + 9 = 34$

(iv) $34 \div 8$, Quotient II digit = 4, Remainder = 2

(v) New dividend or last Remainder $= 23 + 4 = 27$

$\therefore$ Quotient = 94, Remainder = 27

Attention :

1. When the unit digit of the divisor = 9 is converted into Dhwajanka $\bar{1}$, see that Modified dividend = New dividend + Previous quotient digit.
2. When unit digit of the divisor is = 1 i.e. Dhwajanka is also equal = 1, then Modified diviednd = New dividend - Previoud Quotient digit.
3. In both these cases writing of steps can be avoided.

Example 18. $43758972 \div 81$

Mukhyanka = 8, Dhwajanka = 1

1	4	3	7	5	8	9	7	2
8		3	0	1	2	3		2
		5	4	0	2	3	4	$22 - 4 = 18$

Quotient = 540234 Remainder = 18

1.07 Algebra

Solution of Simple Equations (Vedic Method)

Simple equation can be solved quickly by the sutras Paravartya Yojayet and Sunyam Samya samuccaye. The application of these sutras are very simple and based on mental calculation.

Sutra Paravartya Yojayet

The sutra means "transformation and adjustment".Swami Bhartikrishna Ji Teerth has discussed appliation of the sutra. All these sutras give answers by mere metnal arithmetic calculation in one line.

First Application

If $ax + b = cx + d$ then $x = \dfrac{d-b}{a-c}$ (algebraic formula)

Second Application

If $\dfrac{ax+b}{p} = \dfrac{cx+d}{q}$ then $x = \dfrac{dp-bq}{aq-cp}$ (algebraic formula)

Third Application

If $(x+a)(x+b) = (x+c)(x+d)$ then

$$x = \frac{cd-ab}{a+b-c-d} \text{ (algebraic formula)}$$

Example 19 : Solve the simple equation :

$$(x+1)(x+2)=(x-3)(x-4)$$

Solution : By algebraic formula $x=\frac{12-2}{1+2+3+4}=\frac{10}{10}=1$

Fourth Application

If $\frac{m}{x+a}+\frac{n}{x+b}=0$ then $x=-\frac{mb+na}{m+n}$ (algebraic formula)

Example 20 : Solve the eqution $\frac{4}{x+2}+\frac{3}{x+5}=0$

Solution : By algebric formula $x=-\frac{(20+6)}{4+3}=-\frac{26}{7}$

Sutra Sunyam Samyasamuccaye

The meaning of sutra is "when the samuccaya is the same, that samuccaya is zero", i.e., it should be equated to 'zero'. Samuccaya is a technical term which has several meaning under different context and we shall explain them one by one.

First meaing and Application

In x is a common factor in all the terms concerned, then $x=0$ (algebraic formula).

Example 21. Solve the equation $12x+3x=4x+5x$.

Solution : By algebric formula $x=0$.

Example 22. Solve : $2(x+1)=7(x+1)$

Solution : $x+1$ is a common factor, hence $x+1=0$

$\therefore \quad x=-1$

Second meaning and Application

If in a linear equation, the independent terms in both the sides have the same value then value of the variable is 'zero'.

Example 23. Solve : $(x+3)+(2x+5)+4=2(x+6)$.

Solution : Independent terms in both the sides of the eqaution are same = 12, hence x = 0.

Example 24. Solve $(x+1)(x+9)=(x+3)(x+3)$.

Solution : Independent terms in both the sides of the equation are same = 9, hence x = 0.

Third meaning and Application

If in the equation, both the fractions have the same numerical numerator, then the solution of the equation is obtained by putting the sum of denominator equal to 'zero'.

Example 25. Solve $\frac{1}{x+a}+\frac{1}{x+b}=0$.

Solution : Here, the numerator in both the fractions are same = 1

Hence $x+a+x+b=0 \quad \therefore \; x=-\dfrac{a+b}{2}$.

Example 26. Solve $\dfrac{m}{2x+1}+\dfrac{m}{3x+4}=0$

Solution : Here, the numerator in both the fractions are same $=m$

Hence $2x+1+3x+4=0$

or $5x+5=0 \quad \therefore \quad x=-1$

Fourth meaning and Application

If the sum of the numerators and sum of the denominators of the fractions are same or in a definate ratio, then that sum equal to 'zero' will give the solution of the equation.

Example 27. Solve the equation $\dfrac{2x+3}{2x+5}=\dfrac{2x+5}{2x+3}$

Solution : Sum of the numerators $=2x+3+2x+5=4x+8$

Sum of the denominators $=4x+8$

Sum of the numenators and sum of the denominators same, therefore, by the formula $4x+8=0, \; \therefore \; x=-2$

Example 28. Solve the equation $\dfrac{3x+4}{6x+7}=\dfrac{x+1}{2x+3}$

Solution : Sum of the numerators $=3x+4+x+1=4x+5 \quad \cdots$ (i)

Sum of the denominators $=6x+7+2x+3=8x+10 \quad \cdots$ (ii)

Sum of the numerators and sum of the denominators is in the ratio = 1 : 2

Hence, by the formula either $4x+5=0$ or $8x+10=0$

$\therefore \qquad x=-\dfrac{5}{4}$

Fifth meaning and Application

If the difference of numerator and denominator of one fraction be the same as that of second fraction or their differences are in a definate ratio then that difference is equal to 'zero' give the solution of the equation.

Example 29. Solve the equation $\dfrac{3x+4}{2x+1}=\dfrac{x-8}{2x-5}$

Solution : Difference of the numerator and denominator of the L.H.S. fraction

$=3x+4-2x-1=x+3 \quad \cdots$ (i)

Difference of the numerator and denominator of the R.H.S. fraction

$=2x-5-x+8=x+3 \quad \cdots$ (ii)

Differences of numerator and denominator in (i) and (ii) are in the ratio = 2 : 1

Hence either $x+3=0$ and $2x+6=0 \quad \therefore \; x=-3$

Example 30. Solve the equation $\frac{x-8}{3x-2}=\frac{3x+4}{2x+1}$

Solution : Difference of the numerator and denominator of the L.H.S. fraction

$=3x-2-x+8=2x+6 \cdots \text{(i)}$

Difference of the numerator and denominator of the R.H.S. fraction

$=3x+4-2x-1=x+3 \cdots \text{(ii)}$

Difference of numerator and denominator in (i) and in (ii) are in the ratio = 2 : 1

Hence either $x+3=0$ or $2x+6=0$ $\therefore x=-3$

Note : By the fourth and fifth application of the sutra "Sunyam Samyasamuccaye"' we can obtain both the root of a quadratic equation. For example, the equation $\frac{3x+4}{6x+7}=\frac{5x+6}{2x+3}$ has two roots $x=-\frac{5}{4}$ and $x=-1$ obtained by the fourth and fifth application respectively.

Sixth meaning and Application

If in a equation each side contains two fraction and numerator in each fraction be the same, the sum of the denominators in the L.H.S. fraction and sum of the denominator in the R.H.S fraction are the same then that equal to 'zero' will give the solution of the equation.

Example 31. Solve the equation $\frac{1}{x+7}+\frac{1}{x+9}=\frac{1}{x+6}+\frac{1}{x+10}$

Solution : Sum of the denominators in L.H.S.

$=x+7+x+9=2x+16$

Sum of the denominator in R.H.S

$=x+6+x+10=2x+16$

Hence by the formula $2x+16=0$ $\therefore$ $x=-8$

Example 32. Solve the equation $\frac{1}{x-8}+\frac{1}{x-9}=\frac{1}{x-5}+\frac{1}{x-12}$

Solution : Sum of the denominator in L.H.S. $=2x-17$

Hence by the formula $2x-17=0$ $\therefore$ $x=\frac{17}{2}=8\frac{1}{2}$

Example 33. Solve the equation $\frac{1}{x+1}-\frac{1}{x+3}=\frac{1}{x+2}-\frac{1}{x+4}$

Solution : Transferring the negative fraction of both the sides, we have

$$\frac{1}{x+1}+\frac{1}{x+4}=\frac{1}{x+2}+\frac{1}{x+3}$$

Hence by the formula $2x+5=0$ $\therefore$ $x=-\frac{5}{2}=-2\frac{1}{2}$

Methods of checking the results

There are two methods of checking the results received from any operation.

(a) Navanka method

(b) Ekadashanka Method

(a) Navanka method :

In the navanka method we find the beejanka of any number by taking base as digit 9. After subtracting 9 from the digits of a number or sum of the digits of a number remaining single digit is know as Beejanka of that number e.g. Beejanka of 947 = 2.

Checking of the Division results:

$4283 \div 7$, Quotient $= 611$, Remainder $= 6$

We prove that

= Beejanka of Quotient × Beejanka of divisor + Beejanka of Remainder

i.e. $8 = 7 \times 8 + 6$

or $= 62$

or $= 8$

Hence answer is correct

Note: 1. If two digits of any row or two digits of any column interchange their places the error cannot be spotted.

2. In Vedic Mathematics there are several methods to solve a question. The result can be verified by Ekadashanka Method.

(b) Ekadashanka Method or Difference Method:

In this method the difference of the sumof digits at odd places and the sum of digits at even places is called the Beejanka of the number e.g. Beejanka of 63254

$$= 4 - 5 + 2 - 3 + 6 = 4$$

Checking of the Divisions results:

$6789 \div 12$, Quotient = 565, Remainder = 9

Beejanka of Dividend $= 9 - 8 + 7 - 6 = 2$

Beejanka of Quotient $= 5 - 6 + 5 \quad = 4$

Beejanka of Divisor $= 2 - 1 \quad = 1$

Beejanka of Remainder $= 9 \quad = 9$

L.H.S. = Divisor × Quotient + Remainder

$= 4 \times 1 + 9 = 13$

∴ Beejanka of L.H.S. = 3 − 1 = 2

Beejanka of dividend = Beejanka of R.H.S. = 2.

Both are equal, hence answer is correct.

Exercise 1

Divide by the Nikhilam method:

1. $1245 \div 97$ 2. $311 \div 8$ 3. $1013 \div 88$

Square the following numbers:

4. 103 5. 95 6. 204 7. 225

Find the cube of the following numbers:

8. 15 9. 91 10. 32 11. 208

Divide by Dhvajanka method:

12. $4532 \div 112$ 13. $1234 \div 42$ 14. $98765 \div 87$ 15. $2101532 \div 879$

Divide by Paravartya Yojayet method:

16. $1154 \div 103$ 17. $1358 \div 113$ 18. $1432 \div 88$ 19. $14885 \div 123$

Answers

	1.	2.	3.
Quotient →	12	38	11
Remainder →	8	7	45

4. 10609 5. 9025 6. 41616 7. 50625

8. 3375 9. 753571 10. 32768 11. 8998912

	12.	13.	14.	15.
Quotient →	40	29	1135	2390
Remainder →	52	16	20	722

	16.	17.	18.	19.
Quotient →	11	12	16	121
Remainder →	21	2	24	2

❑

02

Real Numbers

2.01 Introduction

In previous classes, we have studied about use of operations on Natural numbers, Integers, Rational and Irrational numbers. Now, we will study about Rational numbers and related fundamental principles and proof of rational and irrational numbers, nature of terminating and non-terminating numbers.

We know that, any positive integer can be expressed as product of two or more than two numbers. Also, we know about division of numbers, that remainder obtained form divison of numbers, that remainder obtained form division of two positive numbers is less than denominator. This important fact is basic fundamental theorem of Arithmetic. In this chapter, we will use these concepts to prove the irrationality of numbers $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ etc. and also study about decimal expansion of rational numbers.

2.02 Euclid's Division Lemma

Euclid was a Greak Mathematician and known by his work for geometry and number system. He had established the principles related to division of real numbers. In number system, Euclid's Division Algorithm is based on division lemma which is established by him.

Let a be any integer $(a \neq 0)$ and b and c are two whole numbers such that $\frac{b}{a} = c$ then number b is called dividend, a as divisor and c as quotient. For division, following properties should be kept in mind.

(i) Any non zero integer can be divided by $\pm$ 1.

(ii) 0 can be divided by any number.

(iii) No number can be divided by 0.

(iv) If none of a and b is zero then division algorithm can be applied.

(v) If a and b are non zero integers and q and r are other integers such that $a = bq + r$. In previous classes we have studied the division algorith. We know that when a positive integer (say a) is divided by another positive integer (say b) we get quotient (say q) and remainder (say r). Now we consider the following pairs of integers :

(i) 56, 16 (ii) 10, 2 (iii) 5, 7

Relations for these pairs can be written as follows :

(i) 56 = 16 × 3 + 8 (16 on dividing 56 by 16 quotient is 3 and remainder is 8)

(ii) 10 = 5 × 2 + 0 (5 on dividing 10 by 2 quotient is 5 and remainder is 0)

(iii) 5 = 7 × 0 + 5 (This relation also holds since 7 is larger than 5)

Therefore, form above examples, it is clear that for each pair of positive integers a and b, we have found whole numbers q and r, satisfying the relation :

$$a = bq + r \text{, where } 0 \leq r < b$$

This result is known as Euclid's divison Lemma in Arithmetics and formally can be expressed by the following way :

Theorem 2.1 (Euclid's Division Lemma)

If a and b are two positive integers then there exist unique integers q and r satisfying

$a = bq + r$, where $0 \le r < b$.

Note : Above lemma can be used for all integers (except zero) and it is kept in mind that q and r may be zero.

Applications of above euclid's division and method can be understood by the following examples :

Example 1. Show that any positive integer can be written in the form $3q$ or $3q + 1$ or, $3q + 2$, where q is some integer.

Solution : Let a is any positive integer and $b = 3$.

Applying division algorithm with a and b.

$a = 3q + r$, where $0 \le r < 3$ and q is any integer. Putting $r = 0, 1, 2$

$a = 3q + 0$, $a = 3q + 1$ $a = 3q + 2$

$\therefore$ $a = 3q$, $a = 3q + 1$ $a = 3q + 2$

Therefore, any integer can be written in the form of $3q, 3q + 1, 3q + 2$.

Example 2. Show that every positive even integer is of the form $2q$ and that every positive odd integer is of the from $2q + 1$, where q is any integer.

Solution : Let a be any positive integer and $b = 2$

Then by Euclid's algorithm,

$a = 2q + r$, where $0 \le r < 2$ and q is any integer

By putting $r = 0, 1$

$a = 2q + 0$, $a = 2q + 1$ ($\because$ r is an integer)

$a = 2q$, $a = 2q + 1$

Since q is an integer and $a = 2q$ then a is an even integer.

We know that any integer can be even or odd. So, if a is an even integer then $a + 1$ i.e. 2q + 1 will be odd integer.

Example 3. Use Euclid's division lemma to show that the square of any positive integer is either of the form $3m$ or $3m + 1$ for some integer m.

Solution : Let a be any positive integer.

We know that it will be of the form of any positive integer

$a = 3q$ or, $a = 3q + 1$ or, $a = 3q + 2$

(i) If $a = 3q$ then, $a^2 = (3q)^2 = 9q^2 = 3(3q) = 3m$, where $m = 3q$

(ii) If $a = 3q + 1$ then $a^2 = (3q + 1)^2 = 9q^2 + 6q + 1$

$= 3q(3q + 2) + 1$

$= 3m + 1$

where $m = q(3q + 2)$

(iii) If $a = 3q + 2$, then

$$a^2 = (3q+2)^2 = 9q^2 + 12q + 4$$

$$= 9q^2 + 12q + 3 + 1 = 3(3q^2 + 4q + 1) + 1$$

$$\Rightarrow \quad = 3m + 1$$

where $m = (3q^2 + 4q + 1)$

from above (i), (ii) and (iii) it is clear that square of any integer a is of the form 3 m or 3m +1.

2.03. Euclid's Division Algorithm (Method)

Here we will study the application of Euclid's division algorithm based on Euclid's division lemma. The word algorithm comes form the name of the 9th century Persian mathematician Al-Khwarizmi. This algorithm is a series of well defined steps which gives a procedure for solving a type of problem.

Euclid's division algorithm is a technique to compute the highest common factor (HCF) of two positive integers. HCF of two positive integers a and b is the largest positive integer that divides both a and b.

To find, highest common factor by Euclid's division algorithm, Euclid's division lemma is used in the following steps.

Step 1 : Let a and b (where a > b) are two positive integers, then apply Euclid's division lemma to a and b so, we find whole numbers q and r such that : $a = bq + r,\ 0 \le r < b$

Step 2 : If $r = 0$, then b is highest common factor of a and b and if $r \ne 0$, then apply the division lemma to b and a to obtained integers q_1 and r_1 such that $b = rq_1 + r_1$

Step 3 : Now, if $r_1 = 0$ then r will be HCF of a and b and if $r_1 \ne 0$ then apply Euclid's division lemma for r and r_1

Step 4 : Continue this process till the remainder becomes zero. In the situation when remainder becomes zero, the divisor at this stage will be the required HCF. This method can easily be understood by the following examples.

Example 1. Use Euclid's algorithm to find the HCF of 81 and 237.

Solution :

Step 1 : Here given integers 81 and 237 are such that 237 > 81, we apply the Euclid's division lemma to these integers, to get

$$237 = 81 \times 2 + 75 \qquad \ldots \text{(i)}$$

Step 2 : Since remainder $75 \ne 0$, we apply the Euclid's division lemma to 81 and 75 to get

$$81 = 75 \times 1 + 6 \qquad \ldots \text{(ii)}$$

Step 3 : From equation (ii) it is clear that, remainder $6 \ne 0$ so again apply Euclid's division lemma to 75 and 6, to get

$$75 = 6 \times 12 + 3 \qquad \ldots \text{(iii)}$$

Step 4 : We will continue this process till remainder becomes zero, here remainder $3 \ne 0$, we apply the division lemma to 6 and 3, to get

$$6 = 3 \times 2 + 0 \qquad \ldots \text{(iv)}$$

from equation (iv) it is clear that remainder is zero so process is completed. Since the divisor at this stage is 3, hence the HCF of 81 and 237 is 3. In brief this division process can be understood as follows

81 | 237 | 2
162
75 | 81 | 1
75
6 | 75 | 12
72
HCF = 3 | 6 | 2
6
0 = Remainder

Example 2. An army contigent of 616 members is to march behind an army band of 32 members in a parade. The two groups are to march in the same number of columns. What is the maximum number of columns in which they can march ?

Solution : The members of an army contingent and an army band are to march in the same number of columns. Therefore a maximum number of columns where two groups can march will be HCF of 616 and 32 . So we apply Euclid's division lemma to find HCF of 616 and 32. So

$$616 = 32 \times 19 + 8 \qquad \text{. . . (i)}$$

Here remainder $8 \neq 0$, we apply Euclid's division lemma to 32 and 8,

$$32 = 8 \times 4 + 0 \qquad \text{. . . (ii)}$$

Now, remainder = 0, so HCF of 616 and 32 is 8. In this way two groups can march in 8 columns.

In brief, this division algorithm can be understood as follows

32 | 616 | 19
608
HCF = 8 | 32 | 4
32
0 = Remainder

Example 3. Find the greatest number which divides 245 and 2053 such that it leaves remainder 5 in each case.

Solution : Given that required number divides 245 and 2053 such that it leaves remainder 5 in each case

$\therefore$ $245 - 5 = 240$ and $2053 - 5 = 2048$

i.e., 240 and 2048 can be completely divisible by the required number. This is possible only when there exist a common factor between them.

It is given that required number is greatest number in common factors. So required number will be HCF of 240 and 2048. Applying Euclid's division algorithm step by step, we get

$$2048 = 240 \times 8 + 128$$
$$240 = 128 \times 1 + 112$$
$$128 = 112 \times 1 + 16$$
$$112 = 16 \times 7 + 0$$

It is clear that last remainder = 0. So required HCF is 16.

In brief, this division algorithm can be understood as follows

240 | 2048 | 8
1920
128 | 240 | 1

128

112 | 128 | 1

112

HCF = 16 | 112 | 7

112

0 = Remainder

Hence

HCF = 16

Exercise 2.1

1. Show that the square of any positive odd integer is of the form $4q + 1$, where q is any integer.
2. Use Euclid's division lemma to show that the cube of any positive integer is of the form $9q$ or $9q + 1$ or $9q + 8$, where q is some integer.
3. Show that any positive odd integer is of the form $6q + 1$, or $6q + 3$, or $6q + 5$, where q is some posetive integer.
4. Use Euclid's division algorithm to find the HCF of :

   (i) 210, 55 (ii) 420, 130 (iii) 75, 243

   (iv) 135, 225 (v) 196, 38220 (vi) 867, 255
5. If HCF of number 408 and 1032 is expressed in the form of $1032x - 408 \times 5$, then find the value of x.

2.04. Fundamental Theroem of Arithmetic

In previous classes, we have studied about prime and composite numbers. We known that any positive prime number is divisible by 1 or itself *i.e.,* factors of a prime number p will be of the form $1 \times p$

Now, we consider any positive integer and express it into factor form. For example

$$5313 = 3 \times 7 \times 11 \times 23$$

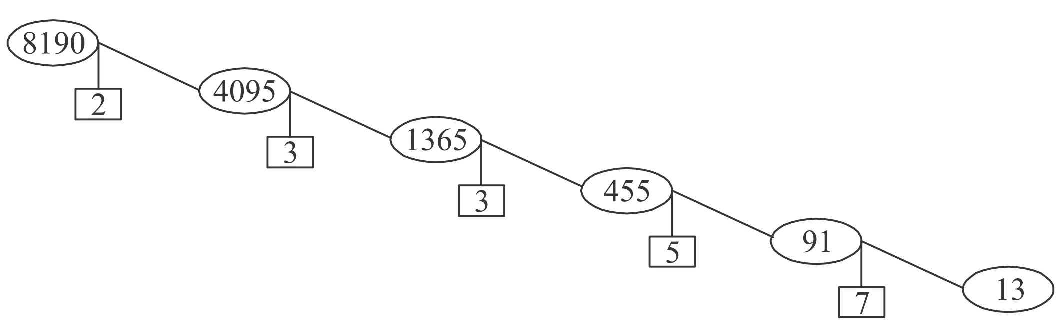
or $$140 = 4 \times 5 \times 7$$ etc.

From the above example it is clear that each factor will be either prime integer or a composite integer. If any factor is composite integer then it can be further factorized, till we get all prime factors. For example, factors of 140 will be as follows :

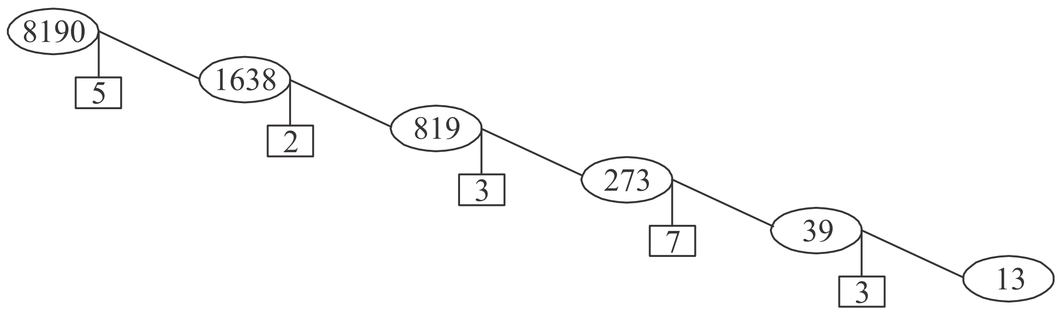
$$140 = 2 \times 2 \times 5 \times 7$$

or $$140 = 2^2 \times 5 \times 7$$

Now, we concentrate on factor tree of any positive integer. Let us take some number say, 8190 and factorize it as shown.



i.e., $$8190 = 2 \times 3 \times 3 \times 5 \times 7 \times 13 \qquad \ldots \text{(i)}$$



i.e., $8190 = 5 \times 2 \times 3 \times 7 \times 3 \times 13$ . . . (ii)

or in this example 8190 is factorized without considering the order of prime numbers in which they are appearing.

So it is clear that every composite number can be expressed (factorised) as a product of primes and this factorisation is unique, apart form the order in which the prime factors occur. If we write factors in ascending order and same prime numbers in the power form then for number 8190, following conjecture is obtained.

$$8190 = 2 \times 3^2 \times 5 \times 7 \times 13$$

This is known as basic or fundamental Theorem of Arithmetic.

Let us now formally state this theorem

Theorem 2.2. (Fundamental Theorem of Arithmetic)

Every composite number can be expressed (factorised) as a product of primes and this factorisation is unique, apart from the order in which the prime factors occur.

This theroem can be understand by the following examples.

Example 1. Examine whether there is any value of n for which 6^n ends with the digit zero ?

Solution : We know that if the number 6^n, for any n were end with the digit zero, then it would be divisible by 5 i.e., the prime factorisation of 6^n would contain the prime factor 5. Here for any n, number 6^n is positive integer which ends with zero. So on factorising, we get

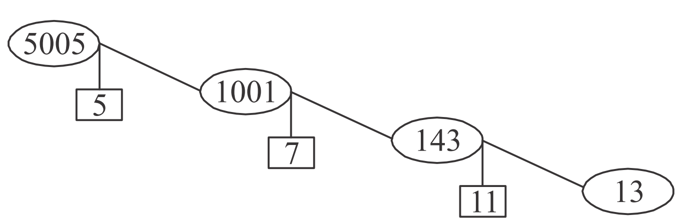
$$6^n = (2 \times 3)^n = 2^n \times 3^n$$

So in the factorisation of 6^n, there is no other prime factor except 2 and 3 i.e., number 5 does not occur in factors. Therefore, there is no natural number n for which 6^n ends with the digit zero.

Example 2. Express following positive integers as a product of its prime factors.

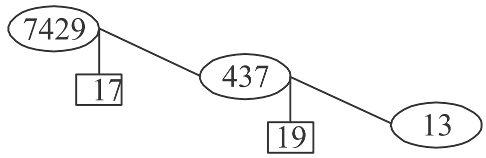
(i) 5005 (ii) 7429

Solution : (i) Factor tree of number 5005 is



So $5005 = 5\times7\times11\times13$ is prime factorisation

(ii) Following will be factor tree of number 7429



So $7429 = 17\times19\times13$ is prime factorisation.

We have already studied how to find the HCF and LCM of two positive integers by prime factorisation method. Here we will find HCF and LCM by fundamental theorem of Arithmetic. This can be understood by the following example. Consider the pair of integers (26 and 91)

Here $26 = 2^1\times13^1$

and $91 = 7^1\times13^1$ are prime factors

$\therefore$ HCF (26, 91) = 13^1

= Product of the smallest power of each common prime factor in the numbers.

and $\text{LCM}(26, 91) = 2^1\times7^1\times13^1$

= Product of the greatest power of each prime factor involved in the numbers.

Here, we see, $\text{HCF}(26, 91)\times\text{LCM}(26, 91) = 26\times91$

Thus, on the basis of fundamental theorem of Arithmetic we concludes that for any two positve integers *a* and *b*.

$$\text{HCF}(a, b)\times\text{LCM}(a, b) = a\times b$$

We can use this result to find the LCM of two positive integers, if we have already found the HCF of the two positive integers.

Example 3. Find the HCF and LCM of 144, 180 and 192 using the prime factorisation method.

Solution : The prime factorisation of 144, 180 and 192 are as follows

$$144 = 2\times2\times2\times2\times3\times3 = 2^4\times3^2$$

$$180 = 2\times2\times3\times3\times5 = 2^2\times3^2\times5^1$$

$$192 = 2\times2\times2\times2\times2\times2\times3 = 2^6\times3^1$$

To find HCF, we write smallest power of each common prime factor as follows

Common factor	Smallest power
2	2
3	1

Here 2^2 and 3^1 are the smallest powers of the common fators 2 and 3 respectively.

$\therefore$ $\text{HCF} = 2^2\times3^1 = 4\times3 = 12$

To find LCM, we write greatest power of each prime factor as follows

Prime factors	Greatest power
2	6
3	2
5	1

$\therefore$ LCM $= 2^6 \times 3^2 \times 5^1 = 64 \times 9 \times 5 = 2880$

Example 4. Find the HCFand LCM of pair of integers (510, 92) and verify whether product of two numbers of pair = HCF × LCM.

Solution : The prime factorisation of 510 and 92 can be written as follows

$$510 = 2 \times 3 \times 5 \times 17 = 2^1 \times 3^1 \times 5^1 \times 17^1$$

$$92 = 2 \times 2 \times 23 = 2^2 \times 23^1$$

To find HCF, we write smallest power of each common prime factor as follows

Common factor	Smallest factor
2	1

$\therefore$ HCF $= 2^1 = 2$

To find LCM, we write greatest power of each factor as follows

Prime factor	Greatest power
2	2
3	1
5	1
17	1
23	1

$\therefore$ LCM $= 2^2 \times 3^1 \times 5^1 \times 17^1 \times 23^1 = 23460$

Verification : Product of two numbers of pair $= 510 \times 92 = 46920$ . . . (i)

and HCF × LCM = 2 × 23460 = 46920 . . . (ii)

from (i) and (ii), we can say that

Product of two numbers = HCF × LCM

Exercise 2.2

1. Express each number as a product of its prime factors :
   (i) 468 (ii) 945 (iii) 140
   (iv) 3825 (v) 20570
2. Find the LCM and HCF of the following pairs of integers and verify that HCF × LCM = Product of the two numbers :
   (i) 96 and 404 (ii) 336 and 54 (iii) 90 and 144
3. Find the HCF and LCM of the following integers by applying the prime factorisation method
   (i) 12, 15 and 21 (ii) 24, 15 and 36 (iii) 17, 23 and 29 (iv) 6, 72 and 120
   (v) 40, 36 and 126 (vi) 8, 9 and 25

4. There is a circular path around a sports field. Raman takes 18 minutes to complete one round of the circular path, while Anupriya takes 12 minutes for the same. Suppose both of them start from the same point and at the same time and go in the same direction. After how many minutes will they meet again at the starting point?
5. In a seminar number of participants in Hindi, English and Mathematics are 60, 84 and 108 respectively. If equal number of participants of same subject are sitting in each room then find the least number of rooms required.

2.05 Proof of irrationality of Numbers

In previous classes, we have studied about irrational numbers in brief. We have also studied the existance of irrational numbers and there representation on number line. Generally, irrational numbers are expressed in the form $\sqrt{p}$, where p is positive prime number. We know that any irrational number cannot be written in the following p / q , Here p and q are integers and $q \neq 0$

For example : $\sqrt{2}, \sqrt{3}, \sqrt{5}, 7\sqrt{5}$ etc. are irrational numbers. In earlier classes we have study properties of irrational numbers that sum and difference of rational and irrational numbers are also irrational numbers and it is also true that multiplication and division of non zero rational and irrational number are also an irrational number.

Here, we will establish the proof of irrational numbers $\sqrt{2}, \sqrt{3}$ and $\sqrt{5}$ *i.e.,* we will prove $\sqrt{2}, \sqrt{3}$ and $\sqrt{5}$ as irrational numbers. We will use the following theroem and proof by contradiction to prove irrationality of these numbers.

Theorem 2.3 Let *p* is a prime number and *'a'* is a positive integer if *p* dividies *a*^{2} then *p* also divides *a*.

Proof: This theorem is obvious result of fundamental theorem of Arithmetic discussed in the preceeding section.

Let the prime factorisation of a be as follows:

$a ? p_1 p_2 \ldots p_n$ where $p_1, p_2, ..., p_n$ are primes, not necessarily distinct.

Now $a^2 ? (p_1 p_2 \ldots p_n)(p_1 p_2 \ldots p_n) ? p_1^2 p_2^2 \ldots p_n^2$

Here, it is given that p is a prime number which divides a^2. However, using the uniqueness part of the fundamental theorem of Arithmetic, we can say that the prime factors of a^2 are $p_1, p_2, ..., p_n$. So p is one of the $p_1, p_2, ..., p_n$. Now, since $a ? p_1 p_2 \ldots p_n$, p divides a.

Theorem 2.4. Prove that $\sqrt{2}$ is irrational.

Proof : Let $\sqrt{2}$ is a rational number.

So, for integers a and b we can write as

$$\sqrt{2} = \frac{a}{b}, \ b \neq 0$$

where a and b are coprime i.e., there is no common factor in a and b other than 1.

$\therefore$ $$\sqrt{2}b = a$$

Squaring on both sides, we get

$$2b^2 = a^2 \qquad \text{. . . (i)}$$

$\because$ $2b^2$ is divided by 2. So, we can say 2 divides a^2

But from theorem 2.3 it is clear that 2 divides *a* so we obtained the first result as 2 devides *a*.

Hence, integer *a* can be written as *a* = *2c*, where *c* is an integer.

$\therefore$ $a^2 = 4c^2$ . . . (ii)

Substituting the value of a^2 from (i) in (ii), we get

$$2b^2 = 4c^2$$

$\Rightarrow$ $b^2 = 2c^2$

Since $2c^2$ is divisible by 2. This means that 2 divides b^2 also is divides *b*.

Hence again from theorem (2.3) we can say that 2 divides *b* so we obtained the second result as 2 divides b.

Therefore from last two results it is evident that *a* and *b* have at least 2 as a common factor.

But this contradicts the fact that *a* and *b* have no common factors other than 1. If means that our hypothesis i.e., $\sqrt{2}$ is rational is wrong. So,we can conclude that $\sqrt{2}$ is irrational.

Theroem 2.5 Prove that $\sqrt{3}$ is irrational.

Proof : Let $\sqrt{3}$ is rational,then for integers *a*and *b* we can write

$$\sqrt{3} = \frac{a}{b},\ b \neq 0$$

where *a* and *b* are co-prime numnber,s means have no common factor other than 1

$\therefore$ $\sqrt{3}b = a$

Squaring on both sides, we get

$$3b^2 = a^2 \qquad \text{. . . (i)}$$

$\therefore$ $3b^2$ is divisibley 3. So, we can say a^2 is divisible by 3 and by theorem 2.3, it is clear that a is also divisible by 3.

So we can write *a* = *3c* where *c* is an integer.

$\therefore$ $a^2 = 9c^2$ . . . (ii)

From equation (i) and (ii), we get

$$3b^2 = 9c^2$$

i.e., $b^2 = 3c^2$

Since $3c^2$ is devisible by 3 this means that b^2 is divisible by 3, and so using by theorem (2.3) *b* is also divisible by 3.

$\therefore$ Thus from these two results it is evident that *a* and *b* have atleast 3 as a common factor.

But this contradicts the fact that *a* and *b* are co-prime numbers.

This means that out hypothesis is wrong. So, we can conclude that $\sqrt{3}$ is irratiuonal.

Theorem, 2.6. Show the $\sqrt{5}$ is irrational

Proof : Let us assumethat $\sqrt{5}$ is rational, then for two integers *a* and *b* we can write as

$$\sqrt{5} = \frac{a}{b},\ b \neq 0$$

where *a* and *b* are co-prime numbers.It means they have no common factor other than 1.

$\therefore \qquad \sqrt{5}\,b = a$

$\Rightarrow \qquad 5b^2 = a^2 \qquad \ldots \text{(i)}$

$\because \quad 5b^2$ is divisible by 5 so a^2 will also be divisible by 5.

By Theorem 2.3, we can say that 5 will divide *a* so integer *a* can be written as $a = 5c$, where *c* is some integer

$\Rightarrow \qquad a^2 = 25c^2 \qquad \ldots \text{(ii)}$

From equation (i) and (ii) we get

$$5b^2 = 25c^2$$

$\Rightarrow \qquad b^2 = 5c^2$

$5c^2$ is devisible by 5 so it is clear that b^2 will also be divisible by 5.

By using theorem 2.3 we can say that 5 will divide *b*.

$\therefore$ Thus from the above discussion it is evident that *a* and *b* have at least 5 as acommon factor but this contradicts the fact that *a* and *b* are co-primes.

This means our hypothesis is wrong. So we can conclude that $\sqrt{5}$ is irrational.

The sum or difference or multiplication and division of rational and irrational numbers can easily be understood by the following examples.

Example 1. Prove that $7\sqrt{5}$ is irrational.

Solution : Let $7\sqrt{5}$ is a rational number

$\therefore \qquad 7\sqrt{5} = \frac{a}{b},\ b \neq 0$, where *a*, *b* are co-prime numbers

or $\qquad \sqrt{5} = \frac{a}{7b} \qquad \ldots \text{(i)}$

since, *a*, *b* are integers, so $\frac{a}{7b}$ is rational and so $\sqrt{5}$ is rational but this contradicts the fact that $\sqrt{5}$ is irrational, so our hypothesis is wrong, so we can conclude that $7\sqrt{5}$ is irrational.

Example 2. Prove that $3 + 2\sqrt{5}$ is irrational.

Solution : Let $3+2\sqrt{5}$ is rational

$$\therefore \qquad 3+2\sqrt{5}=\frac{a}{b},\ b\neq 0 \qquad \text{. . . (i)}$$

where a, b are co-prime. By equation (i) we can say

$$\therefore \qquad 2\sqrt{5}=\frac{a}{b}-3$$

$$\text{or} \qquad \sqrt{5}=\frac{a-3b}{2b} \qquad \text{. . . (ii)}$$

$\because$ a and b are integers so $\frac{a-3b}{2b}$ will be rational. So from equation (ii) we obtain the result that $\sqrt{5}$ is rational but this contradicts the fact that $\sqrt{5}$ is irrational. So our hypothesis is wrong. So, we can conclude that $3+2\sqrt{5}$ is irrational.

Example 3. Show that $\sqrt{2}+\sqrt{5}$ is irrational.

Solution : Let $\sqrt{2}+\sqrt{5}$ is rational.

$$\therefore \qquad \sqrt{2}+\sqrt{5}=\frac{a}{b},\ b\neq 0 \qquad \text{. . . (i)}$$

where integers a, b are co-primes.

Equation (i) can be written as

$$\sqrt{5}=\frac{a}{b}-\sqrt{2}$$

Squaring on both sides, we get

$$5=\left(\frac{a}{b}-\sqrt{2}\right)^2$$

$$\Rightarrow \qquad 5=\frac{a^2}{b^2}+2-2\sqrt{2}\,\frac{a}{b}$$

$$\Rightarrow \qquad 2\sqrt{2}\,\frac{a}{b}=\frac{a^2}{b^2}-3$$

$$\Rightarrow \qquad \sqrt{2}=\frac{a^2-3b^2}{2ab} \qquad \text{. . . (ii)}$$

$\because$ a, b are integers so $\frac{a^2-3b^2}{2ab}$ will be rational so from euation (ii) $\sqrt{2}$ is rational but this contradicts the fact that $\sqrt{2}$ is irrational.So, our hypothesis is wrong. So, we conclude that $\sqrt{2}+\sqrt{5}$ is irrational.

Exericse 2.3

1. Prove that $5-\sqrt{3}$ is irratioanl.
2. Prove that following numbers are irrational

   (i) $\frac{1}{\sqrt{2}}$ (ii) $6+\sqrt{2}$ (iii) $3\sqrt{2}$

3. If p and q are positive prime numbers then prove that $\sqrt{p}+\sqrt{q}$ is irrational.

2.06. Decimal Expansion of Rational Numbers

We know that $\frac{p}{q}, q \neq 0$ is rational, where p and q are co-prime numbers. In previous classes we have studied about decimal expansion of rational numbers. We know that rational numbers have either a terminating decimal expansion or a non-terminating repeating decimal expansion. In this section, we are going to consider a rational number, and explore exactly when the decimal expansion of numbers is terminating and when it is non-terminating. Let us consider the following rational numbers to understand the nature of decimal expansion:

(i) 0.375 (ii) 1.512 (iii) 0.01764 (iv) 23.3408

By changing above decimal numbers in fraction form

(i) $0.375 = \frac{375}{1000} = \frac{375}{10^3}$ (ii) $1.512 = \frac{1512}{1000} = \frac{1512}{10^3}$

(iii) $0.01764 = \frac{1764}{100000} = \frac{1764}{10^5}$ (iv) $23.3408 = \frac{233408}{10000} = \frac{233408}{10^4}$

Denominators of all these numbers are powers of 10. This decimal expansion is of terminating nature. We know that powers of 10 can only have powers of 2 and 5 as factors.So cancelling out the common factors between the numerator and the denominator, we find the following results :

(i) $0.375 = \frac{375}{10^3} = \frac{3\times5^3}{2^3\times5^3} = \frac{3}{2^3} = \frac{3}{2^3\times5^0}$

(ii) $1.512 = \frac{1512}{10^3} = \frac{2^3\times3^3\times7}{2^3\times5^3} = \frac{3^3\times7}{5^3} = \frac{189}{2^0\times5^3}$

(iii) $0.01764 = \frac{1764}{10^5} = \frac{2^2\times3^2\times7^2}{2^5\times5^5} = \frac{3^2\times7^2}{2^3\times5^5} = \frac{441}{2^3\times5^5}$

(iv) $23.3408 = \frac{233408}{10^4} = \frac{2^6\times7\times521}{2^4\times5^4} = \frac{14588}{2^0\times5^4}$

From the above, it is clear that prime factorisation of denominator of rational number is of the form $2^m\times5^n$ where m, n are some non-negative integers.

Theorem 4. Let x be a rational number whose decimal expansion terminates. Then x can be expressed in the form $\frac{p}{q}$, q ? 0, where p and q are co-prime and the prime factorisation of q is of the form $2^m\times5^n$, where m, n are non-negative integers. Decimal expansion of the rational number $a/b,\ b \neq 0$, where a and b are prime integers is possible if b is some power of 10. Now, consider whether the converse of this theorem will be true?

For example:

(i) $\frac{3}{8} = \frac{3 \times 5^3}{2^3 \times 5^3} = \frac{375}{10^3} \Rightarrow \frac{3}{8} = 0.375$

(ii) $\frac{189}{125} = \frac{3^3 \times 7 \times 2^3}{5^3 \times 2^3} = \frac{1512}{10^3} \Rightarrow \frac{189}{125} = 1.512$

(iii) $\frac{441}{25000} = \frac{3^2 \times 7^2 \times 2^2}{2^3 \times 5^5 \times 2^2} = \frac{1764}{10^5} \Rightarrow \frac{441}{25000} = 0.01764$

From above exmaple it is clear that we can convert a rational number of the form $p/q,\ q \neq 0$ where q is of the form $2^m \times 5^n$ to an equivalent rational number of the form $\frac{a}{b}$, where b is some power of 10. Therefore, the decimal expansion of such rational number terminates. Let us write down this result as the following theorem.

Theorem 5. Let $x = \frac{p}{q},\ q \neq 0$ be a rational number, such that the prime factorisation of q is of the form $2^m \times 5^n$, where m, n are non-negative intergers.Then decimal expansion of x terminates.

Now we consider the rational numbers whose decimal expansions are non-terminating recurring.

For example :

Consider the following rational numbers :

(i) $\frac{5}{3}$ (ii) $\frac{29}{343}$ (iii) $\frac{77}{210}$

(i) $\frac{5}{3} = 1.6666...$ (ii) $\frac{29}{343} = 0.0845481...$ (iii) $\frac{77}{210} = 0.36666...$

In the above rational numbers, denominator is not of the form $2^m \times 5^n$ and on dividing the numerator by the denominator the remainder 0 will not be obtained.

It means that for these rational numbers their decimal expansions are non-terminating recurring.

This statement can be expressed in the form of statement of a theorem.

Theorem 6. Let $x = \frac{p}{q},\ q \neq 0$ is rational number such that the prime factorisation of q is not of the form $2^m \times 5^n$, where m, n are non-negative integers, then x has a decimal expansion which is non-terminating recurring.

Example 1. Without actually performing the long division method, state whether the following rational numbers will have a terminating decimal expansion or a non-terminating recurring decimal expasnion

(i) $\frac{17}{8}$ (ii) $\frac{64}{455}$ (iii) $\frac{125}{441}$

Solution : (i) Here, $\frac{17}{8} = \frac{17}{2^3 \times 5^0}$

It is clear that denominator 8 is of the form $2^m \times 5^n$. So $\frac{17}{8}$ has terminating decimal expansion.

(ii) Here $\frac{64}{455} = \frac{64}{5 \times 7 \times 13}$

It is clear that denominator 455 is not of the form $2^m \times 5^n$. So $\frac{64}{455}$ has non-terminating recurring decimal expansion.

(iii) Here $\frac{125}{441} = \frac{5^3}{3^2 \times 7^2}$

It is clear denominator 441 is not of the form $2^m \times 5^n$. So $\frac{125}{441}$ has non-terminating recurring decimal expansion.

Exericse 2.4

1. Without actually performing the long division method, state whether the following rational numbers will have a terminating decimal expansion or a non-terminating recurring decimal expansion.

   (i) $\frac{15}{1600}$ (ii) $\frac{13}{3125}$ (iii) $\frac{23}{2^3 \times 5^2}$ (iv) $\frac{17}{6}$

   (v) $\frac{129}{2^2 \times 5^7 \times 7^5}$ (vi) $\frac{35}{50}$ (vii) $\frac{7}{80}$

2. Write down the decimal expansion of the following rational numbers and show whether these are terminating.

   (i) $\frac{13}{125}$ (ii) $\frac{14588}{625}$ (iii) $\frac{49}{500}$

3. For the followig decimal expansions, decide whether they are rational or not. If they are rational then write the note on prime factors of its denominator.

   (i) 0.120120012000120000 . . . (ii) 43.123456789 (iii) $27.\overline{142857}$

Miscellaneous Exercise -2

1. Sum of powers of prime factors of 196 is :

   (a) 1 (b) 2 (c) 4 (d) 6

2. If two numbers are wirtten in the form $m = pq^3$ and $n = p^3q^2$ then HCF of m, n whereas p, q are prime numbers is :

   (a) pq (b) pq^2 (c) p^2q^2 (d) p^3q^3

3. HCF of 95 and 152 is :

   (a) 1 (b) 19 (c) 57 (d) 38

4. Product of two numbers is 1080 and their HCF is 30 then their LCM is:

(a) 5 (b) 16 (c) 36 (d) 108

5. Decimal expansion of the number $\frac{441}{2^2 \times 5^7 \times 7^2}$ is

(a) Terminating (b) Non-terminating recurring

(c) Terminating and non-terminating both (d) Number is not a rational number

6. In the decimal expansion of number $\frac{43}{2^2 \times 5^3}$ after how many decimal places will terminate?

(a) 1 (b) 2 (c) 3 (d) 4

7. The lowest number, when multiplied by $\sqrt{27}$ gives a natural number, will be

(a) 3 (b) $\sqrt{3}$ (c) 9 (d) $3\sqrt{3}$

8. If HCF = LCM for two rational numbers then numbers should be

(a) Composite (b) Equal (c) Prime (d) Co-prime

9. If LCM of a and 18 is 36 and HCF of a and 18 is 2 then value of a is

(a) 1 (b) 2 (c) 5 (d) 4

10. If n is a natural number, then unit digit in $6^n - 5^n$ is :

(a) 1 (b) 6 (c) 5 (d) 9

11. If $\frac{p}{q}(q \neq 0)$ is a rational number then what condition apply for q whereas $\frac{p}{q}$ is a terminating decimal?

12. Simplify $\frac{2\sqrt{45}+3\sqrt{20}}{2\sqrt{5}}$ and find whether it is rational or irrational number.

13. Prove that any positve odd integer is of the form $4q+1$ or $4q+3$ where q is any integer.

14. Prove that product of two consecutive positive integers is divisible by 2.

15. Find the largest number which is divided by 2053 and 967, left remainder as 5 and 7 respectively.

16. Describe, why $7 \times 11 \times 13 + 13$ and $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 + 5$ are composite numbers?

17. If HCF of two numbers 306 and 657 is 9, then find their LCM.

18. A rectangular varandah is 18 m 72 cm long and 13 m × 20 cm wide. Square tiles of same dimensions are used to cover it.Find the least number of such tiles.

19. Prove that following numbers are irrational numbers :

(i) $5\sqrt{2}$ (ii) $\frac{2}{\sqrt{7}}$ (iii) $\frac{3}{2\sqrt{5}}$ (iv) $4+\sqrt{2}$

20. What can you say about the prime factors of denominator of the followign rational numbers.

(i) 34.12345 (ii) $43.\overline{123456789}$

Important Points

1. Euclid's division lemma : For two positive integers a and b there exist whole numbers q and r satisfying $a = bq + r,\ 0 \le r < b$. This result is also true for r = 0 and q = 0.
2. Euclid's division algorithm : According to this, the HCF of any two positive integers a and b, with $a > b$ is obtained as follows.

   Step 1 : Apply the division lemma to find q and r where $a = bq_1 + r_1,\ 0 \le r_1 < b$

   Step 2 : If $r_1 = 0$, the HCF of a and b is b

   Step 3 : If $r_1 \ne 0$, then apply Euclid's lemma to b and r_1 and obtained the integers q_2 and r_2 where as $b = q_2 r_1 + r_2$.

   Step 4 : If $r_2 = 0$ then r_1 is HCF of a, b.

   Step 5 : If r_2 ? 0 continue the process till the remainder r_n becomes zero.The divisor r_{n-1} at this stage will be HCF of a and b.
3. The fundamental theorem of Arithemtic : Every compositive number can be expressed (factorised) as a product of primes and this factorisation is unique, apart from the order in which the prime factors occur.
4. Each composit number can be uniquelly expressed in ascending and descending order of powers of prime factors.
5. For positive integer a ,prime number p is such that p divided a^2 then p will diivide a also.
6. If p is positive prime number, then $\sqrt{p}$ is an irrational number.
7. For any rational number p / q, its decimal expansion will be terminating if its denominator q can be written in the form of $2^m \times 5^n$, where m, n are non-negative integers. Here p and q are co-prime integers. If q cannot be expressed in the form $2^m \times 5^n$, then decimal expansion will be non-terminating recurring .

Answers

Exericse 2.1

4. (i) 5 (ii) 10 (iii) 3 (iv) 45 (v) 196 (vi) 51 5. 2

Exercise 2.2

1. (i) $2^2 \times 3^2 \times 13$ (ii) $3^3 \times 5 \times 7$ (iii) $2^2 \times 5 \times 7$ (iv) $3^2 \times 5^2 \times 17$ (v) $2 \times 5 \times 11^2 \times 17$
2. (i) HCF = 4, LCM = 9696 (ii) HCF = 6, LCM = 3024 (iii) HCF = 18, LCM = 720
3. (i) HCF = 3, LCM = 420 (ii) HCF = 3, LCM = 360 (iii) HCF = 1, LCM = 11339
   (iv) HCF = 6, LCM = 360 (v) HCF = 2, LCM = 2520 (vi) HCF = 1, LCM = 1800
4. 36 minute 5. 21

Exercise 2.4

1. (i) Terminating (ii) Terminating (iii) Terminating (iv) Non-terminating
   (v) Non-terminating (vi) Termianting (vii) Terminating

2. (i) 0.104 (ii) 23.3408 (iii) 0.098

3. (i) Irrational

(ii) Rational prime factor of denominator is of the form $2^m \times 5^n$ where *m*, *n* are non-negative intergers.

Miscellaneous Exericse-2

1. (c)	2. (b)	3. (b)	4. (c)	5. (a)	6. (d)
7. (b)	8. (b)	9. (d)	10. (a)		

11. Prime factorisation of denominator q is of the form $2^m \times 5^n$ where *m*, *n* are non-negative integers.

12. Rational number 15. 64 17. 22338 18. 4290

20. (i) Since its decimal expansion is terminating so its denominator is of the form $2^m \times 5^n$,where *m*, *n* are non negative integers.

(ii) Since its decimal expansion is non-terminating repeating so its denominator is not of the form $2^m \times 5^n$ where *m*,*n* are nonnegative integers.

03

Polynomials

3.01. Introduction

In previous class, we have studied polynomials in one variable and their degres. We know that in polynomial $f(x)$, the highest powerof x is called the degree of polynomial and on the bases of power, polynomial is recognized as linear, quadratic or cubic polynomial. Generally for variable x , $f(x) = ax + b$ as linear, $f(x) = ax^2 + bx + c$ as quadratic and $f(x) = ax^3 + bx^2 + cx + d$ is taken as cubic polynomials where a, b, c, d are real numbers and $a \neq 0$. In this way for variable x , polynomial of n degree can be defined as follows $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$, where '$n$' is a natural number and $a_n, a_{n-1}, ..., a_1, a_0$ are called coefficients of these terms.

In this chapter we will study the zeroes, coefficients and division algorithm of polynomials and also about the nature of roots and solutions of quadratic equations. In previous classes we have studied HCF and LCM of real numbers. Here we will find HCF and LCM of algebraic expression.

3.02. Zeroes of a Polynomial

Consider the polynomials $f_1(x) = 4x + 2, f_2(x) = 2x^2 + 3x - \frac{2}{5}, f_3(x) = 2 - x^3$

These are examples of linear, quadratic and cubic polynomial respectively. By putting $x = 2$ in polynomials $f_1(x), f_2(x)$ and $f_3(x)$, we get the following values.

$$f_1(2) = 4 \times 2 + 2 = 10$$

$$f_2(2) = 2 \times 2^2 + 3 \times 2 - \frac{2}{5} = 8 + 6 - \frac{2}{5} = \frac{68}{5}$$

$$f_3(2) = 2 - 2^3 = -6$$

So, by putting different values of x we get different value of polynomials, so we can say that :

If $f(x)$ is a polynomial in variable x by 'a' in $f(x)$ is called the value of $f(x)$ at $x = a$ and is denoted by $f(a)$

Now consider the polynomial $f(x) = 2x^2 - 8x + 6$ then putting $x = 1$ and $x = 3$ in the polynomial, we get

$$f(1) = 2 \times 1 - 8 \times 1 + 6 = 0$$

$$f(3) = 2 \times 3^2 - 8 \times 3 + 6 = 0$$

As $f(1)$ and $f(3) = 0$ so 1 and 3 are called zeroes of the quadratic polynomial $f(x) = 2x^2 - 8x + 6$. More generally, a real number 'a' is said to be a zero of a polynomial $f(x)$, If $f(a) = 0$

Let α is zero of linear polynomial $f(x) = ax + b$, then $f(\alpha) = a\alpha + b = 0$

$$\Rightarrow \quad \alpha = \frac{-b}{a} = \frac{-(\text{constant term})}{(\text{cofficient of } x)}$$

Thus, the zero of linear polynomial is related to its coefficients.

3.03 Relationship between Zeroes and Coefficients of a Quadratic Polynomial

In previous class, we have studied about factorisation of polynomials. Quadratic polynomial is factorize by splitting the middle term. in such a way that product of both terms is equal to the product of first and third term. Here we should know that quadratic polynomial has two zeroes (real/imaginary). We can understand the relation between zeros and coefficients of polynomial.

General form : Let α and β are two zeroes of quadratic polynomial $f(x) = ax^2 + bx + c$, then $(x-\alpha)$ and $(x-\beta)$ will be factors of f(x). So for constant k we can write as :

$$f(x) = k(x-\alpha)(x-\beta)$$

i.e., $\qquad ax^2 + bx + c = k\{x^2 - (\alpha+\beta)x + \alpha\beta\}$

or $\qquad ax^2 + bx + c = kx^2 - k(\alpha+\beta)x + \alpha\beta$

equating like powers on both sides, we get

$$\alpha + \beta = \frac{-b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

$\therefore$ For polynomial $f(x) = ax^2 + bx + c$, it is clear that

$$\text{Sum of zeroes} = \frac{-b}{a} = \frac{-(\text{coefficient of } x)}{(\text{coefficient of } x^2)}$$

and $\qquad \text{Product of zeroes} = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$

Example 1 : Find the zeroes of quadratic polynomial $x^2 - 2x - 8$ and verify the relationship between the zeroes and the coefficients.

Solution : Let $f(x) = x^2 - 2x - 8 = x^2 - 4x + 2x - 8$

$$= x^2 - 4x + 2x - 8 = x(x-4) + 2(x-4) = (x+2)(x-4)$$

Now, $\qquad$ by taking $f(x) = 0$, $= (x+2)(x-4) = 0$

or $\qquad x + 2 = 0$ or $x - 4 = 0$

or $\qquad x = -2$ or $x = 4$

Thus, zeroes of polynomial $f(x) = x^2 - 2x - 8$ will be -2 and 4.

Here, $\qquad$ Sum of zeroes $= -2 + 4 = 2$

also $\qquad \text{Sum of zeroes} = \frac{-(\text{coefficient of } x)}{\text{coefficient } x^2} = \frac{2}{1} = 2$

Product of zeroes $= -2 \times 4 = -8$

i.e., Product of zeroes $= \dfrac{\text{constant term}}{\text{coefficient of } x^2} = \dfrac{-8}{1} = -8$

Therefore, relation between zeroes and coefficient for given polynomial is true.

Example 2. Find the zeroes of quadratic polynomial $3x^2 + 5x - 2$ and verify the relationship between the zeroes and the coefficients.

Solution : Let $f(x) = 3x^2 + 5x - 2$

or $f(x) = 3x^2 + 6x - x - 2 = 3x(x+2) - 1(x+2) = (3x-1)(x+2)$

Now, by taking $f(x) = 0$ then $(3x-1)(x+2) = 0$

or $3x - 1 = 0$ or $x + 2 = 0$

or $x = \dfrac{1}{3}$ or $x = -2$

Thus $1/3$ and -2 will be zeroes of given polynomial

Here, Sum of zeroes $= \dfrac{1}{3} + (-2) = \dfrac{-5}{3} = \dfrac{-(\text{coefficient of } x)}{\text{coefficient of } x^2}$

Product of zeroes $= \dfrac{1}{3} \times (-2) = \dfrac{-2}{3} = \dfrac{\text{constant term}}{\text{coefficient of } x^2}$

Therefore, relation between zeroes and coefficient for given polynomial is true.

Example 3. Find a quadratic polynomial, the sum and product of whose zeroes are $1/4$ and -1 respectively.

Solution : Let α and β are zeroes of quadratic equation

$$ax^2 + bx + c$$

$\therefore$ Sum of zeroes $\alpha + \beta = \dfrac{-b}{a} = \dfrac{1}{4}$ (Given) . . . (1)

and, Product of zeroes $\alpha\beta = \dfrac{c}{a} = -1$ (Given) . . . (2)

If $a = k$, where k is real number, then form equation (1) and (2), we have

$$b = -\frac{k}{4} \text{ and } c = -k$$

Thus, quadratic polynomial $ax^2 + bx + c$ is obtained in the following form :

$$kx^2 - \frac{k}{4}x - k \text{ or } \frac{k}{4}(4x^2 - x - 4)$$

Hence, required quadratic polynomial will be $4x^2 - x - 4$.

Exercise 3.1

1. Find the zeroes of the following quadratic polynomial and verify the relationship between the zeroes and the coefficients.

(i) $4x^2+8x$ (ii) $4x^2-4x+1$ (iii) $6x^2-x-2$

(iv) x^2-15 (v) $x^2-(\sqrt{3}+1)x+\sqrt{3}$ (vi) $3x^2-x-4$

2. Find a quadratic polynomial, whose sum and product of the zeroes are the following numbers respectively.

(i) –3, 2 (ii) $\sqrt{2},\frac{1}{3}$ (iii) $-\frac{1}{4},\frac{1}{4}$ (iv) $0,\sqrt{5}$

(v) 4, 1 (vi) 1,1

3. If sum of squares of zeroes of quadratic polynomial $f(x)=x^2-8x+k$ is 40, then find the value of k.

3.04 Division Algorithm for Polynomials with Real Coefficients

In previous chapter, we have studied that division of two integers gives quotient and remainder and the relation between them is

Dividend = Divisor × Quotient + Remainder

Here, we discuss the method of dividing one polynomial by another in same method. We stop the division process when either the remainder is zero or its degree is less than the quotient this process is called division. Algorithm we understand this process by taking a example.

Divide polynomial $f(x)=3x^2-x^3-3x+5$ by polynomial $f(x)=x-1-x^2$ by division algorithm.

Step 1. First arrange dividend and divisor in descending order of x *i.e.,* in standard form. Here, on writing $f(x), g(x)$ in standard form. $f(x)=-x^3+3x^2-3x+5$ and $g(x)=-x^2+x-1$

Step 2. Now divide highest power term of dividend $(-x^3)$ by highest power term of divisor $(-x^2)$ and obtained x as quotient *i.e.,*

$$\begin{array}{r|l} & x \\ -x^2+x-1 & -x^3+3x^2-3x+5 \\ & \mp x^3 \pm x^2 \mp x \\ \hline & 2x^2-2x+5 \end{array}$$

Here remainder is $2x^2-2x+5$

Step 3. Now highest degree term of dividend $(2x^2)$ is divided by highest degree term of divisor $(-x^2)$ and obtained the quotient (–2) *i.e.,*

$$\begin{array}{r|l} & x-2 \\ -x^2+x-1 & -x^3+3x^2-3x+5 \\ & \underset{+}{-x^3} \underset{-}{+} x^2 \underset{+}{-x} \\ \hline & 2x^2-2x+5 \\ & \underset{-}{2x^2} \underset{+}{-2x} \underset{-}{+2} \\ \hline & 3 \end{array}$$

Here remainder is 3 and its degree is less then the degree of divisor $-x^2+x-1$, So stop the division process. Here, we get quotient (x – 2) and remainder (3). In divison algorithm verify dividend = divisor × quotient + remainder.

Here dividend $-x^3+3x^2-3x+5$, divisor $=-x^2+x-1$

quotient = $(x-2)$, remainder = 3

$$\therefore \quad (-x^2+x-1)\times(x-2)+3$$

$$=-x^3+x^2-x+2x^2-2x+2+3$$

$$=-x^3+3x^2-3x+5 \;= \text{dividend}$$

Thus, division algorithm can be expressed by following statement.

Divison algorithm : If $f(x)$ and $g(x)$ are any two polynomials where $g(x)\neq 0$ then from polynomials $q(x)$ and $r(x)$, we get

$$f(x)=q(x)g(x)+r(x)$$

Where $r(x)=0$ or degree $r(x)<$ degree of $g(x)$

Example 1. Using divison algorithm, divide $p(x)=x^4-3x^2+4x+5$ by $g(x)=x^2+1-x$ and find quotient and remainder.

Solution : Arrange polynomial is standard form then applying division algorithm.

$$\begin{array}{r|l}
 & x^2+x-3 \\
\hline
x^2-x+1 & x^4+0x^3-3x^2+4x+5 \\
 & x^4-x^3+x^2 \\
 & \underline{-\quad+\quad-} \\
 & \quad x^3-4x^2+4x+5 \\
 & \quad x^3-x^2+x \\
 & \quad \underline{-\quad+\quad-} \\
 & \qquad -3x^2+3x+5 \\
 & \qquad -3x^2+3x-3 \\
 & \qquad \underline{+\quad-\quad+} \\
 & \qquad\qquad 8
\end{array}$$

Since degree of remainder is less than the degree of divisor so, stop the process.

Here, quotient $=x^2+x-3$ and remainder = 8

Here, divisor × quotiuent + remainder

$$=(x^2-x+1)\times(x^2+x-3)+8$$

$$=x^4-x^3+x^2+x^3-x^2+x-3x^2+3x-3+8$$

$$x^4-3x^2+4x+5 \;= \text{Dividend}$$

Hence, division algorithm is verified.

Example 2. Find all the zeroes of polynomial $f(x) = 3x^4 + 6x^3 - 6x^3 - 2x^2 - 10x - 5$, if its two zeroes are $\sqrt{\frac{5}{3}}$ and $-\sqrt{\frac{5}{3}}$.

Solution : Here $\sqrt{\frac{5}{3}}$ and $-\sqrt{\frac{5}{3}}$ are two zeroes of given polynomial

$\therefore$ $\left(x - \sqrt{\frac{5}{3}}\right)\left(x + \frac{\sqrt{5/3}}{3}\right) = x^2 - \frac{5}{3} = \frac{1}{3}(3x^2 - 5)$ is a factor of polynomial

i.e., $(3x^2 - 5)$ is also a factor of polynomial. Now divide $f(x)$ by $(3x^2 - 5)$

$$
\begin{array}{r|l}
 & x^2 + 2x + 1 \\
\hline
3x^2 - 5 & 3x^4 + 6x^3 - 2x^2 - 10x - 5 \\
 & 3x^4 - 5x^2 \\
 & \underline{-\quad +} \\
 & 6x^3 + 3x^2 - 10x - 5 \\
 & 6x^3 \qquad - 10x \\
 & \underline{-\qquad +} \\
 & 3x^2 - 5 \\
 & -3x^2 - 5 \\
 & \underline{-\quad +} \\
 & 0
\end{array}
$$

Form division algorithm, it is clear that quotient $(x^2 + 2x + 1)$ is a factor of polynomial $f(x)$ because remainder is zero (0), so we can write as

$$x^2 + 2x + 1 = (x+1)^2$$

$\therefore$ Dividend $=$ Divisor $\times$ Quotient $+$ Remainder

$$3x^4 + 6x^3 - 2x^2 - 10x - 5 = (x+1)^2 \times (3x^2 - 5) + 0$$

$$= (x+1)^2(\sqrt{3}x - \sqrt{5})(\sqrt{3}x + \sqrt{5})$$

Since to find zeroes of polynomial $f(x)$, $f(x) = 0$ should satisfy so

$$(x+1)^2(\sqrt{3}x - \sqrt{5})(\sqrt{3}x + \sqrt{5}) = 0$$

or $\quad x + 1 = 0, x + 1 = 0, \sqrt{3}x - \sqrt{5} = 0 \ \sqrt{3}x + \sqrt{5} = 0$

It means $-1, -1, \sqrt{\frac{5}{3}}, -\sqrt{\frac{5}{3}}$ will be zeroes of the polynomial.

Exercise 3.2

1. Using division algorithm, divide $f(x)$ by $g(x)$ and find quotient and remainder.

   (i) $f(x) = 3x^3 + x^2 + 2x + 5,$ $\quad g(x) = 1 + 2x + x^2$

   (ii) $f(x) = x^3 - 3x^2 + 5x - 3,$ $\quad g(x) = x^2 - 2$

   (iii) $f(x) = x^3 - 6x^2 + 11x - 6,$ $\quad g(x) = x + 2$

   (iv) $f(x) = 9x^4 - 4x^2 + 4,$ $\quad g(x) = 3x^2 + x - 1$

2. Divide second polynomial by first and verify that first polynomial is a factor of second polynomial.

   (i) $g(x) = x^2 + 3x + 1,$ $\quad f(x) = 3x^4 + 5x^3 - 7x^2 + 2x + 2$

   (ii) $g(t) = t^2 - 3,$ $\quad f(t) = 2t^4 + 3t^3 - 2t^2 - 9t - 12$

   (iii) $g(x) = x^3 - 3x + 1,$ $\quad f(x) = x^5 - 4x^3 + x^2 + 3x + 1$

3. In the following polynomials, their zerores are given, find all other zeroes.

   (i) $f(x) = 2x^4 - 3x^3 - 3x^2 + 6x - 2;$ $\sqrt{2}$ and $-\sqrt{2}$

   (ii) $f(x) = x^4 - 6x^3 - 26x^2 + 138x - 35;$ $2 \pm \sqrt{3}$

   (iii) $f(x) = x^3 + 13x^2 + 32x + 20;$ -2

4. On dividing polynomial $f(x) = x^3 - 3x^2 + x + 2$ by polynomial $g(x)$, quotient $q(x)$ and remainder $r(x)$ are obtained as $x - 2$ and $-2x + 4$ respectively then find polynomial $g(x)$.

3.05 Standard From of Quadratic Equation

In the beginning of the chapter, we have studied about quadratic polynomial. In general $f(x) = ax^2 + bx + c,\ a \neq 0$ is standard form of polynomial. We have discussed about the zeroes of polynomial $f(x) = ax^2 + bx + c$. We know that value of polynomial is zero at their zeroes. This fact can be expressed as :

"If $f(x)$ is a quadratic polynomial then $f(x) = 0$ is called a quadratic equation *i.e.,* $ax^2 + bx + c = 0,$ is a quadratic equation where *a, b, c* are real numbers and $a \neq 0$." If terms of $f(x)$ is arranged in descending order then $f(x) = 0$ *i.e.,* $ax^2 + bx + c = 0,\ a \neq 0$ is called standard form of quadratic equation.

Let us consider some examples and test whether these are quadratic equations or not.

$$(x-2)(x+1) = (x-1)(x+3)$$

LHS. $= (x-2)(x+1)$

$= x^2 - 2x + x - 2$

$= (x^2 - x - 2)$ . . . (i)

RHS. $= (x-1)(x+3)$

$= x^2 - x + 3x - 3$

$= x^2 + 2x - 3$ . . . (ii)

By equating both sides according to the given equation

$$x^2 - x - 2 = x^2 + 2x - 3$$

or $\quad x^2 - x^2 - x - 2x - 2 + 3 = 0$

$\Rightarrow \quad -x + 1 = 0$ or $(x - 1) = 0$

Here, equation $x - 1 = 0$ has no term of x^2 so given equation is not a quadratic equation.

In another equation, $3x^2 - 5x + 9 = x^2 - 7x + 3$, we have

$$3x^2 - x^2 - 5x + 7x + 9 - 3 = 0$$

$\Rightarrow \quad 2x^2 + 2x + 7 = 0$

Here x^2 is present so given equation is a quadratic equation.

3.06 Solution of Quadratic Equation by Factorisation

Zeroes of quadratic polynomial $f(x)$, from equation $f(x) = 0$. two values of x are obtained. Let $x = \alpha$ is a zero of polynomial $f(x) = ax^2 + bx + c$, then $f(\alpha) = 0$. It means x will satisfy the equation $ax^2 + bx + c = 0$. So we can say that a zero $x = \alpha$ of polynomial $ax^2 + bx + c$ will be root of quadratic equation $ax^2 + bx + c = 0$

So, if $f(x) = 0$ is a quadratic equation is 2. So it has polynomial $f(x)$ are called root of equation $f(x) = 0$

Highest degree of quadratic equation is 2. So it has maximum two roots.

The process to find the roots of any quadratic equation is called solution of an equation. To solve quadratic equation write it as $f(x) = 0$ in standard form, then factorise $f(x)$ and equate each factor equal to zero and then find values of x. These values of x are soluitions of quadratic equtions. Thus values of x so obtained are roots of this equation. This method can be understand by the following examples.

Example 1. Find the roots of quadratic equation $x^2 - 3x - 10 = 0$ by factorisation method.

Solution : Given equation is

$$x^2 - 3x - 10 = 0$$

On factorising,

$$x^2 - 5x + 2x - 10 = 0$$

or $\quad x(x - 5) + 2(x - 5) = 0$

or $\quad (x + 2)(x - 5) = 0$

or $\quad x + 2 = 0 \quad$ or $\quad x - 5 = 0$

or $\quad x = -2 \quad$ or $\quad x = 5$

Thus $x = -2$ and $x = 5$ are required two roots of given equation.

Example 2. Solve quadratic equation $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$ by factorisation method.

Solution : Given equation is

$$\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$$

On factorisation

$$\sqrt{2}x^2+5x+2x+5\sqrt{2}=0$$

or $$x(\sqrt{2}x+5)+\sqrt{2}(\sqrt{2}x+5)=0$$

or $$(\sqrt{2}x+5)(x+\sqrt{2})=0$$

or $$\sqrt{2}x+5=0 \quad \text{or} \quad x+\sqrt{2}=0$$

or $$x=\frac{-5}{\sqrt{2}} \quad \text{or} \quad x=-\sqrt{2}$$

Thus, $x=\frac{-5}{\sqrt{2}}$ and $x=-\sqrt{2}$ are required roots of given equation.

Example 3. Find the roots of the following quadratic equation by factorisation method.

$$\frac{4}{x}-3=\frac{5}{2x+3} \quad \text{where} \quad x\neq 0, \frac{-3}{2}$$

Solution : Given equation is, $\frac{4}{x}-3=\frac{5}{2x+3}$

By taking LCM $$\frac{4-3x}{x}=\frac{5}{2x+3}$$

By cross multiplication, we get,

$$(4-3x)(2x+3)=5x$$

or $$8x-6x^2+12-9x=5x$$

or $$6x^2+6x-12=0$$

On factorising

$$6x^2+12x-6x-12=0$$

or $$6x(x+2)-6(x+2)=0$$

or $$(x+2)(6x-6)=0$$

or $$x+2=0 \quad \text{or} \quad 6x-6=0$$

or $$x=-2 \quad \text{or} \quad x=1$$

Thus, $x=-2$ and $x=1$ are required solutions of quadratic equation.

Exercise 3.3

1. Check whether the following equations are quadratic?

   (i) $x(x+1)+8=(x+2)(x-2)$ (ii) $(x+2)^3=x^3-4$

   (iii) $x^2+3x+1=(x-2)^2$ (iv) $x+\frac{1}{x}+x^2,\ x\neq 0$

2. Solve the following equtions by factorisation method.

   (i) $2x^2-5x+3=0$ (ii) $9x^2-3x-2=0$

(iii) $\sqrt{3}x^2+10x+7\sqrt{3}=0$

(iv) $x^2-8x+16=0$

(v) $\frac{1}{x-2}+\frac{2}{x-1}=\frac{6}{x}$ where $x\neq 1,2$

(vi) $100x^2-20x+1=0$

(vii) $3x^2-2\sqrt{6}\,x+2=0$

(viii) x^2+8x+7

(ix) $\frac{x+3}{x+2}=\frac{3x-7}{2x-3}$

(x) $4x^2-4a^2x+(a^4-b^4)=0$

(xi) $abx^2+(b^2-ac)x-bc=0$

3.07 Solution by Perfect Square Method of a Quadratic Equations

Here given quadratic equations are converted into perfect square form $(x\pm A)^2=k^2$ for variable 'x' and then by taking square root of both sides, required roots of quadratic equation $x=k\pm A$ are obtained. This method will be clear by the following example.

Given equation is $2x^2-5x+3=0$ which have to solve by perfect square method.

So $\quad 2x^2-5x+3=0 \quad \ldots (1)$

or $\quad x^2-\frac{5}{2}x+\frac{3}{2}=0 \quad$ (Taking 1 as coefficient of x^2)

or $\quad x^2-\frac{5}{2}x=-\frac{3}{2} \quad$ (Taking constant term in other side) $\quad \ldots$ (2)

For making perfect square to LHS of equation (2), add half of coefficient of x on both sides, we get.

$$x^2-\frac{5}{2}x+\left(\frac{5}{4}\right)^2=-\frac{3}{2}+\left(\frac{5}{4}\right)^2$$

By writing LHS in perfect square form, simplify RHS. and obtained $(x\pm A)^2=k^2$ form.

i.e., $\quad \left(x-\frac{5}{4}\right)^2=\frac{-24+25}{16}=\frac{1}{16}$

or $\quad \left(x-\frac{5}{4}\right)^2=\left(\frac{1}{4}\right)^2$

Taking square root of both sides, we have

$$x-\frac{5}{4}=\pm\frac{1}{4} \quad \Rightarrow \quad \frac{5}{4}\pm\frac{1}{4}$$

or $\quad x=\frac{5}{4}+\frac{1}{4} \quad$ or $\quad x=\frac{5}{4}-\frac{1}{4}$

or $\quad x=\frac{6}{4}=\frac{3}{2} \quad$ or $\quad x=\frac{4}{4}=1$

So $x=\frac{3}{2}$ and $x=1$ are required roots of equation $2x^2-5x+3=0$.

Here we should note that if we get $(x\pm A)^2=-k^2$ form then value of x will not be real, it mean equation has no real , roots such quadratic equations can be solved by method, proposed by Indian mathematician Shridhar Acharya, quadratic formula.

Let quadratic equation is $ax^2+bx+c=0, a\neq 0$

Here $ax^2+bx+c=0$

$\Rightarrow \quad x^2+\frac{b}{a}x+\frac{c}{a}=0 \qquad \Rightarrow \quad x^2+\frac{b}{a}x=\frac{-c}{a}$

For making perfect square. Adding square of half of coefficient of x on both sides.

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{-c}{a}+\left(\frac{b}{2a}\right)^2$$

or
$$\left(x+\frac{b}{2a}\right)^2=\frac{-4ac+b^2}{4a^2}$$

Taking square root on both sides

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

or
$$x=-\frac{b}{2a}\pm\sqrt{\frac{b^2-4ac}{2a}}$$

It means following are the roots of given equation

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

If $(b^2-4ac)\geq 0$, then value of x will be real, so we can use the quadratic formula for $ax^2+bx+c=0, a\neq 0$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}, \text{ when } (b^2-4ac)\geq 0$$

Example 1. Solve quadratic equation $2x^2-7x+3=0$ by perfect square method and verify the roots by quadratic formula of Shrihar Acharya.

Solution : Given equation is

$$2x^2-7x+3=0$$

or
$$x^2-\frac{7}{2}x+\frac{3}{2}=0$$

or $$x^2 - \frac{7}{2}x = \frac{-3}{2}$$

For making perfect square, adding square of half of coefficient of *x* in both sides, we get

$$x^2 - \frac{7}{2}x + \left(\frac{7}{4}\right)^2 = \frac{-3}{2} + \left(\frac{7}{4}\right)^2$$

or $$\left(x - \frac{7}{4}\right)^2 = \frac{-24+49}{16} = \frac{25}{16}$$

or $$\left(x - \frac{7}{4}\right)^2 = \left(\frac{5}{4}\right)^2$$

Taking square roots on both sides, we get

$$x - \frac{7}{4} = \pm\frac{5}{4}$$

or $x - \frac{7}{4} = \frac{5}{4}$ or $x - \frac{7}{4} = \frac{-5}{4}$

or $x = \frac{7}{4} + \frac{5}{4} = 3$ or $x = \frac{7}{4} - \frac{5}{4} = \frac{1}{2}$

Thus $x = 3$ and 1/2 are solution of given equation.

Verification by Shridhar Acharya quadratic formula.

By comparing equation $ax^2 + bx + c = 0$ with equation $2x^2 - 7x + 3 = 0$, we get $a = 2, b = -7, c = 3$

Here $b^2 - 4ac = (-7)^2 - 4\times2\times3 = 25 \geq 0$ So roots are real. So substituting values of *a, b, c* in quadratic formula, we get.

$$x = \frac{+7 \pm \sqrt{(-7)^2 - 4\times2\times3}}{2\times2} = \frac{7\pm5}{4}$$

Thus, $x = \frac{7+5}{4}$ or $x = \frac{7-5}{4}$

i.e., $x = 3$ and $x = 1/2$ are required roots **Hence Proved**

Hence, solution of the given equation is verified by Shridhar Acharya quadratic formula.

Exercise 3.4

1. Solve the following quadratic equation by the method of perfect the square.

(i) $3x^2 - 5x + 2 = 0$ (ii) $5x^2 - 6x - 2 = 0$ (iii) $4x^2 + 3x + 5 = 0$ (iv) $4x^2 + 4\sqrt{3}x + 3 = 0$

(v) $2x^2 + x - 4 = 0$ (vi) $2x^2 + x + 4 = 0$ (vii) $4x^2 + 4bx - (a^2 - b^2) = 0$

2. Find the roots of the following quadratic equations, if they exist, using the quadratic formula of Shridhar Acharya.

(i) $2x^2 - 2\sqrt{2} + 1 = 0$ (ii) $9x^2 + 7x - 2 = 0$ (iii) $x + \frac{1}{x} = 3, x \neq 0$

(iv) $\sqrt{2}\,x^2 + 7x + 5\sqrt{2} = 0$ (v) $x^2 + 4x + 5 = 0$

(vi) $\frac{1}{x} - \frac{1}{x-2} = 3,\ x \neq 0, 2$

3. Find two consecutive positive odd integers, sum of whose squares is 290.
4. The difference of square of two numbers is 45 and square of smaller number is four times the larger number. Find the two numbers.
5. Divide 16 into two parts such that 2 times the square of larger part is 164 more than the square of smaller part.

3.08 Discriminant and Nature of Roots

In previous appendix we have studied about solving of quadratic equation $ax^2 + bx + c = 0, a \neq 0$ by factorisation, perfect square method and Shridhar Acharya method. In appendix 3.07 we have used the following formula to solve quadratic equation $ax^2 + bx + c = 0$ by Shridhar Acharya method

Formula
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \ldots \text{(i)}$$

Where, $(b^2 - 4ac) \geq 0$ for real roots, from this we obtained two real roots

$(b^2 - 4ac) < 0$ for unreal roots, since $(b^2 - 4ac)$ will be negative and so its square root will be imaginary. So nature of roots depends on $(b^2 - 4ac)$. Thus $(b^2 - 4ac)$ is called discriminant of quadratic equation $ax^2 + bx + c = 0$. So nature of roots can be determined by different values of discriminant as follows :

(i) If $(b^2 - 4ac) > 0$, then roots will be distinct and real.

If α, β are roots then

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

(ii) If $(b^2 - 4ac) = 0$, then roots will be equal and real *i.e.*, $\alpha = \frac{-b}{2a}, \beta = \frac{-b}{2a}$

(iii) If $(b^2 - 4ac) < 0$, then roots will be imaginary. (not real)

Now, we can clearly understand all the three types of nature of roots of quadratic equations by the following examples :

Example 1. Find the nature of the root of the following quadratic equations and if roots exits then find them.

(i) $2x^2 - 6x + 3 = 0$ **(ii)** $3x^2 - 4\sqrt{3}\,x + 4 = 0$ **(iii)** $x^2 + x + 1 = 0$

Solution : (i) Given equation is

$$2x^2 - 6x + 3 = 0$$

On comparing this equation by $ax^2 + bx + c = 0$, we have

$$a = 2, b = -6, c = 3$$

Now, $b^2 - 4ac = 12 > 0$ is positive, so roots of given equation will be distinct and real.

So by Shridhar Acharya quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Two required roots will be $x = \dfrac{+6 \pm \sqrt{12}}{4}$

i.e., $x = \dfrac{3 + \sqrt{3}}{2}$ or $x = \dfrac{3 - \sqrt{3}}{2}$

(ii) Here equation is $3x^2 - 4\sqrt{3}x + 4 = 0$

On comparing given equation by $ax^2 + bx + c = 0$, we have

$$a = 3,\ b = -4\sqrt{3},\ c = 4$$

and Discriminant $(b^2 - 4ac) = (-4\sqrt{3})^2 - 4 \times 3 \times 4$

$$= 48 - 48 = 0$$

So roots of given equation will be equation and real. By Shridhar Acharya Quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

i.e., $x = \dfrac{4\sqrt{3} \pm 0}{2 \times 3} = \dfrac{2}{\sqrt{3}}$

(iii) Given equation is $x^2 + x + 1 = 0$

On comparing given equation by $ax^2 + bx + c = 0$, we have

$$a = 1,\ b = 1,\ c = 1$$

$\therefore$ Discriminant $(b^2 - 4ac) = 1 - 4 = -3 < 0$,so roots will be imaginary.

Exercise 3.5

1. Find the nature of roots of the following quadratic equations.

(i) $2x^2 - 3x + 5 = 0$ (ii) $2x^2 - 4x + 3 = 0$ (iii) $2x^2 + x - 1 = 0$ (iv) $x^2 - 4x + 4 = 0$

(v) $2x^2 + 5x + 5 = 0$ (vi) $3x^2 - 2x + \dfrac{1}{3} = 0$

2. Find the value of k in the following quadratic equations for which roots are real and equal.

(i) $kx(x-2) + 6 = 0$ (ii) $x^2 - 2(k+1)x + k^2 = 0$ (iii) $2x^2 + kx + 3 = 0$

(iv) $(k+1)x^2 - 2(k-1)x + 1 = 0$ (v) $(k+4)x^2 + (k+1)x + 1 = 0$

(vi) $kx^2 - 5x + k = 0$

3. Find the value of k for which following quadratic equations have real and distinct roots :

(i) $kx^2+2x+1=0$ (ii) kx^2+6x+1 (iii) $x^2-kx+9=0$

4. Find the value of k so that equation $x^2+5kx+16=0$ has no real roots.

5. If roots of quadratic equation $(b-c)x^2+(c-a)x+(a-b)=0$ are real and equal then prove that $2b=a+c$.

3.09 Least common Multiple (L.C.M.) and Highest Common Multiple (H.C.M.) of Algebraic Expressions

In previous chapter we have studied LCM and HCF of positive real integers by using fundamental therorem of airthmetic. LCM is product of the greatest power of each prime factor, involved in the number and HCF is product of the smallest power of each common prime factor in the numbers. Here we will study about HCF and LCM of algebraic expressions.

Least Common multiple (LCM) : LCM of given expression $u(x)$ and $v(x)$ is a polynomial which is product of least power polynomial and least power numeric coefficient. Here sign of coefficient of highest power term is same the term of highest power of $u(x)$,$v(x)$.

Highest common factor (HCF) : In two expression $u(x)$ and $v(x)$, product of highest power factors is called HCF and its coefficient is taken as positive. So HCF of two given polynomial is obtained by the product of highest power common factor and maximum divisor of numeric coefficient. Relation between LCM and HCF of any polynomials is written as.

$$\text{LCM} \times \text{HCF} = u(x) \times v(x)$$

In this appendix, common factor means an expression which when divide each expression gives remainder as 0 and common multiple means that if f (x) is a common multiple then it should be completely divided by given polynomials.

Method to find LCM and HCF of expressions and polynomials can be easily understand by the following examples.

Example 1. Find the LCM of following expressions.

(i) $4a^2b^2c$ and $6ab^2d$

(ii) x^2-4x+3 and x^2-5x+6

(iii) $-2(x-1)(x-2)(x+3)$ and $3(x-1)(x-2)(x+3)(x+5)$

Solution : (i) Let given expression $u(x)=4a^2b^2c$ and $v(x)=6ab^2d$

Writing in factorisation form

$u=2^2\times a^2\times b^2\times c$ and $v=2\times 3\times a\times b^2\times d$

So common multiple

$$=2^2\times 3^1\times a^2\times b^2\times c\times d$$

= product of common factor of highest power

This common multiple is required LCM

i.e., LCM = 12 a^2b^2cd

(ii) Let in given polynomial $u(x) = x^2 - 4x + 3$ and $v(x) = x^2 - 5x + 6$

Writing these in factorisation form

$u(x) = x^2 - 4x + 3 = x^2 - 3x - x + 3$

$= x(x-3) - 1(x-3) = (x-3)(x-1)$ . . . (1)

and $v(x) = x^2 - 5x + 6 = x^2 - 3x - 2x + 6$

$= x(x-3) - 2(x-3) = (x-3)(x-2)$ . . . (2)

From equation (1) and (2), It is clear that product of highest power of prime factors

$= (x-1)\times(x-2)\times(x-3)$

$\therefore$ Required LCM $= (x-1)(x-2)(x-3)$

(iii) Let given polynomial

$u(x) = -2(x-1)(x-2)(x+3)$ and

$v(x) = 3(x-1)(x-2)(x+3)(x+5)$

By observation we can write product of common factors

$= -2\times 3\times(x-1)\times(x-2)\times(x+3)\times(x+5)$

In this product, highest power factorization has same sign as of highest power term $-6x^2$ of $u(x)\times v(x)$

Thus, required LCM $= -6(x-1)(x-2)(x+3)(x+5)$

Example 2. Find the highest common factor (HCF) of the folowing.

(i) $8a^2b^2c$ and $18ab^3c^2$ (ii) $20x^2 - 9x + 1$ and $5x^2 - 6x + 1$

(iii) $(x+1)^2(x+2)^2(x+3)^2$ and $(x+1)^3(x-2)^2(x+3)^3$

Solution : (i) Let given expression $u = 8a^2b^2c$ and $v = 18ab^3c^2$

writing in factorisation form $u = 2^3\times a^2\times b^2\times c$ and $v = 2\times 3^2\times a\times b^3\times c^2$

Common divisor of highest power $= 2\times a\times b^2\times c$

Product of common factors of least power

Thus required (HCF) = $2ab^2c$

(ii) Let given polynomial $u(x) = 20x^2 - 9x + 1$ and $v(x) = 5x^2 - 6x + 1$

Writing these in factorisation form

$u(x) = 20x^2 - 9x + 1 = 20x^2 - 5x - 4x + 1$

$= 5x(4x-1) - 1(4x-1) = (4x-1)(5x-1)$ . . . (1)

and $v(x) = 5x^2 - 6x + 1 = 5x^2 - 5x - x + 1$

$= 5x(x-1) - 1(x-1) = (x-1)(5x-1)$ . . . (2)

Form equation (1) and (2) it is clear that common divisor highest power is $(5x-1)$

Thus required HCF $= (5x-1)$

(iii) Let $u(x) = (x+1)^2(x+2)^2(x+3)^2$ and $v(x) = (x+1)^3(x-2)^2(x+3)^3$

$\therefore$ Common divisor of highest power $= (x+1)^2(x+3)^2$

= Product of common factors of least power

Thus required $\text{HCF} = (x+1)^2(x+3)^2$

Exercise 3.6

1. Find the LCM of following expressions :

   (i) $24x^2yz$ and $27x^4y^2z^2$ (ii) x^2-3x+2 and $x^4+x^3-6x^2$

   (iii) $2x^2-8$ and x^2-5x+6 (iv) x^2-1; $(x^2+1)(x+1)$ and x^2+x-1

   (v) $18(6x^4+x^3-x^2)$ and $45(2x^6+3x^5+x^4)$

2. Find the HCF of the following expressions

   (i) a^3b^4, ab^5, a^2b^8 (ii) $16x^2y^2, 48x^4z$

   (iii) $x^2-7x+12$; $x^2-10x+21$ and $x^2+2x-15$ (iv) $(x+3)^2(x-2)$ and $(x+3)(x-2)^2$

   (v) $24(6x^4-x^3-2x^2)$ and $20(6x^6 ? 5x^5 ? x^4)$

3. If $u(x)=(x-1)^2$ and $v(x)=(x^2-1)$ then verify $\text{LCM}\times\text{HCF}=u(x)\times v(x)$.

4. The product of two expressions is $(x-7)(x^2+8x+12)$. If their highest common factor (HCF) is $(x+6)$ then find their least common multiple (LCM).

5. If HCF and LCM of two quadratic expression are $(x-5)$ and $x^3-19x-30$ then find two expressions.

Miscellaneous Exercise 3

1. If one zero of polynomial $f(x)=5x^2+13x+k$ is reciprocal to other zero, then value of k will be :

   (a) 0 (b) 1/5 (c) 5 (d) 6

2. Zeroes of polynomial x^2-x-6 are :

   (a) 1, 6 (b) 2, −3 (c) 3, −2 (d) 1, −6

3. If 3 is a zero of polynomial $2x^2+x+k$, then value of k will be :

   (a) 12 (b) 21 (c) 24 (d) − 21

4. If α, β are zeroes of polynomial $x^2-p(x+1)-c$ such that $(\alpha+1)(\beta+1)=0$, then value of c will be:

   (a) 0 (b) −1 (c) 1 (d) 2

5. If quadratic equation $x^2-kx+4=0$ has equal roots, then value of k will be :

   (a) 2 (b) 1 (c) 4 (d) 3

6. If $x ? 1$ is common root of equations $ax^2+ax+3=0$ and $x^2+x+b=0$, then value of ab will be :

   (a) 1 (b) 3.5 (c) 6 (d) 3

7. Discriminant of quadratic equation $3\sqrt{3}\,x^2+10x+\sqrt{3}=0$ will be :

   (a) 10 (b) 64 (c) 46 (d) 30

8. Nature of roots of quadratic equation $4x^2 - 12x - 9 = 0$ is :

(a) real and equal (b) real and distinct

(c) imaginary and equal (d) imaginary and distinct

9. HCF of expression $8a^2b^2c$ and $20ab^3c^2$ is :

(a) $4ab^2c$ (b) $4abc$ (c) $40a^2b^3c^2$ (d) $40abc$

10. LCM of expressions $x^2 - 1$ and $x^2 + 2x + 1$ is :

(a) $x+1$ (b) $(x^2-1)(x+1)$ (c) $(x-1)(x+1)^2$ (d) $(x^2-1)(x+1)^2$

11. If $30x^2y^4$ is LCM of expressions $6x^2y^4$ and $10xy^2$, then their HCF will be :

(a) $6x^2y^2$ (b) $2xy^2$ (c) $10x^2y^4$ (d) $60x^3y^6$

12. Write Shridhar Acharya's formula to find roots of quadratic equation $ax^2 + bx + c = 0$

13. Find the nature of roots by writing general form of discriminant of equation $ax^2 + by + c = 0$

14. Find the zeroes of qudratic polynomial $2x^2 - 8x + 6$ and verify the relation between zeroes and coefficients.

15. If α and β are zeroes of quadratic polynomial $f(x) = x^2 - px + q$, then find the value of the following.

(i) $\alpha^2 + \beta^2$ (ii) $\frac{1}{\alpha} + \frac{1}{\beta}$

16. If polynomial $x^4 - 6x^3 + 16x^2 - 25x + 10$ is divided by another polynomial $x^2 - 2x + k$ and leaves remainder comes $(x+a)$, then find the value of k and a.

17. The are of a rectangular plot is 528m². Length (in m.) of plot is 1 more than twice the breadth. Determine required quadratic equation and find the lenght and breadth of the plot.

18. Solve the qudratic equation $x^2 + 4x - 5 = 0$ by perfect the square method.

19. Solve the following equation by factorisation method.

(i) $\frac{1}{x} - \frac{1}{x-2} = 3, \quad x \neq 0, 2$ (ii) $\frac{1}{x-1} - \frac{1}{x+5} = \frac{6}{7}, \quad x \neq 1, -5$

(iii) $x - \frac{1}{x} = 3, \quad x \neq 0$ (iv) $\frac{1}{x+4} - \frac{1}{x-7} = \frac{11}{30}, \quad x \neq 4, 7$

20. If -5 is one root of quadratic equation $2x^2 + px - 15 = 0$ and roots of quadratic equation $p(x^2 + x) + k = 0$ is equal, then find the value of k.

21. Solve the following quadratic equation by using Shridhar Acharya quadratic formula.

(i) $p^2x^2 + (p^2 - q^2)x - q^2 = 0$ (ii) $9x^2 - 9(a+b)x + (2a^2 + 5ab + 2b^2) = 0$

22. $x^3 - 7x + 6$ and $(x-1)$ are LCM and HCF of two quadratic expressions. Find the expressions.

23. LCM and HCF of two polynomials are $x^3 - 6x^2 + 3x + 10$ and $(x+1)$ respectively. If one polynomial is $x^2 - 4x + 5$, then find other polynomial.

Important Points

1. Generally $ax + b$ is linear, $ax^2 + bx + c$ is quadratic and $ax^3 + bx^2 + cx + d$ is called cubical polynomial.
2. Polynomial $f(x) = 0$ for which value of x, that value is called zero of $f(x)$.
3. Number of zeroes of a polynomial is equal to its highest power. A quadratic polynomial has maximum two zeroes.
4. If α, β are zeroes of $ax^2 + bx + c$, then $(\alpha + \beta) = \frac{-b}{a}$ and $\alpha\beta = \frac{c}{a}$
5. If α, β are zeroes of any quadratic polynomial, then it can be written as $k[x^2 - (\alpha + \beta)x + \alpha\beta]$
6. **Division Algorithm :** If $f(x)$ and $g(x)$ are any polynomials, then we get $g(x)$ and $r(x)$ such polynomials, so that $f(x) = q(x)\,g(x) + r(x)$, where $r(x) = 0$ of $r(x)$ that power of $g(x)$
7. If $f(x) = ax^2 + bx + c$ is a quadratic polynomial, then $f(x) = 0, a \neq 0$ is a quadratic equation. Zeroes of polynomial $f(x)$ and roots of quadratic equation $f(x) = 0$ are same.
8. **Solving of quadratic equation :** (i) By writing in standard form $f(x) = 0$, factorise into a product of two linear factors then the roots of the quadratic equation can be found by equating each factor to zero.

   (ii) By Shridhar Acharya, roots of quadratic equation $ax^2 + bx + c = 0,\ a \neq 0$ are given by following quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ where } (b^2 - 4ac) > 0$$

9. Nature of roots of a quadratic equation $ax^2 + bx + c = 0,\ a \neq 0$ depends on discrimenant $(b^2 - 4ac)$

   (i) If $(b^2 - 4ac) > 0$, then roots will be real and distinct.

   (ii) If $(b^2 - 4ac) = 0$, then roots will be real and equal.

   (iii) If $(b^2 - 4ac) < 0$, then roots will be imaginary.
10. Highest common factor (HCF) of given expression is common factor of highest power *i.e.,* product of common factors of least power of expressions.
11. LCM of given expression is product of common factors of smallest power i.e., common multiple. Its sign is same as the product of highest power terms have.
12. If $f(x)$ and $g(x)$ are two expressions,then relation between their LCM and HCF is

    LCM × HCF = $f(x) \times g(x)$

Answer Sheet

Exercise 3.1

1. (i) –2, 0 (ii) $\frac{1}{2},\frac{1}{2}$ (iii) $\frac{2}{3},-\frac{1}{2}$ (iv) $-\sqrt{15},\sqrt{15}$ (v) $1,\sqrt{3}$ (vi) $-1,\frac{4}{3}$

2. (i) x^2+3x+2 (ii) $3x^2-3\sqrt{2}x+1$ (iii) $4x^2+x+1$ (iv) $x^2+\sqrt{5}$
(v) x^2-4x+1 (vi) x^2-x+1 3. 12

Exercise 3.2

1. (i) $3x-5;\ 9x+10$ (ii) $x-3;\ \ 7x-9$ (iii) $x^2-8x+27;\ \ -60$ (iv) $3x^2-x;\ \ -x+4$

3. (i) $\frac{1}{2},1$ (ii) –5, 7 (iii) –10, –1 4. x^2-x+1

Exercise 3.3

1. (i) No, (ii) Yes, (iii) No, (iv) No.

2. (i) $1,\frac{3}{2}$ (ii) $-\frac{1}{3},\frac{2}{3}$ (iii) $-\sqrt{3},-\frac{7}{\sqrt{3}}$ (iv) 4, 4 (v) $3,\frac{4}{3}$

(vi) $\frac{1}{10},\frac{1}{10}$ (vii) $\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}}$ (viii) –1, –7 (ix) –1, 5 (x) $\frac{a^2+b^2}{2},\frac{a^2-b^2}{2}$

(xi) $-\frac{b}{a},\frac{c}{b}$

Exercise 3.4

1. (i) $1,\frac{2}{3}$ (ii) $\frac{3\pm\sqrt{19}}{5}$ (iii) not real roots, (iv) $\frac{-\sqrt{3}}{2},-\frac{\sqrt{3}}{2}$ (v) $\frac{-1\pm\sqrt{33}}{4}$

(vi) not real roots (vii) $\frac{-(a+b)}{2},\frac{(a-b)}{2}$

2. (i) $\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}$ (ii) $\frac{2}{9},-1$ (iii) $\frac{3\pm\sqrt{5}}{2}$ (iv) $-\sqrt{2},\frac{-5}{\sqrt{2}}$ (v) not real roots, (vi) $\frac{3\pm\sqrt{3}}{3}$

3. 11, 13 4. 9, 6 and 9, –6 5. 10, 6

Exercise 3.5

1. (i) not real roots (ii) no any real roots (iii) roots are real and distinct
(iv) roots are real and equal (v) roots are not real (vi) roots are real and equal

2. (i) $K=0,6$ (ii) $k=-\frac{1}{2}$ (iii) $k\le-2\sqrt{6},\quad k\ge2\sqrt{6}$

(iv) k = 0, 3 (v) k = 5, –3 (vi) $k=\pm\frac{5}{2}$

3. (i) $k < 1$ (ii) $k < 9$ (iii) $k < -6, \ k > 6$ 4. $\frac{-8}{5} < k < \frac{8}{5}$

Exercise 3.6

1. (i) $216x^4y^2z^2$ (ii) $x^2(x-1)(x-2)(x+3)$ (iii) $2(x^2-4)(x-3)$
   (iv) $(x^4-1)(x^2+x-1)$ (v) $90x^4(x+1)(2x+1)(3x-1)$

2. (i) ab^4 (ii) $16x^2$ (iii) $(x+3)$ (iv) $(x+3)(x-2)$ (v) $4x^2(2x+1)$

3. $\text{LCM} = (x-1)^2(x+1)$; $\text{HCF} = (x-1)$ 4. $\text{LCM} = x^2-5x-14$

5. $x^2-3x-10$ and $x^2-2x-15$

Miscellaneous Exercise 3

1. (c) 2. (c) 3. (d) 4. (c) 5. (c) 6. (d) 7. (b)
8. (b) 9. (a) 10. (c) 11. (b)

12. $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

13. Discriminant (b^2-4ac),

(i) $b^2-4ac > 0$, real and disinct roots, (ii) $b^2-4ac = 0$, real and equal roots

(iii) $b^2-4ac < 0$, imaginary roots

14. 1, 3 15. (i) p^2-2q (ii) $\frac{p}{q}$ 16. $k = 5$ and $a = -5$

17. $2x^2+x-528=0$, breadth $= 16$ m and length $= 33$ m 18. 1, –5

19. (i) 2, –6; (ii) $\frac{3 \pm \sqrt{3}}{3}$; (iii) $\frac{3 \pm \sqrt{13}}{2}$; (iv) 1, 2 20. $k = \frac{7}{4}$

21. (i) $-1, \frac{q^2}{p^2}$; (ii) $\frac{2a+b}{3}, \frac{a+2b}{3}$ 22. x^2+2x-3 and x^2-3x+2 23. x^2-x-2

04

Linear Equation and Inequalities in two variables

4.01. Introduction

To solve any problem it is denoted in mathematical form. We know that problem is based on one or more factors. In previous class we have solved such problems by denoting in equation form. According to problem these equations are based on one variable or two or more variables. If any equation represents a straight line then it is called linear equation. In general form $ax+b=0, a\neq 0$ where *a, b* are real numbers, represents a linear equation in one variable. The value of x which satifies this equation is called solution of the equation. Graph of linear equation in one variable is a line parallel to any axis. In class IX we have draw graph of linear equations of two variables $ax+by+c=0;\ a,b\neq 0$. The values of x and y for which equation satisfies, are called its solutions. Graph of linear equation in two variables is also a straight line. Each point (x, y) on this straight line express solution of this equation. Here, we consider two linear equations in two variables x, y

$$a_1x+b_1y+c_1=0 \qquad \ldots (1)$$

$$a_2x+b_2y+c_2=0, \qquad \ldots (2)$$

Where, a_1, b_1, c_1, a_2, b_2, c_2 all are real numbers and a_1, b_1 and a_2, b_2 are not zero. It means for equations (1) and (2)

$$a_1^2+b_1^2\neq 0 \quad \text{and} \quad a_2^2+b_2^2\neq 0$$

These type of equations are called pair of linear equaitons in two variables. This pair of equations is called simultaneous linear equations in two variables. Here we will discuss in details of the solution of these pair of equation and its consistency and inconsistency.

In this chapter we will also discuss about inequalities. If in linear equations, we replace the = sign by $<$ or $>$or $\leq$ or $\geq$ (sign of inequations) then these equations are called inequations. Here, we will solve the inequations in two variables by graph method. This solution in the form of area. Solution of set of simultaneous linear inequaitons in two variables can be obtained in the form of common area.

4.02. Simultaneous Linear Equation of Two Variables

Pair of linear equations of two variables is called simultaneous linear equation system. For example $5x+2y=17;\ \ 2x-5y=1$ or $x+2y=3;\ \ 2x-y=5$ etc.

Solving of pair of linear equation means to find values of two variables which satisfies both the equations. Solving of pair of linear equation can be understand by the following examples

Example 1. Simultaneous linear equations are

$$3x+2y-5=0;\quad 4x+7y-11=0$$

To find the nature of solution for $x = 1$ and $y = 1$

By Putting $x = 1$ and $y = 1$ in both equations, we get

$$3(1) + 2(1) - 5 = 0;\ 4(1) + 7(1) - 11 = 0$$

then at $x = 1$ and $y = 1$ is required solution of pair of linear equations.

Example 2. $2x+7y=11$; $x-3y=5$ are simultaneous linear pair of equations. to find the nature of solutions for $x = 2$ and $y = 1$.

Putting $x = 2$ and $y = 1$ in above two equations, we get

$$2(2) + 7(1) = 11$$

i.e, first equation satisfies

For second equation put the value of x and y.

$$2-3(1)=-1\neq 5$$

So this equation does not satisfy at $x = 2$, $y = 1$

i.e, $x = 2$, $y = 1$ is not a solution of given pair of linear equations.

Example 3. To find the nature of solution of linear pair $x+2y-5=0$; $2x+4y-10=0$ for $x = 1$, $y = 2$ and $x = 3$, $y = 1$.

Case I : ,at $x = 1$, $y = 2$ simultaneous equations are

$$1 + 2(2) - 5 = 0$$

and $$2(1) + 4(2) - 10 = 0$$

equations are satisfied.

Case II : at $x = 3$, $y = 1$ simultaneous equations are

$$3 + 2(1) - 5 = 0$$

and $$2(3) + 4(1) - 10 = 0$$

So, in this case also, equations are satisfied

i.e, $x = 1$ and $y = 2$ and $x = 3$ and $y = 1$ both are solutions of given pair of linear equations. Similarly many other solutions for these equations are possible.

Here, form above examples it is clear that simultaneous linear equation has unique solutions or many solution or no solution. If any simultaneous linear pair of equations can be solve, either unique or more than one solution, then this type of linear equations are called consistent, and if it has no solution then these pair are called inconsistent. When we draw a linear equation of two variables on graph, a straight line is obtained in a plane, their position can be as follows :

(i) Two lines intersect at a point

(ii) Two lines do not intersect or they are parallel.

(iii) Two lines concides each other.

The linear equation of the lines are consistent or inconsistent. These cases can be understand by graphical representation.

4.03. Graphical Representation of Linear Equation and their Solution

Here, for the following examples we will find the nature of solution by graphical representation of linear pair of equations.

for example, represent the following linear pairs of equations graphically

(i) $2x+3y=13$; $5x-2y=4$

(ii) $2x+4y=10$; $3x+6y=12$

(iii) $4x+6y=18$; $2x+3y=9$

Example 4. Equations

$$2x+3y=13 \qquad \ldots (1)$$

$$5x-2y=4 \qquad \ldots (2)$$

Let us represent these equations graphically. For this, we need at least two solutions for each equation. First we try to get a point at $x = 0$ or $y = 0$. So, these equations reduces to linear equations, so second variable can be find easily.
When we put $x = 0$ or $y = 0$ in equation (1), we do not get value of another variable in integer. So, we put another values,

Putting $x = 2$ in equation (1).

$2 \times 2 + 3y = 13$ or $3y = 13 - 4 = 9$ or $y = 3$

and at $x = 5$

$2 \times 5 + 3y = 13$ or $3y = 13 - 10 = 3$ or $y = 1$

Thus, point are obtained according to the following table

x	2	5
y	3	1

Similarly in equation (2), $x = 0$

$5 \times 0 - 2y = 4$ $y = -2$

and at $x = 2$

$5 \times 2 - 2y = 4$ $-2y = -6$ $y = 3$

Thus, for equation (2), points are obtained as follows :

x	0	2
y	–2	3

Now plot these points on graph paper and obtained following straight lines *i.e.,* draw *XOX′* and *YOY′* axis on graph paper join plotted points and obtained straight line.

In figure, we see that straight lines intersect ar point *P* (2,3)

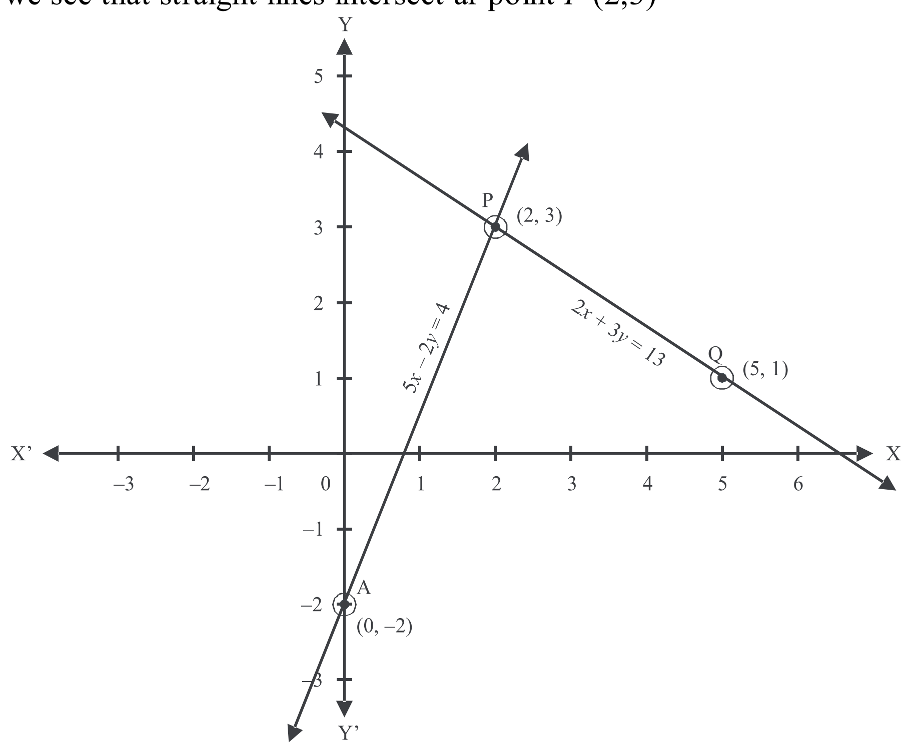


Fig.4.1

Example 5. Pair of linear equations are

$$2x + 4y = 10 \qquad \ldots (1)$$

$$3x + 6y = 12 \qquad \ldots (2)$$

For equivalent graphical representation, we get points as follows :

Put $y = 0$ in equation (1)

$$2x + 4 \times 0 = 10$$

or $x = 5$

and put $x = 1$ in equation (1)

$$2 \times 1 + 4y = 10$$

or $4y = 10 - 2 = 8$

or $y = 2$

For equation (1), points are obtained as follows

x	5	1
y	0	2

Similarly in equation (2), put $x = 0$

$$3 \times 0 + 6y = 12$$

or $6y = 12$

or $y = 2$

and put $y = 0$

$$3x + 6 \times 0 = 12$$

or $3x = 12$

or $x = 4$

Thus for equation (2) obtained as follows points :

x	5	1
y	0	2

Now plot all the points on graph paper and join them, we obtained following graph. In above figure, we see that two straight lines are parallel to each other.

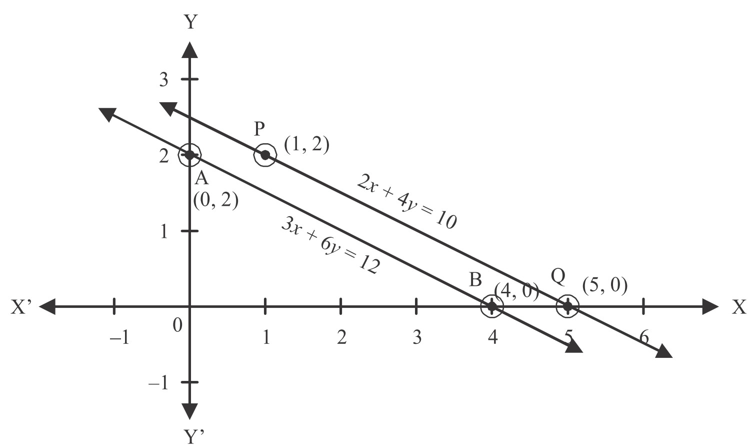


Fig. 4.2

Example 6. Pair of linear equations are

$$4x + 6y = 18 \qquad \text{. . . (1)}$$

$$2x + 3y = 9 \qquad \text{. . . (2)}$$

From equation (1) and (2) points are obtained and plot the lines on graph.

Put $x = 0$ in equation (1)

$$4 \times 0 + 6y = 18$$

or $6y = 18$

or $y = 3$

put $y = 1$

$$4x + 6 \times 1 = 18$$

or $4x = 18 - 6 = 12$

or $x = 3$

Thus, following points are obtained as follows :

x	0	3
y	3	1

For equation (2), put $x = 0$

$$2 \times 0 + 3y = 9$$

or $3y = 9$ or $y = 3$

and put $y = 1$

or $2x + 3 \times 1 = 9$

or $2x = 9 - 3 = 6$

or $x = 3$

Similarly for equation (2) following points are obtained.

x	0	3
y	3	1

Plot all the points on graph paper and join them.

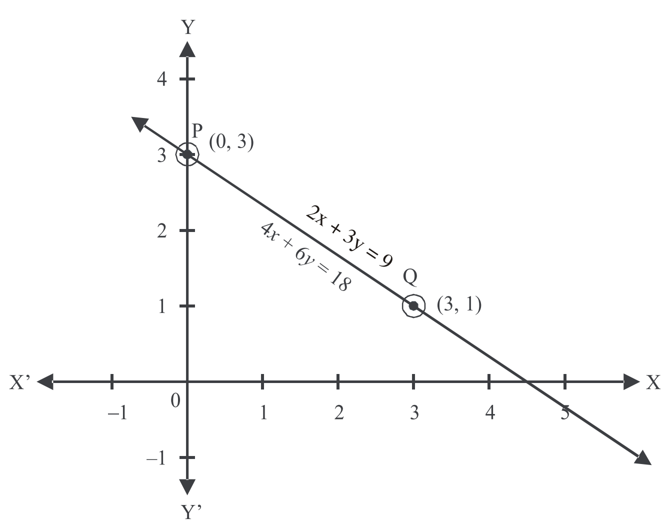


Fig. 4.3

In the above graph fig 4.3 two straight lines coincide each other. It is clear that two equation represents equal lines.

i.e. , equations are equivalent.

In the above examples 4,5,6 equations can be written in general form as :

$$a_1x + b_1y + c_1 = 0 \qquad \dots (1)$$

and

$$a_2x + b_2y + c_2 = 0 \qquad \dots (2)$$

Here, we prepare comparitive table for coefficients of x, y and contents.

Comparitive Table

Example No.	**Pair of linear Equations**	$\frac{a_1}{a_2}$	$\frac{b_1}{b_2}$	$\frac{c_1}{c_2}$	**Compare the the Ratio of Cofficients**	**Nature of lines**	**Algebraic Interp-retation**
(i)	$2x+3y=13$ $5x-2y=4$	$\frac{2}{5}$	$\frac{3}{-2}$	$\frac{13}{4}$	$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$	Intersecting lines	Unique Solution
(ii)	$2x+4y=10$ $3x+6y=12$	$\frac{2}{3}$	$\frac{4}{6}$	$\frac{10}{12}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$	Parallel lines	No Solution
(iii)	$4x+6y=18$ $2x+3y=9$	$\frac{4}{2}$	$\frac{6}{3}$	$\frac{18}{9}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$	Coincident lines	Infinity many Solution

From above table, it is clear that linear pair

$$a_1x + b_1y + c = 0$$

$$a_2x + b_2y + c = 0$$

(i) Intersecting, then relation coefficients is

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

Such linear pair has unique solution and intersection point (x, y) will be required solution.

(ii) Parallel, then relation between cofficients is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

Such linear pair has no solution.

(iii) Coincident, then relation between coefficients is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Such linear pair has infinitely many solutions *i.e.* , for each point value *x, y* will its solutions.

Conversely, if relation in coefficients are given then we can know nature of linear pair of equations represented. Linear equations can be solve graphically by following steps.

Graphical Method :

Step 1 : Write given linear equations in standard form

i.e., $$a_1x + b_1y + c_1 = 0 \quad \text{. . . (1)}$$

$$a_2x + b_2y + c_2 = 0 \quad \text{. . . (2)}$$

Step 2 : Prepare table of corresponding points of two equations.

Draw XOX' and YOY' on graph paper and then plot these points. Let lines L_1 and L_2 corresponds to equations (1) and (2).

Step 3 : If lines L_1 and L_2 intersect at point (α, β) then $x = \alpha$ and $y = \beta$ will be solution of given linear pair of equations.

Step 4 : If lines L_1 and L_2 are parallel then there will be no solutions *i.e.,* linear pair equations will be inconstitent.

Step 5: If lines L_1 and L_2 are consistent then there will be infinite solutions *i.e.,* two lines may be expressed as same line and each point (α, β) of this line will be obtained in the form of many solutions $(x = \alpha, y = \beta)$ of linear pair of equations.

Here, we can understand more from the following examples :

Example 7. Solve the following pairs of linear equations, graphically

(i) $$3x + 2y - 11 = 0$$

$$2x - 3y + 10 = 0$$

(ii) $$2x + 3y = 8$$

$$x - 2y = -3$$

(iii) $$2x + y - 6 = 0$$

$$4x - 2y - 4 = 0$$

Sol : (i) Given equation can be written as

$$3x + 2y = 11 \quad \text{. . . (i)}$$

$$2x - 3y = -10 \quad \text{. . . (ii)}$$

Obtain points table from quation (1)

at $y = 1$

$$3x + 2 \times 1 = 11$$

or $3x = 11 - 2 = 9$

or $x = 3$

Similarly at $x = 1$ in equation (1)

$$3\times1+2y=11$$

or $$2y=11-3=8$$

or $$y = 4$$

Thus point table of equation (1) is as follows

x	3	1
y	1	4

Now prepare point table of equation (2) is as follows

at $$y = 0,$$

$$2x-3\times0=-10$$

or $$x = -5$$

Similarly at $y = 2,\ 2x-3\times2=-10$

or $$2x=-4$$

or $$x = -2$$

So, points tables of equation (2) is as follows

x	–5	–2
y	0	2

Plot all the points on graph paper and join them

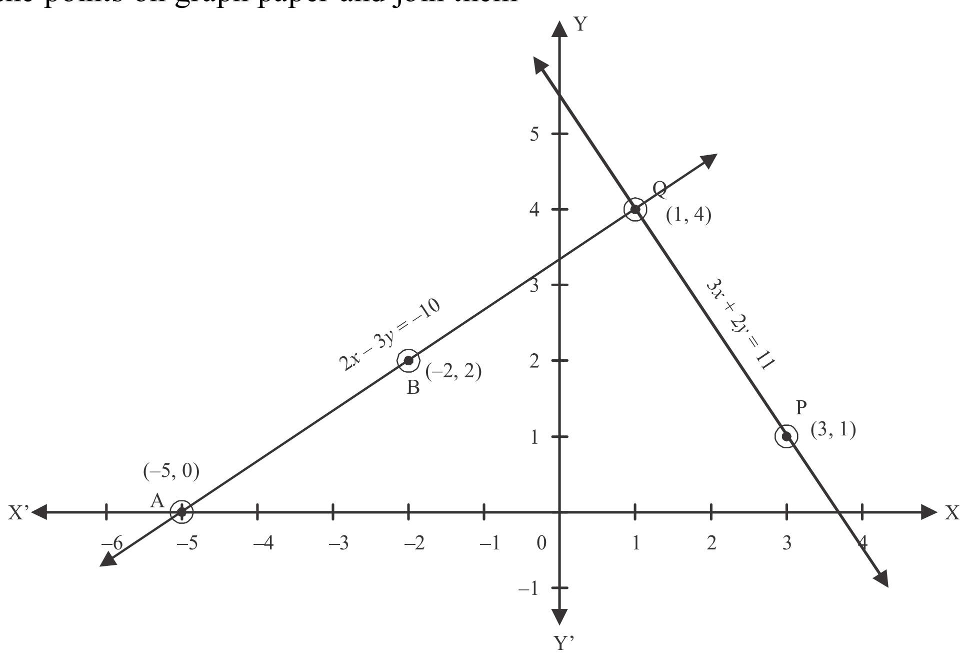


Fig. 4.4

From above representation, it is clear that two lines intersect each other at (1, 4) so, $x = 1$ and $y = 4$ is required solution of given linear pair *i.e.*, $x = 1, y = 4$ satisfy both the equations. So, solution is verified.

Sol : (ii) Given pair of equation is

$$2x+3y=8 \qquad \text{. . . (1)}$$

$$x-2y=-3 \qquad \text{. . . (2)}$$

Obtain the point table for equation (1)

at $x = 1,$ $\quad 2\times1+3y=8$

or $\quad 3y=8-2=6$

or $\quad y = 2$

Similarly $y = 0,$ $\quad 2x+3\times0=8$

or $\quad x = 4$

So, following table is obtained for equation (1)

x	1	4
y	2	0

Now, we obtained point table for equation (2).

In equation (2) at $y = 0,$ $\quad x-2\times0=-3$

$x = -3$

Similarly at $x = 1,$ $\quad 1 - 2y = -3$

$-2y=-4$

$y = 2$

Similarly following table is obtained for equation (2)

x	–3	1
y	0	2

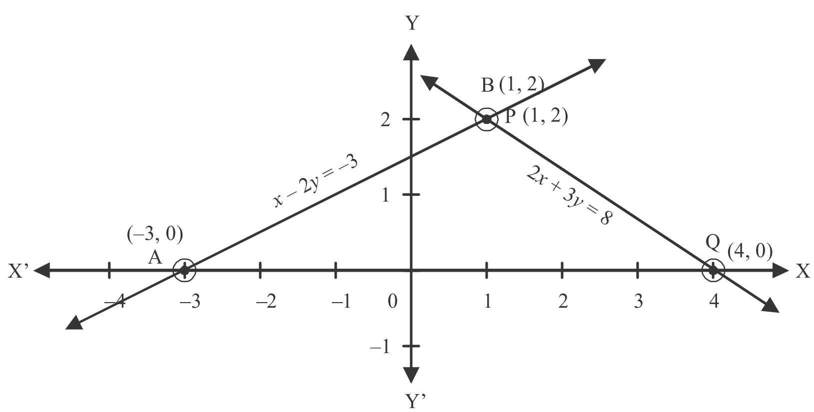


Fig. 4.5

With the help of above tables plot the points on graph paper and join them.

From above graphical representation it is clear that two lines intersect each other at point (1,2). So $x = 1, y = 2$ is required solution of given linear pair. $x = 1$ and $y = 2$ satisfy both the equations.

Sol : (iii) Writing given equations as

$$2x+y=6 \qquad \ldots (1)$$

$$4x-2y=4 \qquad \ldots (2)$$

Obtain point table for equation (1)

at $x = 0,$ $\quad 20+y=6$

or $\quad y = 6$

and at $x = 1,$ $\quad 2\times1+y=6$

or $\quad y = 6 - 2 = 4$

So, following table is obtained for equation (1)

x	0	1
y	6	4

Now obtain point table for equation (2).

In equation (2) at $x = 0$

$$4 \times 0 - 2y = 4$$

$$y = -2$$

and $y = 0$ $\quad 4x - 2 \times 0 = 4$

$$x = 1$$

Following table is obtained for equation (2)

x	0	1
y	–2	0

With the help of above tables linear pair is grtaphically represented.

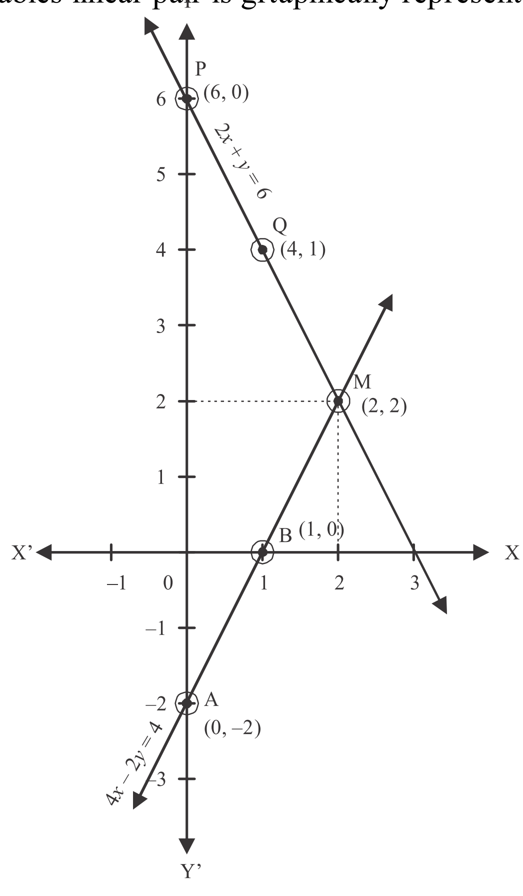


Fig.4.6

From the graph it is clear that two lines intersect each other at point (2, 2). So, $x = 2$, $y = 2$ is required solution of given linear pair and` $x = 2$, $y = 2$ satisfy given equations.

Exercise 4.1

1. On comparing the ratios $\frac{a_1}{a_2}, \frac{b_1}{b_2}$ and $\frac{c_1}{c_2}$, find out whether pairs of linear equations are consistent or inconsistent.

   (i) $2x-3y=8;$ $\quad 4x-6y=9$ $\qquad$ (ii) $3x-y=2;$ $\quad 6x-2y=4$

   (iii) $2x-2y=2;$ $\quad 4x-4y=5$ $\qquad$ (iv) $\frac{4}{3}x+2y=8;$ $\quad 2x+3y=12$

2. Solve the following pairs of linear equations, graphically and find the nature of that solution.

   (i) $x+y=3;$ $\quad 3x-2y=4$ $\qquad$ (ii) $2x-y=4;$ $\quad x+y=-1$

   (iii) $x+y=5;$ $\quad 2x+2y=10$ $\qquad$ (iv) $3x+y=2;$ $\quad 2x-3y=5$

3. Solve the following linear pairs by graphical method and find co-ordinates of point where lines representing by them cuts y– axis.

   (i) $2x-5y+4=0;$ $\quad 2x+y-8=0$ $\qquad$ (ii) $3x+2y=12;$ $\quad 5x-2y=4$

4. Solve the following linear pair by ghraphical method and find coordinated of vertices of triangle, formed by y– axis and lines formed by linear pair.

   $4x-5y=20;$ $\quad 3x+5y=15$

4.04. Linear Inequalities in Two Variables

A mathematical statement in which variable and sign $>, <, \geq$ or $\leq$ are presents, called inequality. Inequality may be of one variable or more than one variable. Let a is a non-zero real number then for variables x, inequalities $ax+b<0$, $ax+b\leq 0$, $ax+b>0$ and $ax+b\geq 0$ and called linear inequalities of one variable.

If number of variables is two then it is called as inequalities of two variables. For example, in general $2x+3y\leq 6$ and $x+y<4$. Inequalities of two variables can be defined as : Let a, b are two non-zero real; numbers for variables x and y inequalities $ax+by<c$, $ax+by\leq c$, $ax+by>c$ or $ax+by\geq c$ are called linear inequalities of two variables.

In this appendix we will study about the solution of linear inequalities of two variables. Many solutions are possible for these inequalities. Set of all possible solutions is called set of a inequality.

4.05. Solution of Linear Inequalities of Two Variables by Graphical Method

Here, we will solve linear inequalities of two variables by graphical method. In co-ordinate geometry we have studied that straight line $ax+by=c$, is represented by joining the points which satisfy the equation in the plane x, y on graph paper that is relative to x – axis and y – axis.

Straight line $ax+by=c$, divides x, y – plane in two parts. *i.e.* these divided areas can be expressed by $ax+by\leq c$ and $ax+by\geq c$.

These are expressed in the following sets in the forms of closed and open semi heavenly area.

In expression of sets :

Set $\{(x,y): ax+by=c\}$ Straight line

Set $\{(x,y): ax+by\leq c\}$ and $\{(x,y): ax+by\geq c\}$ closed semi heavenly area and Set $\{(x,y): ax+by<c\}$ and $\{(x,y): ax+by>c\}$ expose open semi heavenly area. All these semi heavenly areas the solution of inequalities are called solution set.

Hence linear inequalities can be solved by graphical method in following steps.

Step 1 : Write given inequality in equation form it will represent a line.

Step 2 : By putting $x = 0$ and $y = 0$ in the equation of straight line ; we get, meeting point at y and x axis respectively

Step 3 : Join both the obtained points which represents a straight line.

Step 4 : Take a point (may be origin) and put its co-ordinates in inequations. If these coordinates satisfy the inequality, then shade that area form line to point. This shaded part is the required solution of inequality. If origin does not satisfy the inequality then shaded portion will be in opposite side from line and this area will be the required solutions of inequality.

Solution of any inequality can be understood by the following examples.

Example 8. Solve the folowing inequalities graphically.

(i) $x \le 2$ (ii) $2x - y \ge 1$ (iii) $|y - x| \le 3$

Solution : **(i)** Replace inequality $x \le 2$ into equation we get $x = 2$. It is clear that this line is parallel to y-axis and will pass through point (2, 0) of x-axis. Following graph is obtained accordingly.

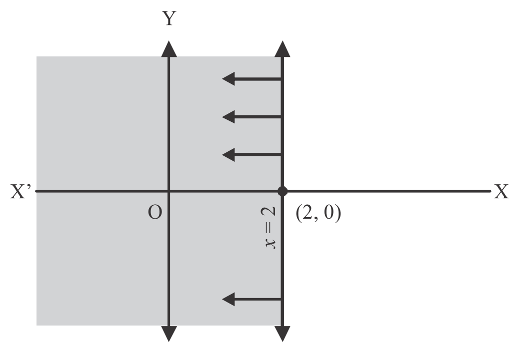


Fig. 4.07

Now inequality $x \le 2$, sastifies by origin (0,0). So, shaded portion from line $x = 2$ to indefinite extent will be the required solution set.

(ii) Replace inequality $2x - y \ge 1$ into equation from we get $2x - y = 1$

By putting $x = 0$, we get $y = -1$. So point (0, –1) cuts x-axis cuts y-axis similarly by puting y=0, $x = \frac{1}{2}$ so point $\left(\frac{1}{2}, 0\right)$ cuts x-axis we get the graph fig 4.08.

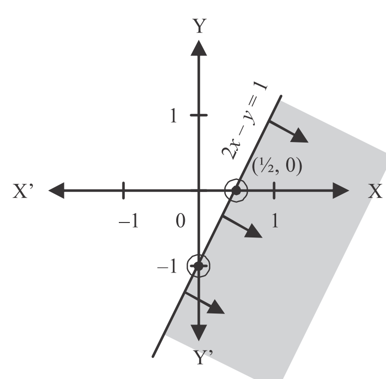


Fig. 4.08

Now inequality $2x - y \ge 1$ does not satisfy with origin (0, 0) *i.e.,* $2 \times 0 - 0 \ge 1$ is not true.

So, shaded portion opposite to\ origin to line $2x - y = 1$ will be the required solution set.

(iii) Here given inequality is $|y - x| \le 3$. It can be written as $-3 \le y - x \le 3$

Again it can be written as following two in equalities

$$-3 \le y - x ; \quad y - x \le 3$$

and $$x - y - 3 \le 0 \qquad \text{. . . (i)}$$

and $$x - y + 3 \geq 0 \qquad \text{. . . (ii)}$$

By planning inequality (1) in equation form $x - y - 3 = 0$ is obtained, we get points (3, 0) and *Y*-axis respectively. Simmilarly by replacing inequality (2) in equation form we get $x - y + 3 = 0$, we get points (0, –3) and (0, 3) at *x* and *y* axis respectively. Now graph of these two lines are obtained in fig 4.09 :

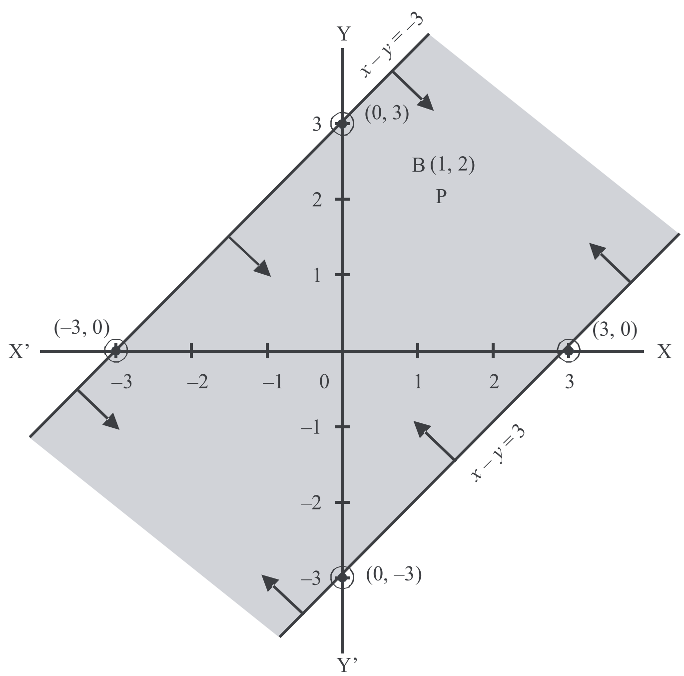


Fig. 4.09

Now inequality $x - y - 3 \leq 0$ satisfies by origin (0, 0) *i.e.*, $0 - 0 - 3 \leq 0$ is true. So its shaded part will be the side of origin from line. Second inequality $x - y + 3 \geq 0$ also satisfy by origin (0, 0) *i.e.*, $0 - 0 + 3 \geq 0$ is ture so its shaded area will be the side of origin form line. Thus shaded area between two lines will be required solution set.

Exercise 4.2

1. Show the solution set of the folowing inequalities, graphically.

   (i) $x \geq 2$ (ii) $y \leq -3$ (iii) $x - 2y < 0$ (iv) $2x + 3y \leq 6$

2. Solve the following inequalities by graphical method :

   (i) $|x| \leq 3$ (ii) $3x - 2y \leq x + y - 8$ (iii) $|x - y| \geq 1$

Miscellaneous Exercise 4

1. For which value of *k*, linear pair $x + y - 4 = 0$; $2x + ky - 3 = 0$ have no solution :

   (a) 0 (b) 2 (c) 6 (d) 8

2. For which value of *k*, linear pair $3x - 2y = 0$ and $kx + 5y = 0$ have infinite solutions :

   (a) $\frac{1}{2}$ (b) 3 (c) $\frac{-5}{3}$ (d) $\frac{-15}{2}$

3. Linear pair $kx - y = 2;\ 6x - 2y = 3$ have unique solution. If

(a) $k = 2$ (b) $k = 3$ (c) $k \neq 3$ (d) $k \neq 0$

4. Inequalities $x \geq 0, y \geq 0$ express corresponding equation to :

(a) x-axis (b) y-axis (c) x and y-axis (d) line

5. For line corresponding to inequality $y - 3 \leq 0$, following statement is true :

(a) parallel to x-axis (b) parallel to y-axis

(c) divides x-axis (d) passes through origin

6. Write the number of solution of following linear pairs

$$x + 2y - 8 = 0; 2x + 4y = 16$$

7. If pair of equations $2x + 3y = 7; (a+b)x + (2a - b)y = 21$ have infinite many solutions then find a and b.

8. Shade the solution set of inequality $|x| \leq 3$

9. Shade the solution set of inequality $2x + 3y \geq 3$

10. Solve the linear pair of equations graphically and with the help of this find value of 'a' where as $4x + 3y = a$, $x + 3y = 6; 2x - 3y = 12$.

11. Solve the linear pair of equations graphically and find the co-ordinates of that points where lines represent them, cut at y–axis $3x + 2y = 12;\ 5x - 2y = 4$.

Important Points

1. If a, b, c are real numbers then linear equation with two variables x and y can be expressed in general form as $ax + by + c = 0$, where $a, b \neq 0$.

2. Pair of linear equation with two variables, in general form can be expresses as $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$

   Values of x, y which satisfy the two equations is solution of simultaneous equation.

3. Pair of linear equation of two variable are called consistent if this pair has at least one solution. If any pair has no solution then it is called inconsistent pair.

4. Consider relation between the coefficient of linear pair

$$a_1x + b_1y + c = 0 \text{ and } a_2x + b_2y + c = 0$$

We can check the nature and existence of solution :

(i) Intersecting when, $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ (Unique, consistent)

(ii) Parallel when, $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ (No solution, inconsistent)

(iii) Coincident when, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ (Infinite many solution consistent)

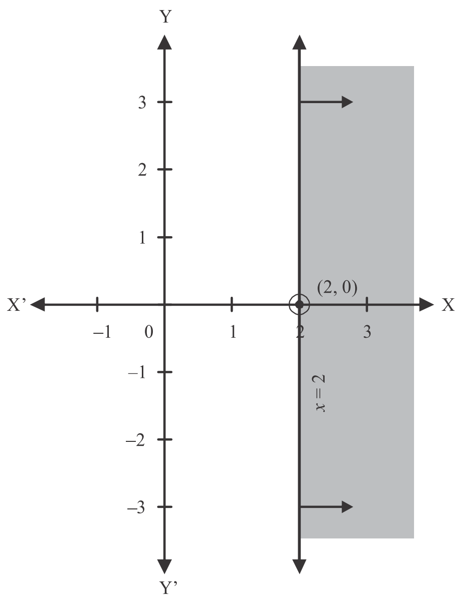
5. A linear pair with two variables can be solved graphically in following steps :
   (i) From the equations of lines we obtain table of corresponding points and then show graphically.
   (ii) If two lines intersect at point (α, β) then $x = \alpha, y = \beta$ will be required solution of given pair.
   (iii) If lines are consistent then they have infininte many solution and two lines can be represent as same line so each point (α, β) will be obtained in the form of solution $x = \alpha, y = \beta$.
   (iv) If lines are parallel then there will be no solution.
6. If a, b are two non-zero real numbers, then for variables x and y, inequalities $ax + by < c$, $ax + by \le c$, $ax + by > c$ or $ax + by \ge c$ are called linear inequalities of two variables.
7. Linear inequalities of two variable can be solved graphically in following steps :
   (i) Write given inequalities in the equation form.
   (ii) Putting $x = 0$ and $y = 0$ in above equations, obtained meeting points at x-axis and y-axis join these points.
   (iii) Now check whether the inequality satisfy by origin (0, 0). If satisfy, then solution set will be shaded part of corresponding side to origin. If origin not satisfies the inequalitiy then solution set will be shaded part of opposite side of origin form line.
   (iv) So common shaded area of all linear inequalities will be required solution of system.
8. Required solution will be common region which will satisfy all the inequalities. This solution set may be null set, bounded or unbounded area.

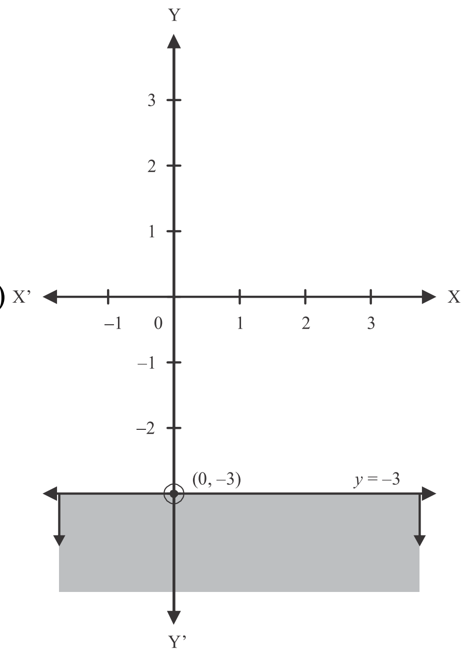
Answer Sheet

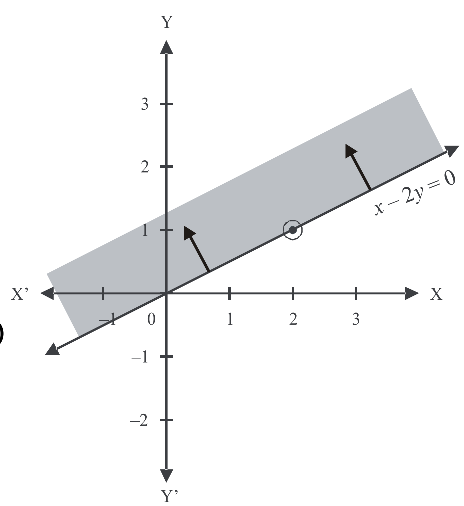
Exercise 4.1

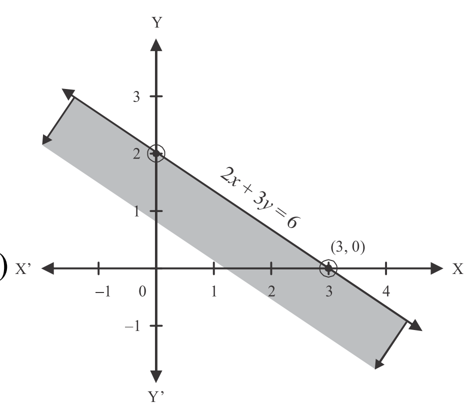
1. (i) Inconsistent, (ii) Consistent, (iii) Inconsistent, (iv) Consistent
2. (i) Unique solution $x = 2, y = 1$ (ii) Unique solution $x = 1, y = -2$
   (iii) Infinite many solutions (iv) Unique solution $x = 1, y = -1$
3. (i) $x = 3, y = 2$; (0, 4/5), (0, 8) (ii) $x = 2, y = 3$; (0, 6), (0, –2)
4. $x = 5$, $y = 0$; (5, 0), (0, 3), (0, –4)

Exercise 4.2

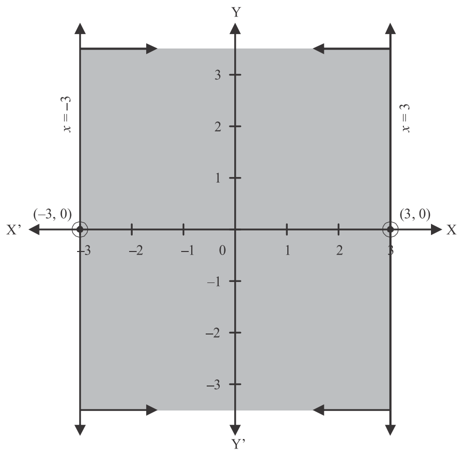
1. (i) 

(ii) 

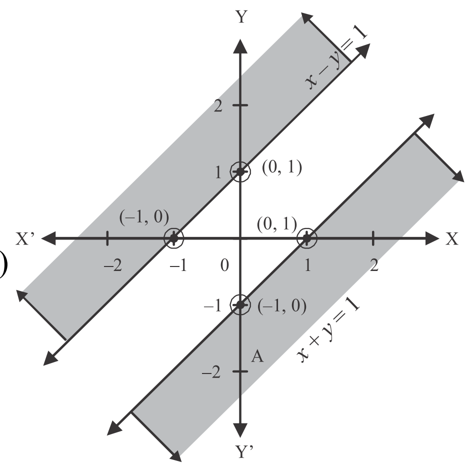
(iii) 

(iv) 

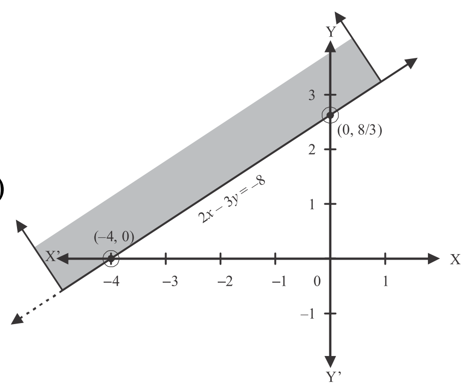
2. (i)



(ii)



(iii)



Miscellaneous Exercise 4

1. (b) 2. (d) 3. (c) 4. (c) 5. (a)

6. Infinite many solutions 7. $a = 5, b = 1$ 10. $x = 6, y = 0$, so $a = 24$

11. $x = 2, y = 3$; (0, 6) and (0 –2)

05

Arithmetic Progression

5.01 Introduction

We must have observed that in nature, many things follows a certain pattern, such as the holes of a beehive represents a definite pattern of such type.

A pipe rod is fixed at definte gap of a steel ladder used in shop or home. In mathematical language we can say that patterns increase or decrease in a fixed number and there is same relation in this first, second or third sequence. This definite series of numbers is called sequence.

For Example : Consider the sequence of following numbers :

(i) $2, 4, 6, 8, 10, ...$

(ii) $8, 5, 2, -1, -4, ...$

(iii) $3^0, 3^1, 3^2, 3^3, 3^4 ...$

In sequence (i) each term is 2 less than the term succeeding it.

In sequence (ii) each term is 3 less than the term preceeding it.

Similarly in sequence (iii) each term is in increasing power of 3.

From above examples, it is clear that all the patterns follows a definite pattern. In this chapter we will discuss such type of pattern in which succeding terms are obtained by adding a fixed number to the preceding terms. Here we will study the general term (n^{th} term) of this pattern and methods to find sum of consecutive terms.

5.02. Arithmetic Progression

Consider the following sequence of numbers:

(i) $1, 4, 7, 10, 13, ...$

(ii) $100, 70, 40, 10, ...$

(iii) $-5, -3, -1, 1, ...$

In the above sequences, each term is obtained by adding a definite number (positive or negative) except the first term. Such sequence of numbers is said to form an arithmetic progression.

Difference between each number with its preceeding number is same. This fixed number is called the common difference of the A.P. Let $a_1, a_2, ..., a_n$ be terms of any sequence if they are in A.P. then each term is obtained by adding a fixed number to the preceding term except the first term.

Let common difference be d, then

$$\begin{aligned} a_2 &= a_1 + d \\ a_3 &= a_2 + d \\ \vdots \quad & \quad \vdots \\ a_n &= a_{n-1} + d \end{aligned}$$

So, $a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1} = d$

In general $a_n - a_{n-1} = d$, where $n = 1, 2, 3, \ldots$

We can say that if first term of sequence is a and common difference is d then general form of A.P. can be written as

$$a, a+d, a+2d, a+3d, \ldots, a+(n-1)d, \ldots$$

Above concepts can be understand by the followign examples

Example 1. Write the first term and common difference of the following A.P.

$$-5, -1, 3, 7, \ldots,$$

Solution : On comparing given A.P. by its general form, we get first term $a = -5$

Common difference d = difference between two consecutive terms

i.e., $-5-(-1)=4$, $3-(-1)=4$

Example 2. Check, which of the following sequence are in the form an A.P.,?

(i) $4, 10, 16, 22, \ldots$

(ii) $-2, 2, -2, 2, -2, \ldots$

Solution : (i) Let us find the common difference to check an A.P. of first sequence i.e., $4, 10, 16, 22, \ldots$

$$a_2 - a_1 = 10 - 4 = 6$$

$$a_3 - a_2 = 16 - 10 = 6$$

$$a_4 - a_3 = 22 - 16 = 6$$

Same difference is obtained each time so this sequence is in on A.P. with common difference = 6

(ii) Let us find out the common difference to check an A.P. of second sequence i.e., $-2, 2, -2, 2, -2, \ldots$

$$a_2 - a_1 = 2 - (-2) = 4$$

$$a_3 - a_2 = -2 - (2) = -4$$

$$a_4 - a_3 = 2 - (-2) = 4$$

Same difference is not obtained each time, so given sequence does not form an A.P.

Example 3. Find the common difference of the followig A.P. and write next four terms.

(i) $0, -3, -6, -9, \ldots$ (ii) $-1, \frac{-5}{6}, \frac{-2}{3}, \ldots$

Solution : (i) Let A.P. is $a_1, a_2, a_3, \ldots$ So, here

$$a_2 - a_1 = -3 - 0 = -3$$

$$a_3 - a_2 = -6 - (-3) = -3$$

$$a_4 - a_3 = -9 - (-6) = -3$$

It is clear that, the difference between two consecutive terms is equal i.e. –3. So, common difference $d = -3$ and next four terms will be as follows:

$$a_5 = a_4 + d = -9 + (-3) = -12$$

$$a_6 = a_5 + d = -12 + (-3) = -15$$

$$a_7 = a_6 + d = -15 + (-3) = -18$$

$$a_8 = a_7 + d = -18 + (-3) = -21$$

(ii) Let A.P. is expressed as $a_1, a_2, a_3, ...$ then

$$a_2 - a_1 = \frac{-5}{6} - (-1) = \frac{1}{6}$$

$$a_3 - a_2 = \frac{-2}{3} - \frac{(-5)}{6} = \frac{-4+5}{6} = \frac{1}{6}$$

It is clear that the difference of two consecutive terms '$\frac{1}{6}$' is same so common difference $d = \frac{1}{6}$

So next four terms will be as follows:

$$a_4 = a_3 + d = \frac{-2}{3} + \frac{1}{6} = \frac{-4+1}{6} = \frac{-3}{6} = \frac{-1}{2}$$

$$a_5 = a_4 + d = \frac{-1}{2} + \frac{1}{6} = \frac{-3+1}{6} = \frac{-2}{6} = \frac{-1}{3}$$

$$a_6 = a_5 + d = \frac{-1}{3} + \frac{1}{6} = \frac{-2+1}{6} = \frac{-1}{6}$$

$$a_7 = a_6 + d = \frac{-1}{6} + \frac{1}{6} = 0$$

Exercise 5.1

1. Find the first term *a* and common difference *d* for the following A.P.

(i) $6, 9, 12, 15, ...$ (ii) $-7, -9, -11, -13, ...$ (iii) $\frac{3}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-3}{2}, ...$

(iv) $1, -2, -5, -8, ...$ (v) $-1, \frac{1}{4}, \frac{2}{3}, ...$ (vi) $3, 1, -1, -3, ...$

(vii) $3, -2, -7, -12, ...$

2. If first term *a* and common difference *d* of A.P. is given then find the first four terms of that progression.

(i) $a = -1,\ d = \frac{1}{2}$ (ii) $a = \frac{1}{3},\ d = \frac{4}{3}$ (iii) $a = 0.6,\ d = 1.1$

(iv) $a = 4,\ d = -3$ (v) $a = 11,\ d = -4$ (vi) $a = -1.25,\ d = -0.25$

(vii) $a = 20,\ d = \frac{-3}{4}$

3. Check the following list of numbers for an A.P. If any of then form an A.P., then find its common difference and write the next four terms:

(i) $2, \frac{5}{2}, 3, \frac{7}{2}, \ldots$

(ii) $\frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \ldots$

(iii) $a, a^2, a^3, a^4, \ldots$

(iv) $\sqrt{3}, \sqrt{6}, \sqrt{9}, \sqrt{12}, \ldots$

(v) $\sqrt{2}, \sqrt{8}, \sqrt{18}, \sqrt{32}, \ldots$

(vi) $a, 2a, 3a, 4a, \ldots$

(vii) 0.2, 0.22, 0.222, ...

(viii) $3, 3+\sqrt{2}, 3+2\sqrt{2}, 3+3\sqrt{2}, \ldots$

5.03 n^{th} term 'general term' of an Arithmetic Progression

In previous section we have studied about finding of consecutive terms of A.P. by first term a and common difference d. Here we consider the following examples.

Let basic salary of an employee is ₹ 1000 and he gets ₹ 300 as annual increment then after 20 yrs what will be his sarly. To find this we calculate salary of first five years.

	Monthly salary in I year	= ₹ 10,000
	Monthly salary in II year	= ₹ (10000 + 300)
i.e.	Salary in II year	= ₹ 10,300
∴	Monthly salary in III year	= ₹ (10300 + 300)
		= ₹ (10,000 + 300 + 300)
		= ₹ [10,000 + 2 × 300]
i.e.	Salary in III year	= ₹ (10,000 + (3–1) × 300)
		= ₹ 10600
∴	Monthly salary in IV year	= ₹ (10600 ? 300)
		= ₹ (10,000 + 300 + 300 + 300)
		= ₹ [10,000 + 3 × 300]
i.e.	Salary in IV year	= ₹ (10,000 + (4 – 1) × 300)
		= ₹ 10,900
	Similarly, salary in V year	= ₹ (10,900 + 300)
		= ₹ (10,000 + 300 + 300 + 300 + 300)
		= ₹ (10,000 + 4 × 300)
		= ₹ (10,000 + (5–1) × 300)
		= ₹ 11,200

Here we write data of 5 years of annual salary in the following order.

10000, 10300, 10600, 10900, 11200 ...

This sequence is an A.P. because common difference between two consecutive terms is 300. From above it is clear that by adding ₹ 300 in previous year salary, we can find salary of required year. From above it is clear that in 20^{th} year salary of employee will be

Salary of 19^{th} year + ₹ 300

= ₹ [10,000 + (300 + 300 + . . . + 300) + 300]

= ₹ [10,000 + (20 – 1) × 300]

i.e. Salary in 20^{th} year = ₹ 15,700

So it is clear that as we have find salary in II, III, IV, V and at last in 20th year so in general this relation can be written in the following form.

Salary for 20th year = I (Basic) salary + (20 –1) annual increment

We can generalize this example as

If first term is a, common difference is d, n^{th} term (general term) is a_n then it can be written as $a_n = a + (n-1)d$

Let A.P. is in the form $a_1, a_2, a_3, ..., a_n, ...$ and $a_1 = a$ = first term and common difference is d then

Second term $\quad a_2 = a + d = a + (2-1)d$

and $\quad a_3 = a_2 + d = (a+d) + d = a + 2d$

or $\quad a_3 = a + (3-1)d$

Similary n^{th} term $\quad a_n = a_{n-1} + d = a + (n-1)d$

So, **General term = first term + (no. of terms –1) × common difference**

Here it is necessary to mention, if there are m terms in A.P., i.e., last term is a_m (or ℓ) then n^{th} term from last will as follows:

n^{th} term from last $\quad = a_{m?n?1}$

$$= a + (m - n + 1 - 1)d$$

$$= a + (m - n)d$$

If we take last term 'ℓ' as first term and reducing common difference as $-d$ then n^{th} term from last can be written as

n^{th} term from last = last term $+(n-1)(-d)$

$$= \ell - (n-1)d$$

By the following examples we can easily understand about the general term of Arithemetic Progression.

Example 4. Find the 30th and n^{th} (general terms) of an A.P. 10, 7, 4, ...

Solution : Given A.P. is

$$10, 7, 4, \ldots$$

Its first term $\quad a = 10$

Common difference $\quad d = 7 - 10 = -3$

So, n^{th} term of given A.P. is a_n

i.e. $\quad a_n = a + (n-1)d$

Similarly, $\quad$ 30th term $\quad a_{30} = 10 + (30-1) \times (-3)$

$$= 10 - 29 \times 3 = -77$$

and general term (n^{th} term) $\quad a_n = 10 + (n-1) \times (-3)$

$$= 10 - 3(n-1) = 13 - 3n$$

Thus, required 30th term $= -77$ and n^{th} term $= 13 - 3n$.

Example 5. Which term of an A.P. 3, 15, 27, 39, ... is 639?

Solution : Give A.P. is 3, 15, 27, 39, ...

$\therefore$ First term $a = 3$ and common difference $d = 12$. Let n^{th} term $= 639$, then general term

$$a_n = a + (n-1)d$$

Here, $639 = 3 + (n-1) \times 12$

or $639 = 3 + 12n - 12$

or $648 = 12n$

or $n = \dfrac{648}{12} = 54$

Hence, 54^{th} term of A.P. is 639.

Example 6. Find the number of terms in A.P. 7, 13, 19, ..., 205.

Solution : Given A.P. is 7, 13, 19, ..., 205. Here first term $a = 7$, and common difference $d = 6$. Let n^{th} term is last term, then

$$a_n = 205.$$

$\therefore$ n^{th} term $a_n = a + (n-1) \times d$

$205 = 7 + (n-1) \times 6$

$205 = 7 + 6n - 6$

or $204 = 6n$

or $n = \dfrac{204}{6} = 34$

Hence there are 34 terms in given A.P.

Example 7. If third term of an A.P. is 12 and 50^{th} term in 106. Find its 29^{th} term.

Solution : General term of A.P. $= n^{th}$ term

$\therefore$ $a_n = a + (n-1)d$

Where a is first term of A.P. and d is common difference

Here $a_3 = 12$ and $a_{50} = 106$

So, $a_3 = a + (3-1)d$

or $12 = a + 2d$ . . . (i)

and $a_{50} = a + (50-1)d$

or $106 = a + 49d$ . . . (ii)

By subtracting equation (i) from (ii), we get

$$106 - 12 = 49d - 2d$$

or $94 = 47d$

or $d = \dfrac{94}{47} = 2$

Putting the value of d in equation (i)

$$12 = a + 2 \times 2$$

or $$a = 8$$

$\therefore$ 29^{th} term $$a_{29} = a + (29-1)\text{d}$$

$$= 8 + 28 \times 2 = 64$$

Thus, 29^{th} term of A.P. will be 64.

Example 8. Is 184 any term of A.P. 3, 7, 11, ... ?

Solution : Given A.P. is 3, 7, 11, ... Here first term $a = 3,$ and common differece $d = 4$

Let n^{th} term of A.P is 184

So, $$a_n = a + (\text{n}-1)\text{d}$$

$\therefore$ $$184 = 3 + (n-1) \times 4$$

or $$184 = 3 + 4n - 4$$

or $$185 = 4n$$

or $$n = \frac{185}{4} = 46\frac{1}{4}$$

Since value of n is not a natural number. So, 184 is not any term of given A.P.

Example 9. How many two digit numbers are divisible by 7?

Solution : We know that smallest number of two digit (positive) which is divisible by 7 is 14. So following will be the sequence of two digit number divisible by 7.

$$14, 21, 28, ..., 98$$

This is an A.P. whose first term is $a = 14$ and common difference $d = 7$.

Let A.P. has n terms then n^{th} term $a_n = 98$ can be expressed as:

$$a_n = a + (\text{n}-1)\text{d}$$

i.e. $$98 = 14 + (\text{n}-1) \times 7$$

or $$98 = 14 + 7n - 7$$

or $$91 = 7n$$

or $$n = \frac{91}{7} = 13$$

Thus, there are 13 two digit number which are divisible by 7.

Example 10. Find the 20^{th} term of A.P. 3, 8, 13, ..., 253 from the last.

Solution : Here last term of A.P. is $\ell = 253$. First term $a = 3$ and common difference $d = 5$. So 20^{th} term from last term

$$= \ell - (20-1)d$$

$$= 253 - 19 \times 5 = 253 - 95 = 158$$

Thus 20^{th} term from the last is 158.

Example 11. How many multiples of 4 between 10 and 250?

Solution : Clearly, first number divisible by 4, between 10 and 250 is 12. When we divide 250 by 4 then remainder 2 obtained. So last term which is divisible by 4 is 250 ? 2 ? 248 i.e., numbers which are divisible by 4 between 10 and 250 makes following A.P.

$$12, 16, \ldots, 248$$

Now we have to find number of multiples of 4. Let this is n, then a_n ? 248 i.e.,

$$a_n = a + (n-1)d$$

or $\quad 248 = 12 + (n-1) \times 4$

or $\quad 248 = 12 + 4n - 4$

or $\quad 240 = 4n$

or $\quad n = 60$

Thus, there will be 60 multiples of 4 between 10 and 250.

5.04 Selection of terms of Arithmetic Progression

To find numbers (odd or even term) in A.P. terms of progression can be conveniently find by the following way

Numbers	**Terms**
3	$a-d, a, a+d$
4	$a-3d, a-d, a+d, a+3d$
5	$a-2d, a-d, a, a+d, a+2d$
6	$a-5d, a-3d, a-d, a+d, a+3d, a+5d$

Here, it is clear that if number of terms is odd then mid term is a and common difference is d and if number of terms is even then $a - d$ and $a + d$ are two mid terms and common difference is $2d$. Problems related to numbers can be clear by following examples.

Examples 12. Three numbers are in A.P. If their sum is –3 and product is 8 then find the numbers.

Solution : Let three numbers in A.P. are

$$a-d, a, a+d$$

Given that sum of numbers is – 3

i.e. $\quad (a-d) + a + (a+d) = -3$

or $\quad 3a = -3$

or $\quad a = -1$

It is also given that product of numbers is 8

$\therefore \quad (a-d) \times a \times (a+d) = 8$

or $\quad (a^2 - d^2) \times a = 8$

Putting $\quad a = -1$

$$[(-1)^2 - \mathrm{d}^2] \times (-1) = 8$$

or $\quad d^2 - 1 = 8$

or $\qquad d^2 = 9$

or $\qquad d = \pm 3$

By putting vlaue of a and d, required numbers are obtained. If d ? 3 then $-1-3, -1, -1+3$ i.e., $-4, -1, +2$ and If $d = -3$ then $-1+3, -1, -1-3$ i.e., $2, -1, -4$ are required numbers

Exercise 5.2

1. Find:
   (i) 10^{th} term of A.P., 2, 7, 12, ...
   (ii) 18^{th} term of A.P. $\sqrt{2}, 3\sqrt{2}, 5\sqrt{2}, ...$
   (iii) 24^{th} term of A.P. 9, 13, 17, 21 ...
2. Solve :
   (i) Which term of A.P., 21, 18, 15, ... is –81?
   (ii) Which term of A.P., 84, 80, 76, ... is zero?
   (iii) Is 301 any term of sequence 5, 11, 17, 23, ...?
   (iv) Is –150 any term of A.P. 11, 8, 5, 2, ...?
3. If 6^{th} and 17^{th} term of an A.P. are 19 and 41 respectively then find 40^{th} term.
4. If 3^{rd} and 9^{th} term of an A.P. are 4 and –8 respectively then which term of its will be zero?
5. If third term of an A.P. is 16 and 7^{th} term is 12 more than 5^{th} term, then find A.P.
6. How many three digits numbers are divisible by 7?
7. Find the 11^{th} term from the last of A.P. 10, 7, 4, ... –62.
8. Find the 12^{th} term from the last of an A.P. 1, 4, 7, 10, ... 88
9. There are 60 terms in an A.P. If its first and last terms are 7 and 125 respectively then find its 32^{th} term.
10. Four numbers are in A.P. If sum of numbers is 50 and largest number is four times the smaller one,then find the numbers.

5.05 Sum of First *n* terms of Arithmetic Progression

In this section, we will obtain the formula of sum of arithemtic progression. To understand this consider an example. Lata's mother gives Rs. 500 on her birthday, Rs 600 on IInd birthday, Rs700 onIIIrd, Rs. 800 on IVth birthday and will continue in the same way upto 18 years of Lata's age. How much money will be collected at the age of 18 years?

Here we see that 500, 600, 700, 800, ... these numbers are in arithemetic progression so for adding these 18 terms i.e., for adding the terms of A.P. we get formula by following method. In this formula, by substituting values of variables we will find sum of A.P. easily.

Let a be first term of A.P. and d is common difference and S_n is sum of its n terms. So first n terms of A.P. are written as

$$a, a+d, a+2d, ..., a+(n-1)d$$

then
$$S_n = a + (a+d) + (a+2d) + ... + \left[a+(n-2)d\right] + \left[a+(n-1)d\right] \qquad ...(i)$$

By reversing the order of terms, sum does not effected so we can write as

$$S_n = [a+(n-1)d]+[a+(n-2)d]+...+(a+2d)+(a+d)+a \quad ...\text{(ii)}$$

On adding corresponding terms of (i) and (ii)

$$2S_n = [2a+(n-1)d]+[2a+(n-1)d]+...+[2a+(n-1)d]+[2a+(n-1)d]$$

($\because$ It contains n terms)

$\therefore$ $$2S_n = n[2a+(n-1)d]$$

or $$S_n = \frac{n}{2}[2a+(n-1)d]$$

This formula shows the sum of n terms,when first term and common difference of an A.P. is given.

If last term of A.P. is ℓ then formula $S_n = \frac{n}{2}[2a+(n-1)d]$ can be written in the following form and sum can be obtained.

i.e. $$S_n = \frac{n}{2}[a+a+(n-1)d]$$

or $$S_n = \frac{n}{2}[a+\ell]$$ [$\because$ $\ell =$ last term = n^{th} term $= a+(n-1)d$]

Thus,there are n terms in A.P. then $a_n = \ell$ will be last term, so

$$\text{Sum of } n \text{ terms} = S_n = \frac{n}{2}\,[\text{First term + last term}]$$

Here, it is necessary to understand that n^{th} term ofA.P. is equal to the difference of sum of first n terms and first $(n-1)$ terms.

i.e., $$a_n = S_n - S_{n-1}$$

By above formula, problems based on sum of terms of A.P., can be easily understand by following examples.

Example 13. Find the sum of

(i) A.P. 1, 4, 7, 10, ... upto 20 terms

(ii) A.P. 2, 7, 12, ... upto 10 terms

Solution : (i) Given A.P. is 1, 4, 7, 10, ...

Here first terms $a = 1$ and common difference $d = 3$

$\therefore$ Sum of n terms $S_n = \frac{n}{2}[2a+(n-1)d]$

Sum of 20 terms $S_{20} = \frac{20}{2}[2\times1+(20-1)\times3]$

$$=10[2+57]=590$$

Thus, required sum = 590.

(ii) Given A.P. is 2, 7, 12, ...

Here first term a = 2 common difference d = 5. Since sum of n terms $S_n = \frac{n}{2}[2a+(n-1)d]$

So, Sum of 10 terms $S_{10} = \frac{10}{2}[2\times 2+(10-1)\times 5]$

or $S_{10} = 5[4+45] = 5\times 49 = 245$

Thus, required sum $= 245$

Example 14. Find the sum of the following :

(i) $34+32+30+....+10$

(ii) $(-5)+(-8)+(-11)+....+(-230)$

Solution : (i) Given series $34+32+30+....+10$ is an A.P. whose first term a = 34 last term $\ell = a_n = 10$ and common difference d = -2

Thus, $a_n = a+(\text{n}-1)\,\text{d}$

or $10 = 34+(n-1)(-2)$

or $10 = 34-2n+2$

or $2n = 26$

or $n = 13$

Sum of series $S_n = \frac{n}{2}[a+\ell]$

Thus, $S_{13} = \frac{13}{2}[34+10] = 13\times 22 = 286$

(ii) Given series $(-5)+(-8)+(-11)+...+(-230)$ is an A.P. whose first term a = -5 and common difference d = -3 last term $a_n = \ell = -230$

$\therefore$ $a_n = a+(n-1)d$

Here $-230 = -5+(n-1)\times(-3)$

or $-230 = -5-3n+3$

or $3n = 228$

or $n = \frac{228}{3} = 76$

$\because$ $S_n = \frac{n}{2}[a+\ell]$

$\therefore$ $S_{76} = \frac{76}{2}[-5+(-230)] = 38\times(-235) = -8930$

Example 15. Sum of how many terms of an A.P. 54, 51, 48, ... will be 513 ?

Solution : First term of an A.P. 54, 51, 48, ... is $a = 54$ and common difference is $d = -3$

Let sum of n terms is $S_n = 513$ then sum of n terms of A.P

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Here $$513 = \frac{n}{2}[2 \times 54 + (n-1) \times (-3)]$$

or $$513 = \frac{n}{2}[108 - 3n + 3]$$

or $$513 \times 2 = n(111 - 3n)$$

or $$3n^2 - 111n + 1026 = 0$$

or $$n^2 - 37n + 342 = 0$$

On factorize, we get

or $$n^2 - 18n - 19n + 342 = 0$$

or $$n(n-18) - 19(n-18) = 0$$

or $$(n-19)(n-18) = 0$$

or $$n = 19 \quad \text{and} \quad n = 18$$

Here common difference $d = -3$ (is negative)

and 19^{th} term $= a_{19} = a + (n-1)d = 54 + (19-1)(-3) = 0$

Here 19^{th} term is zero. So, sum of 18 terms and sum of 19 terms will be same 513.

Example 16. Find the sum of first 15 terms of A.P. whose n^{th} term is $a_n = 9 - 5n$.

Solution : $\because$ n^{th} term of series $a_n = 9 - 5n$

$\therefore$ $$a_1 = 9 - 5 \times 1 = 4$$

$$a_2 = 9 - 5 \times 2 = -1$$

$$a_3 = 9 - 5 \times 3 = -6$$

So sequence of number obtained are $4, -1, -6, ...$

which is an A.P. whose first term $a = 4$ and common difference is $d = -5$.

Thus, sum of n terms of this series

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Here $$S_{15} = \frac{15}{2}[2 \times 4 + (15-1) \times (-5)] = \frac{15}{2}[8 - 70]$$

$$= -(15 \times 31) = -465$$

Thus, sum of first 15 terms of A.P. will be –465.

Example 17. If sum of first 7 terms of A.P. is 49 and sum of first 17 terms is 289, then find the sum of first n terms of A.P.

Solution : Given that $S_7 = 49$ and $S_{17} = 289$

Sum of n terms of series

$\because$ $$S_n = \frac{n}{2}[2a + (n-1)d]$$

Here, $$S_7 = \frac{7}{2}[2a + (7-1)d] = 49$$

and $$S_{17} = \frac{17}{2}[2a + (17-1)d] = 289$$

Thus, on writing above two equations in simplest form, we get first equation

$$2a + 6d = \frac{49 \times 2}{7}$$

or $$a + 3d = 7 \quad \text{. . . (i)}$$

and second equation $$2a + 16d = \frac{289 \times 2}{17}$$

or $$a + 8d = 17 \quad \text{. . . (ii)}$$

On subtracting (ii) from (i), we get

$$5d = 10$$

or $$d = 2$$

Putting value of d in equation (i), we get

$$a + 3 \times 2 = 7$$

or $$a = 7 - 6$$

or $$a = 1$$

Thus, putting value of a and d in the formula of A.P sum of n terms

$$S_n = \frac{n}{2}[2 \times 1 + (n-1) \times 2] = \frac{n}{2}[2 + 2n - 2] = n^2$$

Thus, sum of n terms of A.P is n^2.

Example 18. If sum of n terms of A.P is $4n - n^2$ then what is the first term ? What is the sum of first two terms? What is second term? Similarly find 3rd, 10th and nth term.

Solution : Given that sum of n terms of A.P., $S_n = 4n - n^2$

Putting $n = 1$

$$S_1 = 4 \times 1 - (1)^2 = 4 - 1 = 3$$

So, first term is 3

For sum of two terms

$$S_2 = 4 \times 2 - (2)^2 = 8 - 4 = 4$$

So, sum of first two terms is 4

Second term $a_2 = S_2 - S_1 = 4 - 3 = 1$

i.e., Second term of A.P. is 1.

Here, sum of first three terms $S_3 = 4 \times 3 - (3)^2 = 12 - 9 = 3$

$\therefore$ Third term of A.P. $a_3 = S_3 - S_2 = 3 - 4 = -1$

Thus obtained A.P. is $3, 1, -1, ...$ whose common difference $d = a_3 - a_2 = -1 - 1 = -2$

$$n^{\text{th}} \text{ term } = a_n = a + (n-1)d$$

Here, first term $a = 3$, common difference $d = -2$, So

$$a_n = 3 + (n-1) \times (-2) = 3 - 2n + 2 = 5 - 2n$$

i.e., n^{th} term $a_n = 5 - 2n$ So for 10^{th} term, putting $n = 10$

$$a_{10} = 5 - 2 \times 10 = 5 - 20 = -15$$

Thus, 10th term will be -15.

Example 19. Find the sum of natural number divisible by 3 between 250 and 1000.

Solution : It it clear that between 250 and 1000, numbers divisible by 3 are 252, 255, 258, ... 999 which is in an A.P. Its first term a = 252,last term $a_n = l$ = 999 and common difference d = 3

Here $a_n = a + (n-1)d$

$\therefore$ $999 = 252 + (n-1) \times 3$

or $999 = 252 + 3n - 3$

or $999 = 249 + 3n$

or $3n = 750$

or $n = 250$

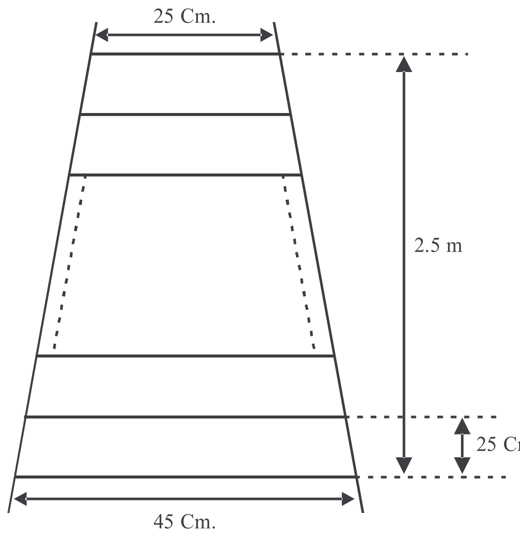
$\therefore$ Required sum $S_n = \dfrac{n}{2}(a + \ell)$

$$\therefore \quad S_{250} = \frac{250}{2}(252 + 999)$$

$$= 125 \times 1251 = 156375$$

Thus required sum will be 156375.

Example 20. A ladder has rungs 25 cm apart (See fig.). Length of lowest rung is 45 cm and it goes to decrease as goes to high and the rungs 25 cm at the top. If the top and the bottom rungs are 2.5m apart, what is the length of the wood required for the rungs ?



Solution : Given that distance between two consecutive rungs is 25 cm and distance between first and last rungs is 2.5 m *i.e.,* 250 cm.

So number of rungs in ladder

$$= \frac{250}{25} + 1 = 10 + 1 = 11$$

It is given that rungs decrease uniformly from bottom to top and length of rungs at bottom is 45 cm and at top 25 cm. So it is clear that length of rungs are in an A.P. whose first term a = 45 cm and 11^{th} term (last term) l = 25 cm.

So, total length of wood required = Sum of 11 terms of A.P.

$$\because \quad S_n = \frac{n}{2}(a + \ell)$$

$$\therefore \quad S_n = \frac{11}{2}(45 + 25)\text{cm} = 11 \times 35 = 385\,\text{cm}$$

Thus, total length of wood will be 3.85 m.

Exercise 5.3

1. Find the sum of the follwing arithmetic progression :

   (i) 1, 3,5, 7, ... upto 12 terms   (ii) 8, 3, -2, ... upto 22 terms

   (iii) $\frac{1}{15}, \frac{1}{12}, \frac{1}{10}, \ldots, \text{upto}\,11\,\text{terms}$

2. Find the sum of the following :

   (i) $3 + 11 + 19 + \ldots + 803$ (ii) $7 + 10\frac{1}{2} + 14 + \ldots + 84$

3. Find the number of terms :

   (i) How many terms of an A.P. $9, 17, 25, \ldots$ taken so that their sum is 636?

   (ii) How many terms of an A.P. $63, 60, 57, \ldots$ taken so that their sum is 693?

4. Find the sum of first 25 terms of following progression whose n^{th} term is given :

   (i) $a_n = 3 + 4n$ (ii) $a_n = 7 - 3n$

5. Find the sum of 51 terms of an A.P. in which 2^{nd} and 3^{rd} terms are 14 and 18 respectively.
6. First and last term of A.P. are 17 and 350 respectively. If common difference is 9 then how many terms are in A.P. what is their sum?
7. Find the sum of all odd numbers divisible by 3 between 1and 1000.
8. First term of an arithmetic progression is 8, n^{th} term is 33 and sum of first n terms is 123, then find n and common difference d.
9. A sum of ₹280 is to be used to give four prizes. If each prize is ₹20 less than its precceding prize, find the value of each of the prizes.
10. A manufacturer of T.V. sets produced 600 sets in the third year and 700 sets in the seventh year. Assuming that the production increases uniformly by a fixed number every year, find
    (i) the production in the 1^{st} year (ii) the production in the 10^{th} year.
    (iii) the total production in first 7^{th} year.

Miscellaneous Exercise-5

1. The common difference of two Arithmetic progression is same out of two, one A.P. has first term is 8 and other is 3. The difference between their 30^{th} terms is :
   (a) 11 (b) 3 (c) 8 (d) 5
2. If $18, a, b, -3$ are in an A.P., then $a+b=$
   (a) 19 (b) 7 (c) 11 (d) 15
3. If 7^{th} and 13^{th} terms of an A.P. are 34 and 64 respectively then its 18^{th} term is :
   (a) 89 (b) 88 (c) 87 (d) 90
4. In an A.P. first term is 2 and common differnce is 8 , if sum of n terms is 90 then value of n will be:
   (a) 3 (b) 4 (c) 5 (d) 6
5. If sum of n terms of an A.P. is $3n^2+5n$, then which term of series is 164?
   (a) 12^{th} (b) 15^{th} (c) 27^{th} (d) 20^{th}
6. If S_n is sum of first n terms of A.P. and $S_{2n}=3S_n$ then $S_{3n}:S_n$ will be :
   (a) 10 (b) 11 (c) 6 (d) 4
7. The first and last term of A.P. are1 and 11 respectively. If sum of its terms is 36 then number of terms will be
   (a) 5 (b) 6 (c) 99 (d) 11
8. Write 5^{th} term form last of A.P. $3, 5, 7, 9, ..., 201$.
9. If three consecutive terms of A.P. are $\frac{4}{5}, a, 2$, then find the value of a.
10. Find the sum of first 1000 positive integers.
11. Is 299 be any terms of sequence of numbers $5, 11, 17, 23, ...$?
12. Which term of an A.P. $20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, ...$ is first negative term?
13. Four numbers are in arithmetic progression. If their sum is 20 and sum of their squares is 120, then find the numbers.
14. If sum of n terms of an A.P. is $\frac{3n^2}{2}+\frac{5n}{2}$, then find its 25^{th} term.

15. Houses in a row are numbered serially from 1 to 49. Show that one value of x in such that sum of numbers of houses preceeding the marked house is equal to the sum of numbers of houses succeeding the marked house. Find the valur of x.

Important Points

1. $a, a+d, a+2d, \ldots$ is general form of an A.P., where a is first term and d is common difference.
2. Sequence of numbers is in arithmetic progression. If difference $a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots$ are same. This is called common difference of A.P.
3. General term of an A.P. (n^{th} term) $a_n = a + (n-1)d$, where a is first term and d is common difference.
4. Sum of n terms of an A.P. $a, a+d, a+2d, \ldots, a+(n-1)d$.

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

or $$S_n = \frac{n}{2}[a + \ell], \text{ where } \ell = \text{last term} = n^{th} \text{ term } = a + (n-1)d.$$

5. Choose the terms of an A.P in the following from

No. of terms	Term
3	$a-d, a, a+d$
4	$a-3d, a-d, a+d, a+3d$
5	$a-2d, a-d, a, a+d, a+2d$

6. If sum of series of an A.P is given then n^{th} term of series can be find by following formula.

$$a_n = S_n - S_{n-1}$$

Answersheet

Exercise 5.1

1. (i) $a = 6, d = 3$ (ii) $a = -7, d = -2$ (iii) $a = \frac{3}{2}, d = -1$ (iv) $a = 1, d = -3$

(v) $a = -1, d = \frac{5}{4}$ (vi) $a = 3, d = -2$ (vii) $a = 3, d = -5$

2. (i) $-1, \frac{-1}{2}, 0, \frac{1}{2}$ (ii) $\frac{1}{3}, \frac{5}{3}, \frac{9}{3}, \frac{13}{3}$ (iii) $0.6, 1.7, 2.8, 3.9$ (iv) $4, 1, -2, -5$

(v) $11, 7, 3, -1$ (vi) $-1.25, -1.50, -1.75, -2.00$ (vii) $20, \frac{77}{4}, \frac{74}{4}, \frac{71}{4}$

3. (i) Yes, $d = \frac{1}{2}; 4, \frac{9}{2}, 5, \frac{11}{2}$ (ii) Yes, $d = 0; \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}$ (iii) No (iv) No

(v) Yes, $d = \sqrt{2}; \sqrt{50}, \sqrt{72}, \sqrt{98}, \sqrt{128}$ (vi) Yes, $d = a; 5a, 6a, 7a, 8a$

(vii) No (viii) Yes, $d = \sqrt{2}; 3+4\sqrt{2}, 3+5\sqrt{2}, 3+6\sqrt{2}, 3+7\sqrt{2}$

Exercise 5.2

1. (i) 47 (ii) $35\sqrt{2}$ (iii) 101 2. (i) 35^{th} (ii) 22^{th} (iii) No (iv) No

3. 87 4. 5^{th} 5. 4, 10, 16, 22 6. 128 7. -32 8. 55 9. 69

10. 5, 10, 15, 20

Exercise 5.3

1. (i) 144 (ii) -979 (iii) $\frac{33}{20}$ 2. (i) 40703 (ii) $1046\frac{1}{2}$ 3. (i) 12 (ii) 21, 22

4. (i) 1375 (ii) -800 5. 5610 6. 38, 6973 7. 83667

8. $n=6, d=5$ 9. ₹ 100, ₹ 80, ₹ 60, ₹ 40 10. (i) 550 (ii) 775 (iii) 4375

Miscellaneous Exercise 5

1. (d) 2- (d) 3. (a) 4. (c) 5. (c) 6. (c) 7. (b)

8. 193 9. $\frac{7}{5}$ 10. 500500 11. Yes 12. 28^{th}

13. 2, 4, 6, 8 or 8, 6, 4, 2 14. 76 15. $x = 35$

06

Trigonometric Ratios

6.01. Introduction

In class IX, We have studied trigonometric ratios of acute angles. In this chapter we will find trigonometric ratios of specific angles of right angled tringle, 0°, 30°, 45°, 60° and 90°.

6.02. Trigonometric Ratios of Angle 0°

Let rotating ray *AC*, makes acute angle $\angle XAC = \theta$ with its initial position *AX*, from point *A* draw perpendicular *CB* on *AX*, which is so small. As line *AC* tends to *AX* then length of *CB* tends to zero. In this case line *AC* and *A* coincides and $\angle XAC = \theta = 0^\circ$ and $AC = AB$, $\therefore$ $CB = 0$ (zero).

Thus, values of trigonometric ratios corresponding to 0° will be as follows :

$$\sin 0^\circ = \frac{CB}{CA} = \frac{0}{CA} = 0$$

$$\cos 0^\circ = \frac{AB}{CA} = \frac{CA}{CA} = 1$$

$$\tan 0^\circ = \frac{CB}{AB} = \frac{0}{AB} = 0$$

$$\cot 0^\circ = \frac{AB}{CB} = \frac{AB}{0} = \infty$$

$$\sec 0^\circ = \frac{CA}{AB} = \frac{CA}{CA} = 1$$

$$\text{cosec}\, 0^\circ = \frac{CA}{CB} = \frac{CA}{0} = \infty$$

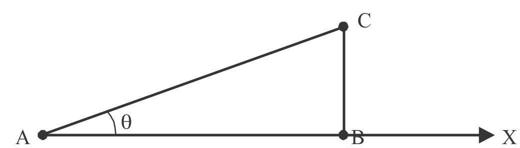


Fig. 6.01

6.03. Trigonometric Ratios of Angle 90°

From ΔCBA, it is clear that as θ increases, length of *CB* decreases and point *B* gets very close to *C* and when θ become equal to 90°, then points *B* and *C* will coincide. In this case $CB = 0$ and $CA = AB$.

$$\sin 90^\circ = \frac{AB}{CA} = \frac{AB}{AB} = 1$$

$$\cos 90^\circ = \frac{CB}{CA} = \frac{0}{AB} = 0$$

$$\tan 90^\circ = \frac{AB}{CB} = \frac{AB}{0} = \infty$$

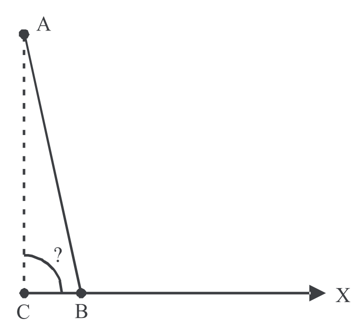


Fig. 6.02

$$\cot 90^\circ = \frac{CB}{AB} = \frac{0}{AB} = 0$$

$$\sec 90^\circ = \frac{CA}{CB} = \frac{CA}{0} = \infty$$

$$\cos ec 90^\circ = \frac{CA}{AB} = \frac{AB}{AB} = 1$$

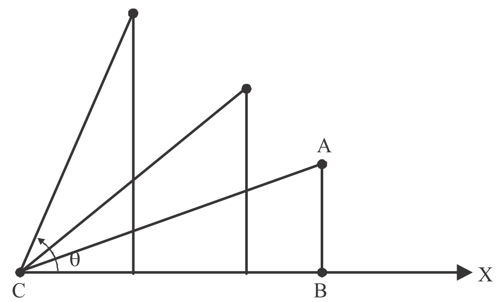


Fig. 6.03

6.04. Trigonometric Ratios of 30° and 60°

Construct an equilateral triangle ABC of each side 2 a. Each angle of an equilateral triangle is of 60°. AD is perpendicular from vertex A to side BC and point D is mid point of side BC.

$BD = DC = a$ and ? BAD ? 30°

∴ $\angle D = 90^\circ$ in ΔABC

$$AB^2 = AD^2 + BD^2$$

$$(2a)^2 = AD^2 + a^2$$

$$AD^2 = 4a^2 - a^2$$

$$AD = \sqrt{3}a$$

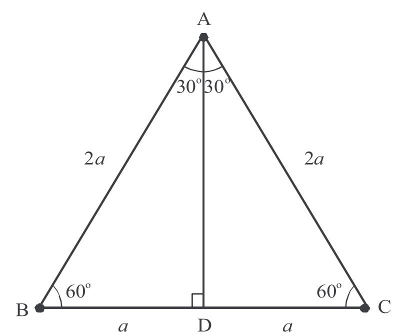


Fig. 6.04

Trigonometric Ratios of 30°

In right angles ADB , base $(AD) = \sqrt{3}a$, perpendicular $(BD) = a$, hypotenuse $(AB) = 2a$ and $= DAB = 30^\circ$

$$\sin 30^\circ = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{AD}{AB} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{BD}{AD} = \frac{a}{\sqrt{3}.a} = \frac{1}{\sqrt{3}}$$

$$\cot 30^\circ = \frac{AD}{BD} = \frac{\sqrt{3}.a}{a} = \frac{\sqrt{3}a}{a} = \sqrt{3}$$

$$\sec 30^\circ = \frac{AB}{AD} = \frac{2a}{\sqrt{3}a} = \frac{2}{\sqrt{3}}$$

$$\cos ec 30^\circ = \frac{AB}{BD} = \frac{2a}{a} = 2$$

Trigonometric Ratios of 60°

In right angled ΔADB, base $(BD) = a$,

Perpendicular $(AD) = a.\sqrt{3}$, hypotenuse $(AB) = 2a$ and $=$ $ABD = 60^\circ$

$$\sin 60^\circ = \frac{AD}{AB} = \frac{a.\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2}$$

$$\tan 60^\circ = \frac{AD}{BD} = \frac{a.\sqrt{3}}{a} = \sqrt{3}$$

$$\cot 60^\circ = \frac{BD}{AD} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\sec 60^\circ = \frac{AB}{BD} = \frac{2a}{a} = 2$$

$$\cos ec 60^\circ = \frac{AB}{AD} = \frac{2a}{a.\sqrt{3}} = \frac{2}{\sqrt{3}}$$

6.05. Trigonometric ratios of 45°

Construct a right tringle ABC in which

$$\angle B = 90^\circ \quad \text{and} \quad \angle A = 45^\circ$$

then in ΔABC,

$$\angle A + \angle B + \angle C = 180^\circ$$

$$45^\circ + 90^\circ + \angle C = 180^\circ$$

$$\angle C = 45^\circ$$

$\therefore$ $\angle A = \angle C$

$\therefore$ $AB = BC$

Let AB = BC = a

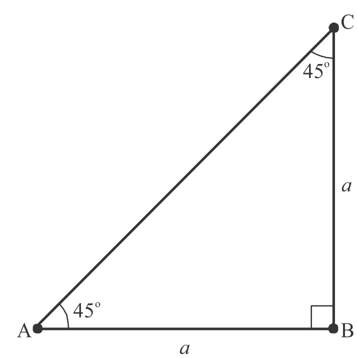


Fig. 6.05

In ΔABC, from Bodhayan theorem

$$AC^2 = AB^2 + BC^2 = a^2 + a^2 = 2a^2$$

$$AC = \sqrt{2a^2} = \sqrt{2}a$$

ΔABC में, $\angle A = 45^\circ$, base $(AB) = a$, perpendicular $(BC) = a$, hypotenuse $(AC) = \sqrt{2}a$

$$\sin 45^\circ = \frac{BC}{AC} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{AB}{AC} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \frac{BC}{AB} = \frac{a}{a} = 1$$

$$\cot 45^\circ = \frac{AB}{BC} = \frac{a}{a} = 1$$

$$\sec 45^\circ = \frac{AC}{AB} = \frac{\sqrt{2}a}{a} = \sqrt{2}$$

$$\operatorname{cos}ec 45^\circ = \frac{AC}{BC} = \frac{\sqrt{2}a}{a} = \sqrt{2}$$

Table of Trigonometric Ratios of Specific Angles

Angle (θ) → (Degree/Radian) Trigonometric Ratio ↓	$0^\circ / 0$	$30^\circ / \pi/6$	$45^\circ / \pi/4$	$60^\circ / \pi/3$	$90^\circ / \pi/2$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
cosec	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Example 1. Find the value of $\tan^2 60^\circ + 3\cos^2 30^\circ$

Solution : $\tan^2 60^\circ + 3\cos^2 30^\circ$ (Putting the values of trignometric ratios)

$$= \left(\sqrt{3}\right)^2 + 3\left(\frac{\sqrt{3}}{2}\right)^2 = 3 + 3 \times \frac{3}{4}$$

$$= 3 + \frac{9}{4} = \frac{12+9}{4} = \frac{21}{4}$$

Example 2. Find the value of $\sin 60^\circ \cos 30^\circ + \sin 30^\circ \cos 60^\circ$

Solution : $\sin 60^\circ \cos 30^\circ + \sin 30^\circ \cos 60^\circ$

$$= \frac{\sqrt{3}}{2} . \frac{\sqrt{3}}{2} + \frac{1}{2} . \frac{1}{2} = \frac{3}{4} + \frac{1}{4} = \frac{3+1}{4} = \frac{4}{4} = 1$$

Example 3. Prove that $4\sin 30^\circ \sin^2 60^\circ + 3\cos 60^\circ \tan 45^\circ = 2\sec^2 60^\circ - \operatorname{cosec}^2 90^\circ$

Solution : $(L.H.S.) = 4\sin 30^\circ \sin^2 60^\circ + 3\cos 60^\circ \tan 45^\circ$

$$= 4.\frac{1}{2}.\left(\frac{\sqrt{3}}{2}\right)^2 + 3\times\frac{1}{2}.1 = \frac{3}{2}+\frac{3}{2} = 3$$

$(R.H.S.) = 2\sec^2 60^\circ - \operatorname{cosec}^2 90^\circ$

$$= 2.\left(\sqrt{2}\right)^2 - (1)^2 = 2\times 2 - 1 = 4 - 1 = 3$$

$\therefore L.H.S. = R.H.S.$

Example 4. Find the value of $\dfrac{\cos 45^\circ}{\sec 30^\circ + \operatorname{cosec} 30^\circ}$

Solution : $\dfrac{\cos 45^\circ}{\sec 30^\circ + \operatorname{cosec} 30^\circ}$

$$= \frac{\frac{1}{\sqrt{2}}}{\frac{2}{\sqrt{3}}+2} = \frac{\frac{1}{\sqrt{2}}}{\frac{2+2\sqrt{3}}{\sqrt{3}}} = \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2+2\sqrt{3}} = \frac{\sqrt{3}}{2\sqrt{2}\left(\sqrt{3}+1\right)}$$

$$= \frac{\sqrt{6}}{4}\left[\frac{\sqrt{3}-1}{\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)}\right] = \frac{\sqrt{6}\left(\sqrt{3}-1\right)}{4(3-1)} = \frac{\sqrt{6}\left(\sqrt{3}-1\right)}{8}$$

Example 5. Prove that $3\tan^2 30^\circ - \frac{4}{3}\sin^2 60^\circ - \frac{1}{2}\operatorname{cosec}^2 45^\circ + \frac{4}{3}\sin^2 90^\circ = \frac{1}{3}$

Solution : $(L.H.S.) = 3\tan^2 30^\circ - \frac{4}{3}\sin^2 60^\circ - \frac{1}{2}\operatorname{cosec}^2 45^\circ + \frac{4}{3}\sin^2 90^\circ$

$$= 3\left(\frac{1}{\sqrt{3}}\right)^2 - \frac{4}{3}\left(\frac{\sqrt{3}}{2}\right)^2 - \frac{1}{2}\left(\sqrt{2}\right)^2 + \frac{4}{3}(1)^2 = 3.\left(\frac{1}{3}\right) - \frac{4}{3}.\left(\frac{3}{4}\right) - \frac{1}{2}.(2) + \frac{4}{3}$$

$$= 1 - 1 - 1 + \frac{4}{3} = \frac{1}{3} \quad (R.H.S.)$$

Example 6. If $\tan 3x = \sin 45^\circ \cos 45^\circ + \sin 30^\circ$, then find the value of x. $(x < 90^\circ)$

Solution : Given, $\tan 3x = \sin 45^\circ \cos 45^\circ + \sin 30^\circ$

$$\tan 3x = \frac{1}{\sqrt{2}}.\frac{1}{\sqrt{2}} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2}$$

or $\quad \tan 3x = 1 \Rightarrow \tan 3x = \tan 45^\circ$ or $\tan 3x = \tan 225^\circ$.

or $\quad 3x = 45^\circ$ or $x = 15^\circ$ and $3x = 225^\circ$ or $x = 75^\circ$

Example 7. If $\sin(A+B)=1$ and $\cos(A-B)=\frac{\sqrt{3}}{2}$ here $0^\circ<(A+B)\le 90^\circ$, $A>B$ Find the value of A and B.

Solution : Given $\sin(A+B)=1$

or $\sin(A+B)=\sin 90^\circ$

or $A+B=90^\circ$ . . . (1)

and $\cos(A-B)=\frac{\sqrt{3}}{2}$

or $\cos(A-B)=\cos 30^\circ$

or $A-B=30^\circ$ . . . (2)

On adding equations (1) and (2), we get

$$(A+B)+(A-B)=90+30^\circ$$

$$2A=120^\circ \qquad \text{or} \qquad A=60^\circ$$

Putting value of A in equation (1), we get

$$60^\circ+B=90^\circ$$

$$B=30^\circ$$

$\therefore$ $A=60^\circ, B=30^\circ$

Example 8. Find the value of $\frac{\sin 30^\circ+\tan 45^\circ-\cos ec60^\circ}{\sec 30^\circ+\cos 60^\circ+\cot 45^\circ}$

Solution : $\frac{\sin 30^\circ+\tan 45^\circ-\cos ec60^\circ}{\sec 30^\circ+\cos 60^\circ+\cot 45^\circ}$

$$=\frac{\frac{1}{2}+1-\frac{2}{\sqrt{3}}}{\frac{2}{\sqrt{3}}+\frac{1}{2}+1}=\frac{\frac{3}{2}-\frac{2}{\sqrt{3}}}{\frac{2}{\sqrt{3}}+\frac{3}{2}}$$

$$=\frac{\frac{3\sqrt{3}-4}{2\sqrt{3}}}{\frac{4+3\sqrt{3}}{2\sqrt{3}}}=\frac{3\sqrt{3}-4}{2\sqrt{3}}\times\frac{2\sqrt{3}}{4+3\sqrt{3}}=\left(\frac{3\sqrt{3}-4}{4+3\sqrt{3}}\right)\times\left(\frac{4-3\sqrt{3}}{4-3\sqrt{3}}\right)$$

[Multiplying Numerator and Denominator by $\left(4-3\sqrt{3}\right)$

$$= \frac{-\left(4-3\sqrt{3}\right)\left(4-3\sqrt{3}\right)}{(4)^2-\left(3\sqrt{3}\right)^2} = \frac{-\left(4-3\sqrt{3}\right)^2}{16-27}$$

$$= \frac{-\left(16+27-24\sqrt{3}\right)}{-11} = \frac{43-24\sqrt{3}}{11}$$

$$\therefore \qquad \frac{\sin 30^\circ + \tan 45^\circ - \operatorname{cosec} 60^\circ}{\sec 30^\circ + \cos 60^\circ + \cot 45^\circ} = \frac{43-24\sqrt{3}}{11}$$

Exercise 6.1

Find the value of the following :

1. $2\sin 45^\circ \cos 45^\circ$
2. $\cos 45^\circ \cos 60^\circ - \sin 45^\circ \sin 60^\circ$
3. $\sin^2 30^\circ + 2\cos^2 45^\circ + 3\tan^2 60^\circ$
4. $3\sin 60^\circ - 4\sin^3 60^\circ$
5. $\dfrac{5\cos^2 60^\circ + 4\sin^2 30^\circ + \tan^2 45^\circ}{\sin^2 30^\circ + \cos^2 45^\circ}$
6. $4\cot^2 45^\circ - \sec^2 60^\circ + \sin^2 60^\circ + \cos^2 90^\circ$
7. $\dfrac{4}{\cot^2 30^\circ} + \dfrac{1}{\sin^2 30^\circ} - \cos^2 45^\circ$
8. $\dfrac{\tan^2 60^\circ + 4\sin^2 45^\circ + \sin^2 90^\circ}{3\sec^2 30^\circ + \cos ec^2 60^\circ - \cot^2 30^\circ}$
9. $\dfrac{\sin 30^\circ - \sin 90^\circ + 2\cos 0^\circ}{\tan 30^\circ \tan 60^\circ}$
10. $\dfrac{2\tan 30^\circ}{1-\tan^2 30^\circ}$
11. Find the value of x in the following :
    (i) $\cos x = \cos 60^\circ \cos 30^\circ + \sin 60^\circ \sin 30^\circ$
    (ii) $\sin 2x = \sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ$
    (iii) $\sqrt{3}\tan 2x = \sin 30^\circ + \sin 45^\circ \cos 45^\circ + 2\sin 90^\circ$

Prove that :

12. $\dfrac{\cos 30^\circ + \sin 60^\circ}{1+\cos 60^\circ + \sin 30^\circ} = \dfrac{\sqrt{3}}{2}$

13. $4\cot^2 45^\circ - \sec^2 60^\circ - \sin^2 30^\circ = -\dfrac{1}{4}$

14. $4\sin 30^\circ \sin^2 60^\circ + 3\cos 60^\circ \tan 45^\circ = 2\sec^2 45^\circ - \operatorname{cosec}^2 90^\circ$

15. $\operatorname{cos}ec^2 45^\circ \sec^2 30^\circ \sin^3 90^\circ \cos 60^\circ \ ? \ \dfrac{4}{3}$

16. $\dfrac{\sin 60^\circ + \sin 30^\circ}{\sin 60^\circ - \sin 30^\circ} = \dfrac{\tan 60^\circ + \tan 45^\circ}{\tan 60^\circ - \tan 45^\circ}$

17. $2\left(\cos^2 45^\circ + \tan^2 60^\circ\right) - 6\left(\sin^2 45^\circ - \tan^2 30^\circ\right) = 6$

18. $\left(\sec^2 30^\circ + \operatorname{cos}ec^2 45^\circ\right)\left(2\cos 60^\circ + \sin 90^\circ + \tan 45^\circ\right) = 10$

19. $\left(1 - \sin 45^\circ + \sin 30^\circ\right)\left(1 + \cos 45^\circ + \cos 60^\circ\right) = \dfrac{7}{4}$

20. $\cos^2 0^\circ - 2\cot^2 30^\circ + 3\operatorname{cos}ec^2 90^\circ = 2\left(\sec^2 45^\circ - \tan^2 60^\circ\right)$

21. If $x = 30^\circ$ then prove that

(i) $\sin 3x = 3\sin x - 4\sin^3 x$

(ii) $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

(iii) $\sin x = \sqrt{\dfrac{1-\cos 2x}{2}}$

(iv) $\cos 3x = 4\cos^3 x - 3\cos x$

22. If $A = 60^\circ$ and $B = 30^\circ$ then prove that :

$$\cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

Miscellaneous Exercise 6

Multiple Choice Questions (from 1 to 5)

1. The value of $\tan^2 60^\circ$ is:

(a) 3 (b) $\dfrac{1}{3}$ (c) 1 (d) ∞

2. The value of $2\sin^2 60^\circ \cos 60^\circ$ will be:

(a) $\dfrac{4}{3}$ (b) $\dfrac{5}{2}$ (c) $\dfrac{3}{4}$ (d) $\dfrac{1}{3}$

3. If $\operatorname{cosec}\theta = \frac{2}{\sqrt{3}}$ then value of θ is:

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{6}$

4. The value of $\cos^2 45^\circ$ will be :

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{3}}$

5. If $\theta = 45^\circ$ then, value of $\frac{1-\cos 2\theta}{\sin 2\theta}$ is :

(a) 0 (b) 1 (c) 2 (d) ∞

Prove that :

6. $\cos 60° = 2\cos^2 30° - 1$

7. $\sin 60° = \frac{2\tan 30°}{1+\tan^2 30°}$

8. $\cos 60° = \frac{1-\tan^2 30°}{1+\tan^2 30°}$

9. $(\sin 45° + \cos 45°)^2 = 2$

10. $4\tan 30° \sin 45° \sin 60° \sin 90° = \sqrt{2}$

11. Find the value of $\sin^2 60^\circ \cot^2 60^\circ$.

12. Find the value of $4\cos^3 30^\circ - 3\cos 30^\circ$

13. If $\cot\theta = \frac{1}{\sqrt{3}}$ then prove that $\frac{1-\cos^2\theta}{2-\sin^2\theta} = \frac{3}{5}$

14. Prove that $3\left(\tan^2 30^\circ + \cot^2 30^\circ\right) - 8\left(\sin^2 45^\circ + \cos^2 30^\circ\right) = 0$

15. Prove that $4\left(\sin^4 30^\circ + \cos 60^\circ\right) - 3\left(\cos^2 45^\circ - \sin^2 90^\circ\right) = \frac{15}{4}$

16. Prove that $\frac{\cos 30^\circ + \sin 60^\circ}{1+\cos 60^\circ + \sin 30^\circ} = \frac{\sqrt{3}}{2}$

17. Prove that $2\left(\cos^2 45^\circ + \tan^2 60^\circ\right) - 6\left(\sin^2 45^\circ - \tan^2 30^\circ\right) = 6$

Answer Sheet

Exercise 6.1

(1) 1 **(2)** $\frac{1-\sqrt{3}}{2\sqrt{2}}$ **(3)** $10\frac{1}{4}$ **(4)** 0 **(5)** $\frac{67}{12}$ **(6)** $\frac{3}{4}$

(7) $\frac{13}{6}$ **(8)** $\frac{18}{7}$ **(9)** $\frac{3}{2}$ **(10)** $\sqrt{3}$ **(11)** (i) 30^{o} (ii) 15^{o} (iii) 30^{o}

Miscellaneous Exercise 6

1. (a) 2. (c) 3. (b) 4. (c) 5. (b) 11. $\frac{1}{4}$ 12. 0

Trigonometric Indentities

7.01. Introduction

In the previous chapter we have studied trigonometric ratios and their mutual relations. In this chapter we will discuss trigonometric identities.

Trigonometric Identities

Trigonometric identities are true for all angles involved. Here we discuss proof of following identities.

According to figure, in Δ ABC , $\angle B$ is right angle. For angle θ, BC is perpendicular AB is base and AC will be hypotenuse.

$\therefore \quad BC^2 + AB^2 = AC^2 \quad \ldots (1)$

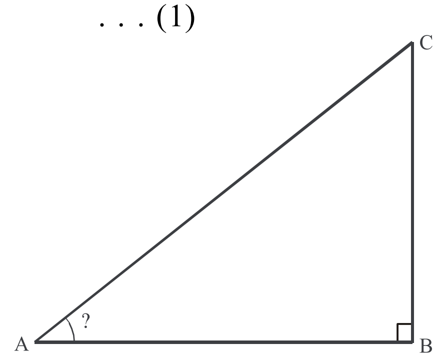


Fig. 7.01

Dividing both sides of equation (1) by AC^2

$$\frac{BC^2}{AC^2} + \frac{AB^2}{AC^2} = \frac{AC^2}{AC^2}$$

$$\Rightarrow \quad \left(\frac{BC}{AC}\right)^2 + \left(\frac{AB}{AC}\right)^2 = \left(\frac{AC}{AC}\right)^2$$

$$\Rightarrow \quad (\sin\theta)^2 + (\cos\theta)^2 = 1$$

$$\Rightarrow \quad \sin^2\theta + \cos^2\theta = 1 \quad \ldots (2)$$

For all θ where $0^o \le \theta \le 90^o$ is true value

This is an identity.

Now dividing each term of equation (1) by AB^2

$$\frac{BC^2}{AB^2} + \frac{AB^2}{AB^2} = \frac{AC^2}{AB^2}$$

$$\left(\frac{BC}{AB}\right)^2 + \left(\frac{AB}{AB}\right)^2 = \left(\frac{AC}{AB}\right)^2$$

$$(\tan\theta)^2 + 1 = (\sec\theta)^2$$

$$\Rightarrow \quad \tan^2\theta + 1 = \sec^2\theta$$

$$\Rightarrow \quad 1 + \tan^2\theta = \sec^2\theta \quad \ldots (3)$$

Now dividing each term of equation (1) by BC^2

$$\frac{BC^2}{BC^2} + \frac{AB^2}{BC^2} = \frac{AC^2}{BC^2}$$

$$\left(\frac{BC}{BC}\right)^2+\left(\frac{AB}{BC}\right)^2=\left(\frac{AC}{BC}\right)^2$$

$$1+(\cot\theta)^2=(\operatorname{cosec}\theta)^2$$

$$1+\cot^2\theta=\operatorname{cosec}^2\theta \qquad \text{. . . (iv)}$$

Above identities can be written as

1. $\sin^2\theta+\cos^2\theta=1$ or $\sin^2\theta=1-\cos^2\theta$ or $\cos^2\theta=1-\sin^2\theta$
2. $1+\tan^2\theta=\sec^2\theta$ or $\sec^2\theta-\tan^2\theta=1$ or $\tan^2\theta=\sec^2\theta-1$
3. $1+\cot^2\theta=\operatorname{cosec}^2\theta$ or $\operatorname{cosec}^2\theta-\cot^2\theta=1$ or $\cot^2\theta=\operatorname{cosec}^2-1$

Table

	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\operatorname{cosec}\theta$
$\sin\theta$	$\sin\theta$	$\sqrt{1-\cos^2\theta}$	$\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\frac{1}{\sqrt{1+\cot^2\theta}}$	$\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\frac{1}{\operatorname{cosec}\theta}$
$\cos\theta$	$\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\frac{1}{\sqrt{1+\tan^2\theta}}$	$\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$	$\frac{1}{\sec\theta}$	$\frac{\sqrt{\operatorname{cosec}^2\theta-1}}{\operatorname{cosec}\theta}$
$\tan\theta$	$\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\frac{1}{\cot\theta}$	$\sqrt{\sec^2\theta-1}$	$\frac{1}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\cot\theta$	$\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\frac{1}{\tan\theta}$	$\cot\theta$	$\frac{1}{\sqrt{\sec^2\theta-1}}$	$\sqrt{\operatorname{cosec}^2\theta-1}$
$\sec\theta$	$\frac{1}{\sqrt{1-\sin^2\theta}}$	$\frac{1}{\cos\theta}$	$\sqrt{1+\tan^2\theta}$	$\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$	$\sec\theta$	$\frac{\operatorname{cosec}\theta}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\operatorname{cosec}\theta$	$\frac{1}{\sin\theta}$	$\frac{1}{\sqrt{1-\cos^2\theta}}$	$\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\sqrt{1+\cot^2\theta}$	$\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\operatorname{cosec}\theta$

Illustrative Examples

Example 1. Prove that $\cot\theta+\tan\theta=\operatorname{cosec}\theta\sec\theta$

Solution : LHS $=\cot\theta+\tan\theta$

$$=\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}=\frac{\cos^2\theta+\sin^2\theta}{\sin\theta\cos\theta} \qquad (\because \sin^2\theta+\cos^2\theta=1)$$

$$=\frac{1}{\sin\theta\cos\theta}=\frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta}$$

$$=\operatorname{cosec}\theta\cdot\sec\theta = \text{RHS}$$

Example 2. Prove that $\left(1+\tan^2\theta\right)\left(1+\sin\theta\right)\left(1-\sin\theta\right)=1$

Solution : LHS $=\left(1+\tan^2\theta\right)\left(1+\sin\theta\right)\left(1-\sin\theta\right)$

$$=\left(1+\tan^2\theta\right)\left(1-\sin^2\theta\right)$$

$$=\sec^2\theta\cos^2\theta$$

$$=\frac{1}{\cos^2\theta}\cdot\cos^2\theta$$

$= 1 =$ RHS

Example 3. Prove that $\frac{1}{1+\sin\theta}+\frac{1}{1-\sin\theta}=2\sec^2\theta$

Solution : LHS $=\frac{1}{1+\sin\theta}+\frac{1}{1-\sin\theta}$

$$=\frac{1-\sin\theta+1+\sin\theta}{\left(1+\sin\theta\right)\left(1-\sin\theta\right)}$$

$$=\frac{2}{1-\sin^2\theta}=\frac{2}{\cos^2\theta}=2\cdot\frac{1}{\cos^2\theta}$$

$=2\sec^2\theta$ = RHS

Example 4. Prove that $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}=\operatorname{cosec}\theta+\cot\theta$

Solution : LHS $=\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$

Multiplying numerator and denominator by $\sqrt{1+\cos\theta}$

$$\sqrt{\frac{\left(1+\cos\theta\right)}{\left(1-\cos\theta\right)}\times\frac{\left(1+\cos\theta\right)}{\left(1+\cos\theta\right)}}$$

$$=\sqrt{\frac{\left(1+\cos\theta\right)^2}{1-\cos^2\theta}}=\sqrt{\frac{\left(1+\cos\theta\right)^2}{\sin^2\theta}}$$

$$=\frac{1+\cos\theta}{\sin\theta}=\frac{1}{\sin\theta}+\frac{\cos\theta}{\sin\theta}$$

$=\operatorname{cosec}\theta+\cot\theta$ = RHS

Example 5. Prove that $(\sec\theta-\tan\theta)^2=\dfrac{1-\sin\theta}{1+\sin\theta}$

Solution : LHS $=(\sec\theta-\tan\theta)^2$

$$=\left(\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}\right)^2=\left(\frac{1-\sin\theta}{\cos\theta}\right)^2$$

$$=\frac{(1-\sin\theta)^2}{\cos^2\theta}=\frac{(1-\sin\theta)^2}{1-\sin^2\theta}$$

$$=\frac{(1-\sin\theta)(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}$$

$$=\frac{1-\sin\theta}{1+\sin\theta}=\text{RHS}$$

Example 6. Prove that $\dfrac{\sin\theta}{1+\cos\theta}+\dfrac{1+\cos\theta}{\sin\theta}=2\operatorname{cosec}\theta$

Solution : LHS $=\dfrac{\sin\theta}{1+\cos\theta}+\dfrac{1+\cos\theta}{\sin\theta}$

$$=\frac{\sin^2\theta+(1+\cos\theta)^2}{\sin\theta(1+\cos\theta)}$$

$$=\frac{\sin^2\theta+1+\cos^2\theta+2\cos\theta}{\sin\theta(1+\cos\theta)}$$

$$=\frac{1+\sin^2\theta+\cos^2\theta+2\cos\theta}{\sin\theta(1+\cos\theta)}$$

$$=\frac{1+1+2\cos\theta}{\sin\theta(1+\cos\theta)}=\frac{2+2\cos\theta}{\sin\theta(1+\cos\theta)}$$

$$=\frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)}=\frac{2}{\sin\theta}=2\cdot\frac{1}{\sin\theta}$$

$$=2\operatorname{cosec}\theta=\text{RHS}$$

Example 7. Prove that $\dfrac{\sin\theta-2\sin^3\theta}{2\cos^3\theta-\cos\theta}=\tan\theta$

Solution : LHS $=\dfrac{\sin\theta-2\sin^3\theta}{2\cos^3\theta-\cos\theta}$

$$= \frac{\sin\theta\left(1-2\sin^2\theta\right)}{\cos\theta\left(2\cos^2\theta-1\right)}$$

$$= \frac{\sin\theta\left[\sin^2\theta+\cos^2\theta-2\sin^2\theta\right]}{\cos\theta\left[2\cos^2\theta-\left(\sin^2\theta+\cos^2\theta\right)\right]} \quad \left(\because\ \sin^2\theta+\cos^2\theta=1\right)$$

$$= \frac{\sin\theta\left[\cos^2\theta-\sin^2\theta\right]}{\cos\theta\left[\cos^2\theta-\sin^2\theta\right]} = \frac{\sin\theta}{\cos\theta}$$

$$= \tan\theta = \text{RHS}$$

Example 8. Prove that $\dfrac{1+\sec A}{\sec A} = \dfrac{\sin^2 A}{1-\cos A}$

Solution : LHS $= \dfrac{1+\sec A}{\sec A}$

$$= \frac{1+\frac{1}{\cos A}}{\frac{1}{\cos A}} = \frac{\frac{\cos A+1}{\cos A}}{\frac{1}{\cos A}} = \frac{\cos A+1}{\cos A}\times\frac{\cos A}{1}$$

$$= \frac{1+\cos A}{1}$$

$$= \frac{(1+\cos A)(1-\cos A)}{1-\cos A}$$

Multiplying numerator and denominator bya (1 – cosA)

$$= \frac{(1)^2-(\cos A)^2}{1-\cos A} = \frac{1-\cos^2 A}{1-\cos A}$$

$$= \frac{\sin^2 A}{1-\cos A} = \text{RHS} \quad \left(\because\ 1-\cos^2 A=\sin^2 A\right)$$

Example 9. Prove that $\cos^4\theta-\sin^4\theta = 1-2\sin^2\theta$

Solution : LHS $= \cos^4\theta-\sin^4\theta$

$$= \left(\cos^2\theta\right)^2-\left(\sin^2\theta\right)^2$$

$$= \left(\cos^2\theta+\sin^2\theta\right)\left(\cos^2\theta-\sin^2\theta\right) \qquad \because \quad a^2-b^2=(a+b)(a-b)$$

$$= 1\cdot\left(\cos^2\theta-\sin^2\theta\right) \qquad \because \quad \sin^2\theta+\cos^2\theta=1$$

$$= 1\cdot\left(1-\sin^2\theta-\sin^2\theta\right) = 1-2\sin^2\theta = \text{RHS}$$

Example 10 . If $\sin\theta+\cos\theta=p$ and $\sec\theta+\operatorname{cosec}\theta=q$, then prove that $q\left(p^2-1\right)=2p$.

Solution : LHS $=q\left(p^2-1\right)$

Putting value of p and q

$$=\left(\sec\theta+\operatorname{cosec}\theta\right)\left[\left(\sin\theta+\cos\theta\right)^2-1\right]$$

$$=\left(\frac{1}{\cos\theta}+\frac{1}{\sin\theta}\right)\left[\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta-1\right]$$

$$=\left(\frac{\sin\theta+\cos\theta}{\sin\theta\cos\theta}\right)\left[1+2\sin\theta\cos\theta-1\right]$$

$$=\left[\frac{\sin\theta+\cos\theta}{\sin\theta\cdot\cos\theta}\right]\times\left(2\sin\theta\cos\theta\right)$$

$$=2\left[\sin\theta+\cos\theta\right]=2p = \text{RHS}$$

Example 11 . Prove that $\dfrac{\cot A+\operatorname{cosec}A-1}{\cot A-\operatorname{cosec}A+1}=\dfrac{1+\cos A}{\sin A}$

Solution : LHS$=\dfrac{\cot A+\operatorname{cosec}A-1}{\cot A-\operatorname{cosec}A+1}$

$$=\frac{\left(\cot A+\operatorname{cosec}A\right)-\left(\operatorname{cosec}^2A-\cot^2A\right)}{\cot A-\operatorname{cosec}A+1} \qquad \left(\because\ \operatorname{cosec}^2A-\cot^2A=1\right)$$

$$=\frac{\left(\operatorname{cosec}A+\cot A\right)-\left[\left(\operatorname{cosec}A+\cot A\right)\left(\operatorname{cosec}A-\cot A\right)\right]}{\cot A-\operatorname{cosec}A+1}$$

$$=\frac{\left(\operatorname{cosec}A+\cot A\right)\left[1-\left(\operatorname{cosec}A-\cot A\right)\right]}{\cot A-\operatorname{cosec}A+1}$$

$$=\frac{\left(\operatorname{cosec}A+\cot A\right)\left[\cot A-\operatorname{cosec}A+1\right]}{\left(\cot A-\operatorname{cosec}A+1\right)}$$

$$=\operatorname{cosec}A+\cot A$$

$$=\frac{1}{\sin A}+\frac{\cos A}{\sin A}=\frac{1+\cos A}{\sin A} = \text{RHS}$$

Example 12. Prove that $\left(\dfrac{1+\tan^2A}{1+\cot^2A}\right)=\left(\dfrac{1-\tan A}{1-\cot A}\right)^2=\tan^2A$

Solution : LHS $=\dfrac{1+\tan^2A}{1+\cot^2A}=\dfrac{\sec^2A}{\operatorname{cosec}^2A}$

$$=\left[\frac{\sec A}{\operatorname{cosec} A}\right]^2=\left[\frac{1/\cos A}{1/\sin A}\right]^2=\left[\frac{1}{\cos A}\times\frac{\sin A}{1}\right]^2$$

$$=\left[\frac{\sin A}{\cos A}\right]^2=[\tan A]^2=\tan^2 A \ = \text{RHS}$$

Now, $$\left[\frac{1-\tan A}{1-\cot A}\right]^2=\left[\frac{1-\frac{\sin A}{\cos A}}{1-\frac{\cos A}{\sin A}}\right]^2=\left[\frac{\frac{\cos A-\sin A}{\cos A}}{\frac{\sin A-\cos A}{\sin A}}\right]^2$$

$$=\left[\frac{\cos A-\sin A}{\cos A}\times\frac{\sin A}{\sin A-\cos A}\right]^2=\left[-\frac{(\sin A-\cos A)}{\cos A}\times\frac{\sin A}{(\sin A-\cos A)}\right]^2$$

$$=\left[-\frac{\sin A}{\cos A}\right]^2=[-\tan A]^2=\tan^2 A \ = \text{RHS}$$

Exercise 7.1

1. Express all trigonometric ratios in term of $\sec\theta$, for angle θ.
2. Express trigonometric ratios $\sin\theta$, $\sec\theta$, $\tan\theta$ in terms of $\cot\theta$.

Prove the following with the help of identities.

3. $\cos^2\theta+\cos^2\theta\cot^2\theta=\cot^2\theta$
4. $\sec\theta(1-\sin\theta)(\sec\theta+\tan\theta)=1$
7. $\operatorname{cosec}^2\theta+\sec^2\theta=\operatorname{cosec}^2\theta\sec^2\theta$
6. $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}=\sec\theta-\tan\theta$
7. $\sqrt{\sec^2\theta+\operatorname{cosec}^2\theta}=\tan\theta+\cot\theta$
8. $\dfrac{\tan\alpha+\tan\beta}{\cot\alpha+\cot\beta}=\tan\alpha\tan\beta$
9. $\dfrac{1+\sin\theta}{\cos\theta}=\dfrac{\cos\theta}{1+\sin\theta}=2\sec\theta$
10. $\dfrac{\sin^4\theta-\cos^4\theta}{\sin^2\theta-\cos^2\theta}=1$
11. $\cot\theta-\tan\theta=\dfrac{1-2\sin^2\theta}{\sin\theta\cos\theta}$
12. $\cos^4\theta+\sin^4\theta=1-2\cos^2\theta\sin^2\theta$

13. $\left(\sec\theta-\cos\theta\right)\left(\cot\theta+\tan\theta\right)=\tan\theta\sec\theta$

14. $\dfrac{1-\tan^2\alpha}{\cot^2\alpha-1}=\tan^2\alpha$

15. $\dfrac{\sin\theta}{1-\cos\theta}=\dfrac{1+\cos\theta}{\sin\theta}$

16. $\sin^6\theta+\cos^6\theta=1-3\sin^2\theta\cos^2\theta$

17. $\dfrac{\tan\theta}{1-\cot\theta}+\dfrac{\cot\theta}{1-\tan\theta}=1+\tan\theta+\cot\theta$

18. $\sin\theta\left(1+\tan\theta\right)+\cos\theta\left(1+\cot\theta\right)=\operatorname{cosec}\theta+\sec\theta$

19. $\sin^2\theta\cos\theta+\tan\theta\sin\theta+\cos^3\theta=\sec\theta$

20. $\dfrac{\tan\theta}{1-\cot\theta}+\dfrac{\cot\theta}{1-\tan\theta}=1+\sec\theta\operatorname{cosec}\theta$

21. $\left(\sin A+\operatorname{cosec}A\right)^2+\left(\cos A+\sec A\right)^2=7+\tan^2A+\cot^2A$

22. $\sin^8\theta-\cos^8\theta=\left(\sin^2\theta-\cos^2\theta\right)\left(1-2\sin^2\theta\cos^2\theta\right)$

23. $\sqrt{\dfrac{\sec\theta+1}{\sec\theta-1}}=\cot\theta+\operatorname{cosec}\theta$

24. $\dfrac{\left(1+\cot\theta+\tan\theta\right)\left(\sin\theta-\cos\theta\right)}{\sec^3\theta-\operatorname{cosec}^3\theta}=\sin^2\theta\cos^2\theta$

27. $\dfrac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta}+\dfrac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}=\dfrac{2}{1-2\cos^2\theta}=\dfrac{2}{2\sin^2\theta-1}$

26. $\dfrac{\cos A}{1-\tan A}+\dfrac{\sin A}{1-\cot A}=\sin A+\cos A$

27. $\left(\operatorname{cosec}A-\sin A\right)\left(\sec A-\cos A\right)=\dfrac{1}{\tan A+\cot A}$

28. $\dfrac{\cos^2\theta}{1-\tan\theta}+\dfrac{\sin^3\theta}{\sin\theta-\cos\theta}=1+\sin\theta\cos\theta$

29. If $\sec\theta+\tan\theta=P$ then, prove that $\dfrac{P^2-1}{P^2+1}=\sin\theta$

30. If $\dfrac{\cos A}{\cos B}=m$ and $\dfrac{\cos A}{\sin B}=n$ then, prove that $\left(m^2+n^2\right)\cos^2B=n^2$

7.02. Trigonometric Ratios of Complementary Angles

Complementary Angles

If sum of two angle is 90° then two angles are called complementary angles. Complementary angle of any acute angle θ will be (90° – θ). If in a right angled Δ ABC, $\angle B$ = 90° then sum of $\angle A$ and $\angle C$ will be 90°.

$$\angle A + \angle C = 90^\circ$$

If $\quad \angle A = \theta$ then

$$\angle C = 90^\circ - \theta$$

Thus θ and 90° – θ are complementary angles of each other.

In right angled triangle ABC for angle θ, side BC and AB will be Perpendicular and base respectively. Thus in ΔABC, trigonometric ratios for angle θ and (90° – θ).

$$\sin(90^\circ - \theta) = \frac{AB}{AC} \qquad \sin\theta = \frac{BC}{AC}$$

$$\cos(90^\circ - \theta) = \frac{BC}{AC} \qquad \cos\theta = \frac{AB}{AC}$$

$$\tan(90^\circ - \theta) = \frac{AB}{BC} \qquad \tan\theta = \frac{BC}{AB}$$

$$\cot(90^\circ - \theta) = \frac{BC}{AB} \qquad \cot\theta = \frac{AB}{BC}$$

$$\sec(90^\circ - \theta) = \frac{AC}{BC} \qquad \sec\theta = \frac{AC}{AB}$$

$$\operatorname{cosec}(90^\circ - \theta) = \frac{AC}{AB} \qquad \operatorname{cosec}\theta = \frac{AC}{BC}$$

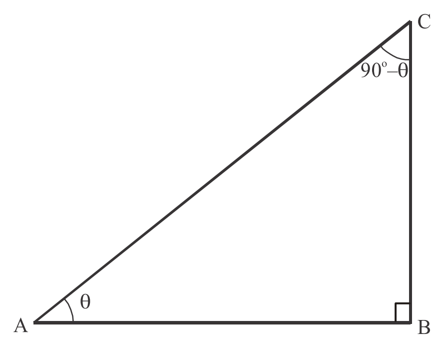


Fig. 7.02

Comparing above equation, we get

$$\sin(90^\circ - \theta) = \cos\theta \qquad \cos(90^\circ - \theta) = \sin\theta$$

$$\tan(90^\circ - \theta) = \cot\theta \qquad \cot(90^\circ - \theta) = \tan\theta$$

$$\sec(90^\circ - \theta) = \operatorname{cosec}\theta \qquad \operatorname{cosec}(90^\circ - \theta) = \sec\theta$$

$$0^\circ \le \theta \le 90^\circ$$

Note : We can say that

sin of any angle = cos of its complementary angle

tan of any angle = cot of its complementary angle

sec of any angle = cosec of its complementary angle

Its converse is also true

Illustrative Examples

Example 13. Find the value of $\frac{\tan 49^\circ}{\cot 41^\circ}$

Solution : $\tan 49^\circ = \cot(90^\circ - 49^\circ) = \cot 41^\circ$ $\quad \{\tan\theta = \cot(90^\circ - \theta)\}$

$\therefore \quad \frac{\tan 49^\circ}{\cot 41^\circ} = \frac{\cot 41^\circ}{\cot 41^\circ} = 1$

Example 14. Find the value of $\sin^2 50^\circ + \sin^2 40^\circ$

Solution : $\because \quad 40^\circ = 90^\circ - 50^\circ$

$\therefore \quad \sin 40^\circ = \sin(90^\circ - 50^\circ) = \cos 50^\circ$

Thus, $\sin^2 50^\circ + \sin^2 40^\circ = \sin^2 50^\circ + \cos^2 50^\circ = 1$ $\quad (\because \ \sin^2\theta + \cos^2\theta = 1)$

Example 15. Find the value of $\tan 39^\circ - \cot 51^\circ$

Solution : $\tan 39^\circ = \cot(90 - 39^\circ) = \cot 51^\circ$

Thus, $\quad \tan 39^\circ - \cot 51^\circ = \cot 51^\circ - \cot 51^\circ = 0$

Example 16. Find the value of $\sec 50^\circ \sin 40^\circ + \cos 40^\circ \operatorname{cosec} 50^\circ$

Solution : $\sec 50^\circ \sin 40^\circ + \cos 40^\circ \operatorname{cosec} 50^\circ$

$= \operatorname{cosec} 40^\circ \sin 40^\circ + \cos 40^\circ \operatorname{cosec}(90^\circ - 40^\circ)$

$= \operatorname{cosec} 40^\circ \sin 40^\circ + \cos 40^\circ \sec 40^\circ$

$= \frac{1}{\sin 40^\circ} \cdot \sin 40^\circ + \cos 40^\circ \cdot \frac{1}{\cos 40^\circ} = 1 + 1 = 2$

Example 17. Prove that $\tan 15^\circ \tan 20^\circ \tan 70^\circ \tan 75^\circ = 1$

Solution : LHS $= \tan 15^\circ \tan 20^\circ \tan 70^\circ \tan 75^\circ$

$= \tan 15^\circ \tan 20^\circ \tan(90^\circ - 20)\tan(90^\circ - 15^\circ)$

$= \tan 15^\circ \tan 20^\circ \cdot \cot 20^\circ \cdot \cot 15^\circ$

$= \tan 15^\circ \tan 20^\circ \cdot \frac{1}{\tan 20^\circ \tan 15^\circ} = 1$ (RHS)

Example 18. Prove that $\tan 2A = \cot(A - 18^\circ)$ then find the value of A.

Solution : $\tan 2A = \tan[90^\circ - (A - 18^\circ)]$

$\tan 2A = \tan(108^\circ - A)$

$\therefore \quad 2A = 108^\circ - A$

$3A = 108^\circ \Rightarrow A = 36^\circ$

Example 19. Find the value of x form the following equation.

$$\operatorname{cosec}(90^\circ-\theta)+x\cos\theta\cot(90^\circ-\theta)=\sin(90^\circ-\theta)$$

Solution : $\operatorname{cosec}(90^\circ-\theta)+x\cos\theta\cot(90^\circ-\theta)=\sin(90^\circ-\theta)$

$\sec\theta+x\cos\theta\tan\theta=\cos\theta$

$x\sin\theta=\cos\theta-\sec\theta$ $\qquad\left(\because \tan\theta=\dfrac{\sin\theta}{\cos\theta}\right)$

$x=\dfrac{\cos\theta-\sec\theta}{\sin\theta}=\dfrac{\cos^2\theta-1}{\sin\theta\cos\theta}$ $\qquad\left(\because \sec\theta=\dfrac{1}{\cos\theta}\right)$

$=-\left[\dfrac{1-\cos^2\theta}{\sin\theta\cos\theta}\right]$ $\qquad\left(\because 1-\cos^2\theta=\sin^2\theta\right)$

$=-\dfrac{\sin^2\theta}{\sin\theta\cos\theta}$ $\qquad x=-\tan\theta$

Exercise 7.2

Find the value of the following :

1. (i) $\dfrac{\cos 37^\circ}{\sin 53^\circ}$ (ii) $\dfrac{\operatorname{cosec} 32^\circ}{\sec 58^\circ}$ (iii) $\dfrac{\tan 10^\circ}{\cot 80^\circ}$ (iv) $\dfrac{\cos 19^\circ}{\sin 71^\circ}$

2. (i) $\operatorname{cosec} 25^\circ-\sec 65^\circ$ (ii) $\cot 34^\circ-\tan 56^\circ$

   (iii) $\dfrac{\sin 36^\circ}{\cos 54^\circ}-\dfrac{\sin 54^\circ}{\cos 36^\circ}$ (iv) $\sin\theta\cos(90^\circ-\theta)+\cos\theta\sin(90^\circ-\theta)$

3. (i) $\sin 70^\circ\sin 20^\circ-\cos 20^\circ\operatorname{cosec} 70^\circ$ (ii) $\dfrac{2\cos 67^\circ}{\sin 23^\circ}-\dfrac{\tan 40^\circ}{\cot 50^\circ}-\cos 60^\circ$

4. (i) $\left(\dfrac{\sin 35^\circ}{\cos 55^\circ}\right)^2+\left(\dfrac{\cos 55^\circ}{\sin 35^\circ}\right)^2-2\cos 60^\circ$ (ii) $\left(\dfrac{\sin 27^\circ}{\cos 63^\circ}\right)^2+\left(\dfrac{\cos 63^\circ}{\sin 27^\circ}\right)^2$

5. (i) $\cot 12^\circ\cot 38^\circ\cot 52^\circ\cot 60^\circ\cot 78^\circ$ (ii) $\tan 5^\circ\tan 25^\circ\tan 30^\circ\tan 45^\circ\tan 65^\circ\tan 85^\circ$

6. Express the following in terms of trigonometric ratios of angles between 0° and 45°

   (i) $\sin 81^\circ+\sin 71^\circ$ (ii) $\tan 68^\circ+\sec 68^\circ$

   Prove the following :

7. $\sin 65^\circ+\cos 25^\circ=2\cos 25^\circ$

8. $\sin 35^\circ\sin 55^\circ-\cos 35^\circ\cos 55^\circ=0$

9. $\dfrac{\cos 70^\circ}{\sin 20^\circ}+\dfrac{\cos 59^\circ}{\sin 31^\circ}-8\sin^2 30^\circ=0$

10. $\sin(90^\circ-\theta)\cos(90^\circ-\theta)=\dfrac{\tan\theta}{1+\tan^2\theta}$

11. $\dfrac{\cos(90^\circ-\theta)\cos\theta}{\tan\theta}+\cos^2(90^\circ-\theta)=1$

12. $\dfrac{\tan(90^\circ-\theta)\cot\theta}{\operatorname{cosec}^2\theta}-\cos^2\theta=0$

13. $\dfrac{\cos(90^\circ-\theta)\sin(90^\circ-\theta)}{\tan(90^\circ-\theta)}=\sin^2\theta$

14. $\dfrac{\sin\theta\cos(90^\circ-\theta)\cos\theta}{\sec(90^\circ-\theta)}+\dfrac{\cos\theta\sin(90^\circ-\theta)\sin\theta}{\operatorname{cosec}(90^\circ-\theta)}=\sin\theta\cos\theta$

15. If $\sin 3\theta=\cos(\theta-6^\circ)$, where 3θ and $(\theta-6^\circ)$ are acute angles, then find the value of θ .

16. If $\sec 5\theta=\operatorname{cosec}(\theta-36^\circ)$, where 5θ is an acute angle, then find the value of θ .

17. If *A, B* and *C* are interior angles of a ΔABC, then prove that $\tan\left(\dfrac{B+C}{2}\right)=\cot\dfrac{A}{2}$.

18. If $\cos 2\theta=\sin 4\theta$ where 2θ and 4θ are acute angles,then find the value of θ.

Answer

Exercise 7.1

1. $\sin\theta=\dfrac{\sqrt{\sec^2\theta-1}}{\sec\theta}$, $\cos\theta=\dfrac{1}{\sec\theta}$, $\tan\theta=\sqrt{\sec^2\theta-1}$, $\cot\theta=\dfrac{1}{\sqrt{\sec^2\theta-1}}$, $\operatorname{cosec}\theta=\dfrac{\sec\theta}{\sqrt{\sec^2\theta-1}}$

2. (i) $\sin\theta=\dfrac{1}{\sqrt{1+\cot^2\theta}}$, $\tan\theta=\dfrac{1}{\cot\theta}$, $\sec\theta=\dfrac{\sqrt{1+\cot^2\theta}}{\cot\theta}$

Exercise 7.2

1. (1) 1 (ii) 1 (iii) 1 (iv) 1

2. (i) 0 (ii) 0 (iii) 0 (iv) 1

3. (i) 0 (ii) 1 / 2

4. (i) 1 (ii) 2

5. (i) $\dfrac{1}{\sqrt{3}}$ (ii) $\dfrac{1}{\sqrt{3}}$

6. (i) $\cos 9^\circ+\cos 19^\circ$ (ii) $\cot 22^\circ+\operatorname{cosec} 22^\circ$

15. $\theta=24^\circ$

16. $\theta=21^\circ$

18. $\theta=15^\circ$

08

Height and Distance

8.01. Introduction

In the preceding chapter, we have studied trigonometric identities and trigonometric ratios for complementary angles. In this chapter we will study problems based on height and distace by using trigonometric ratios, we will see how trigonometry is used for finding the heights and distances of various objects without actually measuring them. Before this, we will study some definitations.

8.02. Important Definitions

Line of sight : The line of sight is the line drawn from the eye of an observer to the point in the object viewed by the observer.

In figure 8.01, if eye is at point O and object is at point P then OP is the line of sight.

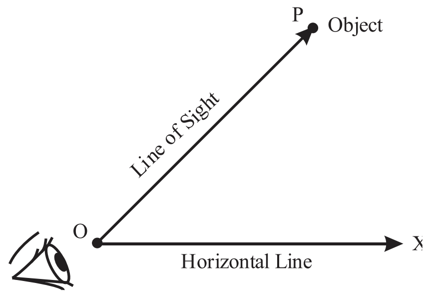


Fig. 8.01

Angle of Elevation

If an oject is above the eye then we raise our head to look at the object. The angle formed by the line of sight with the horizontal line is called angle of elevation.

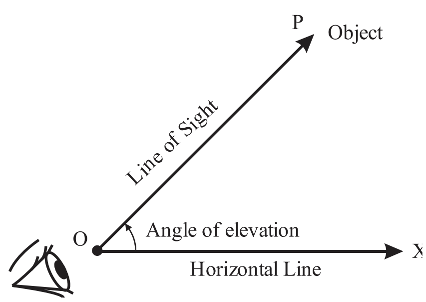


Fig. 8.02

In fig. 8.02, eye is at point O and object is at point P then line of sight OP makes an angle $\angle XOP$ with horizontal line OX then

Angle of elevation $= \angle XOP$

Note : Angle of elevation is also called as angular height of the object.

Angle of depression

The angle of depression of a point on the object being viewed is the angle formed by the line when the point is below the horizontal line.

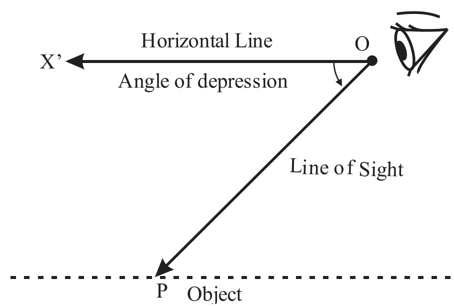


Fig. 8.03

In figure 8.03, eye is at point O and object is at point P then OP is line of sight which makes an angle $X'OP$ with horizontal line OX' then angle of depession $= \angle XO'P$

In solving the problems related to height and distance, following points are to be kept in mind.

(i) First read the question carefully then draw figure and prepare right angled triangle.

(ii) In right angled triangle, express trignometric ratios (sin, cos, tan etc.) of given angle in the terms of given sides.

Note : Complemetary angles : If sum of two angles is 90^o then they are called complelmentary angles.

Following are the Examples with figure of angle of depression subtended at the eye of the observer by the objects.

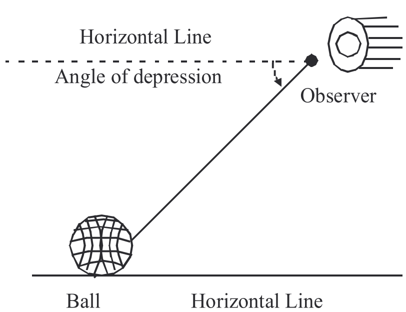


Fig. 8.04

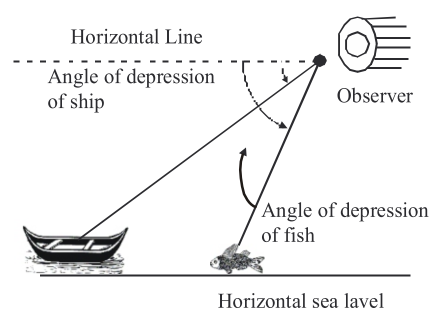


Fig. 8.05

Illustrative Examples

Example 1. The angle of elevation of the top of a tower form a point on the ground which is $10\sqrt{3}$ m away from the foot of the tower is 60°. Find the height of the tower.

Solution : Let AB is a tower. The angle of elevation of the top of a tower from point C on the ground, which is $10\sqrt{3}$ m away from the foot of the tower is 60°. Let h is the height of the tower AB.

In right angled ΔABC

$$\tan 60^\circ = \frac{AB}{BC}$$

$$\Rightarrow \quad \sqrt{3} = \frac{h}{10\sqrt{3}}$$

or $\quad h = 10\sqrt{3} \times \sqrt{3}$

or $\quad h = 10 \times 3 = 30$

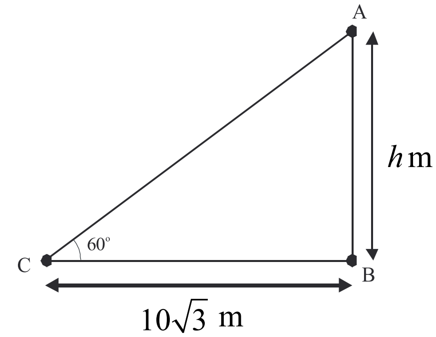


Fig. 8.06

Thus, height of the tower AB is 30 m.

Example 2. The angle of drepression of any boat from a 50 m high bridge is 30°. Find the horizontal distance between boat and bridge.

Solution : Let horizontal distance between boat and bridge is x m.

Given, Angle of depression is 30°

Here $\quad PQ = 50$ m

$$\angle XPO = \angle POQ = 30^\circ \text{ (Alternate angles)}$$

In right angled ΔPQO

$$\because \quad \tan 30^\circ = \frac{PQ}{OQ}$$

$$\therefore \quad \frac{1}{\sqrt{3}} = \frac{50}{x}$$

or $\quad x = 50\sqrt{3} = 50 \times 1.732 \quad \left(\because \sqrt{3} = 1.732\right)$

or $\quad x = 86.60$

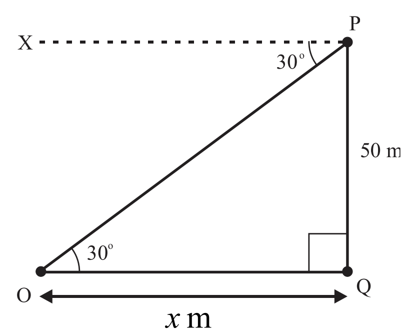


Fig. 8.07

Thus, horizontal distance between boat and bridge is 86.60 m.

Example 3. The shadow of a 1.5 m tall student standing on a plane ground in found to be 1 m and at the same time shadow of a tower on ground is 5 m, then find height of the tower.

Solution : Given, Length of student AC = 1.5 m

Shadow of student BC = 1 m

In right angled ΔACB

$$\tan\theta = \frac{AC}{BC} \quad \Rightarrow \quad \tan\theta = \frac{1.5}{1}$$

or $\quad \tan\theta = 1.5 \qquad \ldots (1)$

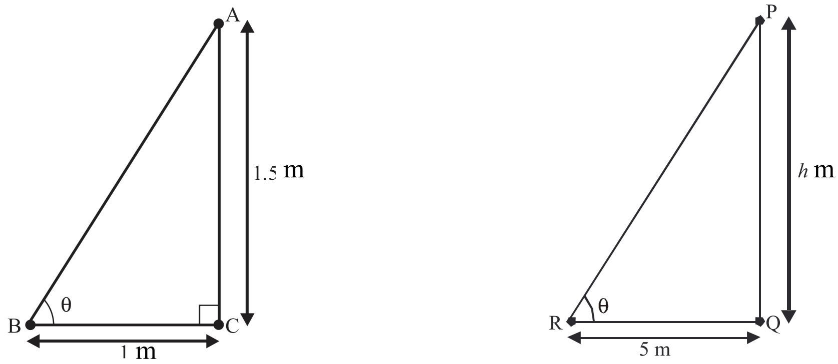


Fig 8.08

Now it is given that

Length of shadow of the tower $QR = 5$ m

Let height of the tower $= PQ = h$ m

In right angled ΔPQR

or $\tan\theta = \dfrac{PQ}{QR}$

or $\dfrac{h}{5} = 1.5$ $\quad$ [$\because \tan\theta = 1.5$ from equation (i)]

or $h = 5 \times 1.5$

or $h = 7.5$

Thus, height of tower is 7.5 m.

Example 4. There is a small island in 100 m broad river and there is a tall tree on this island. P and Q lie on the opposite banks of the river such that P, Q and tree are in the same line. If angle of elevation form P and Q at top of the tree are 30^{o} and 45^{o} respectivaly then find height of the tree.

Solution : Let OA is tree whose height is h m.

In figure, $PQ = 100$ m

$\angle APO = 30^\circ$ and $\angle AQO = 45^\circ$

Now, in right angled ΔPOA and ΔQOA

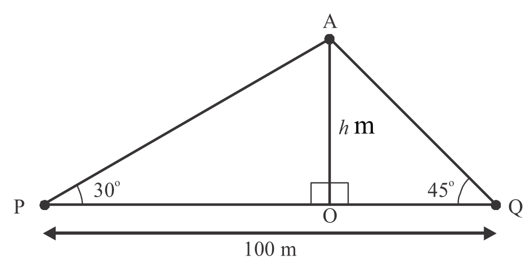


Fig 8.09

$\tan 30^\circ = \dfrac{OA}{OP}$ and $\tan 45^\circ = \dfrac{OA}{OQ}$

$\dfrac{1}{\sqrt{3}} = \dfrac{h}{OP}$ and $1 = \dfrac{h}{OQ}$

$OP = h\sqrt{3}$ and $OQ = h$

$\therefore$ From figure $PQ = OP + OQ$

$100 = h\sqrt{3} + h$

$$100 = h\left(\sqrt{3}+1\right)$$

$$\therefore \qquad h = \frac{100}{\sqrt{3}+1} = \frac{100}{\left(\sqrt{3}+1\right)} \times \left(\frac{\sqrt{3}-1}{\sqrt{3}-1}\right)$$

$$h = \frac{100\left(\sqrt{3}-1\right)}{2}$$

$$h = 50\left(\sqrt{3}-1\right) = 36.6 \text{ m} \qquad \left(\because \sqrt{3} = 1.732\right)$$

Thus, height of the tree is 36.6 m.

Example 5. A car is moving on a straight road which goes towards a tower. At a distance of 500 m from tower, driver of car observe that angle of elevation of top of tower is 30° after driving the car for 10 sec. towards tower then he observe that the angle of elevation of top of tower became 60°. Find the speed of the car.

Solution : Let height of tower $AB = h$ m and distance covered by car in 10 sec. (DC) = x m.

$$BD = 500 \text{ m}$$

$$\therefore \qquad BC = (500 - x) \text{ m}$$

$$\angle ADB = 30^\circ, \angle ACB = 60^\circ$$

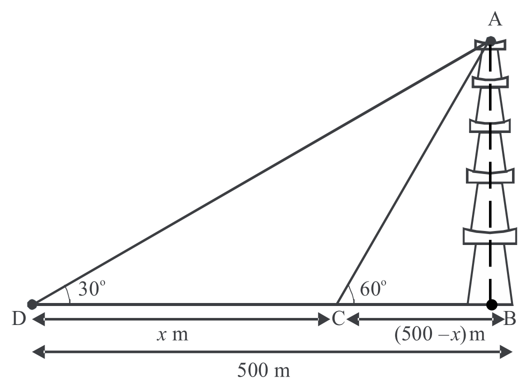


Fig 8.10

Now, in right angled ? ABD

$$\frac{AB}{BD} = \tan 30$$

$$\frac{h}{500} = \frac{1}{\sqrt{3}} \Rightarrow h = \frac{500}{\sqrt{3}} \ldots(1)$$

Now, in right angled ΔABC

$$\frac{AB}{BC} = \tan 60$$

or $$\frac{h}{500-x} = \sqrt{3} \Rightarrow h = (500-x)\sqrt{3} \ldots(2)$$

From equation (1) and (2)

$$\frac{500}{\sqrt{3}} = (500-x)\sqrt{3} \Rightarrow 500 = (500-x)\sqrt{3}.\sqrt{3}$$

or $$500 = (500-x).3$$

or $$500 = 1500 - 3x$$

or $$3x = 1500 - 500 = 1000$$

or $$x = \frac{1000}{3}$$

Distance covered by car in 10 sec $= \frac{1000}{3}$ m

? Distance covered by car in 1 min. $= \frac{1000 \times 60}{3 \times 10} = 2000$ m $= 2$ km

Thus speed of car = 2 km/min.

Example 6. The angle of elevations of the top of a tower form two pointrs C and D from base of tower and in the same straight line at a distance a and b respectively , are complement to each other. Prove that height of tower is $\sqrt{ab}$.

Solution : Let the height of tower $AB = h$ meter and points C and D are in such a way that $BC = a,\ BD = b$.

If $\angle ACB = \theta$ then $\angle ADB = 90^\circ - \theta$

Again, in right angled ΔABC

$$\tan\theta = \frac{AB}{BC} = \frac{h}{a} \qquad \text{... (1)}$$

Again, in right angled ΔABD

$$\tan\left(90^\circ - \theta\right) = \frac{AB}{BD}$$

or $$\cot\theta = \frac{h}{b} \qquad \text{... (2)}$$

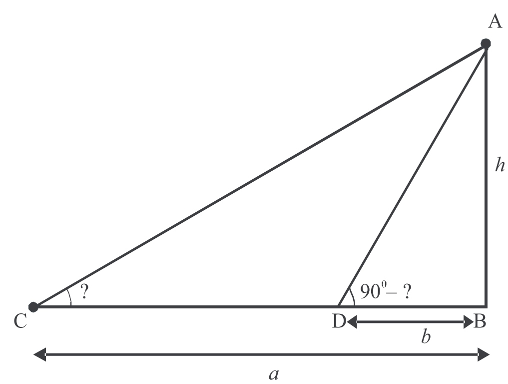


Fig. 8.11

On multiplying equation (1) and (2), we get

$$\tan\theta \times \cot\theta = \frac{h}{a} \times \frac{h}{b}$$

or $$1 = \frac{h^2}{ab} \Rightarrow h^2 = ab$$

or $$h = \sqrt{ab}$$

Example 7. Two poles of equal height are standing opposite each other on either side of the road which is 80 m wide. From a point between them on the road, the angle of elevation of the top of the poles are 60° and 30° respectively. Find the height of the poles and the distances of the point from the poles.

Solution: Let BC and DE are two poles of same height (in meter). From a point between the poles,on the road BD,the angles of elevations of the top of the poles are 60° and 30°

Thus, $\angle CAB = 60^\circ$ and $\angle EAD = 30$ $BC = DE = h$ m. $BD = 80$ m.

Let $AD = x$ m

$\therefore$ $AB = BD - AD = (80 - x)$ m

In right angle ΔADE

$$\tan 30^\circ = \frac{DE}{AD}$$

$$\frac{1}{\sqrt{3}} = \frac{h}{x}$$

$$\therefore \qquad h = \frac{x}{\sqrt{3}} \qquad \ldots (1)$$

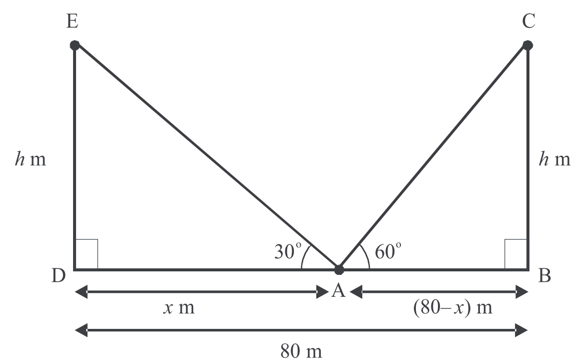


Fig. 8.12

Again, in right angled ΔABC

$$\tan 60^\circ = \frac{BC}{AB}$$

$$\sqrt{3} = \frac{h}{(80-x)}$$

$$h = (80-x)\sqrt{3} \text{ m} \qquad \ldots (2)$$

From equation (1) and (2)

$$\frac{x}{\sqrt{3}} = \sqrt{3}(80-x)$$

$$x = \sqrt{3}.\sqrt{3}(80-x)$$

$$x = 3(80-x)$$

$$x \,?\, 240 ? \,3x$$

$$\Rightarrow \quad x + 3x = 240$$

$$4x = 240$$

$$x = \frac{240}{4} = 60$$

From equation (1)

$$h = \frac{60}{\sqrt{3}} = \frac{60}{\sqrt{3}}.\frac{\sqrt{3}}{\sqrt{3}} = \frac{60\sqrt{3}}{3}$$

$$h = 20\sqrt{3}$$

Thus height of poles $(h) = 20\sqrt{3}$ m and distance of the point form the poles is 20 m and 60 m.

Example 8. The angle of elevation of a cloud form a point *'h'* m above a lake is and the angle of depression of its reflection in the lake is β. Prove that height of the cloud from surface of water is $\dfrac{h(\tan\beta + \tan \propto)}{\tan\beta - \tan\propto}$.

Solution : Let AB is surface of lake and P is point of observation. Given $AP = h$ m. Let position of cloud is c and c' is its shadow in lake $\therefore CB = C'B$. Let PM is perpendicular from P to CB, it is given that

$$\angle CPM = \alpha \text{ and } \angle MPC' = \beta \text{ Let } CM = x$$

It is clear that $CB = CM + MB = CM + PA = x + h$

In ΔCMP $\tan\alpha = \frac{CM}{PM}$

or $\tan\alpha = \frac{x}{AB}$ $(\because PM = AB)$

$\therefore$ $AB = x\cot\alpha$ . . . (1)

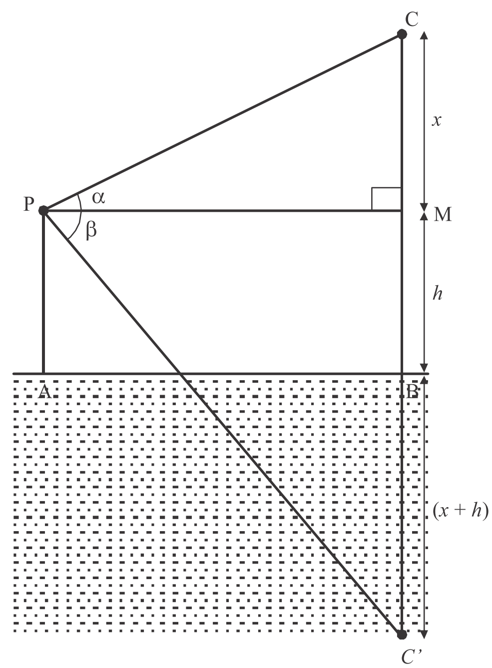


Fig. 8.13

In $\Delta PMC'$, $\tan\beta = \frac{C'M}{PM} = \frac{x+2h}{AB}$

$\therefore$ $AB = (x+2h)\cot\beta$ . . . (2)

From equations (1) and (2), we get

$$x\cot\alpha = (x+2h)\cot\beta$$

$$x(\cot\alpha - \cot\beta) = 2h\cot\beta$$

or $$x\left(\frac{1}{\tan\alpha} - \frac{1}{\tan\beta}\right) = \frac{2h}{\tan\beta}$$

or $$x\left[\frac{\tan\beta - \tan\alpha}{\tan\alpha\tan\beta}\right] = \frac{2h}{\tan\beta}$$

or $$x = \frac{2h\tan\alpha}{\tan\beta - \tan\alpha}$$

Thus height of cloud form surface of water

$$CB = x + h = \frac{2h\tan\alpha}{\tan\beta - \tan\alpha} + h = \frac{h(\tan\alpha + \tan\beta)}{\tan\beta - \tan\alpha}$$

Exercise 8

1. The shadow of a verticle pillar is same the height of pillar, then angle of elevation of Sun will be :

   (a) 45° (b) 30° (c) 60° (d) 50°

2. From a point on the ground which is 100 m away from the foot of the tower, the angle of elevation of the top of the tower is 60°, then height of tower is :

   (a) $100\sqrt{3}$ m (b) $\frac{100}{\sqrt{3}}$ m (c) $50\sqrt{3}$ m (d) $\frac{200}{\sqrt{3}}$ m

3. A 15 m long ladder touches the top of a vertical wall. If this ladder makes an angle of 60° with the wall then height of the wall is :

(a) $15\sqrt{3}$ m (b) $\frac{15\sqrt{3}}{2}$ m (c) $\frac{15}{2}$ m (d) 15 m

4. From the top of 10 m height tower, angle of depression at a point on earth is 30°. Distance of point from base of tower is

(a) $10\sqrt{3}$ m (b) $\frac{10}{\sqrt{3}}$ m (c) 10 m (d) $5\sqrt{3}$ m

5. A bridge above the river makes an angle of 45° with the bank of river. If length of bridge above the river is 150 m then breadth of river will be

(a) 75 m (b) $50\sqrt{2}$ m (c) 150 m (d) $75\sqrt{2}$ m

6. Top of two towers of height 20 m and 14 m are joined by a wire. If wire makes an angle of 30° with horizontal line then length of wire is :

(a) 12 m (b) 10 m (c) 8 m (d) 6 m

7. The angle of elevation of the top of the tower form two points distance a and b from the base of tower $(a > b)$ are 30° and 60° then height of tower is :

(a) $\sqrt{a+b}$ (b) $\sqrt{a-b}$ (c) $\sqrt{ab}$ (d) $\sqrt{\frac{a}{b}}$

8. From the top of a 25 m high pillar the angle of elevation of top of the tower is same as the angle of depression of foot of tower then height of tower is :

(a) 25 m (b) 100 m (c) 75 m (d) 50 m

9. If ratio of length of a verticle rod and length of its shadow is $1:\sqrt{3}$ then angle of elevation of sun is :

(a) 30° (b) 45° (c) 60° (d) 90°

10. The slope of a hill makes an angle of 60° with horizontal . If to reach at top, 500 m distance have to covered then height of the hill is :

(a) $500\sqrt{3}$ m (b) $\frac{500}{\sqrt{3}}$ m (c) $250\sqrt{3}$ m (d) $\frac{250}{\sqrt{3}}$ m

11. A tower is vertically placed on a horizontal plane. If angle of elevation of sun is 30° and length of shadow of tower is 45 m the find height of the tower.

12. The upper part of a tree is broken by windstrom and it makes an angle of 60° with the ground. The distance form the bottom of the tree to the point where the top touches the ground is 10 m. Find the original height of the tree $\left(\sqrt{3}=1.732\right)$

13. From a point on the ground which is 120 m away form the foot of the unfinished tower, the angle of elevation of the top of the tower is found to be 30°. Find how much height of tower have to increased so that its angle of elevation at same point become 60° ?

14. The angle of elevation of the top of a tower form a point situated at 100 m far form the foot of tower is 30°. Find the height of the tower.

15. The angle of elevation of the top of a pillar form a point on the ground is 15° on walking 100 m towards the tower, the angle of elevation is found to be 30°. Find the height of the tower (where $\tan 15^0 = 2 - \sqrt{3}$)
16. The shadow of a verticle tower on level ground is increased by 40 m, when the altitude of the sun changes form 60° to 30°. Find the height of the tower.
17. The angle of depression of two ships from the top of light house situated at 60 m heigtht from sea level, are 30° and 45° if two ships are on the same side of the light house then find the distance between two ships.
18. A 1.5 m tall boy is standing at some distance away form a 30 m high building when he moves towards the building then angle of elevation from his eye become 60° to 30°. Find how much distance he covered towards the building ?
19. Angle of elevation of top of a tower form a 7 m high building is 60° and angle of depression of its foot is 45°. Find the height of the tower.
20. From the top of a hill, in east side at two points of angle of depression are 30° and 45°. If distance between two points is 1 km, then find height of the hill.
21. The angle of elevation of a cloud from a point 20 m above a lake (point A) is 30°. If the angle of depression of its reflection form point A is 60° then find the distance of cloud form point A.
22. From a point on a bridge across a river, the angles of depression of the banks on opposite side of the river are 30° and 45° respectively. If the bridge is at height of 4 m form the bank, find the width of the river.
23. A man on the deck of the ship is 10 m above water -level. He observes that the angle of elevation of the top of hill is 60° and the angle of depression of the base is 30° then find the distance of the hill form the ship and height of the hill.
24. A verticle straight tree 12 m high is broken by strong wind in such a way that its top touches the ground and makes an angle of 60° with the ground. Find at what height from the ground did the tree break ? $\left(\sqrt{3} = 1.732\right)$
25. A straight highway leads to the foot of a tower. A man standing at the top of the tower observes a car at an angle of depression of 30°, which is approaching the foot of the tower with a uniform speed. Six seconds later,the angle of depression of the car is found to be 60°. Find the time taken by the car to reach the foot of the tower from this point.
26. The angles of elevation of the top of the tower form two points at a distance of 4 m and 9 m form the base of the tower in the same straight line are complementary. Prove that the height of tower is 6 m.
27. A tower and a building on the opposite side of road are situated. The angles of depression form the top of tower at the roof and base of building are 45° and 60° respectively. If height of building is 12 m then find the height of the tower $\left(\sqrt{3} = 1.732\right)$
28. If angle of elevation of sun changes from 30° to 60°. Then at these angles of elevation find the difference in the length of shadow of 15 m high pillar.

Important Points

1. The line of sight is the line drawn from the eye of an observer to the point in the object viewed by the observer.
2. Angle subtended by an eye with the horizontal to see an object in the upward direction is called angle of elevation.
3. Angle subtended by an eye with the horizontal to see an object in the downward direction is called angle of depression.
4. $\sin 30^\circ = 0.5774 = \cos 60^\circ$

   $\sin 45^\circ = 0.7071 = \cos 45^\circ$

   $\sin 60^\circ = 0.8660 = \cos 30^\circ$

   $\sqrt{2} = 1.4141,\ \sqrt{3} = 1.732$

Answers
Exercise 8

1. (a) 45^o	2. (a) $100\sqrt{3}$ m	3. (c) $\frac{15}{2}$ m	4. (a) $10\sqrt{3}$ m
5. (d) $75\sqrt{2}$	6. (a) 12 m	7. (c) $\sqrt{ab}$	8. (d) 50 m
9. (a) 30^o	10. (c) $250\sqrt{3}$ m	11. $15\sqrt{3}$ m	12. 37.32 m
13. 138.56 m	14. 57.73 m	15. 50 m	16. 34.64 m
17. 43.92 m	18. $19\sqrt{3}$ m	19. $7\left(\sqrt{3}+1\right)$m	20. 1.366 km
21. 40 m	22. 10.92 m	23. $10\sqrt{3}$ m, 40 m	24. 5.569 m
25. 3 m	27. 28.392 m	28. 17.32 m	

09

Co-ordinate Geometry

9.01 Introduction

In earlier classes we have studied geometry which is called Euclidean geometry. Now we will study the analytic geometry. Where position of the point is expressed by specific numbers which are called co-ordinates, lines and curves so formed are represented by algebraic equations. **Due to the use of co-ordinates in Analytic geometry, this is called as co-ordinate geometry.**

9.02 Cartesian co–ordinates

Let $X'OX$ and $Y'OY$ be two perpendicular lines in any plane, which intersects each other at point O. These lines are called coordinate axes and O is called origin. $X'OX$ and $Y'OY$ are perpendicular to each other. Thus $X'OX$ and $Y'OY$ are called rectangular axes.

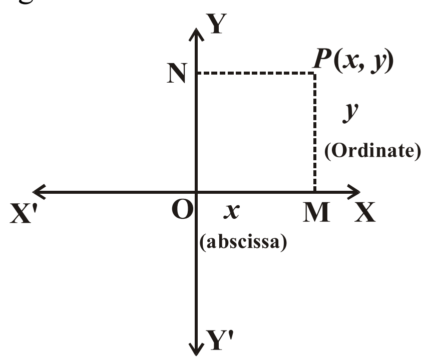


Fig. 9.01

Now to find co-ordinates of P, draw perpendiculars PM and PN from P on x and y axes respectively. The length of the segment OM ($OM=x$) is called the x-coordinate or abscissa of point P. Similarly, the length, of line segment ON is called the y-coordinate or ordinate of point. These coordinates are written in ordered pair (x,y) *i.e.,* while writing the co-ordinates of a point, write co-ordinate first and then y co-ordinate in parentheses.

9.03 Sign of Co–ordinates in quadrants

In figure 9.02, two axes $X'OX$ and $Y'OY$ divide the plane into four equal parts which are called quadrants. XOY, YOX', $X'OY'$ and $Y'OX$ are called respectively I,II,III, and IV quadrants. We always take OX and OY as + ve and OX' and OY' as - ve directions.

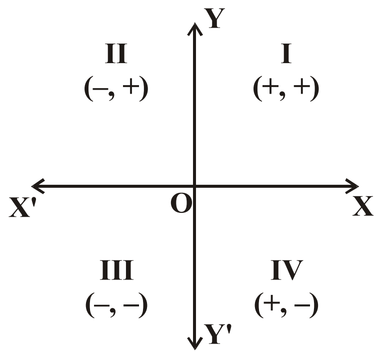


Fig.9.02

If (x,y) be coordinates of any point P in plane, then in

I Quadrant $x > 0, y > 0$; Coordinates(+, +)

II Quadrant $x < 0, y > 0$; Coordinates (–, +)

III Quadrant $x < 0, y < 0$; Coordinates (–, –)

IV Quadrant $x > 0, y < 0$; Coordinates (+, –)

Note:(i) If coordinate of any point P is (x,y) then we can write as P (x,y).

(ii) The abscissa of any point is at a perpendicular distance from y - axis.

(iii) The ordinate of any point is at a perpendicular distance form x-axis.

(iv) The abscissa of any point is positive at R.H.S of y axis and negative at LHS of y axis.

(v) The ordinate of any point is positive above the x-axis and negative below the x-axis.

(vi) If $y = 0$, then point lies on x-axis.

(vii) If $x = 0$, then point lies on y-axis.

(viii) If $x = 0, y = 0$ then point is origin.

9.04 Distance between two points

Let XOX' and YOY' are co-ordinate axes and two points in the plane are $P(x_1, y_1)$ and $Q(x_2, y_2)$ we have to find distance between these two points. From point P and Q draw perpendicular PM and QN on x-axis, respectively and draw perpendicular PR form P to QN.

$\therefore$ OM = Abscissa of $P = x_1$

Similarly $ON = x_2$, $PM = y_1$

and $QN = y_2$

According to figure $PR = MN = ON - OM = x_2 - x_1$

and $QR = QN - RN = QN - PM = y_2 - y_1$

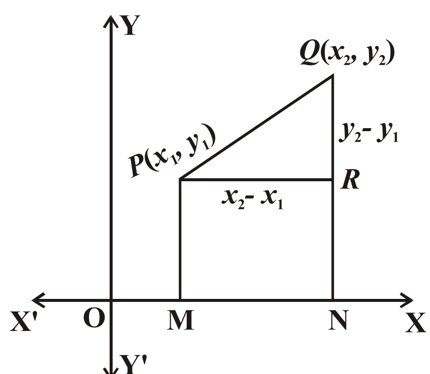


Fig. 9.03

Thus, by Bodhayan formula in right angled ΔPRQ

$$PQ^2 = PR^2 + QR^2$$

or $$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$\therefore$ $$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(\text{difference of } x - \text{coordinates})^2 + (\text{difference of } y - \text{coordinate})^2}$$

which is a formula to find distance between two points.

Special case : Distance of point P(x,y) from origin O (0, 0)

$$OP = \sqrt{x^2 + y^2}$$

Illustrative Examples

Example 1. Plot the points (2, 4), (–2, 3), (–4, –3) and (5, –2) in the rectangular co-ordinate system.

Solution :

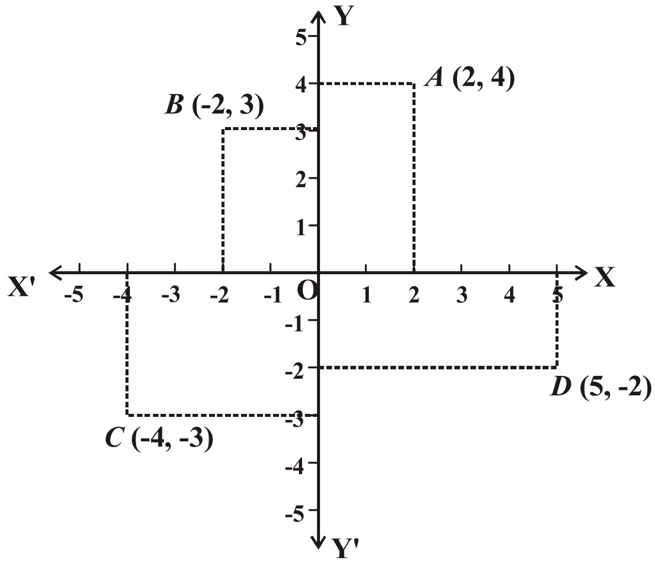


Fig. 9.04

Example 2. If $2a$ is a side of an equilateral triangle, then find the co-ordinates of its vertices.

Solution : According to figure 9.05

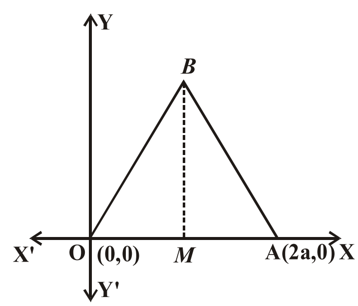


Fig. 9.05

$\because$ OAB is an equilateral triangle of side $2a$

$\therefore$ $OA = AB = OB = 2a$

Draw perpendicular BM from point B to OA

$\therefore$ $OM = MA = a$

In rigtht angle ΔOMB,

$$OB^2 = OM^2 + MB^2$$

or $$(2a)^2 = (a)^2 + MB^2$$

or $$MB^2 = 3a^2$$

$\therefore$ $$MB = \sqrt{3}a$$

Thus, co-ordinates of vertices of an equilateral triangle are O (0, 0), $A(2a, 0)$ and $B(a, \sqrt{3}a)$ since $OM = a$ and $MB = \sqrt{3}a$.

Example 3. Find the distance between the points (2,3) and (5,6).

Solution : Let points (2, 3) and (5, 6) are P and Q respectively so distance between them

$$PQ = \sqrt{(5-2)^2 + (6-3)^2}$$

$$= \sqrt{(3)^2 + (3)^2}$$

$$= \sqrt{9+9}$$

$$= \sqrt{18} = 3\sqrt{2}$$

Example 4. If distance between points $(x, 3)$ and $(5, 7)$ is 5, then find the value of x.

Solution : Let $P(x, 3)$ and $Q(5, 7)$ are given points then according to question.

$$PQ = 5$$

$$\sqrt{(x-5)^2+(3-7)^2} = 5$$

Squaring both sides,

$$(x-5)^2+(-4)^2 = 25$$

or $$x^2-10x+25+16 = 25$$

or $$x^2-10x+16 = 0$$

or $$(x-2)(x-8) = 0$$

$\therefore$ $$x = 2, 8$$

Example 5. Prove that points $(-2, -1)$, $(-1, 1)$, $(5, -2)$ and $(4, -4)$ taken in order, are vertices of a rectangle.

Solution : Let given points are $P(-2, -1)$, $Q(-1, 1)$, R $(5, -2)$ and $S(4, -4)$

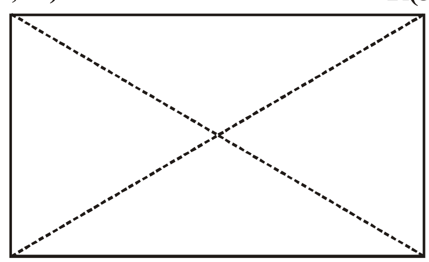


Fig. 9.06

$$PQ = \sqrt{[-2-(-1)]^2+[-1-1]^2} = \sqrt{(-1)^2+(-2)^2} = \sqrt{5}$$

$$QR = \sqrt{[5-(-1)]^2+[-2-1]^2} = \sqrt{(6)^2+(-3)^2} = \sqrt{45}$$

$$RS = \sqrt{[4-5]^2+[-4-(-2)]^2} = \sqrt{(-1)^2+(-2)^2} = \sqrt{5}$$

$$SP = \sqrt{[4-(-2)]^2+[-4-(-1)]^2} = \sqrt{(6)^2+(-3)^2} = \sqrt{45}$$

$\therefore$ $PQ = RS$ and $QR = SP$

So opposite sides are equal

Again, diagonal $PR = \sqrt{[5-(-2)]^2+[-2-(-1)]^2} = \sqrt{(7)^2+(-1)^2} = \sqrt{50}$

$$QS = \sqrt{[4-(-1)]^2+[-4-1)]^2} = \sqrt{(5)^2+(-5)^2} = \sqrt{50}$$

Thus, diagonals are equal. So given points are vertices of rectangle $PQRS$.

Example 6. If points (x, y) lies at equal distance form points $(a+b, b-a)$ and $(a-b, a+b)$ then prove that $bx = ay$.

Solution : Let $P(x, y)$, $Q(a+b, b-a)$ and $R(a-b, a+b)$ are given points, so according to question

$$PQ = PR$$

or $$PQ^2 = PR^2$$

or $\quad [x-(a+b)]^2+[y-(b-a)]^2=[x-(a-b)]^2+[y-(a+b)]^2$

or $\quad x^2-2(a+b)x+(a+b)^2+y^2-2(b-a)y+(b-a)^2$

$\quad =x^2-2(a-b)x+(a-b)^2+y^2-2(a+b)y+(a+b)^2$

or $\quad -2(a+b)x-2(b-a)y=-2(a-b)x-2(a+b)y$

or $\quad ax+bx+by-ay=ax-bx-ay-by$

or $\quad 2bx=2ay \Rightarrow bx=ay$

Exercise 9.1

1. Find the co-ordinates of points P, Q, R and S from given figure.

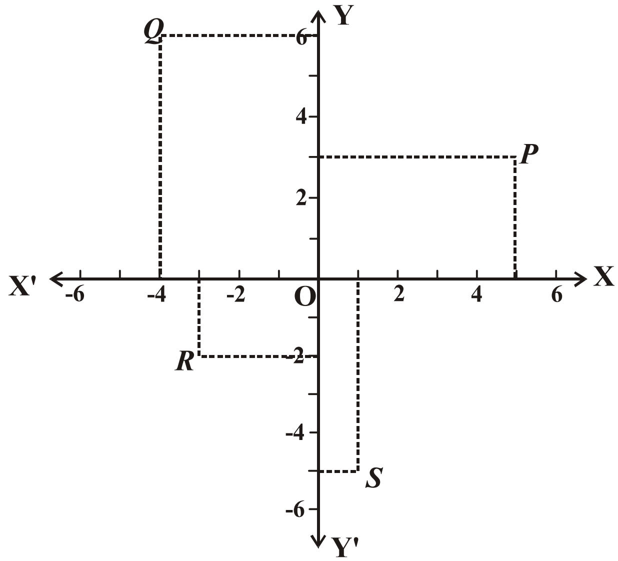


Fig. 9.07

2. Plot the points of the following co-ordinates.
   (1, 2), (–1, 3), (–2, –4), (3, –2), (2, 0), (0, 3)
3. By taking rectangular coordinate axis plot the points O (0,0), P (3, 0) and R(0, 4). If $OPQR$ is rectangle then find coordinates of Q.
4. Plot the points (–1, 0), (1, 0), (1, 1), (0, 2), (–1, 1). Which figure is obtained, by joining them serially?
5. Draw quadrilateral, if its vertices are following :
   (i) (1,1), (2, 4), (8, 4) and (10, 1)
   (ii) (–2, –2), (–4, 2), (–6, –2) and (–4, –6)
   Also, mention type of obtained quadrilateral.
6. Find the distance between the following points :
   (i) (–6, 7) and (–1, –5)
   (ii) (–1, –1) and (8, –2)
   (iii) $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$
7. Prove that the points (2, –2), (–2, 1) and (5, 2) are vertics of a right angled triangle.
8. Prove that points (1, –2), (3, 0), (1, 2) and (–1, 0) are vertices of a square.
9. Prove that points (a, a), $(-a, -a)$ and $(-\sqrt{3}a, \sqrt{3}a)$ are vertices of a equilateral triangle.

10. Prove that points (1, 1), (–2, 7) and (3, –3) are collinear.
11. Find that point on x-axis which is equidistant form points (–2, –5) and (2, –3).
12. Find that point on y-axis which is equidistant form points (–5, –2) and (3, 2).
13. If points (3, K) and (K, 5) are equidistant from a point (0, 2), then find the value of K.
14. If co-ordinates of P and Q are $(a\cos\theta, b\sin\theta)$ and $(-a\sin\theta, b\cos\theta)$ respectively, then show that

$OP^2 + OQ^2 = a^2 + b^2$, where O is origin.

15. If (0, 0) and $(3, \sqrt{3})$ are two vertices of an equilateral triangle then find third vertex.

9.05 Internal and external division of distance between two points

Let A and B are two points in plane. If point P lies in the middle of line AB then this type of division is called internal division.If point P is not in the middle of A and B, but it lies either left of A or right of B then such division is called external division.

(i) Internal division :

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points in plane and point $P(x, y)$ divides line segment AB in the ratio $m_1 : m_2$, internally. AL, PM and BN are perpendicular drawn form A, P and B on x-axis, respectively. Draw perpendicular AQ and PR form A to PM and from P to BN. Then

$OL = x_1,\ OM = x,\ ON = x_2$

$AL = y_1,\ PM = y$ and $BN = y_2$

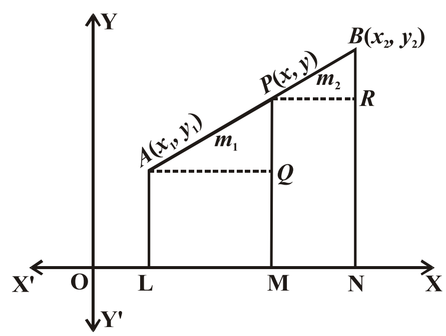


Fig. 9.8

$\therefore \quad AQ = LM = OM - OL = x - x_1$

$PR = MN = ON - OM = x_2 - x$

$PQ = PM - QM = PM - AL = y - y_1$

$BR = BN - RN = BN - PM = y_2 - y$

In Fig. 9.8, ΔAQP and ΔPRB are similar triangles.

$$\therefore \quad \frac{AP}{BP} = \frac{AQ}{PR} = \frac{PQ}{BR}$$

$$\text{or} \quad \frac{m_1}{m_2} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y_1}$$

$$\text{Now} \quad \frac{m_1}{m_2} = \frac{x - x_1}{x_2 - x}$$

$$\text{or} \quad m_1x_2 - m_1x = m_2x - m_2x_1$$

$$\text{or} \quad (m_1 + m_2)x = m_1x_2 + m_2x_1$$

$$\therefore \quad x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}$$

$$\text{Again} \quad \frac{m_1}{m_2} = \frac{y - y_1}{y_2 - y}$$

or $\quad m_1y_2 - m_1y = m_2y - m_2y_1$

or $\quad (m_1 + m_2)y = m_1y_2 + m_2y_1$

$$\therefore \quad y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$$

Thus, required coordinate of $P\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$

(ii) External division :

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ are points lie in plane. Point P externally divides line segment AB it the ratio $m_1 : m_2$. AL, BN and PM are perpendiculars drawn form A, B and P respectively. AQ and BR are perpendicular from point A and B on PM and BN respectively. Then $OL = x_1$, $ON = x_2$, $OM = x$, $AL = y_1$, $BN = y_2$ and $PM = y$

$\therefore \quad AQ = LM = OM - OL = x - x_1$

$BR = NM = OM - ON = x - x_2$

$PQ = PM - QM = PM - AL = y - y_1$

and $\quad PR = PM - RM = PM - BN = y - y_2$

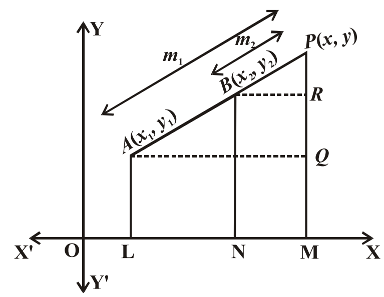


Fig. 9.9

In fig. 9.09, ΔAPQ and ? BPR are smilar triangles

$$\therefore \quad \frac{AP}{BP} = \frac{AQ}{BR} = \frac{PQ}{PR}$$

$$\text{or} \quad \frac{m_1}{m_2} = \frac{x - x_1}{x - x_2} = \frac{y - y_1}{y - y_2}$$

$$\text{Now} \quad \frac{m_1}{m_2} = \frac{x - x_1}{x - x_2}$$

or $\quad m_1x - m_1x_2 = m_2x - m_2x_1$

or $\quad (m_1 - m_2)x = m_1x_2 - m_2x_1$

$$\therefore \quad x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2}$$

$$\text{Again} \quad \frac{m_1}{m_2} = \frac{y - y_1}{y - y_2}$$

or $\quad m_1y - m_1y_2 = m_2y - m_2y_1$

or $\quad (m_1 - m_2)y = m_1y_2 - m_2y_1$

$$\therefore \quad y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$$

Thus, required coordinates of P

$$\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}\right)$$

Special case : If point P lie in the mid of line segment AB *i.e., P* divides AB in the ratio 1 : 1, then co-ordinates of P are $\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$

Note :

(i) Internal division formula is replaced by external division formula by putting –ve sign of m_1 or m_2.

(ii) In external division, if $|m_1| > |m_2|$ then division point is in right of B and if $|m_1| < |m_2|$ then, division point is in left of A.

(iii) If point $P(x_1, y_1)$ divides line segment AB in ratio λ : 1 then co-ordinates of P are $\left(\frac{x_1 + \lambda x_2}{1+\lambda}, \frac{y_1 + \lambda y_2}{1+\lambda}\right)$.

So co-ordinates of any point of the line joining the points (x_1, y_1) and (x_2, y_2) can be expressed as above.

Illustrative Examples

Example 1. Find the co-ordinates of that point which divides the line joining the points (–2, 1) and (5, 4) internally in the ratio2 : 3.

Solution : Let required point is (x, y), then by formula

$$x = \frac{2\times 5 + 3\times(-2)}{2+3} = \frac{10-6}{5} = \frac{4}{5}$$

and $$y = \frac{2\times 4 + 3\times 1}{2+3} = \frac{8+3}{5} = \frac{11}{5}$$

Therefore, co-ordinates of required point are $\left(\frac{4}{5}, \frac{11}{5}\right)$

Example 2. Find the co-oridinates of that point which externally divides the line joining the points (–4, 4) and (7, 2) in the ratio 4 : 7

Solution : Let co-ordinates of requiredpoint is (x, y), then

$$x = \frac{4\times 7 - 7\times(-4)}{4-7} = \frac{28+28}{-3} = -\frac{56}{3} = -18\frac{2}{3}$$

and $$y = \frac{4\times 2 - 7\times 4}{4-7} = \frac{8-28}{-3} = \frac{20}{3} = 6\frac{2}{3}$$

Thus, coordinates of required point are $\left(-18\frac{2}{3}, 6\frac{2}{3}\right)$

Example 3. In which ratio x–axis divides the line joining the points $A(3, -5)$ and $B(-4, 7)$?

Solution : Ordinate of each point on x–axis is zero.So, let point $P(x, 0)$ internally divides the given line segment in the ratio $m_1 : m_2$.

$$\therefore \qquad 0 = \frac{m_1 \times 7 + m_2 \times (-5)}{m_1 + m_2}$$

or $\qquad 7m_1 - 5m_2 = 0$

or $\qquad \frac{m_1}{m_2} = \frac{5}{7}$

Therefore, x-axis internally divides the line joiniung the given points in the ratio 5: 7.

Example 4. In which ratio, point (–2,3) divides the line joining the points (–3, 5) and (4, –9).

Solution : Let point (–2, 3) inetrnally divides the line joining the given points in the ratio λ : 1. So by internal division formula

$$-2 = \frac{\lambda \times 4 + 1 \times (-3)}{\lambda + 1}$$

or $\qquad -2 = \frac{4\lambda - 3}{\lambda + 1}$

or $\qquad -2\lambda - 2 = 4\lambda - 3$

or $\qquad 6\lambda = 1 \Rightarrow \lambda = \frac{1}{6}$

Thus, required ratio is 1 : 6

Note : By ordinate same ratio will be obtained.

Example 5. If point P (–1, 2) divides the line joining the points A(2, 5) and B in the ratio 3: 4 internally, then find co-ordinates of B.

Solution : Let co-ordinates of B are (x_1, y_1) and given that $AP : BP = 3 : 4$

By internal division formula

$$-1 = \frac{3 \times x_1 + 4 \times 2}{3 + 4} = \frac{3x_1 + 8}{7}$$

or $\qquad -7 = 3x_1 + 8 \Rightarrow x_1 = -\frac{15}{3} = -5$

and $\qquad 2 = \frac{3 \times y_1 + 4 \times 5}{3 + 4} = \frac{3y_1 + 20}{7}$

or $\qquad 14 = 3y_1 + 20$

$\Rightarrow \qquad y_1 = -\frac{6}{3} = -2$

Therefore, co-ordinates of B are (–5, –2)

Example 6. Find in which ratio line $x + y = 4$ divides the line joining the points (–1, 1) and (5, 7)?

Solution : Let given line divides line joining the points A(–1, 1) and B(5, 7) in the ratio λ : 1. So co-ordinates of P will be

$$\left(\frac{5\lambda-1}{\lambda+1},\frac{7\lambda+1}{\lambda+1}\right)$$

But point P lies on line $x + y = 4$

$\therefore$ $$\frac{5\lambda-1}{\lambda+1}+\frac{7\lambda+1}{\lambda+1}=4$$

or $$5\lambda-1+7\lambda+1=4\lambda+4$$

or $$8\lambda=4$$

$\Rightarrow$ $$\lambda=\frac{1}{2}$$

or $$\lambda:1=1:2$$

Exercise 9.2

1. Find the coordinates of the point which divides the line segment joining the points (3,5) and (7,9) in the ratio 2 : 3 internally.
2. Find the coordinates of the point which divides the line segment joining the points (5, –2) and $\left(-1\frac{1}{2},4\right)$ in the ratio 7 : 9 externally.
3. Prove that origin O divides the line joining the points $A(1, -3)$ and $B(-3, 9)$ in the ratio 1 : 3 internally. Find the coordinartes of the points which devides the line AB externally in the ratio 1 : 3.
4. Find the mid point of line joining the points (22, 20) and (0, 16).
5. In which ratio, x-axis divides the line segment which joins points (5, 3) and (–3, –2)?
6. In which ratio, y-axis divides the line segment which joins points (2, –3) and (5, 6)?
7. In which ratio, point (11, 15) divides the line segment which joins (15, 5) and (9, 20)?
8. If point P (3, 5) divides line segment which joins $A(-2, 3)$ and B (x,y) in the ratio 4 : 7 intermally, then find the co-ordinates of B.
9. Find the co-ordinates of point which trisects the line joining point (11, 9) and (1, 2).
10. Find the co-ordinates of point which quartersects the line joining point (–4, 0) and (0, 6).
11. Find the ratio in which line $3x + y = 9$ divides the line segment which joins points (1, 3) and (2, 7)
12. Find the ratio where point (–3, p), divides intermally the line segment which joins points (–5, –4) and (–2, 3). Also find p.

Miscellaneous Exercise-9

Objective Question [1 to 10]

1. Distance of point (3, 4) form y-axis will be :

   (a) 1 (b) 4 (c) 2 (d) 3
2. Distance of point (5, –2) from x-axis will be :

   (a) 5 (b) 2 (c) 3 (d) 4
3. Distance between points (0, 3) and (–2, 0) will be :

   (a) $\sqrt{14}$ (b) $\sqrt{15}$ (c) $\sqrt{13}$ (d) $\sqrt{5}$

4. Triangle having vertices (–2, 1), (2, –2) and (5, 2) is :

(a) Right triangle (b) Equilateral (c) Isoceles (d) None of these

5. Quadrilateral having vertices (–1, 1), (0, –3), (5, 2) and (4, 6) will be :

(a) Square (b) Rectangle (c) Rhombus (d) Parallelogram

6. Point equidistance from (0, 0), (2, 0) and (0, 2) is :

(a) (1, 2) (b) (2, 1) (c) (2, 2) (d) (1, 1)

7. *P* divides the line segment which joins points (5, 0) and (0, 4) in the ratio of 2 : 3 internally. Co-ordinates of *P* are :

(a) $\left(3, \frac{8}{5}\right)$ (b) $\left(1, \frac{4}{5}\right)$ (c) $\left(\frac{5}{2}, \frac{3}{4}\right)$ (d) $\left(2, \frac{12}{5}\right)$

8. If points (1, 2), (–1, *x*) and (2, 3) are collinear, then *x* will be :

(a) 2 (b) 0 (c) −1 (d) 1

9. If distance between point (3, *a*) and (4, 1) is $\sqrt{10}$, then *a* will be :

(a) 3, −1 (b) 2, −2 (c) 4, −2 (d) 5, −3

10. If point (x, y) is at equidistant from (2, 1) and (1, –2), then the true statement is :

(a) $x+3y=0$ (b) $3x+y=0$ (c) $x+2y=0$ (d) $2y+3x=0$

11. Find the type of quadrilateral, if its vertices are (1, 4), (–5, 4), (–5, –3) and (1,–3).

12. Which shape will be formed on joining (–2, 0), (2, 0), (2, 2), (0, 4), (–2, 2) in the given order?

13. Find the ratio in which point (3, 4) divides the line segment which joins points (1, 2) and (6, 7).

14. Opposite vetrices of any square are (5, –4) and (–3, 2), then find the length of diagonal.

15. If co-ordinate of one end and mid point of a line segment are (4, 0) and (4, 1) respectively, then find the co-ordinates of other end of line segment.

16. Find the distance between of point (1, 2) from mid point of line segment which joints the points (6, 8) and (2, 4).

17. If in any plane, there are four points *P*(2, –1), *Q*(3, 4), *R*(–2, 3) and *S*(–3, –2), then prove that *PQRS* is not a square but a rhombus.

18. Prove that mid point (*C*) of hypotaneous in a right angled triangle *AOB* is situated at equal distance form vertics *O*, *A* and *B* of triangle.

19. Find the length of median of a triangle whose vertices are (1,–1), (0, 4) and (–5, 3).

20. Prove that mid point of a line segment which joins points (5, 7) and (3, 9) is the same as mid point of line segment which joins points (5,7) and (3, 9).

21. If mid points of sides of a triangle is (1, 2), (0, –1) and (2, –1), then find its vertices.

Important Points

1. Formula of distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

or

$$PQ = \sqrt{(\text{difference of abscissas})^2 + (\text{difference of ordinates})^2}$$

2. The coordinates of the point $P(x,y)$ which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $m_1 : m_2$ are

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}$$

and

$$y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$$

3. The coordinates of the point P (x, y) which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ externally in the ratio $m_1 : m_2$ are

$$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2}$$

and

$$y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$$

4. The mid point of the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$$

or

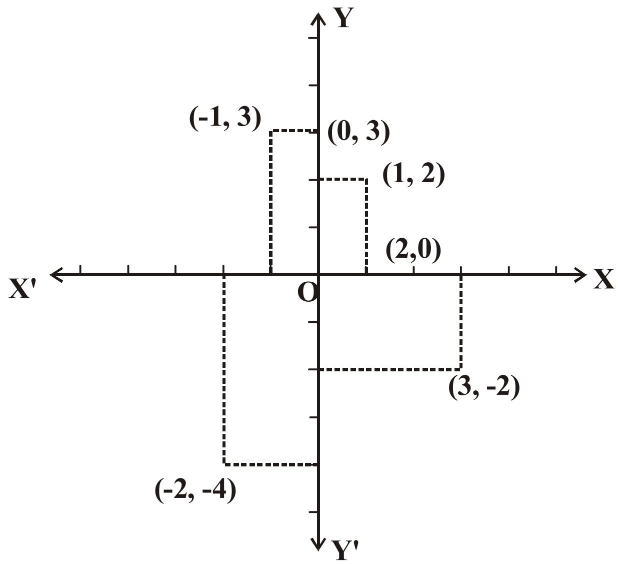
$$\left(\frac{\text{Sum of } x \text{ co-ordinates}}{2}, \frac{\text{Sum of } y \text{ co-ordinates}}{2}\right)$$

Answer Sheet

Exercise 9.1

1. $P(5,3), Q(-4,6), R(-3,-2), S(1,-5)$

2.



3. (3, 4) 4. Pentagon 5. (i) Trapezium (ii) Rhombus

6. (i) 13 (ii) $\sqrt{82}$ (iii) $a(t_2 - t_1)\sqrt{(t_2 + t_1)^2 + 4}$ 11. (–2, 0) 12. (0, –2)

13. 1 15. $(0, 2\sqrt{3})$ or $(3, -\sqrt{3})$

Exercise 9.2

1. $\left(\frac{23}{5}, \frac{33}{5}\right)$ 2. $\left(27\frac{3}{4}, -23\right)$ 3. (3, –9) 4. (11, 18)

5. 3 : 2 6. 2 :5 external division 7. 2 :1 8. $\left(\frac{47}{4}, \frac{17}{2}\right)$

9. $\left(\frac{13}{3}, \frac{13}{3}\right), \left(\frac{23}{3}, \frac{20}{3}\right)$ 10. $\left(-3, \frac{3}{2}\right), (-2, 3), \left(-1, \frac{9}{2}\right)$ 11. 3 : 4 12. $2:1, p = \frac{2}{3}$

Miscellaneous Exercise-9

1. (d) 2. (b) 3. (c) 4. (a) 5. (d) 6. (d) 7. (a)
8. (b) 9. (c) 10. (a) 11. Rectangle 12. Pentagon 13. 2 :3. 14. 10

15. (4, 2) 16. 5 19. $\frac{\sqrt{130}}{2}, \frac{\sqrt{130}}{2}, \sqrt{13}$ 21. (1, –4), (3, 2), (–1, 2)

10

Locus

10.01. Introduction

As certained, you have seen many unique scenes around you in your life many times. All certain have you ever seen some birds flying in the sky in a rare shape? Or have you ever seen an army of ants moving on the wall or on the floor in special or unusual shape? There is a particular condition which is in their nature in the motion of these tiny creatures. All of them follow these rules accordingly their nature and living conditions. If every creature of the two incident considered a point,then the unique figure made by them is a set of many points who obey these rules. In fact in geometrical figures, the set of the points, with a determination by one or more specified condition from the figures. i.e., for a particular figure, a set of all the necessary points in the space is called the locus.

10.02. Defination

Locus is a special set of the points which satisfies the certain given conditions. To understand it, we shall take some example :

(*a*) Let O be the fixed point in a plane and r be a positive integer. All points in the plane which are at a distance of 'r' units from 'O' describe a locus. This locus is a circle with centre O and radius r (Fig. 10.01)

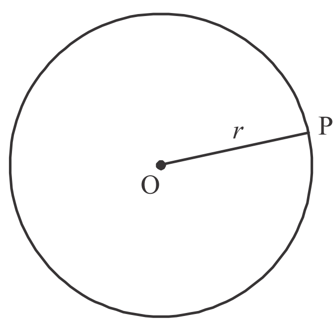


Fig. 10.01

(*b*) Take two parallel lnes l and m. Consider all those points which are equidistant from l and m. A figure of a line n is forward by all these points and the obtained is equidistant from l and m. see (Fig. 10.02).

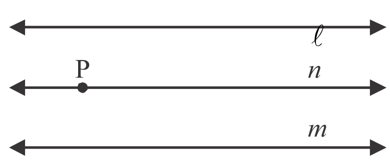


Fig. 10.02

(*c*) Let l be a line and d be a positive real number. consider all of the points which are at the d distance away form l. In this way two lines m and n parallel to l are obtained which are at d distant from l (Fig.10.03)

It is considerable that all three above situations the points follow the certain condition/conditions.

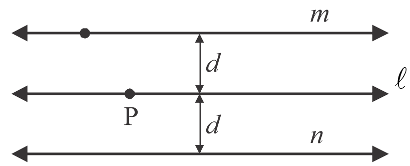


Fig. 10.03

Thus the loucs of the given points is the set of these points which satisfy the certain condition/conditions. Remember in this defination there are two complimentary statements.

(i) The points which satisfies the given conditions is the point of the locus.

(ii) Every point of the locus has to satisfy the given conditions.

In this way the locus and its deciding points may be considered same. When one of two is described, other automatically cleared.

Let us study two important locii, which are very useful in other theorems and in some geometrical constructions.

10.03. Locus of points equidistant from two given points

Let A and B be two given points. Consider path of the point P which satisfies the condition $AP = BP$.

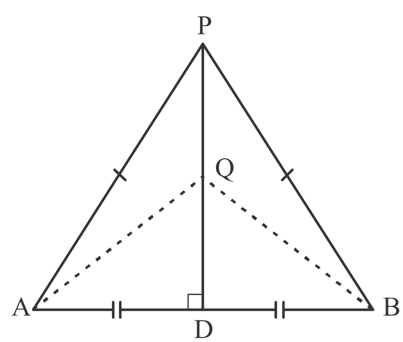


Fig. 10.04

If D is the mid point of AB, then AD=DB therefore D is also situated at the locus. Let besides D, P is another point such that AP = BP. Now is P is joined to D, then AP and BP become the sides of two triangles ADP and BDP respectively. What can we say about these two triangles? We observe that these follow the SSS congurency theorem. So, $\Delta ADP \cong \Delta BDP$ in which $\angle ADP = \angle BDP$, that $\angle ADP = 90^\circ$ or $PD \perp AB$ can be proved easily. So, is the perpendicular bisector of AB. PD is also a straight line. Since, P is equidistant from A and B. If another point Q is also taken at PD, it will also be proved equidistant from A and B. Similarly, all the points on are equidistant form A and B. So, above description gives the following results.

Theorem 10.1.

The locus of a point equidistant form two given points is the perpendicular bisector of the line segment joining the two points.

Theorem 10.2. (Converse of theorem : 10.1)

Every point on the perpendicular bisector of line joining the two points is equidistant form these points.

Example 1 : ΔPBC, ΔQBC and ΔRBC three isoceles triangles are formed at the same base BC.Prove that the points P, Q and R are collinear.

Solution : Given : The $\Delta PBC, \Delta QBC$ and ΔRBC are such that PB = PC, QB = QC, RB = RC.

Proof: ΔPBC is an isosceles triangle

∴ $PB = PC$

And the locus of the point equidistant from B and C is the perpendicular bisector of B. Let it is l.

∴ P lies on l . . . (1)

Similarly Q and R also lie on l . . . (2)

From 1, and 2 we get

Hence, points P, Q and R collinear.

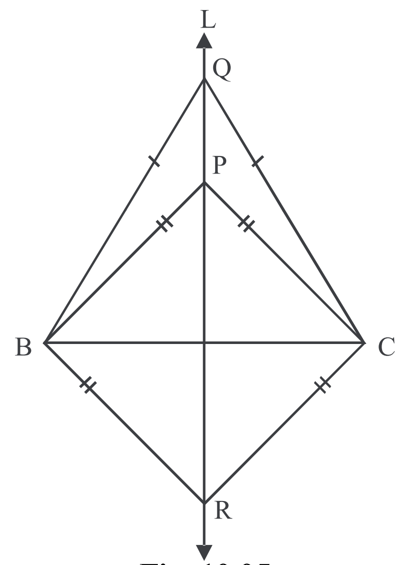


Fig. 10.05

10.04. Locus of points equidistant from two intersecting lines

Let two straight lines m and l intersect each other at O. And we are to find the locus equidistant from line l and m. If a point P which is not on l and m and its distance from p will be equal to the length of perpendicular drawn on l and m from P. At the other hand. If P lies on the sides l and m both then its distance from these lines will be zero.

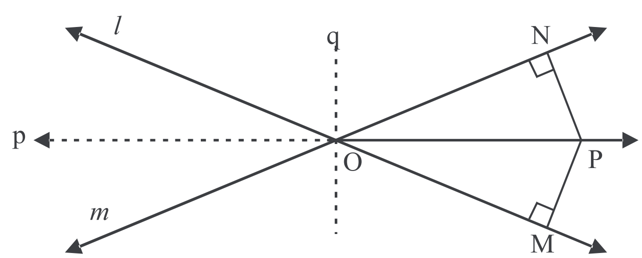


Fig. 10.06

If $d = 0$ then P is on l and m both *i.e.,* P will be coincidental to O and will be on the locus.

If d ? 0 then P will be neither on l nor on m. So it will be some where in four the angle formed by l and m.

Now, if $PM \perp \ell$ and $PN \perp m$, then

$PM = PN = d$ (by given conditions)

In ΔOPM and ΔOPN

$\angle M = \angle N$ ($90°$ each)

$OP = OP$ (Common)

$PM = PN (PM = PN = d)$

$\therefore$ $\Delta OPM \cong \Delta OPN$ (right angle, hypontaneous, side)

$\therefore$ $\angle POM = \angle PON$

Clearly, P is located inside $\angle MON$ andOP is the bisector of $\angle MON$ or P is on the bisector of $\angle MON$. Similarly, the point P may lie on the bisector of other three angles. The bisectors of these four angles form a straight lines. Let these lines be p and q. P will be an eliments of all points located on p and q. In this way we can say lines p and q are the locus of point P.

Hence, we obtain the following result form above explantion.

Theorem 10.2

The locus of a point equidistant from two intersecting lines is the pair of bisectors of the angles at that lines.

Example 2: The bisectors of $\angle B$ and $\angle C$ of a quadrilateral $ABCD$ meet each other at P. Prove that point P is equidistant form opposite sides AB and CD.

Solution :

Given : In quadrilateral $ABCD$ in which bisectors of $\angle B$ and $\angle C$ meet at P.

Also $PM \perp AB$ and $PN \perp CD$

To prove : $PM=PN$.

Construction : Draw $PL \perp BC$.

Proof : Point P lies on the bisector of $\angle B$

$\therefore$ $PM = PL$ . . . (1)

Point P lies on the bisector of $\angle C$

$\therefore$ $PL = PN$ . . . (2)

from (i) and (ii) we get

PM=PN **Proved.**

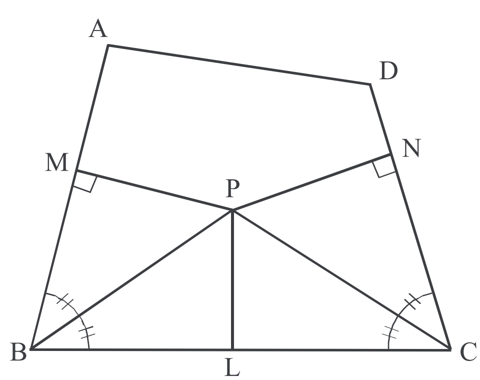


Fig. 10.07

Exercise 10.1

1. State whether following statements are true or false and justify your answer.
   (i) The set of the points equidistant form the given line is also a line.
   (ii) A circle is a locus of the points which are equidistant form a certain point.
   (iii) Three points are collinear only when they are not the elements of the set of the points of a line.
   (vi) The locus of the points equidistant from two lines will be a parallel line.
   (v) The locus of a point equidistant from two given points, the perpendicular bisector of the line joining the two points.
2. The diagonals of a quadrilateral bisect each other. Prove that the given quadrilateral is a parallelogram.
3. What will be the locus of a point equiditant from three non-collinear points A, B and C ? Justify your answer.
4. What will the locus of a points, equidistant form three collinear points? Justify your answer.
5. Prove that the locus of the centres of the circles passing through the points A and B is the bisector of line segment AB.
6. In the fig. 10.08. ΔPBC and ΔQBC are situated on the opposite sides of common base BC. Prove that the line joining P and Q. is a perpendicular bisector of BC.

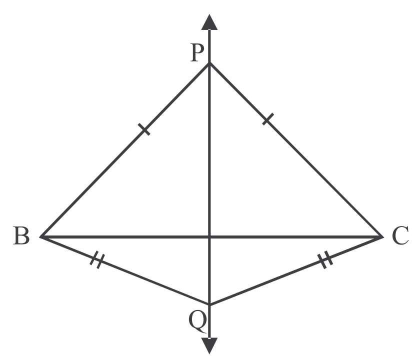


Fig. 10.08

7. In the fig.10.09, ΔPQR and ΔSQR are on the same side of common base QR. Prove that line SP is the perpendicular bisector of QR.

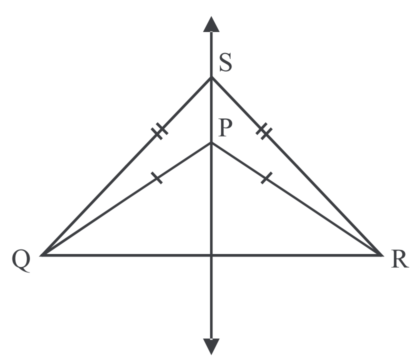


Fig. 10.09

8. In the given fig. 10.10 PS is the bisector of the angle $\angle P$ intersect QR at S. $SN \perp PQ$ and $SM \perp PR$ are drawn. Prove that $SN = SM$.

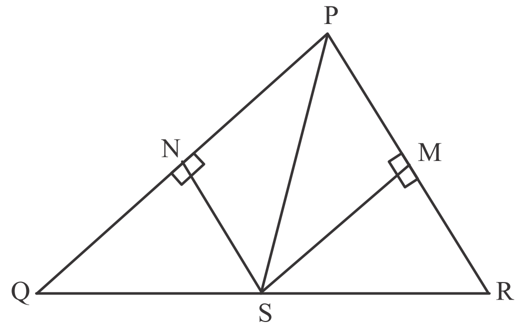


Fig. 10.10

9. In the fig. 10.11, find the locus of the points inside the $\angle ABC$ and equidistant form two sides BA and BC.

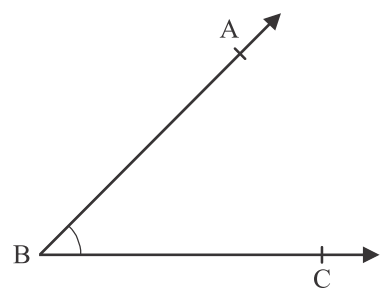


Fig. 10.11

10.5. Concurrent lines

In the previous class you have studied some terms related to triangles, which will be used in this section of the chapter. It is necesary to remind them.

1. **Median:** The line joining the vertex and the mid point of the opposite side is called the median of triangle.
2. **Perpendicular bisector:** Perpendicular drawn at the mid-point of the line segment is called the perpendicular bisector.
3. **Angle bisector:** Angle bisector is line which divides an angle into two equal parts.
4. **Altitude:** The line-segment which is obtained by drawing the perpendicular form a vertex of a triangle to its opposite side.
5. **Concurrent lines:** When three or more straight lines pass through a same point, they are called the concurrent lines. In this condition the common point of these lines is called the (point of concurrency). Let us discuss the points of cocurrency of above lines with which we get some certain results. That can be proved through the following theorems. All of these results are very useful in geometry.

Theorem 10.3

Perpendicular bisectors of the sides of a triangle are concurrent.

Given : In ΔABC, the perpendicular bisectors of AB and AC meet at O and $OD \perp BC$

To prove : OD is the perpendicular bisector of BC

Construction : Join OA, OB and OC

Proof :

OE and OF are the respectively perpendicular bisectors.

$\therefore$ $OA = OB = OC$ (from converse of 10.1)

$\because$ $OD \perp BC$

$\therefore$ OB = OC (By theorem 10.1)

OD is perpendicular bisector of BC

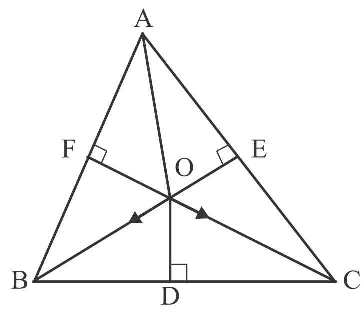


Fig. 10.12

Circumcentre : The perpendicular bisectors of three side of a triangle intersect each other at the same point, this point of intersection is called circumcentre of the triangle.

Theorem 10.4

Bisectors of three angles of a triangle are concurrent.

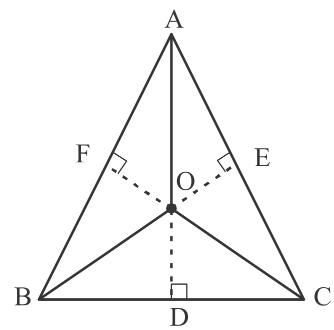


Fig. 10.13

Given : In ΔABC, bisector of $\angle B$ and $\angle C$ meet at O.

To prove : OA bisects $\angle A$

Construction : From O, draw $OD \perp BC, OE \perp AC$

and $OF \perp AB$

Proof : OB bisects $\angle B$ and OC bisects $\angle C$

$\therefore$ $OD = OE$ . . . (i) (from 10.2)

and $OD = OF$ . . . (ii)

from (i) and (ii) $OE = OF$

Thus O is equidistant form AB and AC.

Therefore OA bisects $\angle A$ **Proved.**

Circumcenter : The perpendicular bisectors of three side of a triangle intersect each other at the same point, this point of intersection is called circumcenter of the triangle.

Theorem 10.5

Three altitudes of a triangle are concurrent.

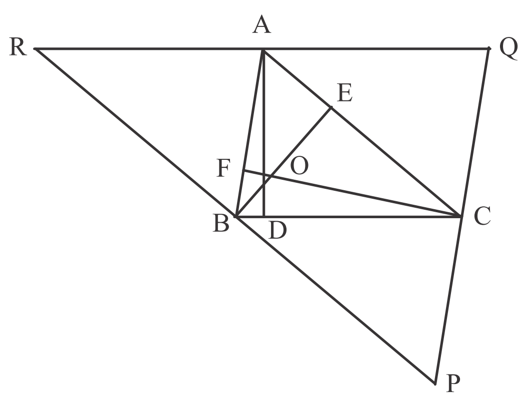


Fig. 10.14

Given : In ΔABC AD, CF and BE are vertex altitudes.

To prove : AD, CF and BE pass through a certain point.

Construction : From each vertex draw $QP \parallel AB, RQ \parallel BC$ and $RP \parallel AC$ (see fig. 10.14)

Proof : In BCAR, $AC \parallel RB$ and $BC \parallel RA$ (By construction)

$\therefore$ BCAR is a parallelogram.

$\therefore$ $RA = BC$ . . . (i)

Similarly ABCQ is a parallelogram.

$\therefore$ $AQ = BC$ . . . (ii)

From (i) and (ii) $AR = AQ$ . . . (iii)

And $AD \perp BC$, also $BC \parallel QR$

$\therefore$ $AD \perp QR$ . . . (iv)

AD is perpendicular bisector of QR similarly BE and CF are respectively perpendicualr bisectors of PR and PQ.

Since perpendicular bisectors of sides of the triangles are concurrent

$\therefore$ AD,CF and BE pass through a fixed point.

Ortho centre : The concurrent point of the altitudes of a triangles is called ortho centre.

Theorem 10.6

The medians of a triangle pass through a fixed point which divides every median in 2 : 1.

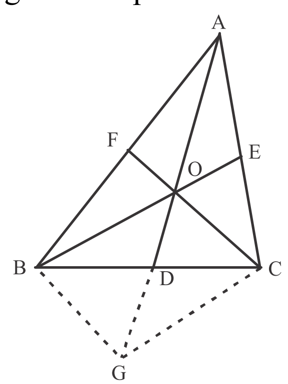


Fig. 10.15

Given : The medians BE and CF of a ΔABC intersect each other at O.

To prove : (i) The line AO is produced which bisects BC at D. *i.e.*, $BD = DC$.

(ii) $AO : OD = BO : OE = CO : OF = 2 : 1$

Construction : produce AD to G such that we get

$AO = OG$. Join BG and CG.

Proof : We know that the line joining of mid points of two sides in a triangle is parallel and half of the third side.

$\therefore$ In ΔABG, F mid point of AB, (given)

And O is the mid point of AG. (by constructiuon)

$\therefore$ $OF \quad BG$ and $CO \quad BG$ (since CO and OF are the part of CF) . . . (i)

and $OF = \frac{1}{2} BG$ . . . (ii)

Similarly in ΔACG , E and O are respectively mid points of AC and AG ,

So, $OE \quad GC$ and $BO \quad GC$ (Since, BO and OE are parts of BE) . . . (iii) and $\quad OE = \frac{1}{2}GC$ . . . (iv)

From (i) and (iii) BOCG is a parallelogram. Since, diagonals of a parallelogram bisect each other.

$\therefore$ BD=DC

Hence, the line AD form the vertex A is also a median of ΔABC.

(ii) Since D is the point of intersection of diagonal of the parallelogram BOCG

$\therefore \quad OD = DG$ . . . (v)

and $\quad OD = \frac{1}{2}OG$ . . . (vi)

Also AO = OG (by construction)

from (v) and (vi)

$$OD = \frac{1}{2}AO$$

or $\quad \frac{AO}{OD} = \frac{2}{1} \Rightarrow AO : OD = 2 : 1$

Similarly, we can prove that

BO : OE = 2 : 1 and CO : OF = 2 : 1

Hence AO : OD = BO : OE = CO : OF = 2 : 1 **Hence Proved.**

Illustrative Examples

Example 1 : The medians AD, BE and CF of a triangle ABC pass through the point G. If $AG = 6$ cm, BE = 12.6 cm and FG = 3cm then find the length of AD, GE and GC.

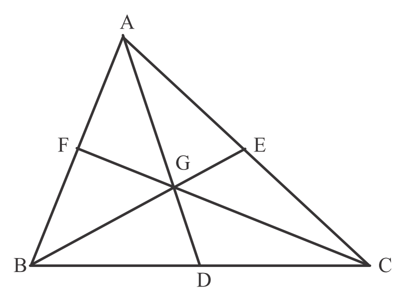


Fig. 10.16

Solution : We know that centroid of a triangle divides its medians in the ratio of 2 : 1

$$\therefore \quad \frac{AG}{GD} = \frac{2}{1} \quad \Rightarrow \quad \frac{GD}{AG} = \frac{1}{2}$$

Adding 1 to both sides

$$\frac{GD}{AG} + 1 = \frac{1}{2} + 1 \Rightarrow \quad \frac{GD + AG}{AG} = \frac{1+2}{2}$$

$$\Rightarrow \quad \frac{AD}{AG} = \frac{3}{2} \quad \Rightarrow \quad \frac{AD}{6} = \frac{3}{2}$$

$\Rightarrow \quad AD = \frac{3}{2} \times 6 \qquad \Rightarrow \quad AD = 9$ cm.

Similarly, $\frac{BG}{GE} = \frac{2}{1}$ or $\frac{BG}{GE} + 1 = \frac{2}{1} + 1$

or $\frac{BG + GE}{GE} = \frac{2+1}{1}$ or $\frac{BE}{GE} = \frac{3}{1}$

or $GE = \frac{1}{3} BE$ or $GE = \frac{12.6}{3}$

or $GE = 4.2$ and $\frac{FG}{GC} = \frac{1}{2}$

or $2FG = GC$ or $GC = 2 \times 3 = 6$

Hence $AD = 9$ cm, $GE = 4.2$ cm and $GC = 6$ cm.

Example 2. If the medians of a triangle are equal, then it is equilateral.

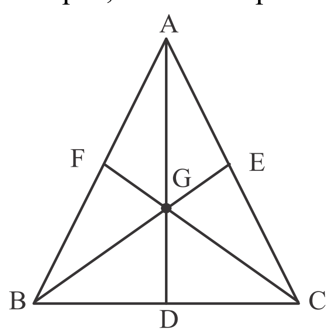


Fig. 10.17

Solution :

Given : In ΔABC, medians *AD, BE* and *CF* meet at *G* and $AD = BE = CF$.

To Prove : ΔABC is equilateral triangle.

Proof : We know that the centroid divides the medians of a triangle in 2 : 1.

So, $AD = BE = CF$ (given)

$\therefore \quad \frac{2}{3} AD = \frac{2}{3} BE = \frac{2}{3} CF$

Or $AG = BG = CG$ . . . (i)

Similarly, $\frac{1}{3} AD = \frac{1}{3} BE = \frac{1}{3} CF$

or $GD = GE = GF$ . . . (ii)

Now in ΔBGF and ΔCGE

BG = CG (from (i))

GF = GE (form (ii))

and $\angle BGF = \angle CGE$ (vertically opposite angle)

$\therefore \quad \Delta BGF \cong \Delta CGE$ (side, angle side congruency)

And corresponding sides of two congruent triangles are equal.

$\therefore \quad BF = CE$

And $\quad 2BF = 2CE$

$\Rightarrow \quad AB = AC$ . . . (iii)

Similarly, $\quad \Delta CGD \cong \Delta AGF$

So $\quad BC = AB$

from (iii) and (iv) . . . (iv)

$AD = BC = CF$

Hence ΔABC is a equilateral triangle. **Proved.**

Example 3 : If two medians of a triangle are equal in length, the triangle is an isosceles triangle.

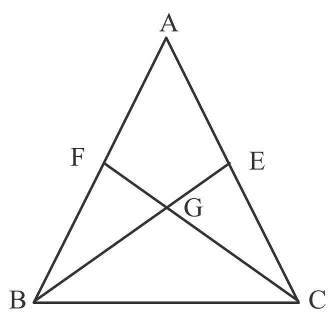


Fig. 10.18

Given : In ΔABC, $BE = CF$

and F and E are the mid-points of AB and AC

To prove : ΔABC is an isosceles triangle

Proof : In ΔABC, G is the centroid.

$\therefore \quad BG : GE = CG : GF = 2:1$ (given)

$\therefore \quad BG = \frac{2}{3} BE$ . . . (i)

$GE = \frac{1}{3} BE$ . . . (ii)

and $\quad CG = \frac{2}{3} CF$ . . . (iii)

$GF = \frac{1}{3} CF$ . . . (iv)

but $\quad BE = CF$ (given)

From (i) and (iii)

$BG = CG$

and from (ii) and (iv)

$GE = GF$

Now in ΔBGF and ΔCGE

BG = CG (Proved)

GE = GF (Proved)

$\angle BGF = \angle CGE$ (Vertically opposite angles)

∴ $\Delta FBG \cong \Delta ECG$ (SAS congurency)

Thus BF = CE or $2BF = 2CE$

∴ $AB = AC$

Hence *ABC* is an isosceles triangle **Proved.**

Exercise 10.2

1. Find the locus of a point equidistant form the three vertices and also find the locus of a pointed equidistant from the sides of a triangle.
2. In the ΔABC, medians *AD, BE* and *CF* intersect each other at G. If *AG* = 6 cm, *BE* = 9 cm, and *GF* = 4.5 cm, find the length of *GD*, *BG*, and *CF*.
3. In a triangle medians *AD, BE* and *CF* interesect each other at *G* prove that $AD + BE > \frac{3}{2}AB$.

   [Hint $AG + BG > AB$]
4. In a triangle ABC, the sum of two medians is greater that third one.
5. In a *ΔABC*, medians *AD*, *BE*, and *CF* passses through the point G.

   $4(AD + BE + CF) > 3(AB + BC + CA)$
6. In a *ΔABC, P* is the orthocentre. Prove that orthocentre of *ΔPBC* is *A*.
7. In *ΔABC*, the medians *AD*, *BE* and *CF* passses through the point *G*.
   (a) If *GF* = 4 cm then, find *GC*.
   (b) If *AD*=7.5 cm, then find *GD*.
8. In an isosceles *ΔABC, AB = AC,* and *D* is the mid-point of BC. Prove that circum centre, in centre, ortho centre and centroid all are collinear.
9. In a triangle *ABC, H* is the orthocentre. *X, Y* and *Z* are mid-points of *AH*, *BH* and *CH* respectively prove that orthocentre of *ΔXYZ* is *H*.
10. How will you find a point *P* inside *BC* of *ΔABC* which is equidistant form *AB* and *AC*.

Miscellaneous Exercise 10

Objective Questions (from 1 to 7)

1. The point equidistant from the vertices of triangle is called :
   (a) Centre of gravitation (b) Cicrcumcentre (c) Orthocentre (d) Incentre
2. Gravitation centre of a triangle :
   (a) Point of concurrency of prepandicular bisectors of the sides of the triangle .
   (b) Concurrent point of angles bisectors of the triangle.
   (c) Point of concurrency of medians of the triangle.
   (d) The orthocentre.

3. Locus of the centre of rolling circle in a plane is :

   (a) Circle (b) Curve

   (c) a line parallel to the plane (d) perpendicular to the plane.

4. If two medians of a triangle are equal, then the triangle is :

   (a) right triangle (b) isoceles triangle (c) equilateral triangle (d) none of these.

5. If *AB* and *CD* are two non-parallel lines, then the locus of the point *P* equidistant form these lines will be

   (a) The line parallel to AB, and passing through the point P.

   (b) The bisector of the angle subtended by lines *AB* and *CD* passing through point *P*.

   (c) A parallel line to AB and CD both and passing through point P.

   (d) The altitiude drawn on the sides *AB* and *CD* both and passsing through point *P*.

6. The triangle whose orthocentre, circumcentre and incentre coincides is known as

   (a) equilateral triangle (b) right triangle (c) isosceles triangle (d) none of these

7. The triangle whose ortho centre is its vertex point is called.

   (a) Right triangle (b) equilateral triangle (c) isosceles triangle (d) none of these

8. Find the locus of an end of a pendulam of the clock.

9. In a triangle *ABC, D,E* and *F* are the mid-points of side *BC, CA* and respectively then prove that *EF* bisects *AD*.

Important Points

1. Under some conditions the locus of a point is such a geometrical figure whose all points satisfy the given conditions.
2. The locus of a point equidistant form two points is the perpendicular bisector of line joining two given points.
3. The locus of the points equidistant form two inntersecting lines is the bisector of the angle formed by two given lines.
4. The perpendicluar bisectors of the sides are concurrent and this point of concurrency is called the circumcentre of triangle.
5. The angle bisector of a triangle are concurrent and this point of concurrency is called the incentre of triangle.
6. The altitudes of a triangle are concurrent and this point of concurrency is called the orthocentre of triangle.
7. Three medians of a triangle are concurrent this point of concurrency divides the median in to the ratio of 2 : 1 and called the orthocentre.

Answer Sheet

Exercise 10.1

(i) False, The locus of the points equidistant from the given line is the parallels either side this line.

(ii) True

(iii) False, three points may colinear only when they are the elements of the set of points which lies on that line.

(iv) False, it depends on the location of two lines (If two given lines are parallel, then it will parallel to them and if they intrersect each other then it will on the bisector of angle formed by them)

(v) True.

Exercise 10.2

1. Circumcentre, incentre 2. 3 cm, 6 cm, 13.5 cm, 7.8 cm, 2.5 cm

Miscellaneous Exercise 10

1. b 2. c 3. c 4. b 5. b 6. c 7. a

8. an arc

11

Similarity

11.01. Introduction

Do you ever think how the distance of the moon, height of the Gaurishankar peak (mount Everest), Guru shikhar. (The hightest peak of mount Abu) has been measured? Will they measured by using measuring tape? Infact to measure all these distances and heights the concept of measurement was used. This indirect concept of measurment is based on the principle of similarity of figures. In this chapter we shall study about the similarity, specially similarity of triangles in detail.

11.02. Similar Figures

Remember, in class IX you have discussed the figures (congurent figures) of same size and of same measurment figures under which, you have seen that all the circles of same radius are congurent, all the squares of same side are congurent.Similarly all equilateral triangles of same length are congurent. Let us consider the following figures.

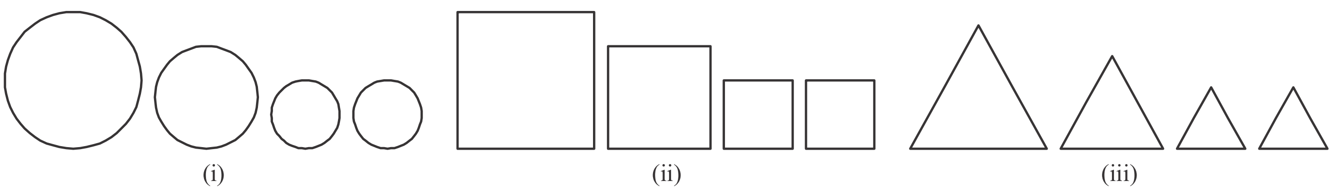


Fig. 11.1

Take two or more circles from the above fig. 11.1 (i). Are they congurent? Since all these do not have same radius, so they are not congurent. consider, Some of these are congurent while other are not. But all of them have same shape. So all of these figures are called similar. Two similar figures have the same shape but it is not essential having same size or measures. So all the circles are similar. Similarly in the fig. 11.1 (ii), (iii) all the squares and all the equilateral triangles can be called like similar like the circles mentioned above. At the bases of above explanation necessarily. We can say all the congurent figures are similar but all similar figures are not congurent.

Now, see the above figures 11.1 (i), (ii), (iii) and tell whether a circle and a square similar or a square and an equilateral triangle similar to each other? Definitely your answer will be negative as the shapes of these figures are not same. Now, what do you think about the figures of pentagons shown in the fig. 12.2? Are these similar? Although these two figures look similar but their similarity is doubtfull

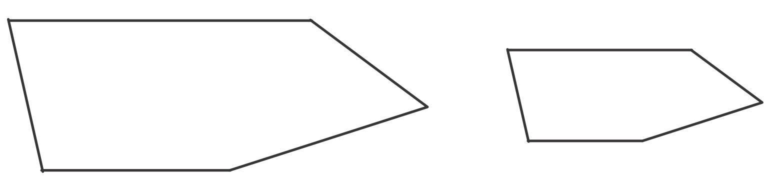


Fig. 12.2

Now see the fig. 11.3.

Fig. 11.3

In these figures, there are three pictures of great Indian mathematician Srinivas Ramanujan (22 December 1887-18 April 1920) in different size. Are all these pictures similar? Undoubtfully these are similar figures. Can you state after seeing these pictures why were you doubtfull about their similarity? So let's find a defination of smilarity so that we can decide whether the given figures are similar or not.

You will make the photocopy of Your documents *i.e.,* mark-sheet, birth certificate etc.

Similarly you would also have photographs your photo, stamp size, passport size or postcard size. All the photographs clicked at the same time are similar even they may be different in size. Draw a figure on a white paper and make it enlarged. Now you have two figures. Measure the sides and angles of these figures with the help of scale and protector and write them with names. See the fig. 11.4 given below.

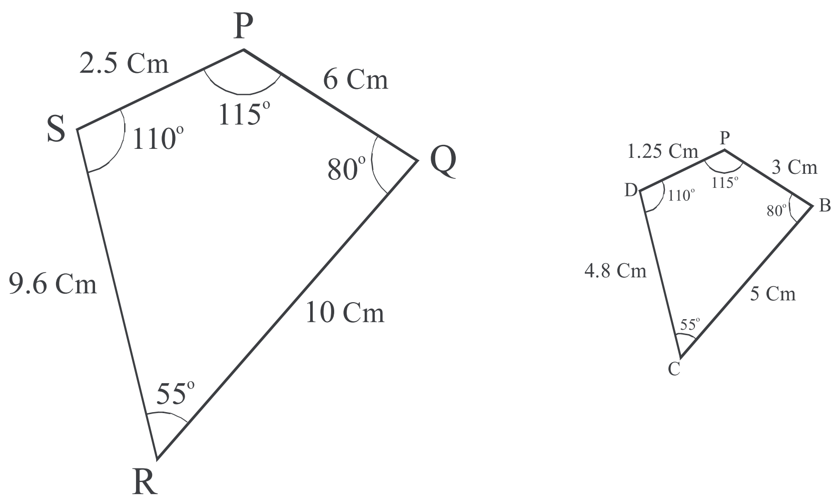


Fig. 11.4

Now compare the corresponding sides and angles of these two figures. You will get that the sides of larger figures have become doubled in the comparison of sides of smaller figure. In other words we can say that the ratio of sides of larger figure and that of smaller one is 2 : 1. In the same way corresponding angles of two figures are equal. There two results are considered the conclusion of similarity .Thus the conditions for the similarity of two polygons with equal sides are (i) The corresoponding angles are equal and (ii) The ratio of corresponding sides is equal.

In the fig. 11.4, there are two quadrilateral *ABCD* and *PQRS*, where vertex *A* is corresponding to vertex *P*, vertex *B* is corresponding to vertex *Q*, vertex *C* is corresponding to vertex *R* as well as vertex *D* is corresponding to vertex *S*. In short these correspondings can be expressed as $A \leftrightarrow P, B \leftrightarrow Q, C \leftrightarrow R$ and $D \leftrightarrow S$. So

(i) $\angle A = \angle P, \angle B = \angle Q, \angle C = \angle R$ and $\angle D = \angle S$.

(ii) $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CD}{RS} = \frac{DA}{SP} = \frac{1}{2}$

Hence quadrilateral *ABCD* is similar to quadrilateral *PQRS*, For the pentagons *ABCDE* and *PQRST* see fig. 11.5. we get.

(i) $\angle A = \angle P, \angle B = \angle Q, \angle C = \angle R, \angle D = \angle S$ and $\angle E = \angle T$

(ii) $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CD}{RS} = \frac{DE}{ST} = \frac{EA}{TP} = \frac{1}{2}$

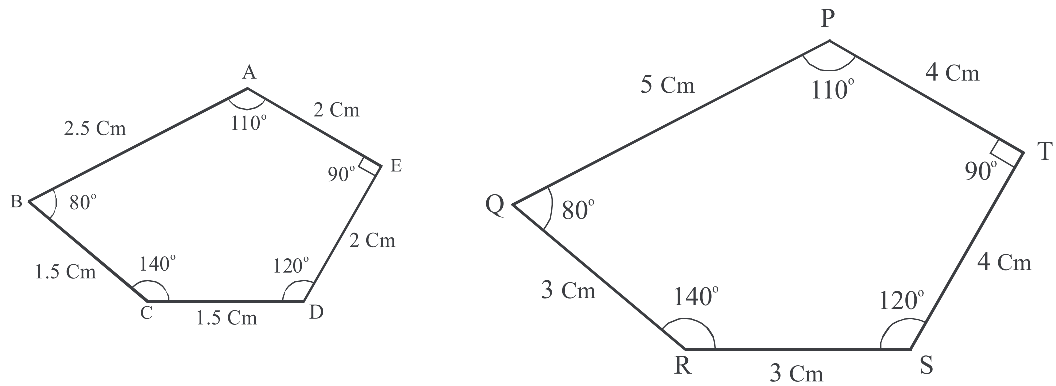


Fig. 11.5

Hence pentagon ABCDE is similar to pentagon PQRST

In the fig. 11.6 (i) corresponding angles of the square and rectangle are same but their sides are not in the same ratio. So they are not similar.

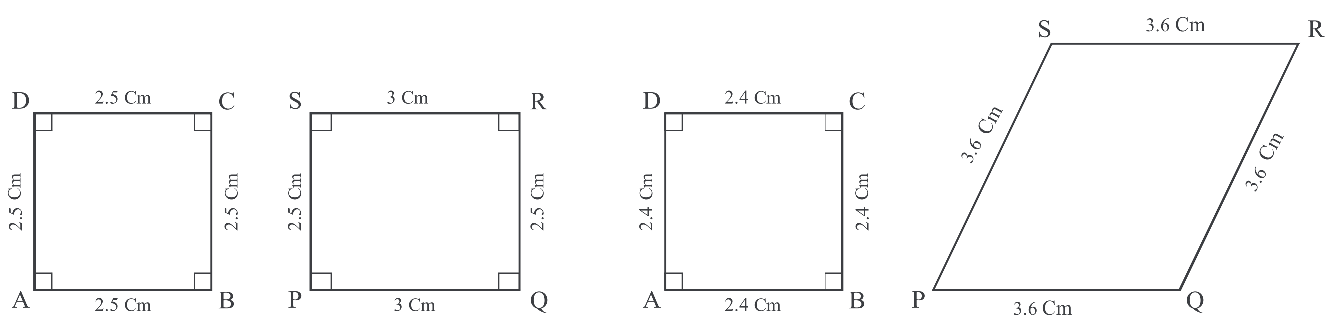


Fig. 11.6

Similarly in fig. 11.6 there is a square and a quadrilateral with opposite sides equal. Their corresponding sides are proportional but angles are not equal, hence both quadrialterals are not similar.

Exercise 11.1

1. **Fill in the blanks :**

   (i) All the circles are ....

   (ii) All the squares are ....

   (iii) All the equilateral triangles are ....

   (iv) The polygons having same number of sides are similar.

   (a) .... (b) ....

2. **State whether following statements are true and false.**

   (i) Two congruent figures are similar.

   (ii) Two similar figures are congruent.

   (iii) Two polygons having same ratio of their corresponding sides are similar.

   (iv) Two polygons are similar if their corresponding sides are proportional and corresponding angles are equal.

   (v) Two polygons having corresponding angles equal are similar.

3. **Give any two examples of simliar figures with their diagrams.**

Similarity of triangles and equiangular triangles

In this chapter we have studied about the conditions for the smilarity of two or more polygons. Since triangles are also polygons, so we can state the same conditions for the similarity of two triangles that is two triangles are similar, if

(i) their corresponding angles are equal and

(ii) their corresponding sides are in the same ratio

In fig. 11.7 ΔABC and ΔDEF are similar if

(i) $\angle A = \angle D, \angle B = \angle E$ and $\angle C = \angle F$ and

(ii) $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

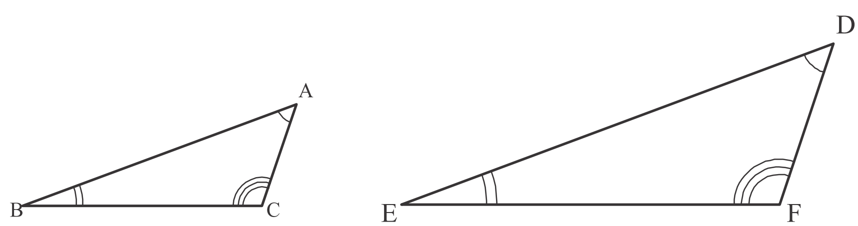


Fig. 11.7

Equiangular Triangles

If corresponding angles of two triangles are equal, then both triangles are called equiangular triangles.

Results related to the basic proportionality theorem :

(a) Draw an angle XAY, On its one arm AX mark five points P_1, P_2, D, P_3 and B at the equal distance. In this way we get $AP_1 = P_1P_2 = P_2D = DP_3 = P_3B = 1$cm (If these points are marked at the same distance of 1 cm. it will be conventional)

(b) Take a point C on the side AY and meet B to C. Now form the point D, draw $DE \quad BC$ to meet AY at E. In this way we get a triangle.

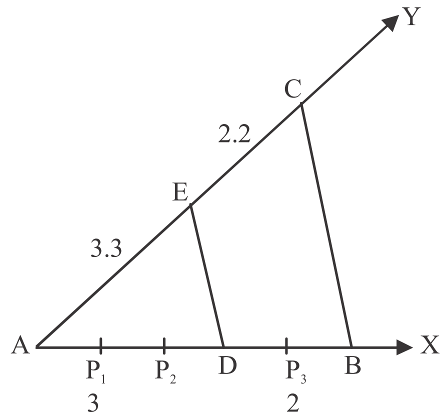


Fig. 11.8

According to the fig. 11.8.

$AD = AP_1 + P_1P_2 + P_2D = 3$ untis (since distance between to consecutive points is 1 cm)

$DB = DP_3 + P_3B = 2$ Units (2 cm)

$$\therefore \quad \frac{AD}{DB} = \frac{3}{2} \qquad \ldots \text{(i)}$$

Now measure the length of AE and AC (Here $AE = 3.3$ cm and $EC = 2.2$ cm)

$$\text{So} \quad \frac{AE}{EC} = \frac{3.3}{2.2} = \frac{3}{2} \qquad \ldots \text{(ii)}$$

Comparing (i) and (ii) we get

$$\frac{AD}{DB} = \frac{AE}{EC}$$

If in ΔABC, D and E are two points on the sides AB and AC such that $DE \parallel BC$ then we get,

$$\frac{AD}{DB} = \frac{AE}{EC}$$

This result was obtained by famous Greek Mathematician Thales first so it is also called the Thales theroem. This result is known as basic proportionality theroem.

Theorem 11.1 : (Basic proportionally theroem/Thales theorem)

If a line is drawn parallel to one side of a triangle intersecting the other two sides, then it divides the two sides in the same ratio.

Given : ΔABC, in which $DE \parallel BC$ and DE intersects AB and AC at D and E.

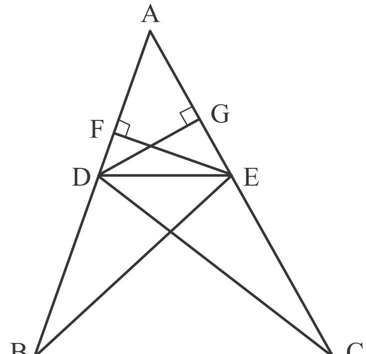


Fig. 11.9

To prove : $\dfrac{AD}{DB} = \dfrac{AE}{EC}$

Construction : Join B to E and C to D,

Draw $EF \perp BA$ and $DG \perp CA$

Proof : Since EF is the height of ΔADE and ΔABE.

$\therefore$ Area of ΔADE $= \dfrac{1}{2}\text{base} \times \text{height}$

$= \dfrac{1}{2}\ \text{AD} \times \text{EF}$

and Area of $\Delta DBE = \dfrac{1}{2}\text{base} \times \text{height}$

$= \dfrac{1}{2} DB \times EF$

$$\therefore \quad \frac{Area\ of\ \Delta\ ADE}{Area\ of\ \Delta DBE} = \frac{\frac{1}{2} AD \times EF}{\frac{1}{2} DB \times EF} = \frac{AD}{DB} \qquad \text{. . . (i)}$$

$$\text{Similarly,} \quad \frac{Area\ of\ \Delta\ ADE}{Area\ of\ \Delta DEC} = \frac{\frac{1}{2} AE \times DG}{\frac{1}{2} EC \times DG} = \frac{AE}{EC} \qquad \text{. . . (ii)}$$

But ΔDBE and ΔDEC are at the same base DE and between lines $DE \parallel BC$

$\therefore$ Area of ΔDBE = Area of ΔDEC . . . (iii)

From (i), (ii) and (iii) we get

$$\frac{Area\ of\ \Delta\ ADE}{Area\ of\ \Delta DBE} = \frac{Area\ of\ \Delta ADE}{Area\ of\ \Delta DEC}$$

or $\dfrac{AD}{DB} = \dfrac{AE}{EC}$ **Hence Proved**

With the help of basic proportionality theorem the following usefull results can also be obtained.

(i) $\dfrac{AB}{AD} = \dfrac{AC}{AE}$ (ii) $\dfrac{AB}{DB} = \dfrac{AC}{EC}$

Theorem 11.2 : (Converse of the theorem 11.1)

If a line divides any two sides of a triangle in the same ratio, then this line is parallel to the third side.

Given : Line l intersects two sides BC and AC of a triangle ABC at D and E respectively.

To prove : $l \quad BC \Rightarrow DE \quad BC$

Proof : Assume, line DE is not parallel to side BC, then another line $DF \parallel BC$

$\because$ DF || BC

$\therefore \quad \dfrac{AD}{DB} = \dfrac{AF}{FC}$ ...(i)

(By basic proportionality theorem)

But $\quad \dfrac{AD}{DB} = \dfrac{AE}{EC}$ (Given) ...(ii)

So $\quad \dfrac{AF}{FC} = \dfrac{AE}{EC}$ from (i) and (ii)

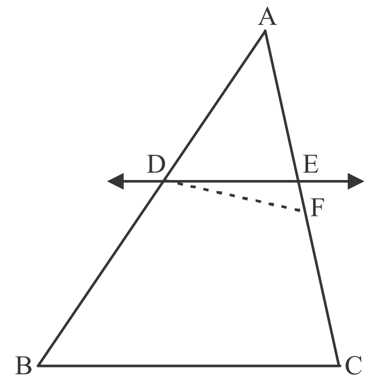


Fig. 11.10

Adding 1 to both the sides, we get

$$\frac{AF}{FC} + 1 = \frac{AE}{EC} + 1$$

or $\quad \dfrac{AF + FC}{FC} = \dfrac{AE + EC}{EC}$

or $\quad \dfrac{AC}{FC} = \dfrac{AC}{EC}$

or $\quad \dfrac{1}{FC} = \dfrac{1}{EC}$

or $\quad FC = EC$, this result can be obtained only when points F and E is coincides and also fall on DF and DE respectively

Hence DE || BC **Hence Proved**

Illustrative Examples

Examples 1 : In ΔABC, $DE \parallel BC$ and $\dfrac{AD}{DB} = \dfrac{3}{5}$. If $AC = 5.6$ unit, then find measure of AE.

Solution : In ΔABC, $DE \parallel BC$ (given)

$\therefore \quad \dfrac{AD}{DB} = \dfrac{AE}{EC}$

(By basic proportionality theorem)

or $\quad \dfrac{AD}{DB} = \dfrac{AE}{(AC - AE)}$

or $\quad \dfrac{3}{5} = \dfrac{AE}{5.6 - AE}$ $\quad \left(\because \dfrac{AD}{DB} = \dfrac{3}{5} \text{ and } AC = 5.6\right)$

or $\quad 3\,(5.6 - AE) = 5AE$

or $\quad 16.8 - 3AE = 5AE$

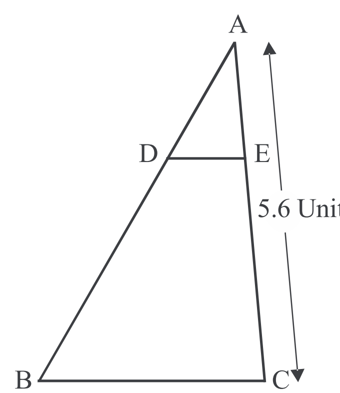


Fig. 11.11

or $\quad 5AE + 3AE = 16.8$

or $\quad 8AE = 16.8$

or $\quad AE = \dfrac{16.8}{8} = 2.1$ unit

Hence, the measure of $AE = 2.1$ unit.

Examnple 2 : In the given fig. 11.12 $DE \parallel BC$. If $AD = x$, $DB = x - 2$, $AE = x + 2$ and $EC = x - 1$, then find the value of x.

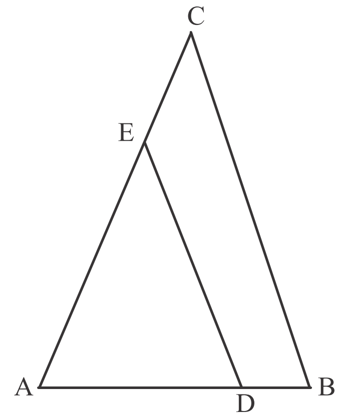


Fig. 11.12

Solution : In ΔABC, $DE \parallel BC$

Thus $\quad \dfrac{AD}{DB} = \dfrac{AE}{EC}$ (By B.P.T)

or $\quad \dfrac{x}{x-2} = \dfrac{x+2}{x-1}$

or $\quad x(x-1) = (x+2)(x-2)$

or $\quad x^2 - x = x^2 - 4$

or $\quad x = 4$

Example 3 : In a trapezium $ABCD$, $AB \parallel DC$, E and F are points on AD and BC respectively such that $EF \parallel AB$. Prove that $\dfrac{AE}{ED} = \dfrac{BF}{FC}$

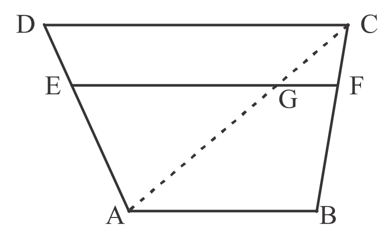


Fig. 11.13

Solution : Join A and C such that interest EF at G.

Now, $\quad AB \parallel DC$ and $EF \parallel AB$ (Given)

$\therefore \quad EF \parallel DC$ (all parallel to the sameline are parallel)

In ΔADC, $EG \parallel DC$ (here $EF \parallel DC$ and EG is a part of EF)

So $\quad \dfrac{AE}{ED} = \dfrac{AG}{GC}$ (By basic proportionality theorem)

or $\quad \dfrac{AG}{CG} = \dfrac{AE}{ED}$ . . . (1)

Similarly, in ΔCAB, $\dfrac{CG}{AG} = \dfrac{CF}{BF}$

or $\quad \dfrac{AG}{CG} = \dfrac{BF}{CF}$ . . . (2)

From (i) and (ii), we get

$\dfrac{AE}{ED} = \dfrac{BF}{FC}$ **Hence proved.**

Example 4 : In quadrilateral $ABCD$, Points P, Q, R and S are on the sides AB, BC, CD and DA respecively in such a way that they trisect the vertics A and C, prove that $PQRS$ is a parallelogram.

Solution : To prove $PQRS$ a parallelogram, we have to prove $PQ \parallel SR$ and $QR \parallel PS$.

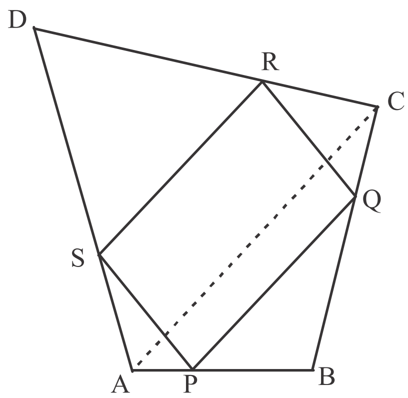


Fig. 11.14

Given : Points P, Q, R and S on the sides AB, BC, CD and DA in such a way that

$BP = 2\,PA$, $BQ = 2\,QC$, $DR = 2RC$ and $DS = 2SA$

Construction : Join A to C.

Proof : In ΔADC, $\dfrac{DS}{SA} = \dfrac{2SA}{SA} = 2$

and $\quad \dfrac{DR}{RC} = \dfrac{2RC}{RC} = 2$ (given)

$\Rightarrow \quad \dfrac{DS}{SA} = \dfrac{DR}{RC} \Rightarrow$ SR || AC . . . (i)

$\dfrac{BP}{PA} = \dfrac{2PA}{PA} = 2$ (By basic proportionality theroem)

In ΔABC

$\dfrac{PQ}{QC} = \dfrac{2QC}{QC} = 2$ (given)

$\Rightarrow \quad \dfrac{BP}{PA} = \dfrac{BQ}{QC}$

$\therefore$ PQ || AC (By basic proportionality theroem) . . . (ii)

Now from (i) and (ii), we get

$SR \quad AC$ and $PQ \quad AC \Rightarrow SR \quad PQ$.

Similarly, joining B to D, we can also prove that $QR \parallel PS$.

Hence $PQRS$ is a parallelogram. **Hence proved.**

Example 5 : Diagonal of a quadrilateral $ABCD$, intersect each other at the point O such that $\dfrac{AO}{BO} = \dfrac{CO}{DO}$ prove that $ABCD$ is a trapezium.

Solution :

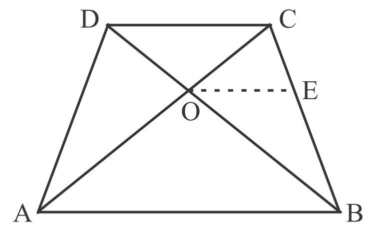


Fig. 11.15

Given : In ΔABCD, $\frac{AO}{BO}=\frac{CO}{DO}$ (See fig. 11.15)

To Prove : *ABCD* is a trapezium *e.g., AB* || *DC.*

Construction : Draw a line *OE*, such that *OE* || *AB.*

Proof : $\frac{AO}{BO}=\frac{CO}{DO}$ (given)

or $\frac{AO}{CO}=\frac{BO}{DO}$ . . . (i)

In ΔABC, OE || AB

∴ $\frac{CO}{OA}=\frac{CE}{EB}$ (By B.P.T)

or $\frac{OA}{CO}=\frac{EB}{CE}$ . . . (ii)

from (i) and (ii), we get

$\frac{BO}{OD}=\frac{EB}{CE}$

or $\frac{BO}{OD}=\frac{BE}{EC}$

or OE || DC (by converse of BPT) . . . (iii)

and OE || AB (by construction) . . . (iv)

from (iii) and (iv), we get

AB || DC **Hence Proved.**

Hence ABCD is a trapezium.

Exercise 11.2

1. The points *D* and *E* are on the sides *AB* and *AC* in a *ΔABC*, Such that *DE* || *BC*, then
   - (i) If *AD* =6 cm, *DB* = 9 cm and *AE* = 8 cm, then find the measure of *AC*.
   - (ii) If $\frac{AD}{DB}=\frac{4}{13}$ and $AC = 20.4$ cm, find the length of *EC*
   - (iii) If $\frac{AD}{DB}=\frac{7}{4}$ and $AE = 6.3$ cm, then find *AC*.
   - (iv) If $AD = 4x-3$, $AE = 8x-7$, $BD = 3x-1$ and $CE = 5x-3$, then find the value of *x*.
2. Points *D* and *E* are on the sides *AB* and *AC* respectively in the *ΔABC* on the bases of the measures given below, state whether *DE* || *BC* or not :
   - (i) *AB* = 12 cm, *AD* = 8 cm, *AE* = 12 cm and *AC* = 18 cm
   - (ii) *AB* = 5.6 cm, *AD* = 1.4 cm, *AC* = 9.0 cm and *AE* = 1.8 cm
   - (iii) *AD* = 10.5 cm, *BD* = 4.5 cm, *AC* = 4.8 cm and *AE* = 2.8 cm
   - (iv) *AD* = 5.7 cm, *BD* = 9.5 cm, *AE* = 3.3 cm and *EC* = 5.5 cm

3. In the fig 11.16 points L. M and N are situsted on OA, OB and OC respectively such that $LM \parallel AB$ and $MN \parallel BC$, then show $LN \parallel AC$

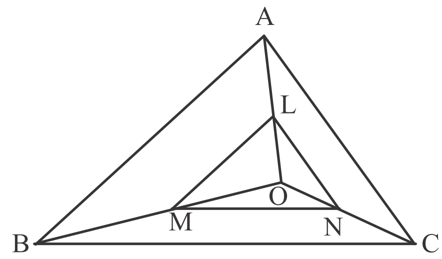


Fig. 11.16

4. In ΔABC, points D and E are respectively situated on AB and AC such that $BD = CE$. If $\angle B = \angle C$, then show that $DE \parallel BC$.

5. In the fig. 11.17. If $DE \parallel BC$ and $CD \parallel EF$, then prove $AD^2 = AB \times AF$.

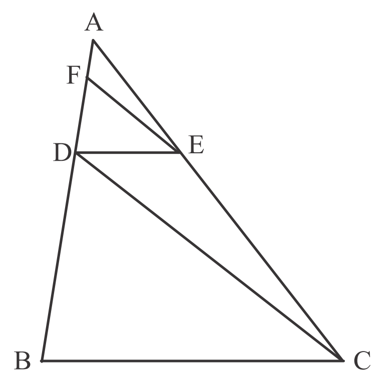


Fig. 11.17

6. In the fig. 11.18 if $EF \parallel DC \parallel AB$, then prove that $\dfrac{AE}{ED} = \dfrac{BF}{FC}$

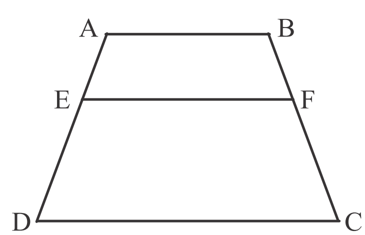


Fig. 11.18

7. $ABCD$ is a parallelogram in which point P is situated on the side BC. If DP and AB are increased then they meet at L then prove

(i) $\dfrac{DP}{PL} = \dfrac{DC}{BL}$ (ii) $\dfrac{DL}{DP} = \dfrac{AL}{DC}$

8. Point D and E are situated in the side AB in a ΔABC such that $AD = BE$. If $DP \parallel BC$ and $EQ \parallel AC$, then prove that $PQ \parallel AB$.

9. In trapezium $ABCD$, diagonals intersects each other at O and $AB \parallel DC$. Show that $\frac{AO}{BO} = \frac{CO}{DO}$

10. If the points D and E are situated on the sides AB and AC respectively in a triangle ABC. Such that $BD = CE$, then prove that ΔABC is an isosceles triangle.

11.4. Bisector of Internal and External Angles of a Triangle

We have studied about the basic proportionaity theorem and the related results. Now if any line divides the angles of a triangle, what results may be obtained. Let's study the theorems and their results given below.

Theorem 11.3

The internal bisector of an angle of a triangle divides the opposite side intertnally in the ratio of the sides containing the angle.

Given : AD is the bisector of $\angle A$ in ΔABC and so $\angle 1 = \angle 2$.

To Prove : $\frac{BD}{DC} = \frac{AB}{AC}$

Construction : Draw a line $CE \parallel DA$ and extend it to meet at E on produced BA.

Proof : $CE \parallel DA$ and AC and BE are transversals.

$\therefore \quad \angle 2 = \angle 3$ (alternate angles) . . . (i)

$\angle 1 = \angle 4$ (corresponding angles) . . . (ii)

But $\angle 1 = \angle 2$ (given)

from (i) and (ii), we get

$\angle 3 = \angle 4$

$\therefore$ In ΔACE

AE = AC . . . (iii)

In ΔBCE, DA $\parallel$ CE

$\therefore \quad \frac{BD}{DC} = \frac{BA}{AE}$ (By B.P.T)

or $\quad \frac{BD}{DC} = \frac{BA}{AC}$ from (iii)

Hence $\quad \frac{BD}{DC} = \frac{BA}{AC}$ **Hence Proved.**

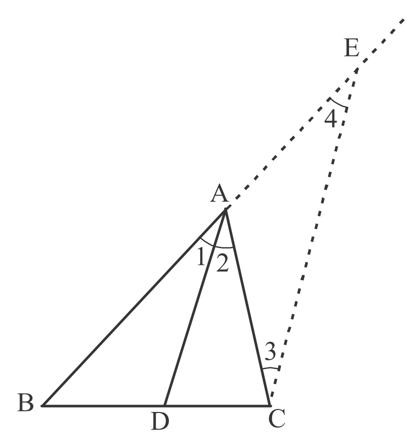


Fig. 11.19

Theorem 11.4 : (Converse of theorem 11.3)

If a line through one vertex of a triangle divides the opposite sides in the ratio of other two sides, then the line bisects the angle at the vertex.

Given : In ΔABC, D is a point on BC such that

$$\frac{BD}{DC} = \frac{AB}{AC}$$

To prove : AD is bisector of $\angle A$ and $\angle 1 = \angle 2$

Construction : Extend BA to E and such that $AE = AC$ and join E to C.

Proof : In ΔACE we have

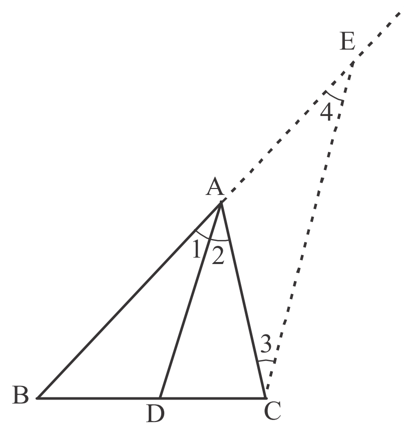


Fig. 11.20

$AE = AC$ (By construction)

$\therefore \quad \angle 3 = \angle 4 \quad$ . . . (i)

$\because \quad \dfrac{BD}{DC} = \dfrac{AB}{AC}$ (given)

$\therefore \quad \dfrac{BD}{DC} = \dfrac{AB}{AE}$ ($\because$ $AE = AC$ by construction)

Similarly, In ΔABC, If $\dfrac{BD}{DC} = \dfrac{AB}{AE}$ (conversed BPT)

$DA \parallel CE$ thus $\angle 1 = \angle 4$ (corresponding angles)

and $\quad \angle 2 = \angle 3$ (alternate angles)

But $\quad \angle 3 = \angle 4$ from (i)

$\therefore \quad \angle 1 = \angle 2$

Hence AD is the bisector of $\angle A$ **Hence proved.**

Theorem 11.5

The external bisector of an angle of a triangle divides the opposite side externally in the ratio of the sides containing the angle.

Given : In ΔABC, AD is the bisector of the exterior $\angle A$ and intersects BC produced in D.

To prove: $\dfrac{BD}{CD} = \dfrac{AB}{AC}$

Construction : Draw $CE \parallel DA$ meeting AB at E.

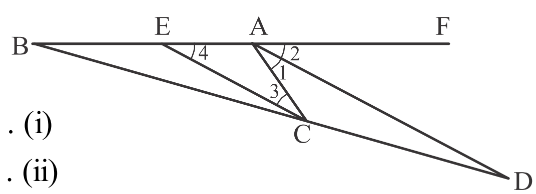


Fig. 11.21

Proof : $CE \parallel DA$ (by constructions)

$\angle 1 = \angle 3$ (alternate angles) . . . (i)

$\angle 2 = \angle 4$ (Corresponding angles) . . . (ii)

DA and BF are transversal

$\angle 1 = \angle 2$ (given)

$\therefore \quad \angle 3 = \angle 4$ . . .(iii)

In ΔAEC, AE = AC (Opposite sides of equal angles)

Now in ΔBAD, EC || AD

$\therefore \quad \dfrac{BD}{CD} = \dfrac{BA}{EA}$ (properties of BPT)

$\Rightarrow \quad \dfrac{BD}{CD} = \dfrac{BA}{AC}$ from (iii)

$\Rightarrow \quad \dfrac{BD}{CD} = \dfrac{AB}{AC}$ **Hence proved.**

11.5. Similarity of Triangles

In the section 11.3, we havce studied that two triangles are similar if :

(i) their corresponding angles are equal and

(ii) corresponding sides are proportional. In the fig. 11.22. We have two triangles ABC and DEF in which

(i) $\angle A = \angle D, \angle B = \angle E$ and $\angle C = \angle F$ and

(ii) $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{CA}{FD}$

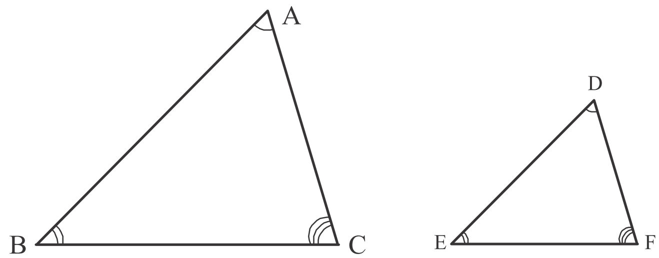


Fig. 11.22

Then ΔABC and ΔDEF are similar.

In the figures. you can see that A corresponds to D, B corresponds to E and C corresponds to F. Symbolicaly. we write the similarityof these two triangles as ΔABC ~ ΔDEF and read it as triangle ABC is similar to triangle DEF." Remember symbol ' ' stands for **'is similar to'.** Recall that you have used the symbol "≅" for 'is congurent to' in your previous class.

It must be remembered that the similarly of two triangles should also be expressed symbolically, using correct correspondence of their vertices. For example, for the ΔABC and DEF of fig. 11.22, we can not be written ΔABC ~ ΔEFD or ΔABC ~ ΔFED. We can write them as. ΔABC ~ ΔDEF or ΔBAC ~ ΔEDF or ΔBCA ~ ΔEFD.

In our previous class. You learnt about some criteria for congurency of two triangles involving only three pairs of corresponding parts of two triangles. Similarly here, we try certain criteria for similarity of two triangles involving relationship between some number of corresponding parts of the two triangles instead of all the six pairs of corresponding parts. Let us perform the following activity and see what result we obtain.

First of all we draw two line segment $BC = 6$ cm and $EF = 4$ cm. Then at the points B and E of line segments respectively BC and EF construct the angles measure 65° each and on the points C and F construct the angles 45°. In this way two triangles respectively ABC and DEF are obtained (see fig 11.23). Can you measure the third angle of these triangles? Since, sum of three internal angles of a triangle is 180°

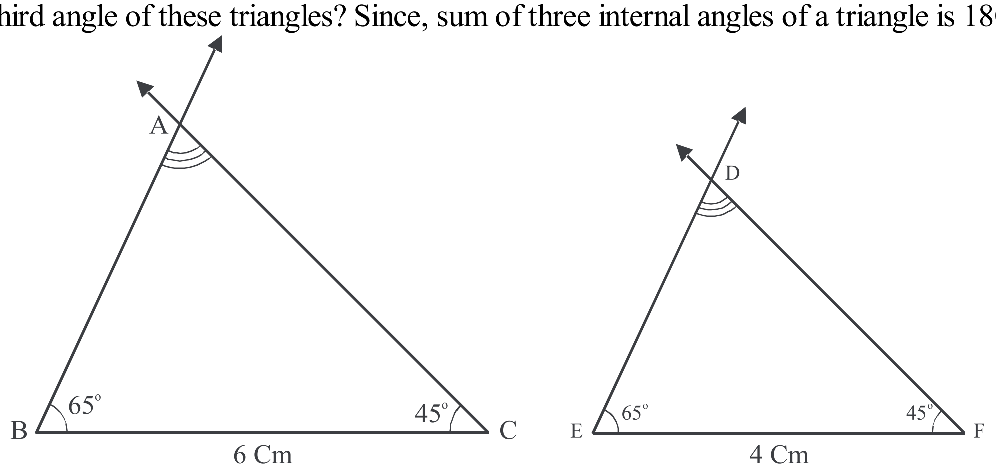


Fig. 11.23

$$\angle A = 180° - (\angle B + \angle C) = 180° - (65° + 45°)$$

$$\angle A = 180° - 110° = 70°$$

$$\angle D = 180° - (\angle E + \angle F) = 180 - (65° + 45°)$$

$$\angle D = 180° - 110° = 70°$$

At the basis of above figures, we get

$$\angle A = \angle D, \angle B = \angle E \text{ and } \angle C = \angle F$$

In this way we conclude that corresponding angles of ΔABC and ΔDEF are equal thus two triangle are equiangular. Now we measure the sides of these triangles and find out the ratio between corresponding sides.

$$\frac{BC}{EF} = \frac{6}{4} = 1.5, \quad \frac{AB}{DE} = \frac{4.5}{3} = 1.5 \text{ and } \frac{AC}{DF} = \frac{5.85}{3.9} = 1.5$$

Here, AB = 4.5 cm, DE = 3 cm, AC = 5.85 cm and DF = 3.9 cm.

Thus we get $\dfrac{BC}{EF} = \dfrac{AB}{DE} = \dfrac{AC}{DF}$

By constructing several pairs of triangle having their corresponding angles equal, you can repeat this activity as many times as you wish but every time, you will find the same result. This activity leads us to the criterion " The ratio of corresponding sides of two equiangular triangles is always same."

Now according to the defination of similar triangles.If corresponding angles of two triangles ABC and DEF are equal and the ratio of their corresponding sides is also equal. Then two triangles are similar hence $\Delta ABC \sim \Delta DEF$.

Theorem 11.6 (AAA Criterion)

Two equiangular triangles are similar.

Given : ΔABC and ΔDEF such that

$$\angle A = \angle D, \angle B = \angle E \text{ and } \angle C = \angle F$$

To prove : $\Delta ABC \sim \Delta DEF$

Construction : Cut $DP = AB$ and $DQ = AC$ and join P to Q.

Proof : $AB = DP$ and $AC = DQ$ (by construction)

$\angle A = \angle D$ (given)

$\therefore$ $\Delta ABC \cong \Delta DPQ$ (SAS criterion)

$\therefore$ $\angle B = \angle DPQ$ $\therefore$ $\angle C = \angle DQP$

But $\angle B = \angle E$ and $\angle C = \angle F$ (given)

so $\angle DPQ = \angle E$ and $\angle DQP = \angle F$ (Corresponding angles)

$\therefore$ PQ || EF

Thus, $\dfrac{DP}{PE} = \dfrac{DQ}{QF}$ (Using the B.P.T.)

$\Rightarrow$ $\dfrac{PE}{DP} = \dfrac{QF}{DQ}$

$$\Rightarrow \quad \frac{PE}{DP}+1=\frac{QF}{DQ}+1$$

$$\Rightarrow \quad \frac{PE+DP}{DP}=\frac{QF+DQ}{DQ}$$

$$\Rightarrow \quad \frac{DE}{DP}=\frac{DF}{DQ}$$

$$\Rightarrow \quad \frac{DP}{DE}=\frac{DQ}{DF}$$

$$\Rightarrow \quad \frac{AB}{DE}=\frac{AC}{DF}$$

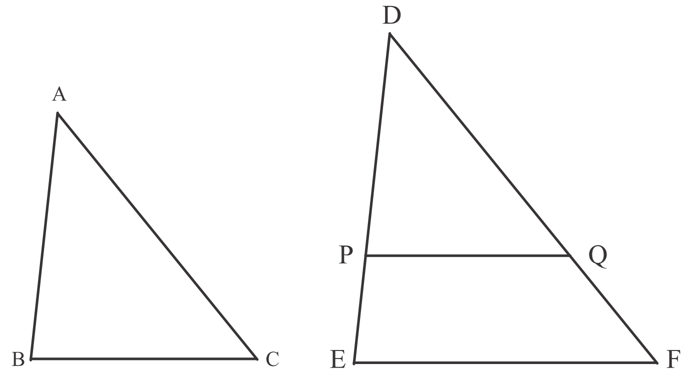


Fig. 11.24

Similarly, we can find

$$\frac{AB}{DE}=\frac{BC}{EF}$$

Thus $\frac{AB}{DE}=\frac{BC}{EF}=\frac{AC}{DF}$

Hence, ΔABC and ΔDEF follow the criterion of similarity of two triangles.

$\therefore$ $\Delta ABC \sim \Delta DEF$ **Hence proved,**

If two angles of a triangle are respectively equal to two angles of another triangle, then by the angle sum property of a triangle their third angles will also be equal. Thus AAA similarly criterion can also be stated as AA similarity.

If it is possible that if corresponding sides of two triangles are proportion then their corresponding angles are equal. Let us check with the help of an activity given below.

Activity : Draw two triangles *ABC* and *DEF* such that *AB* = 3cm, *BC* = 6 cm and *CA* = 8 cm, also *DE* = 4.5 cm *EF* = 9 cm. and *FD* = 12 cm and measure the degree of angles with the help of a protector. Now compare their corresponding angles.

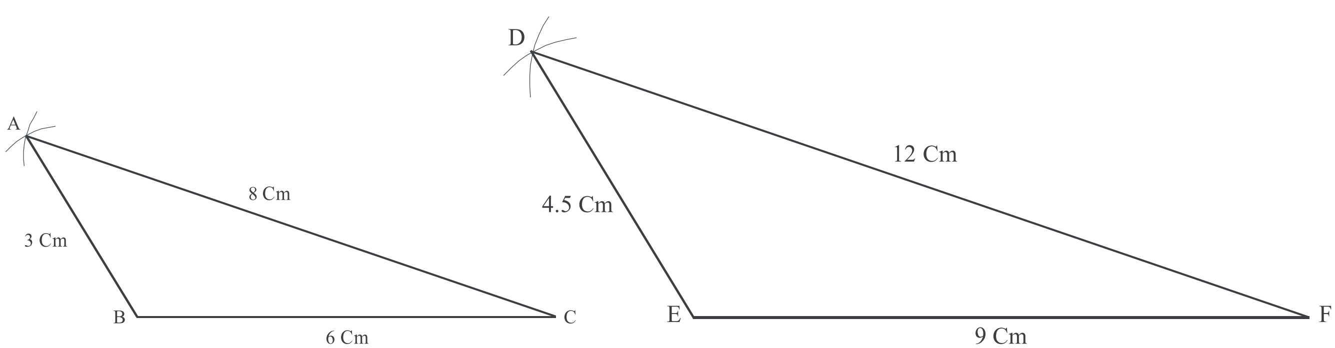


Fig. 11.25

Here, $\frac{AB}{DE}=\frac{BC}{EF}=\frac{CA}{FD}=\frac{2}{3}$ (ratio of the corresponding sides)

When angles are measured, we get $\angle A = \angle D = 40°, \angle B = \angle E = 120°, \angle C = \angle F = 20°$. It means the ratio of the corresponding sides of two triangles is same. their corresponding angles will also be equal. Therefore $\Delta ABC \sim \Delta DEF$.

You can repeat this activity by drawing several pairs of triangles having their sides in the same ratio. And every time you will find their corresponding angles are equal.

Let us prove this result of similarity through the theorem given below.

Theorem 11.7 : (SSS similarity criterion)

If the ratio of the corresponding sides of two triangles is equal, then the triangles are similar.

Given : In ΔABC and ΔDEF, $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{CA}{FD}$

To prove : $\Delta ABC \sim \Delta DEF$

Construction : Cut $DP = AB$ and $DQ = AC$ and join P to Q.

Proof : $\dfrac{AB}{DE} = \dfrac{AC}{DF}$ (given)

$\Rightarrow$ $\dfrac{DP}{DE} = \dfrac{DQ}{DF}$ (By construction)

$\Rightarrow$ $PQ \parallel EF$ (Using converse of B.P.T))

$\therefore$ By AA similarity criterion

$\Delta DPQ \sim \Delta DEF$ . . . (i)

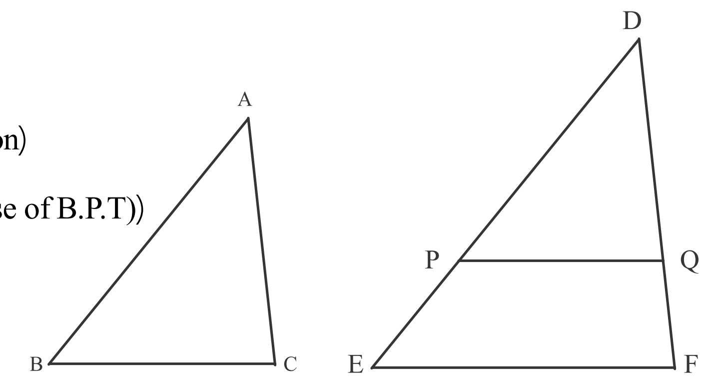


Fig. 11.26

$\Rightarrow$ $\dfrac{DP}{DE} = \dfrac{PQ}{EF}$

$\Rightarrow$ $\dfrac{AB}{DE} = \dfrac{PQ}{EF}$ ($\because$ $AB = DP$ by construction)

But $\dfrac{AB}{DE} = \dfrac{BC}{EF}$ (given)

$\therefore$ $\dfrac{PQ}{EF} = \dfrac{BC}{EF}$

$\Rightarrow$ $PQ = BC$

Similarly, in ΔABC and ΔDPQ

$AB = DP$, $BC = PQ$, and $AC = DQ$

From SSS similaity criterion,

$\Delta ABC \cong \Delta DPQ$ . . . (ii)

From (i) and (ii) we get

$\Delta ABC \cong \Delta DPQ$ or $\Delta DPQ \sim \Delta DEF$ (congurent ΔS are also similar)

So $\Delta ABC \sim DPQ$ and $\Delta DPQ \sim \Delta DEF$

Hence, $\Delta ABC \sim \Delta DEF$ **Hence proved,**

Theorem 11.8 : (SAS similarity criterion)

If two corresponding sides in two triangles are proportional and the angles between them are also equal, the two triangles are similar.

Given : In ΔABC and ΔDEF, $\frac{AB}{DE} = \frac{AC}{DF}$ and $\angle A = \angle D$

To prove : $\Delta ABC \sim \Delta DEF$

Construction : Cut DP from DE and DQ from DF such that $DP = AB$ and $DQ = AC$ in ΔDEF

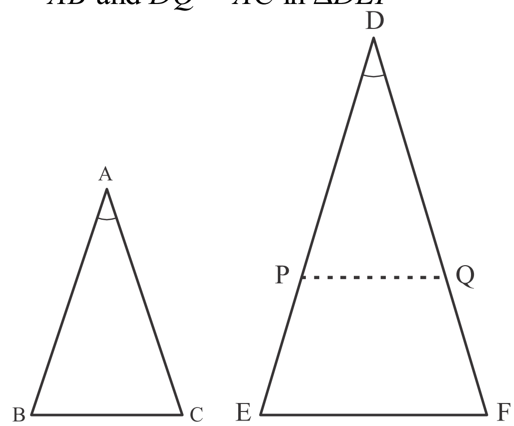


Fig. 11.27

Proof : In ΔABC and ΔDPQ

$AB = DP$

and $AC = DQ$ (by construction)

$\angle A = \angle D$ (given)

So, by SAS criterion of congurency

$\Delta ABC \cong \Delta DPQ$ . . . (i)

Now $\frac{AB}{DE} = \frac{AC}{DF}$ (given)

$\Rightarrow \frac{DP}{DE} = \frac{DQ}{DF}$ ($\because AB = DP$ and $AC = DQ$ by construction)

$\Rightarrow$ PQ || EF (converse of Thales theorem)

$\Rightarrow \angle DPQ = \angle E$ and $\angle DQP = \angle F$ (corresponding angles)

From AA similarity

$\therefore$ $\Delta DPQ \sim \Delta DEF$ . . . (ii)

from (i) and (ii), we get

$\Delta ABC \cong \Delta DPQ$ and $\Delta DPQ \sim \Delta DEF$

(All congurent triangles are similar)

$\Rightarrow$ $\Delta ABC \sim \Delta DEF$ **Hence proved**

Illustrative Examples

Example 1 : Find out the simillar pair in the following pairs of triangles given below, write the symbolic notation with similatriy criterion for them.

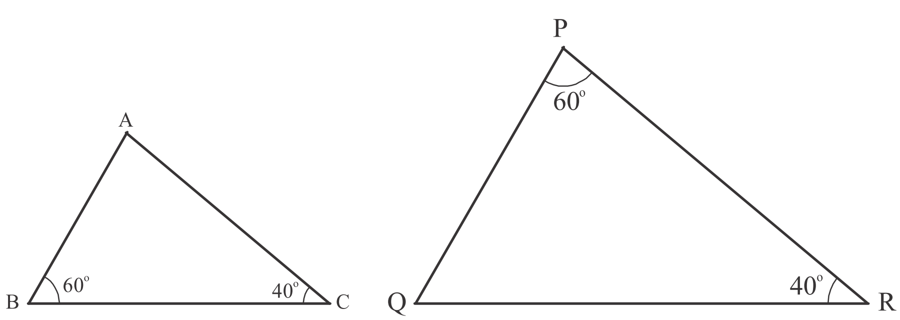


Fig. 11.28 (i)

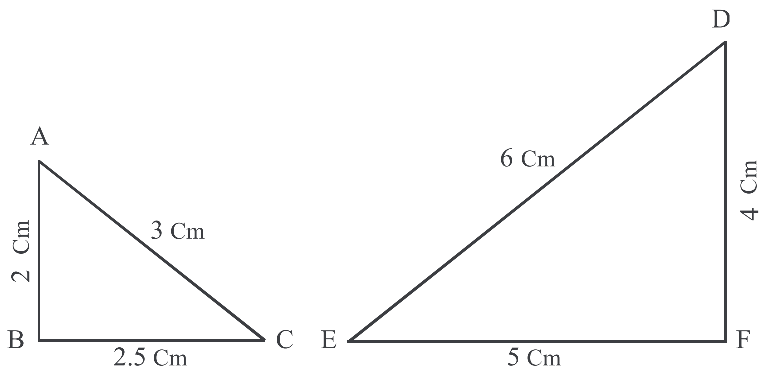


Fig. 11.28 (ii)

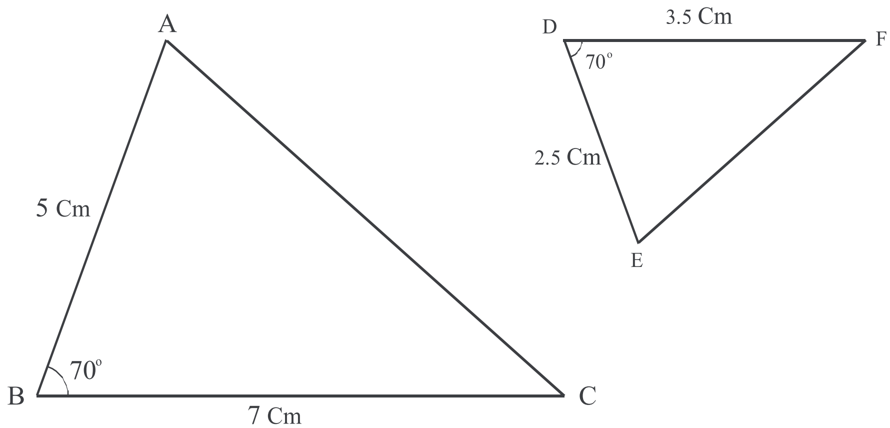


Fig. 11.28 (iii)

Solution : (i) $\Delta BCA \sim \Delta PQR$

$\because$ $\angle B = \angle P = 60°$, $\angle C = \angle R = 40°$

$\therefore$ $\angle A = 180° - (60° + 40°) = \angle Q = 80°$

$\therefore$ $\Delta BCA \sim \Delta PRQ$ (By AAA similarity criterion)

(ii) In ΔABC and ΔDEF

$$\frac{AB}{DF} = \frac{BC}{FE} = \frac{CA}{ED} = \frac{1}{2}$$

$\therefore$ $\Delta ABC \sim \Delta DEF$ (By SSS similarity criterion)

(iii) In ΔABC and ΔEDF

$$\frac{AB}{DE} = \frac{BC}{DF} = 2 \text{ and } \angle ABC = \angle EDF = 70°$$

$\Delta ABC \sim \Delta EDF$ (By *SAS* Similarity criterion)

Example 2 : Compare ΔABC and ΔDEF and find the degree measures $\angle D, \angle E$ and $\angle F$.

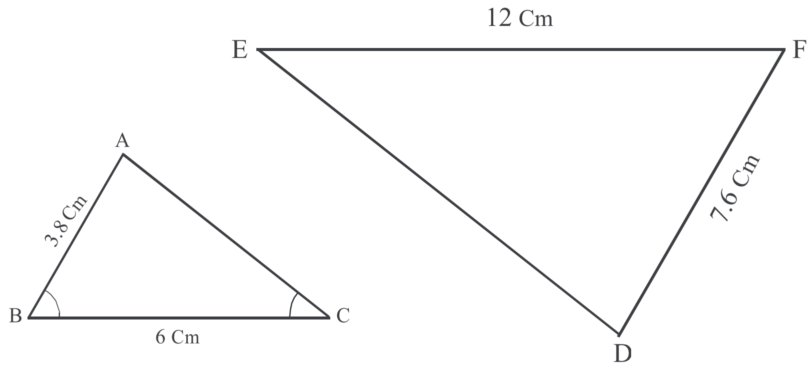


Fig. 11.29

Solution : In ΔABC and ΔDEF

$$\frac{AB}{DF} = \frac{BC}{FE} = \frac{CA}{ED} = \frac{1}{2}$$

$\therefore$ By SSS similarity criterion

$\Delta ABC \sim \Delta DEF$

$\Rightarrow \quad \angle A = \angle D, \ \angle B = \angle F$ and $\angle C = \angle E$

$\Rightarrow \quad \angle F = 60°, \angle E = 40°$

$\Rightarrow \quad \angle D = 180° - (60° + 40°) = \angle A = 80°$

Hence $\quad \angle D = 80°, \angle E = 40°$ and $\angle F = 60°$

Example 3 : In the fig. 11.30, if $OA.OB = OC.OD$ then show that $\angle A = \angle C$ and $\angle B = \angle D$.

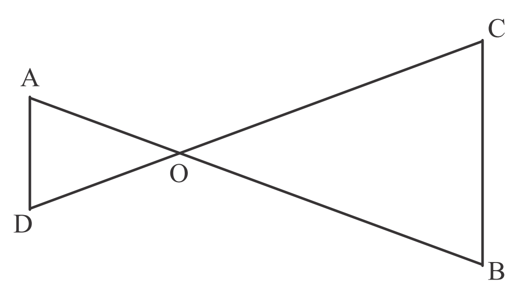


Fig. 11.30

Solution : In ΔAOD and ΔBOC, we have

$OA.OB = OC.OD$ (given)

$\therefore \quad \frac{OA}{OD} = \frac{OC}{OB}$ . . .(i)

and $\quad \angle AOD = \angle COB$ (vertically opposite angles) . . . (ii)

From (i) and (ii), we get

$\Delta AOD \sim \Delta COB$

$\therefore \quad \angle A = \angle C$ and $\angle D = \angle B$

(Corresponding angles of similar triangles) **Hence proved.**

Example 4 : In the fig. 11.31 QA and PB are perpendicular to AB. If AB = 16 cm, $OQ = 5\sqrt{3}$ cm and $OP = 3\sqrt{13}$ cm, then find the length of AO and BO.

Solution : In ΔAOQ and ΔBOP

$\angle OAQ = \angle OBP = 90°$

$\angle AOQ = \angle BOP$

(vertically opposite angles)

$\therefore$ By AAA similarity criterion.

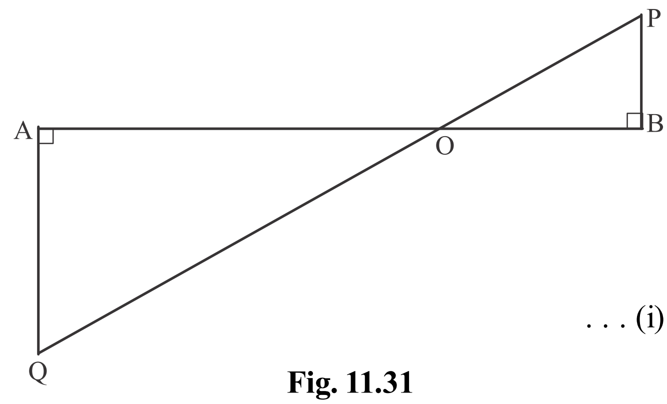


Fig. 11.31

$$\frac{AO}{BO} = \frac{OQ}{OP} = \frac{AQ}{BP} \quad \ldots (i)$$

But AB = AO + BO = 16 cm

Let AO = x $\Rightarrow$ BO = 16 – x.

$$\frac{x}{16-x} = \frac{OQ}{OP} \quad \text{(form (i))}$$

$$\Rightarrow \quad \frac{x}{16-x} = \frac{5\sqrt{13}}{3\sqrt{13}}$$

$\Rightarrow$ $3x = 80 - 5x$

$\Rightarrow$ $8x = 80$

$\Rightarrow$ $x = 10$

$\therefore$ AO = 10 cm and BO = 16 – 10 = 6 cm.

Hence AO = 10 cm and BO = 6 cm.

Example 5 : In the adjoining fig. 11.32, $\angle ADE = \angle B$ and AD = 3.8 cm, AE = 3.6 cm, BE = 2.1 cm and BC = 4.2 cm. find DE.

Solution : In triangles ADE and ABC

$\angle ADE = \angle B$ (given) and

$\angle A = \angle A$ (common)

$\therefore$ By AA similarity criterion

$\Delta ADE \sim \Delta ABC$

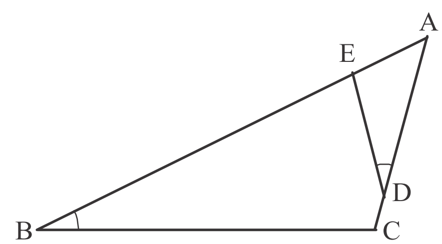


Fig. 11.32

$$\Rightarrow \quad \frac{AD}{AB} = \frac{DE}{BC}$$

or $$\frac{AD}{AE+EB} = \frac{DE}{BC}$$

or $$\frac{3.8}{3.6+2.1} = \frac{DE}{4.2}$$

or $$DE = \frac{3.8 \times 4.2}{5.7} = \frac{15.96}{5.7} = 2.8$$

Hence, the length of DE = 2.8 cm

Example 6 : In the given figure 11.33 $ABCD$ is a trapezium in which $AB \parallel DC$. If $\Delta AED \sim \Delta BEC$ then prove that $AD = BC$

Solution : In the triangles ΔEDC and ΔEBA

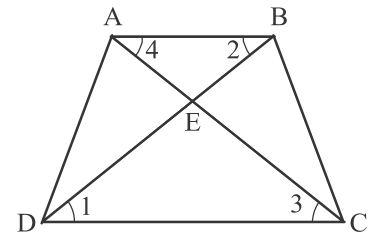


Fig. 11.33

$\angle 1 = \angle 2$ and $\angle 3 = \angle 4$ (alternate angles)

$\therefore$ by AAA similarity criterion

$\Delta EDC \sim \Delta EBA$

So $\dfrac{ED}{EB} = \dfrac{EC}{EA}$

or $\dfrac{ED}{EC} = \dfrac{EB}{EA}$ . . . (i)

Since, $\Delta AED \sim \Delta BEC$

$\therefore$ $\dfrac{AE}{BE} = \dfrac{ED}{EC} = \dfrac{AD}{BC}$ . . . (ii)

From (i) and (ii) $\dfrac{EB}{EA} = \dfrac{AE}{BE}$

$\Rightarrow$ $(BE)^2 = (AE)^2$

$\Rightarrow$ $BE = AE$

Substituting this value in (ii)

$\dfrac{AE}{AE} = \dfrac{AD}{BC}$

$\Rightarrow$ $\dfrac{AD}{BC} = 1$

or $AD = BC$ **Hence Proved.**

Example 7 : In the parallelogram $ABCD$, M is the mid point of CD and joining line B to M intersects AC at L. If BM meets AD produced at E, then prove that $EL = 2\ BL$.

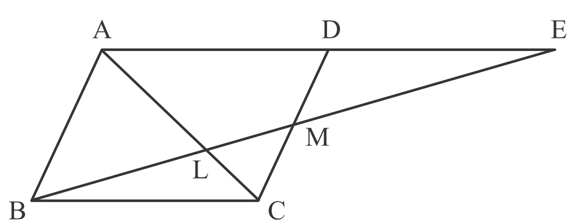


Fig. 11.34

Solution : In ΔBMC and ΔEMD

$MC = MD$ ($\because M$ is the mid point of CD)

$\angle CMB = \angle DME$ (vertically opposite angles)

$\angle MCB = \angle MDE$ (alternate angles)

$\therefore$ By ASA congruency

$\Delta BMC \cong \Delta EMD$

$\therefore$	BC = ED AD = BC	($\because ABCD$ is a parallelogram)	
and	AE = AD + DE		
or	AE = BC + BC		
or	AE = 2BC		. . . (1)

Now, in ΔAEL and ΔCBL

$\angle ALE = \angle CLB$ (vertically opposite angles)

$\angle EAL = \angle BCL$ (alernate angles)

So. By AA simialrity criterion

$$\frac{EL}{BL} = \frac{AE}{CB}$$

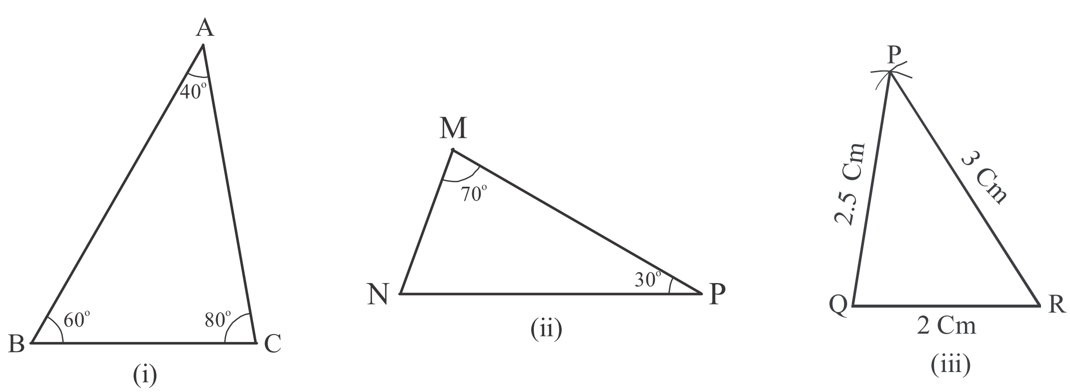
$$\Rightarrow \quad \frac{EL}{BL} = \frac{2BC}{BC}$$

$$\Rightarrow \quad \frac{EL}{BL} = 2$$

$\Rightarrow$ EL = 2 BL **Hence proved.**

Exercise 11.3

1. In ΔABC and ΔPQR, $\frac{AB}{PQ} = \frac{BC}{QR}$, State any two angles of two triangles which must be equal so that both triangles may be similar. Justify your answer.
2. In ΔABC and ΔDEF, $\angle A = \angle D, \angle B = \angle F$, state whether $\Delta ABC \sim \Delta DEF$? Give proper reason in support to your answer.
3. In $\Delta ABC \quad \Delta FDE$, state whether $\frac{AB}{DE} = \frac{BC}{EF}$ and $\frac{CA}{FD}$ is true or not? Justify your answer.
4. If two sides of a triangle are proportational to two sides of another triangles also one angle of first triangle is equal to one angle of other triangle. Then both triangles are similar. Is this statement true? Give reasons to support your answer.
5. What do you understand by equiangular triangles? What relation may be among them?
6. Find out the similar triangles out of the following figure 11.35 given below and write their similarity in symbolic notation.



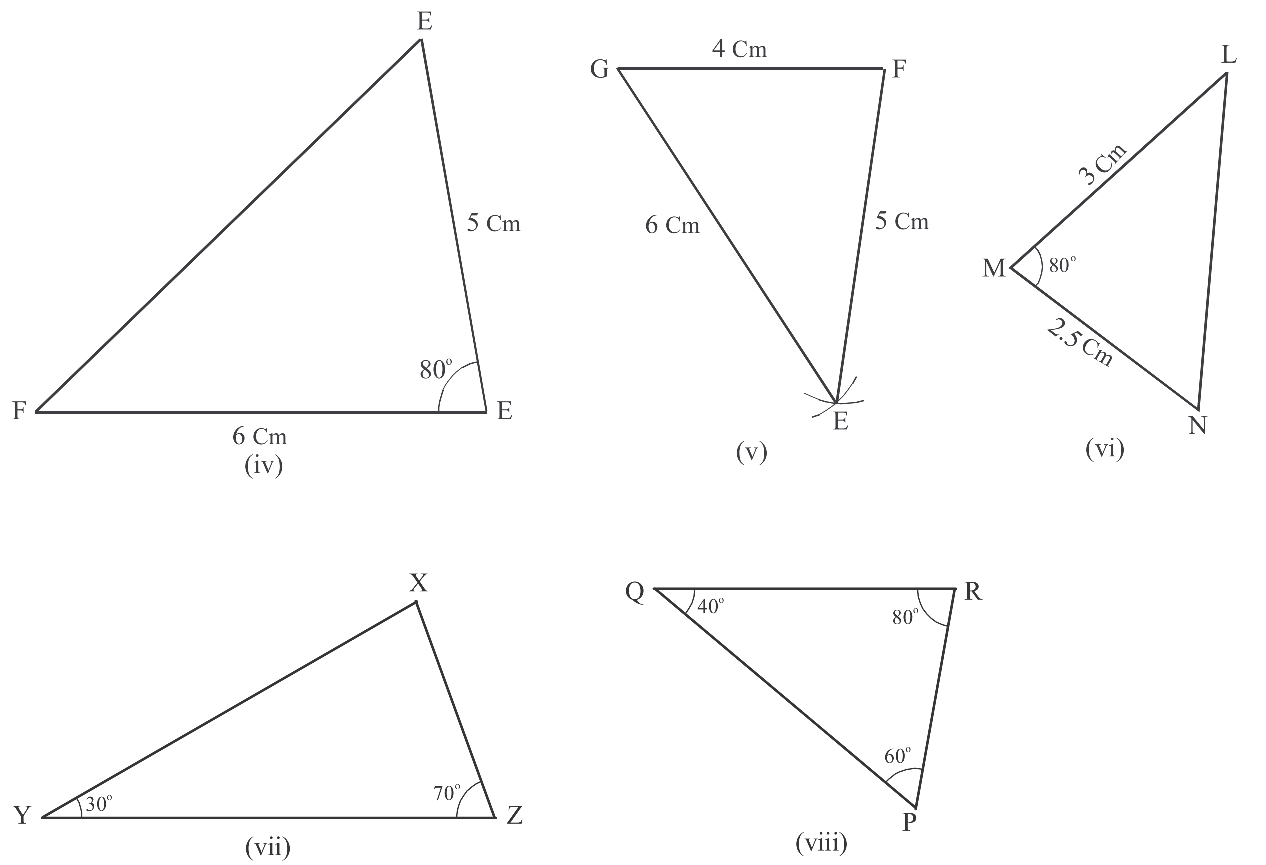


Fig. 11.35

7. In the adjoiniung fig. 11.36 $\Delta PRQ \sim \Delta TRS$, state which angles should be equal to each other in the given pair of similar triangles?

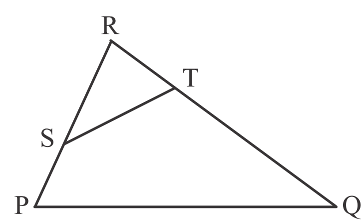


Fig. 11.36

8. You have to choose two similar triangles out of the following triangles which are similar to each other if $\angle CBE = \angle CAD$.

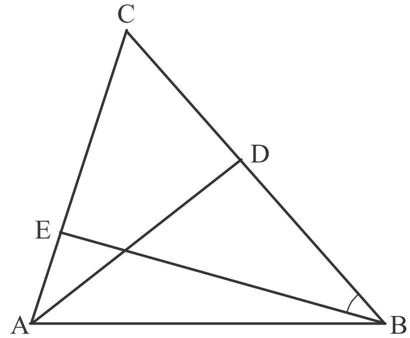


Fig. 11.37

9. In the fig. 11.38 given below PQ   RS, then prove that $\Delta POQ \sim \Delta SOR$

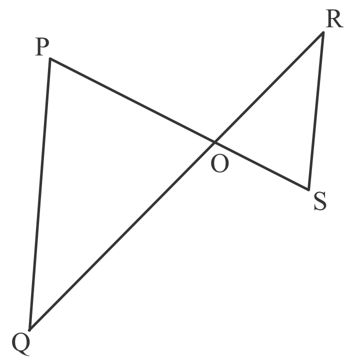


Fig. 11.38

10. A girl of height 90 cm is walking away from the base of a long bulbs put pole at a speed of 1.2 m/s. If bulb is put at the height of 3.6 m above the ground, find the length of her shadow after 4 minutes.
11. The length of the shadow of a vertical pole 12 m high from the ground is 8 m. At the same time the shadow of the vertical minaret is 56 meter long. Find the height of the minaret.
12. If perpendicular is drawn from vertex A to opposite side BD in a triangle ABC, we get $AD^2 = BD \times DC$, then prove that ABC is a right angled triangle.
13. Prove that four triangles obtained by joining the mid-points of three sides ofA triangle are similar to the original triangle.
14. According to the fig. if $AB \perp BC, DC \perp BC$ and $DE \perp AC$, then prove $\Delta CED \sim \Delta ABC$.

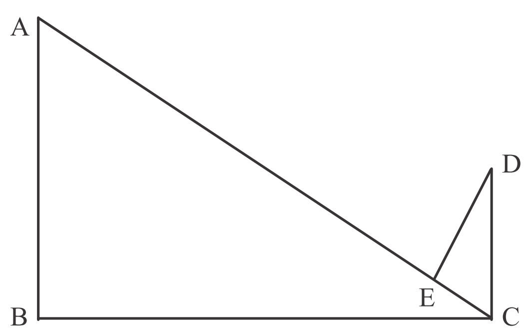


Fig. 11.39

15. In ΔABC, D is the mid-point BC. If a line is drawn in such a way that it bisects AD and cut AD and AC at E and X respectively, then prove that $\frac{EX}{BE} = \frac{1}{3}$.

11.5.2. Areas of Two Similar Triangles

You have learnt that in two similar triangles, the ratio of their corresponding sides is same. Now we shall study the relationship between the ratio of their area and the ratio of the corresponding sides. The area is always measured in square units, hence this ratio is the square of the ratio of their corresponding sides.

Theorem 11.13

The ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides.

Given : ΔABC ~ ΔDEF

To prove : $\dfrac{Area\ of\ \Delta ABC}{Area\ of\ \Delta DEF}=\dfrac{AB^2}{DE^2}=\dfrac{BC^2}{EF^2}=\dfrac{AC^2}{DF^2}$

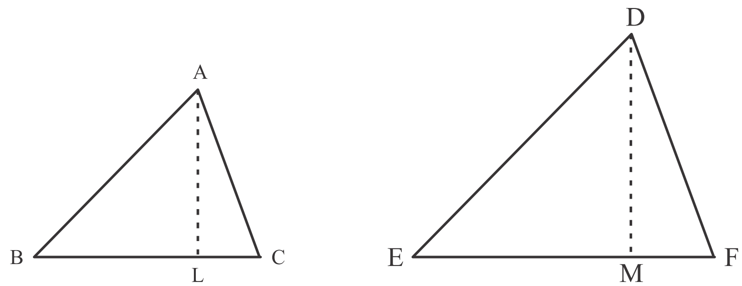


Fig. 11.40

Construction : Draw $AL \perp BC$ and $DM \perp EF$

Proof : ΔABC ~ ΔDEF (given)

$\therefore \quad \angle A=\angle D, \angle B=\angle E,$ and $\angle C=\angle F$ . . . (i)

Also $\dfrac{AB}{DE}=\dfrac{BC}{EF}=\dfrac{CA}{FD}$ . . . (ii)

Now in ΔALB and ΔDME

$\angle ALB=\angle DME$ (90° each)

$\angle B=\angle E$ (corresponding angles of similar triangles)

$\therefore \quad \Delta ALB \sim \Delta DME$ (*AA* similarity criterion)

$\Rightarrow \quad \dfrac{AL}{DM}=\dfrac{AB}{DE}$ . . . (iii)

from (ii) and (iii) we get

$$\frac{AB}{DE}=\frac{BC}{EF}=\frac{AC}{DF}=\frac{AL}{DM} \qquad . . . (iv)$$

Now, $\dfrac{Area\ of\ \Delta ABC}{Area\ of\ \Delta DEF}=\dfrac{\frac{1}{2}BC\times AL}{\frac{1}{2}EF\times DM}$ (area of triangle $=\frac{1}{2}\times$ base $\times$ height)

$$=\frac{BC}{EF}\times\frac{AL}{DM}=\frac{BC}{EF}\times\frac{BC}{EF}=\frac{BC^2}{EF^2}$$

But $\dfrac{AB}{DE}=\dfrac{BC}{EF}=\dfrac{AC}{DF}$

$$\Rightarrow \quad \frac{AB^2}{DE^2} = \frac{BC^2}{EF^2} = \frac{AC^2}{DF^2}$$

$$\Rightarrow \quad \frac{\text{area of } \Delta ABC}{\text{area of } \Delta DEF} = \frac{AB^2}{DE^2} = \frac{BC^2}{EF^2} = \frac{AC^2}{DF^2}$$ **Hence Proved.**

With the help of this theorem we can also find the other results.

Corollary 11.2. The ratio of the areas of two similar triangles is equal to the ratio of the square of the perpendiculars drawn form corresponding vertex to the opposite sides.

Corollary 11.3. The ratio of areas of two similar triangles is equal to the square of their corresponding medians.

Corollary 11.4. The ratio of the areas of two similar triangles is equal to the ratio of the square of their corresponding angles bisectors.

11.5.3. Justification of Bodhayan theorem (pythagoras theorem) By similarity criterion

In your earlier classes, you have learnt about the Bodhayan theorem. You have proved it in class IX and solved many related problems. Now, using the similarity criterion of triangles we shall prove it again.

Theorem 11.9

In a right triangle, the square of hypotenuse is equal to the sum of the square of remaining two sides.

Given : ΔABC right angled at B.

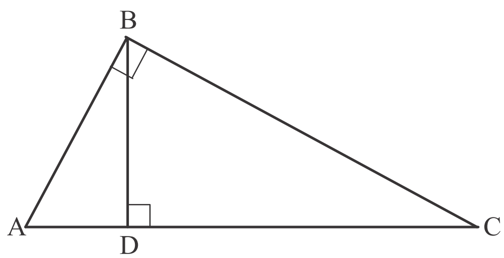


Fig. 11.41

To prove : $AC^2 = AB^2 + BC^2$

Construction : Draw $BD \perp AC$

Proof : In ΔADB and ΔABC

$\angle ADB = \angle ABC$ ($90°$ each, by construction)

$\angle A = \angle A$ (common)

By AA similarly criterion

$\Delta ADB \sim \Delta ABC$

$\Rightarrow \quad \frac{AD}{AB} = \frac{AB}{AC}$ (By BPT)

$\Rightarrow \quad AB^2 = AC \times AD$ .....(i)

In ΔABC and ΔCDB

$\angle ABC = \angle CDB$ ($90°$ each by construction)

$\angle C = \angle C$ (common)

By AA similarity criterion

$\Delta ABC \sim \Delta BCD$

$\frac{BC}{AC} = \frac{CD}{CB}$ (By BPT)

$\Rightarrow \frac{BC}{AC} = \frac{DC}{BC}$

$\Rightarrow$ $BC^2 = AC \times DC$ .....(ii)

By adding (i) and (ii)

$AB^2 + BC^2 = AC \times AD + AC \times DC$

$\Rightarrow$ $AB^2 + BC^2 = AC\ (AD + DC)$

$\Rightarrow$ $AB^2 + BC^2 = AC \times AC$

$\Rightarrow$ $AC^2 = AB^2 + BC^2$ **Hence Proved.**

Let us prove to converse of this theorem by using same criterion.

Theorem 11.10

Converse of Bodhayan theorem :

In a triangle, if square of one side is equal to the sum of the square of the other sides, then the angle opposite the first side is a right angle.

In a triangle, if sum of the square of two sides is equal to the square of third side the triangle is right angled.

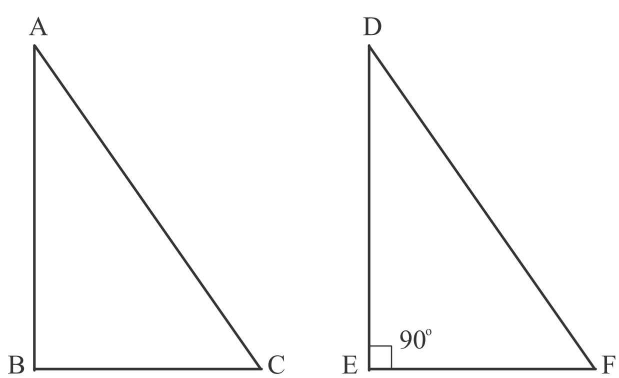


Fig. 11.42

Given : In a ΔABC $AC^2 = AB^2 + BC^2$

To prove : ΔABC is a right angled triangle.

Construction : Construct a ΔDEF, such that

$DE = AB,\ EF = BC$ and $\angle E = 90°$

Proof : $DF^2 = DE^2 + EF^2$ (By Bodhayan theorem)

$\Rightarrow$ $DF^2 = AB^2 + BC^2$ (By construction)

But $AC^2 = AB^2 + BC^2$ (given)

$\therefore$ $AC^2 = DF^2$

$\Rightarrow$ $AC = DF$ ...(1)

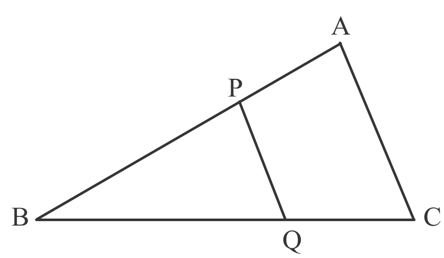


Fig. 11.43

Now, in ΔDEF

$AB = DE,\ BC = EF$ (By construction)

and $AC = DF$ (By i)

$\Delta ABC \cong \Delta DEF$

$\Rightarrow$ $\angle B = \angle D = 90°$

Hence ΔABC is right angled.

11.5.4. Some Important results based on Bodhayan theorem

Theorem11.11

In an obtuse angled ΔABC in which $\angle B$ is the obtuse angle and $AD \perp BC$, then

$$AC^2 = AB^2 + BC^2 + 2BC \times BD$$

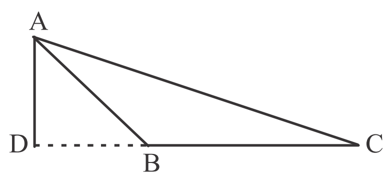


Fig. 11.43

In an obtuse triangle, the square of the side opposite to obtuse angle is equal to the sum of the squares of other two side plus twice the product of one side and the projection of other on first.

Given : ΔABC, obtuse angled at B.

To prove : $AC^2 = AB^2 + BC^2 + 2BC \times DB$

Proof : In $\Delta ADB,\ \angle D = 90°$ (given)

$\therefore$ $AB^2 = AD^2 + DB^2$ .....(i)

Now in ΔADC

$AC^2 = AD^2 + DC^2$

$\Rightarrow$ $AC^2 = AD^2 + (DB + BC)^2$

$\Rightarrow$ $AC^2 = AD^2 + DB^2 + BC^2 + 2DB \times BC$

$\Rightarrow$ $AC^2 = [AD^2 + DB^2] + BC^2 + 2DB \times BC$

$\Rightarrow$ $AC^2 = AB^2 + BC^2 + 2BC \times DB$ **Hence Proved**

Theorem 11.12 :

In an acute triangle ABC, $AD \perp BC$ then $AC^2 = AB^2 + BC^2 - 2BC \times BD$.

In an acute triangle, the square of the side opposite to an acute angle is equal to the sum of the squares of other two side minus twice the product of one side and the projection of other on first.

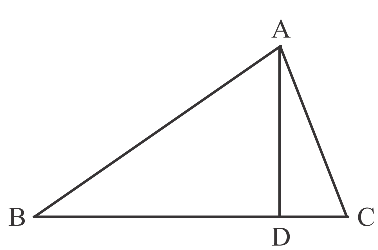


Fig. 11.44

Given : Acute ΔABC in which $AD \perp BC$

To prove : $AC^2 = AB^2 + BC^2 - 2BC \times BD$

Proof : In ΔABD

$AB^2 = AD^2 + BD^2$ .....(i)

Similarly $AC^2 = AD^2 + DC^2$

$\Rightarrow AC^2 = AD^2 + (BC - BD)^2$

$\Rightarrow AC^2 = AD^2 + BC^2 + BD^2 - 2BC \times BD$ From (i)

$\Rightarrow AC^2 = (AD^2 + BD^2) + BC^2 - 2BC \times BD$

$\Rightarrow AC^2 = AB^2 + BC^2 - 2BC \times BD$

Hence $AC^2 = AB^2 + BC^2 - 2BC \times BD$ **Hence Proved.**

Corollary : In any triangle the sum of the square of any two sides is equal to twice the square of half of the third side together with twice the square of the median which bisects the third side.

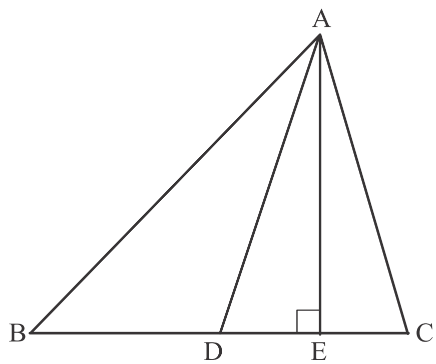


Fig. 11.45

Given : AD is median in the ΔABC.

To Prove : $AB^2 + AC^2 = 2\left[AD^2 + \left(\frac{BC}{2}\right)^2\right]$

Construction : Draw $AE \perp BC$

Proof : Since $\angle AED = 90°$, therefore in ΔADE

$\angle ADE < 90° \Rightarrow \angle ADB > 90°$

Thus ΔADB is an obtuse angled triangle

and ΔADC is an acute angled triangle

$\therefore$ In obtuse ΔABD, Producing BD and draw $AE \perp BD$.

$AB^2 = AD^2 + BD^2 + 2BD \times DE$ (By theorem 11.11) . . .(i)

In acute ΔACD, $AE \perp CD$

$AC^2 = AD^2 + DC^2 - 2DC \times DE$

$\Rightarrow AC^2 = AD^2 + BD^2 - 2BD \times DE$ [$\because$ CD = BD] . . . (2)

Adding (i) and (ii), we get

$AB^2 + AC^2 = AD^2 + BD^2 + 2BD \times DE + AD^2 + BD^2 - 2BD \times DE$

$\Rightarrow \quad AB^2 + AC^2 = 2AD^2 + 2BD^2$

$\Rightarrow \quad AB^2 + AC^2 = 2AD^2 + 2\left(\frac{BC}{2}\right)^2$

$\Rightarrow \quad AB^2 + AC^2 = 2\left[AD^2 + \left(\frac{BC}{2}\right)^2\right]$

Hence $\quad AB^2 + AC^2 = 2\left[AD^2 + \left(\frac{BC}{2}\right)^2\right]$

or $\quad AB^2 + AC^2 = 2(AD^2 + BD^2)$ **Hence Proved**

Illustrative Examples

Example 1 : A ladder 10 meter long reaches a window 8 m above the ground. Find the distane of the foot of the ladder form the base of the wall.

Solution : According to the fig. ΔABC is right angled triangle in which $\angle B = 90°$

Now,By the Bodhayan theorem

$AC^2 = AB^2 + BC^2$

$\Rightarrow \quad BC^2 = AC^2 - AB^2$

$\Rightarrow \quad BC^2 = 10^2 - 8^2$

$\Rightarrow \quad BC^2 = 100 - 64$

$\Rightarrow \quad BC^2 = 36$

$\Rightarrow \quad BC = \sqrt{36} = 6$ m.

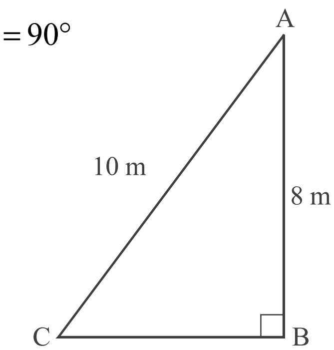


Fig.11.46

Example 2 : An aeroplane leaves an airport and flies due to north at a speed of 1000 km per hour. At the same time, another aeroplane leaves the same airport and flies due west at a speed of 1200 km per hour. How far apart the two planes after $1\frac{1}{2}$ hours?

Solution : Speed of the first aeroplane = 1000 km/h

Distance covered by it in $1\frac{1}{2}$ hour to north $= 1000 \times \frac{3}{2} = 1500$ km.

Speed of the second areoplane = 1200 km/h

Distance covered by it in $1\frac{1}{2}$ hour to east and = $1200 \times \frac{3}{2} = 1800$ km.

In this way we obtain a right triangle as shown in the fig. 11.47. using Bodhayan theorem, we get.

$AB^2 = AC^2 + BC^2$

$AB^2 = 1500^2 + 1800^2$

$= 2250000 + 3240000$

$= 5490000$

$AB = \sqrt{5490000} = 2343.1$ km. (approx.)

Hence, the distance between the two areoplanes

= **2343.1 km.** (approx.)

Example 3 : If $\Delta ABC \sim \Delta DEF$ in which AB = 2.2 cm and DE = 3.3 cm find the ratio of the areas of ΔABC and ΔDEF.

Solution : We know that the ratio of the areas of two similar triangles is equal to squares of their corresponding sides.

So, $$\frac{area\ of\ \Delta ABC}{area\ of\ \Delta DEF} = \frac{AB^2}{DE^2} = \frac{(2.2)^2}{(3.3)^2} = \left(\frac{22}{33}\right)^2$$

$$\Rightarrow \quad \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

Hence, the ratio of the areas of two triangles = 4 : 9

Example 4 : If areas of two similar triangles ABC abd PQR are 36 cm^2 and 43 cm^2 respectively, find the ratio of their corresponding sides.

Solution : We know that the ratio of corresponding sides of two similar triangles is equal to the ratio of their areas.

$$\therefore \quad \frac{AB^2}{PQ^2} = \frac{Area\ of\ \Delta ABC}{Area\ of\ \Delta PQR}$$

$$\Rightarrow \quad \frac{AB^2}{PQ^2} = \frac{36}{49} \Rightarrow \frac{AB}{PQ} = \sqrt{\frac{36}{49}} = \frac{6}{7}$$

Hence, the ratio of the corresponding sides = 6 : 7

Example 5 : $\Delta ABC \sim \Delta PQR$ area ΔABC = 16 cm and area ΔPQR = 9 cm also if AB = 2.1 cm, then find the length of the side PQ.

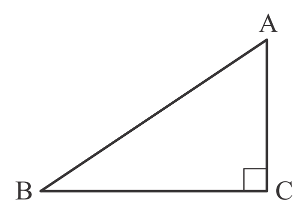


Fig. 11.47

Solution : We know, $$\frac{Area\ of\ \Delta ABC}{Area\ of\ \Delta PQR} = \frac{(AB)^2}{(PQ)^2}$$

$$\Rightarrow \quad \frac{16}{9} = \frac{(2.1)^2}{PQ^2}$$

Taking square root of two sides, we get

$$\Rightarrow \quad \frac{4}{3} = \frac{2.1}{PQ}$$

$$\Rightarrow \quad PQ = \frac{2.1 \times 3}{4} = \frac{6.3}{4} = 1.575 \text{ cm.}$$

Hence, the length of PQ = 1.6 cm (approx)

Example 6 : In fig. 11.48, in $\Delta ABC,$ line parallel to BC intersects AB and AC at D and E respectively and divides AB in the ratio 1:2, find the ratio of the areas of trapezium BDEC and ΔADE formed.

Solution : Since $\ell \parallel BC$

So, $\angle ADE = \angle B$ and $\angle AED = \angle C$ (corresponding angles)

In ΔADE and ΔABC

$\angle ADE = \angle B$

and $\angle AED = \angle C$

$\therefore$ $\Delta ADE \sim \Delta ABC$ (By AA similarity criterion)

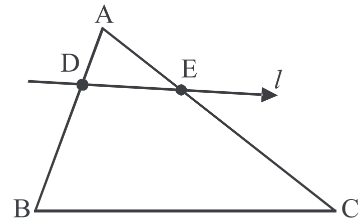


Fig. 11.48

So, $$\frac{Area\ of\ \Delta ADE}{Area\ of\ \Delta ABC} = \frac{AD^2}{AB^2} \quad \text{. . . (i)}$$

But $$\frac{AD}{AB} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{AD}{AD+DB} = \frac{1}{1+2} = \frac{1}{3} = \frac{AD}{AB} \quad \text{. . . (ii)}$$

From (i) and (ii), we get

$$\frac{Area\ of\ \Delta ADE}{Area\ of\ \Delta ABC} = \frac{1^2}{3^2} = \frac{1}{9}$$

$\Rightarrow$ Area ΔABC = 9 × Area ΔADE . . . (iii)

But area trapezium $BDEC$ = area ΔABC – area ΔADE

from (iii) we get

Area trapezium $BDEC$ = 9 × area ΔADE – area ΔADE = 8 × area ΔADE

$$\Rightarrow \quad \frac{area\ trapezium\ BDEC}{area\ \Delta ADE} = \frac{8}{1}$$

Hence, the ratio of areas of trapezium BDEC and ΔADE = 8:1

Example 7 : In fig. 11.49, in a ΔABC, a line segment PQ is drawn parallel to side AC, which intersects AB such that $\frac{BP}{BA} = \frac{1}{\sqrt{2}}$, then prove that PQ, also divides the triangle ABC in the same ratio.

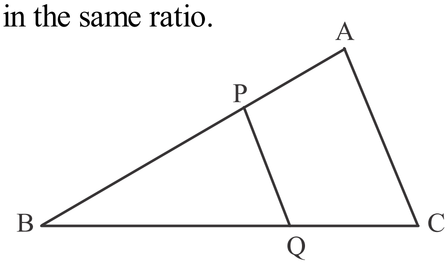


Fig. 11.49

Solution : **Given :** $PQ \parallel AC$

$\therefore$ $\angle A = \angle BPQ$

and $\angle C = \angle BQP$ (corresponding angles)

Also. $$\frac{BP}{BA} = \frac{1}{\sqrt{2}}$$

So, $\Delta BAC \sim \Delta BPQ$ (By AA similarity criterion)

To prove: area ΔBPQ = area trapezium $PACQ$

or area trapezium $PACQ$ $= \frac{1}{2} \Delta\, BAC$

= area of ΔBPQ (given)

Hence, we are to prove $2 \times$ area ΔBPQ = Area of ΔBAC

Proof : Since $\Delta BAC \sim \Delta BPQ \Rightarrow \Delta BPQ \sim \Delta BAC$

$$\therefore \quad \frac{Area\ of\ \Delta BPQ}{Area\ of\ \Delta BAC} = \frac{(1)^2}{(\sqrt{2})^2}$$

$$\Rightarrow \quad \frac{Area\ of\ \Delta BPQ}{Area\ of\ \Delta BAC} = \frac{1}{2}$$

$\Rightarrow \quad 2\times$ area ΔBPQ = area ΔBAC **Hence Proved**

Example 8 : Prove that if area of two simialr triangles are equal, they are congurent.

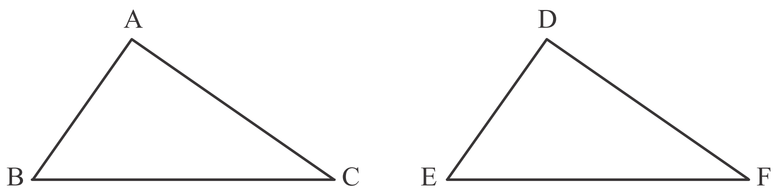


Fig. 11.50

Solution : **Given :** $\Delta ABC \sim \Delta DEF$

and area ΔABC = area ΔDEF

To prove : $\Delta ABC \cong \Delta DEF$

Proof : $\because$ $\Delta ABC \sim \Delta DEF$

$\therefore$ ΔABC and ΔDEF are equilateral triuangles

$$\frac{Area\ of\ \Delta ABC}{Area\ of\ \Delta DEF} = \frac{BC^2}{EF^2}$$

$\Rightarrow \quad 1 = \dfrac{BC^2}{EF^2}$ (Areas of two triangles are equal)

$\Rightarrow \quad BC^2 = EF^2$ or $BC = EF$ . . . (i)

Now in triangles ΔABC and ΔDEF, we have

$\angle B = \angle C$ (given)

$BC = EF$ (proved)

$\angle C = \angle F$ (given)

$\therefore$ $\Delta ABC \cong \Delta DEF$ (By *SAS* congurency)

Example 9 : ΔABC is a isoceles triangle with $\angle C = 90°$. Prove that $AB^2 = 2AC^2$.

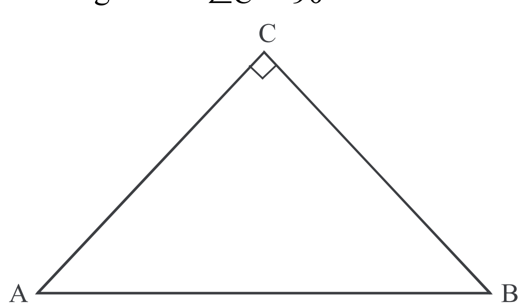


Fig. 11.51

Solution : We have ΔABC in which

$$\angle C = 90°, AC = BC$$

Now using Bodhayan theorem, we get

$AB^2 = AC^2 + BC^2$

$\Rightarrow$ $AB^2 = AC^2 + AC^2$ (since $BC = AC$)

$\Rightarrow$ $AB^2 = 2AC^2$ **Hence Proved.**

Example 10 : If D is a point on the side BC of a equilateral ΔABC such that $BD = \frac{1}{3}BC$ then prove that $9AD^2 = 7AB^2$.

Solution : Given : ΔABC is equilateral triangle

Construction : $AE \perp BC$ from A

To Prove : $9AD^2 = 7AB^2$

Proof : $\because$ In an equilateral triangle perpendicular drawn form a vertex bisects the opposite side.

$\Rightarrow$ $BE = EC = \frac{1}{2}BC$ (By construction)

and $BD = \frac{1}{3}BC$ (given)

Also $AB = BC = CA$ (given)

In right ΔABE, $AB^2 = AE^2 + BE^2$

$\Rightarrow$ $AE^2 = AB^2 - BE^2$

$\Rightarrow$ $AE^2 = AB^2 - \left(\frac{1}{2}BC\right)^2$ $\left[\because BE = \frac{1}{2}BC\right]$

$\Rightarrow$ $AE^2 = AB^2 - \frac{BC^2}{4}$

$\Rightarrow$ $AE^2 = \frac{4AB^2 - BC^2}{4}$ . . . (i)

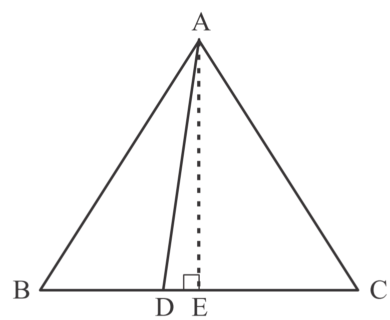


Fig. 11.52

In right angled ΔADE.

$\Rightarrow$ $AD^2 = AE^2 + DE^2$

$\Rightarrow$ $AE^2 = AD^2 - DE^2$

$\Rightarrow$ $AE^2 = AD^2 - (BE - BD)^2$

$\Rightarrow$ $AE^2 = AD^2 - \left(\frac{1}{2}BC - \frac{1}{3}BC\right)^2$ [$\because$ $BE = \frac{1}{2}$ BC एवं $BD = \frac{1}{3}$ BC]

$\Rightarrow$ $AE^2 = AD^2 - \left(\frac{BC}{6}\right)^2$

$\Rightarrow$ $AE^2 = \frac{36AD^2 - BC^2}{36}$ .....(ii)

Now from (i) and (ii) we get.

$$\frac{4AB^2 - BC^2}{4} = \frac{36AD^2 - AB^2}{36}$$

$$\Rightarrow \quad \frac{4AB^2 - AB^2}{4} = \frac{36AD^2 - AB^2}{36} \qquad [\because \text{ AB = BC = CA}]$$

$$\Rightarrow \quad \frac{3AB^2}{4} = \frac{36AD^2 - AB^2}{36}$$

$\Rightarrow$ $27\ AB^2 = 36\ AD^2 - AB^2$

$\Rightarrow$ $28\ AB^2 = 36\ AD^2$

$\Rightarrow$ $7\ AB^2 = 9\ AD^2$ (Dividing bothside by 4)

Hence $9AD^2 = 7AB^2$ **Hence proved.**

Example 11 : In a right angled ΔABC, $\angle B = 90°$. If points D and E on the AB and BC respectively then prove $AE^2 + CD^2 = AC^2 + DE^2$ (see fig. 11.53)

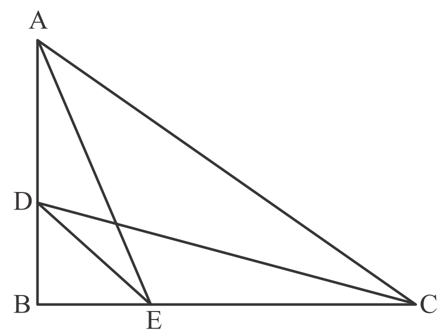


Fig. 11.53

Solution : ΔABE is a right angled with $\angle B = 90°$

$\therefore$ $AE^2 = AB^2 + BE^2$ . . . (i)

again in right angled $\Delta DBC, \angle B = 90°$

$CD^2 = BD^2 + BC^2$ . . . (ii)

On adding (i) and (ii) we get

$AE^2 + CD^2 = (AB^2 + BC^2) + (BE^2 + BD^2)$ . . . (iii)

Similarly in right angled triangles ΔABC and ΔDBE

$AC^2 = AB^2 + BC^2$ and $DE^2 = BE^2 + BD^2$ . . . (iv)

From $AE^2 + CD^2 = AC^2 + DE^2$ **Hence proved.**

Exercise 11.4

1. State whether the following statements are true or false? And justify your answer if possible.
   (i) If ratio of corresponding sides of two similar triangles then the ratio of their areas will be 4 : 9.
   (ii) If in two ΔABC and ΔDEF, in $\frac{area\,\Delta ABC}{area\,\Delta DEF} = \frac{AB^2}{DE^2} = \frac{9}{4}$ then $\Delta ABC \cong \Delta DEF$.

(iii) Ratio of the areas of two similar triangles is equal to the ratio of the squares of their corresponding sides.

(iv) If $\Delta ABC \sim \Delta AXY$ and their areas are equal then the sides XY and BC may coinsides.

2. If $\Delta ABC \sim \Delta DEF$ also area ΔABC = 64 cm^2, area ΔDEF = 121 cm^2 and EF = 15.4 cm. find BC.

3. ΔABC and ΔDBC are at the same base BC. If AD and BC intersect each other at O. then prove.

$$\frac{Area\,\Delta ABC}{Area\,\Delta DBC} = \frac{AO}{DO}$$

4. Solve the following problems.

(i) In ΔABC ,DE || BC and AD : DB = 2 : 3, find the ratio of the areas of ΔADE and ΔABC

(ii) Perpendiculars PA and QB are drawn to two ends of a line segment AB. If P and Q are at the opposite sides of AB and their joining line intersects AB at O also PO = 5 cm, QO = 7 cm, area ΔPOB = 150 cm^2, find the area ΔQOA.

(iii) In the fig. 11.54 given below, find the value of x in the terms of a, b and c.

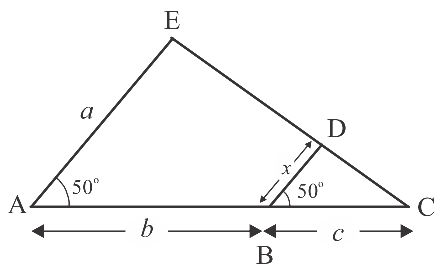


Fig. 11.54

5. In $\Delta ABC,\ \angle B = 90°$ and $BD \perp AC$ then prove $\Delta ADB \sim \Delta BDC$.

6. Prove that area of an equilateral triangle formed on one side of a square is equal to the half of the area of the equilateral triangles constructed on it diagonal.

Miscellaneous Exercise 11

1. In the fig. 11.55 $DE \parallel AD$, AD = 4 cm. DB = 6 cm, and AE = 5 cm, then measure of EC will be :

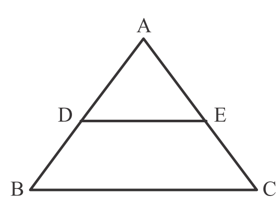


Fig. 11.55

(a) 6.5 cm. (b) 7.0 cm. (c) 7.5 cm. (d) 8.0 cm.

2. In the fig. 11.56 AD is the bisector of $\angle A$, if $AB = 6$ cm, $BD = 8$ cm, $DC = 6$ cm, then the length of AC will be :

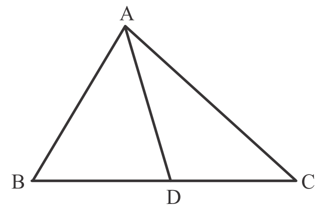


Fig. 11.56

(a) 4.0 cm (b) 4.5 cm (c) 5 cm (d) 5.5 cm.

3. In the fig. 11.57, if $DE \parallel BC$ the value of x is :

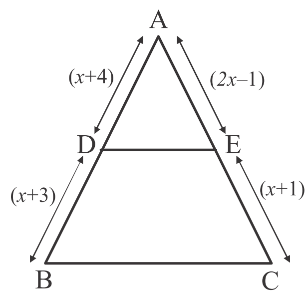


Fig. 11.57

(a) $\sqrt{5}$ (b) $\sqrt{6}$ (c) $\sqrt{3}$ (d) $\sqrt{7}$

4. In the fig. 11.58m if $AB = 3.4$ cm, $BD = 4$ cm, $BC = 10$ cm, then the measure of AC is.

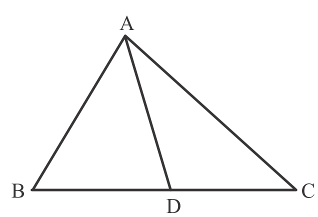


Fig. 11.58

(a) 5.1 cm (b) 3.4 cm (c) 6 cm (d) 5.3 cm.

5. The areas of two triangles are 25.25 cm^2 and 36 cm^2 respectively. If a median of smaller triangles is 10 cm, the corresponding median of larger triangle is :

(a) 12 cm (b) 15 cm (c) 10 cm (d) 18 cm

6. In a trapezium $ABCD$, $AB \parallel DC$ and its diagonals intersects each other at O.If $AB = 6$ cm, the ratio of the areas DAOB and DCOD is :

(a) 4 : 1 (b) 1 : 2 (c) 2 : 1 (d) 1 : 4

7. In DABC and DDEF, $\angle A = 50°, \angle B = 70°, \angle C = 60°, \angle D = 60°, \angle E = 70°$ and $\angle F = 50°$. State which of the following is true :

(a) $\Delta ABC \sim \Delta DEF$ (b) $\Delta ABC \sim \Delta EDF$ (c) $\Delta ABC \sim \Delta DEF$ (d) $\Delta ABC \sim \Delta FED$

8. If $\Delta ABC \sim \Delta DEF$, and $AB = 10$ cm, $DE = 8$ cm, then area of DABC : area of DDEF will be :
(a) 25 % 16 (b) 16 % 25 (c) 4 % 5 (d) 5 % 4

9. If DABC DDEF , also D and E points on the lines AB and AC such that $DE \parallel BC$ and AD = 8 cm, AB= 12 cm and AE = 12 cm, the measure of CE is :
(a) 6 cm (b) 18 cm (c) 9 cm (d) 15 cm.

10. The length of the shadow of a vertical rod of length 12 cm on the ground is 8 cm. If at the same time the length of the shadow of a minaret is 40 cm. The height of the minerat is :
(a) 60 m (b) 60 cm (c) 40 cm (d) 80 cm.

11. In a DABC, if point D is on the BC such that $\frac{AB}{AC} = \frac{BD}{DC}$ anda $\angle B = 70°, \angle C = 50°$. Find $\angle BAD$.

12. In DABC, $DE \parallel BC$ also $AD = 6$ cm. $DB = 9$ cm and $AE = 8$, find the length of AC.

13. In DABC, AD is the bisector of $\angle A$, also $AB = 8$ cm. BD = 5 cm and DC = 4 cm, find the length of AC.

14. If the ratio of the heights of two similar triangles is 4 : 9,find the ratio of their areas.

Important Points

1. It is not necessary that two similar figures have same shape and size.
2. Two polygons are similar if their corresponding sides are proportion and corresponding angles are equal.
3. Two triangles are similar if their corresponding sides are proportional and corresponding angles are equal.
4. **Thales theorem (basic proportional theorem) :** If a line is drawn parallel to one side of a triangle to intersect the other two sides at distinct points, then the other two sides are divided in the same ratio.
5. If a line divides two sides of a triangle in the same ratio, then the line is parallel to the third.
6. Two straight line figures are similar if their corresponding angles are equal and corresponding sides are in the same ratio.
7. ***AAA* similarity criterion :** If corresponsing angles of two triangles are equal, then the triangle are similar.
8. ***AA* similarity criterion :** If two angles of a triangle are equal to two corresponding angles of another triangle, the triangle are simialr.
9. **SAS simiilarity criterion :** If an angle of a triangle is equal to one angle of another triangle and the sides including this angle are in the same ratio, then the triangles are similar.
10. The ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides.
11. The ratio of the area of two similar triangles is equal to the ratio of the square of their height.
12. In obtuse triangles ABC, $\angle B > 90°$ and $AD \perp BC$ then $AC^2 = AB^2 + BC^2 + 2\ BC \times BD$.
13. In acute triangle ABC $\angle B < 90°$ and $AD \perp BC$ then
$$AC^2 = AB^2 + BC^2 - 2BC \times BD$$

Answer Sheet

Exercise 11.1

1. (i) Similar, (ii) Similar, (iii) Equilateral (iv) (a) Corresponding angles are equal (b) Ratio of corresponding sides is equal.
2. (i) True, (ii) False (iii) False (since, it is not sufficient to be proportional of corresponding sides. (iv) True, (v) False.

Exercise 11.2

1. (i) 20 cm. (ii) 15.6 cm. (iii) 9.9 cm. (iv) $x = 1, \frac{-1}{2}$
2. (i) Parallel, (ii) Not parallel, (iii) Not parallel, (iv) Parallel.

Exercise 11.3

1. If $\angle A = \angle P$ and $\angle C = \angle R$, then $\angle B$ and $\angle Q$ will be automatically equal.
2. $\Delta ABC \sim \Delta DEF$. (since angles must be taken consecutively

   $\therefore \quad \Delta ABC \sim \Delta DFE$
3. The given ratio can not be expressed for $\Delta ABC \sim \Delta FDE$ the serial must be

$$\frac{AB}{FD} = \frac{BC}{DE} = \frac{CA}{EF}$$

4. The statement is not true, for similar triangles because one angle must be equal and the raito of the sides including this angle must be same.
5. In two equiangular triangles, corresponding angles are equal. If corresponding angles are equal, then both triangles are similar.
6. (i) and (viii) $\Delta ABC \sim \Delta QRP$, (ii) and (vii) $\Delta MPN \sim \Delta ZYX$, (iii) and (v) $\Delta PQR \sim \Delta EFG$, (iv) and (vi) $\Delta EDF \sim \Delta NML$
7. $\angle P = \angle RTS, \angle Q = \angle RST$
8. $\Delta ADC \sim \Delta BEC$
10. 1.6 m.
11. 84 m.

Exercise 11.4

1. (i) False the ratio of the squares of the sides i.e. 16 : 8.

   (ii) False because the ratio of the corresponding sides is $= \frac{3}{2}$ where for similarity the ratio is 1 : 1

   (iii) False

   (iv) True
11. 11.2 cm
4. (i) 4 : 25 (ii) 294 cm^2 (iii) $x = \frac{ac}{b+c}$

Miscellaneous Exercise 11

1. (c)	2. (b)	3. (d)	4. (a)	5. (a)	6. (a)	7. (d)
8. (a)	9. (a)	10. (a)	11. 20	12. 20 cm	13. 6.4 cm	14. 16: 81

12

Circle

12.01. Introduction

We have seen many objects form childhood those are round in shape such as bangles, coins, wheels of a vehicles, plate, buttons of shirts etc.

Also, the path of the tip of a needle in a wrist is round in shape. The path traced by the tip of the needle is known as circle.

In this chapter, we will discuss about circle and its properties .

12.02. Circle and its parts

Take a compass, and fix a pencil in it, Put its pointed leg on a point on a sheet of paper, rotate the other leg through one revolution. You will see a shape on a paper, it is a circle. You kept in mind that tip of the pencil is produced a point. Actually a circle is group (set) of infinite points which make after rotate other leg (pencil) through revolution. or the collection of all points in a plane, which are at a fixed (constant) distance from a fixed point in a plane, is called a circle.

The fixed point is called a centre of the circle and the fixed distance is called the radius of the circle. In a figure 12.01, O is the centre of circle and Length OP is the radius of circle.

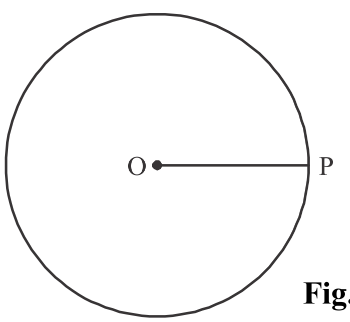


Fig. 12.01

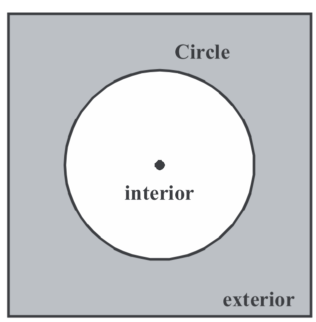


Fig. 12.02

A circle divides the plane on which it lies into three parts. In figure 12.02, They are

(i) inside the circle, which is called the interior of the circle

(ii) the circle

(iii) outside the circle, which is called exterior part of the circle.

the circle and its interior make up the circular region.

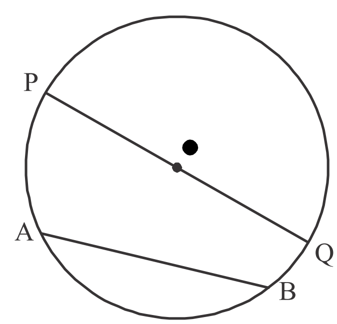


Fig. 12.03

Chord and diameter - If you take two points A and B on a circle, then the line segment AB is called a chord. If the chord, which passes through the centre of the circle, is called diameter of the circle. see the Fig 12.03, chord PQ is a diameter of the circle

Activities –

(i) You make circles with different-different radius in your exercise book and in every circle draw two chords. then measure of all chords. Do you find any other chord of the circle longer than a diameter, exactly No, a diameter is the longest chord of circle, which is equal to two times the radius.

(ii) Draw a circle and see how many diameters can be drawn. Does there exist more than one diamtere? Yes, you can draw infinite numbers of diameters.(see figure 12.04)

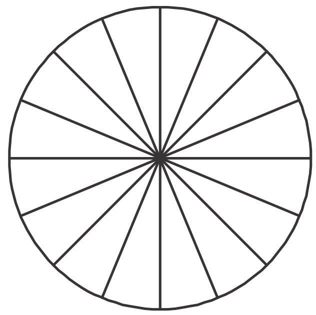


Fig.12.04

Arc : In a figure12.05. There are two points P and Q on the circle. which divides the circle into two parts. You find that one part is longer and the other part is smaller.

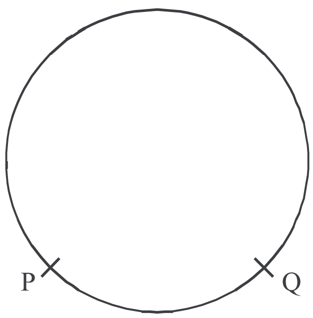


Fig. 12.05

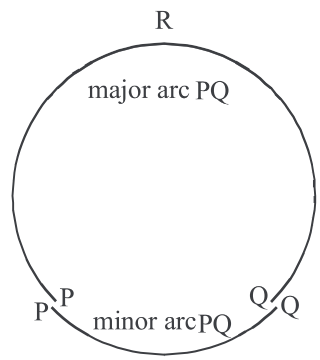


Fig. 12.06

If both the arc viewed sparately according figure 12.06 the shorter one is called minor arc PQ and denoted by PQ. and the longer one is called major arc PQ and denoted by PRQ where R is some point on the arc between P and Q.

When P and Q are ends of a diameter, then both arcs are equal and each is called a semi circle. see Figure 12.07

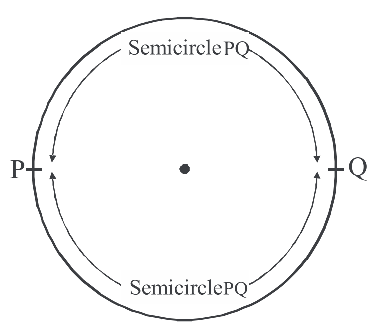


Fig. 12.07

Exercise 12.1

1. **Fill in the blanks :**

(i) Centre of a circle is situated in .... of the circle. (exterior / interior)

(ii) A point whose distance from the center of the circle is grearter then its radius lies in .... of the circle.

(exterior / interior)

(iii) The longest chord of a circle is a .... of the circle.

(iv) An arc is a .... when its ends are the ends of a diameter of circle.

(v) A circle divides the plane, on which it lies in .... parts.

2. **Write true or False : Give reasons for your answers.**

(i) Line segment joining the centre to any point on the circle is a radius of the circle.

(ii) In a circle number of equal chords are finite.

(iii) If a circle is divided into three equal arcs, each is a major arc.

(iv) A chord of a circle,which is twice as long as its radius, is a diameter of the circle.

(v) A circle is a plane figure.

(vi) In a plane the set of points are constant distance from the fixed point on the same plane, it is diamter.

(vii) The chord which lies on the centre, it is known as radius.

12.03 Angle Subtended by a Chord at a Point

If P , Q and R lie on the plane which are not in a straight line. Join PR and QR. Then $\angle PRQ$ is called the angle subtended by the line segment PQ at a point R. (see figure 12.08). In Figure 12.09, $\angle AOB$ is the angle subtended by the chord AB at a centre O, $\angle ADB$ and $\angle ACB$ are respectively the angles subtended by AB at points D and C on the major and minor arcs AB.

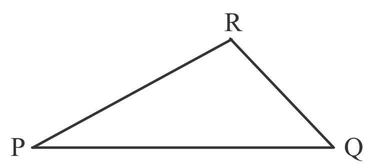


Fig. 12.08

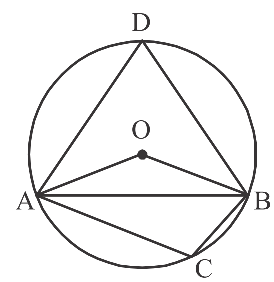
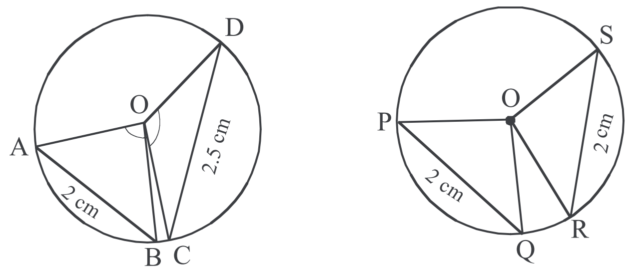


Fig. 12.09

Activity : Let us examine the relationship between the size of the chord and the angle subtended by it at the centre.



(i) Figure 12.10 (ii)

In Figure 12.10 (i) Two chords AB and CD have length 2 cm and 2.5 cm. Here AB < CD then $\angle AOB$ angle subtended by the chord AB on the centre and $\angle COD$ angle subtended by the chord CD on the centre. Have you observed any relation between these angles? Yes, $\angle AOB < \angle COD$. But in Figure 12.10 (ii) What you have seen, You see that $PQ = RS$ then $\angle POQ = \angle ROS$.

or In a circle, the longer chord subtended angle at the centre is greater then the shorter chord subtended angle at the centre.

Let us see to prove these result by the previous results as a theorems.

Theorem 12.1 : Equal chords of a circle subtends equal angles at the centre.

Given : $AB = CD$

To prove : $\angle AOB = \angle COD$

Proof : In ΔAOB and ΔCOD

$OA = OC$ (radii of a circle)

$OB = OD$ (radii of a circle)

$AB = CD$ (given)

$\therefore$ $\Delta AOB \cong \Delta COD$ (SSS rule)

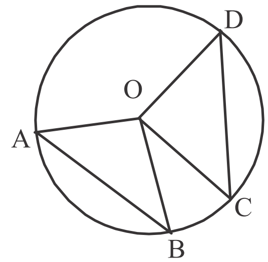


Fig. 12.11

Thus gives $\angle AOB = \angle COD$ (Hence Prove)

Now we can prove its converse.

Theorem 12.2 : If the angles subtended by the chords of a circle at the centre are equal, then chords are equal.

Given : $\angle AOB = \angle COD$

To prove : $AB = CD$

Proof : ΔAOB and ΔDOC

OA = OD (radius of circle)

$\angle AOB = \angle COD$ (given)

OB = OC (radius of a circle)

$\therefore$ $\Delta AOB \cong \Delta DOC$ (SAS rule)

Hence $AB = CD$ **Hence Proved.**

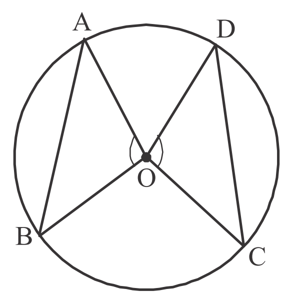


Fig. 12.12

Because the radius of two congruent circles are equal therefore theorem 12.1 and 12.2, we can prove by two congruent circles also.

12.04. Perpendicular from the Centre to a Chord

Theorem 12.3 : The perpendicular from the centre of a circle to the chord bisects the chord.

Given : $OM \perp AB$, AB is a chord

To prove : $AM = BM$

Construct : Join O to A and B

Proof : In ΔOAM and ΔOBM

$OA = OB$ (radius of circles)

$\angle AMO = \angle OMB = 90°$, $OM \perp AB$ (Given)

OM is common

$\Rightarrow$ ΔOAM and ΔOBM are right angle triangles

$\Rightarrow$ $\Delta OAM \cong \Delta OBM$ (RHS rule)

Therefore $AM = BM$ **Hence Proved.**

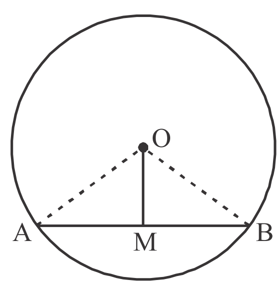


Fig. 12.13

Theorem 12.4 : The line joining the centre of circle to the mid point of chord is perpendicular to the chord.

Given : $AM = BM$

To prove : $OM \perp AB$

Construct : Join O to A and B

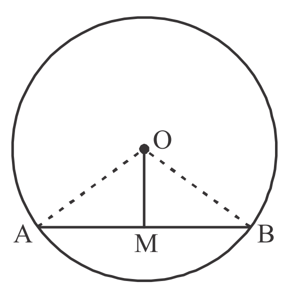


Fig. 12.14

Proof : In ΔOMA and ΔOMB

$OA = OB$ (radius of a circle)

OM is common

AM = BM (given)

i.e. $\Delta OMA \cong \Delta OMB$ (*SSS* rule)

i.e. $\angle OMA = \angle OMB$ (Pair of linear angle)

$\Rightarrow$ $\angle OMA = \angle OMB = 90^{\circ}$

Hence, $OM \perp AB$ **Hence Proved.**

12.05. Circle passes through three points

You have learnt axiom of class IX that two points are sufficient to draw one and only one straight line. A question arises that do you know that how many points are sufficient to draw one and only one circle?

Take a point P. You have seen that there may be as many circles as you like passing through this point (see fig. 12.15 (i))

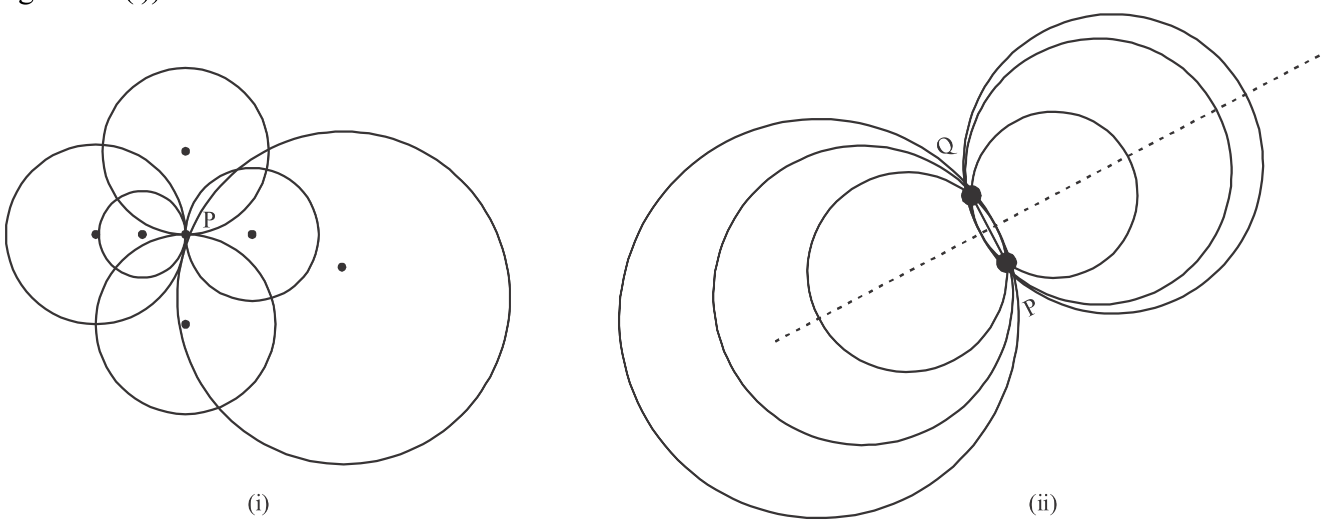


Fig 12.15

Now take two points P and Q. You again see that there may be an infinite number of circles passing through P and Q (see fig. 12.15 (ii)). But you can see that centres of every circlers lie on a straight line which is perpendicular bisector of the line PQ.

In this order, take three points P, Q and R which is collinear points. Can you draw a circle passing through three collinearr points. No, If the points on a line, then the third point will lie inside or outside the circle which passing through two points (see fig. 12.16)

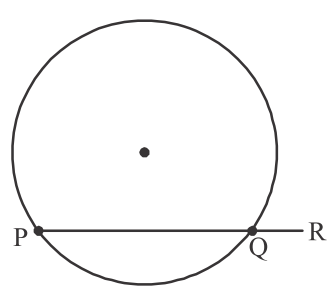


Fig. 12.16

So, let us take three points P, Q and R, which are not on the same line (see fig. 12.17)

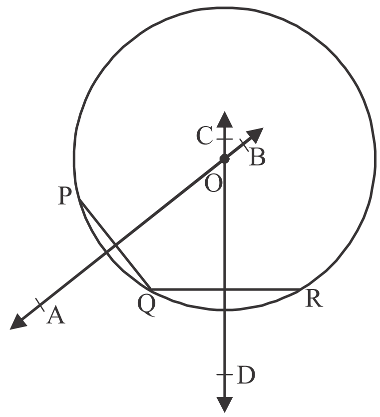


Fig. 12.17

In a fig. 12.15 (ii), You have seen that centres of the circles passes through the two points are lying on the perpendicular bisector.

The centre of the circle which passes through points P, Q and R is lying on the perpendicular bisectors of line segment PQ and QR. So draw perpendicular bisectors of PQ and QR. In fig. 12.17 AB and CD are perpendicular bisectors of PQ and QR respectively.

You have seen that AB and CD are intersect at O. therefore draw the circle form O to P, Q and R (why?) we have studied in class 9 that as every point on the perpendicular bisector of a line segment is equidistant from its end points. i.e., OP = QO ...(i)

and OQ = OR ...(ii)

from (i) and (ii) $\Rightarrow$ OP = OQ = OR

So if draw a circle from O is take OP as radius then it will passes through P, Q and R.

So three points which are not collinear only one circle will pass through them. You know that two lines (perpendicular bisectors) can intersect at only one point which means that the points P, Q and R are at equal distances from the point O. So In other words, there is a unique circle passing through P, Q and R.

This result can be written in the form of following theorem.

Theorem 12.5 There is one and only one circle passing through three given non-collinear points.

12.06. Equal Chords and Their Distances form the Centre

You have studied in class 9, that, Draw infinite line segment from the exterior point to the any line segment, perpendicular will be least . This is the distance from the exterior point to the line segment.

Note : If the point lies on the line, the distance of the line from the point is zero.

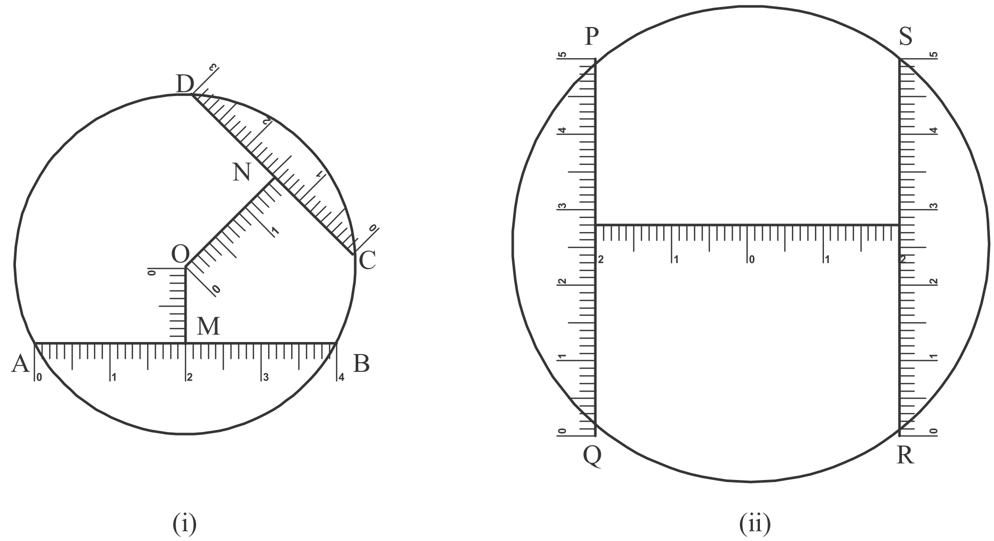


Fig. 12.18

Many chords can be drawn in a circle. You construct a circle and draw more than one chords in the circle. Find the distance of every chord from the centre of circle. What have you observed?

Let us think on an activity-

In fig. 12.18 (i), two chords have 4 cm and 3 cm and distance form the centre is 1 cm and 1.5 cm respectively. so that larger chord is nearer to the centre, then the smaller chord.

In Fig. 12.18 (ii), two chords of equal length 5 cm and having equal distance 2 cm from the centre in othr words equal length of chords are having equal distance form the centre of a circle.

Now again we have to prove for a circle and two congurence circles by the following theorem.

Theorem 12.6 : Equal Chords of a circle are equidistant from the centre.

Given : chord AB = chord CD

To prove : $OM = ON$

Construction : Join OA and OD

Proof : $AM = BM = \frac{1}{2}AB \ldots$ (perpendicular drawn from the centre to the chord is bisecting the chord)

$DN = CN = \frac{1}{2}CD$ (perpendicular drawn from the centre to the chord is bisecting the chrod)

But $AB = CD$ (given)

$AM = DN \ldots$ (i)

In ΔOMA and ΔOND

AM = DN (from (i))

OA = OD (radii of circle)

$\angle OMA = \angle OND = 90^\circ$

$\therefore$ RHS Rule $\Delta OMA \cong \Delta OND$

$\Rightarrow$ OM = ON Hence Proved

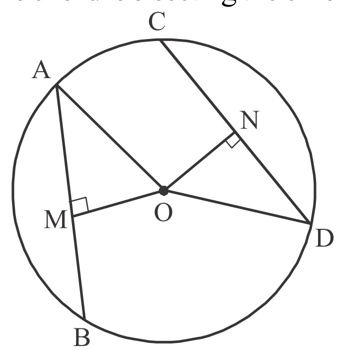


Fig. 12.19

Sub Theorem : In congurent circles equal chords are equidistant from their corresponding centres.

Theorem 12.7 : (Converse of theorem) The chords equidistant from the centre of a circle are equal in length.

Given : Chord AB and CD are having equal distance from the centre O. *i.e.* $OM = ON$

To prove : $AB = CD$

Construction : Join A and D to O

Proof : In ΔOMA and ΔOND

$OM = ON$ (given)

$OA = OD$ (radii of a circle)

$\angle OMA = \angle OND$ ($OM \perp AB$ and $ON \perp CD$)

$\Rightarrow$ RHS rule $\Delta OMA \cong \Delta OND$

$\therefore$ AM = ND

$\Rightarrow$ 2 AM = 2 ND

$\Rightarrow$ AB = CD Hence proved

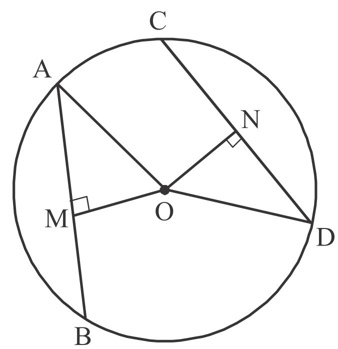


Fig. 12.20

Sub Theorem : Chords of congurent circles are equidistant from the corresponding centres are equal.

Illustrative Example

Example 1. In fig. 12.21, centre of a circle O with radius 5 cm. If $OP \perp AB$, $OQ \perp CD$, AB|| CD, AB = 8 cm and CD = 6 cm. Then find PQ.

Solution : Given : $OP \perp AB$ and $OQ \perp CD$

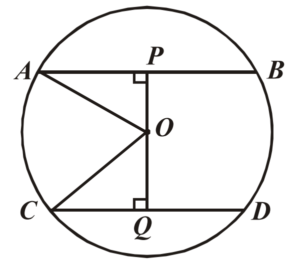


Fig. 12.21

$$\Rightarrow \quad AP = PB = \frac{1}{2}AB = 4\,\text{cm}$$

$$CQ = QD = \frac{1}{2}CD = 3\,\text{cm}$$

and $OA = OC = 5$ cm (radius)

In ΔOPA by bodhayan theorem

$OP^2 = OA^2 - AP^2$

$OP^2 = 5^2 - 4^2$

$= 25 - 16 = 9$

$\therefore$ $OP = 3$ cm

Similarly In ΔOQC,

$OQ^2 = OC^2 - CQ^2$

$OQ^2 = 5^2 - 3^2 = 25 - 9 = 16$

$\therefore$ $OQ = 4$ cm

$\Rightarrow$ Therefore $PQ = OP + OQ = 3 + 4 = 7$ cm

Example 2 : In fig.12.22, arc AB = arc CD, To prove $\angle A = \angle B$.

Solution : Given : arc AB = arc CD

To prove : $\angle A = \angle B$

Proof : We know that equal arcs of a circle subtended equal angles at the centre.

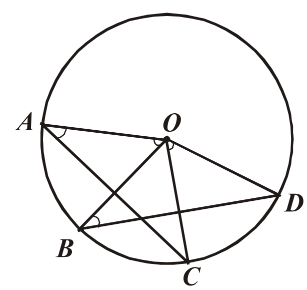


Fig. 12.22

$\therefore$ $\angle AOB = \angle COD$

Adding $\angle BOC$ both sides

$\angle AOB + \angle BOC = \angle BOC + \angle COD$

$\Rightarrow$ $\angle AOC = \angle BOD$ . . . (i)

Again ΔAOC and ΔBOD

$OA = OB$ (radii of circle)

$OC = OD$ (radii of circle)

$\angle AOC = \angle BOD$ (from (i))

$\Delta AOC \cong \Delta BOD$ (SAS Rule)

$\therefore$ (Corresponding angles are equal in congruent triangles)

$\Rightarrow$ $\angle A = \angle B$ **Hence Proved.**

Example 3 : In a circle, if two chords AB and AC are equal, prove that, centre of a circle is lie on the bisector of the $\angle BAC$.

Solution : Given : a circle whose centre O, and have equal chord AB and AC

To prove : centre O, lies on the bisector of $\angle BAC$.

Construction : Join O to C and B.

Proof : In ΔAOB and ΔAOC

$BO = OC$ (radii of circle)

$OA = OA$ (common side)

$AB = AC$ (given)

$\Delta AOB \cong \Delta AOC$ (SSS rule)

Therefore In congruent triangles corresponding angles are equal.

$$\angle OAB = \angle OAC$$

centre O, lies on the bisector of $\angle BAC$

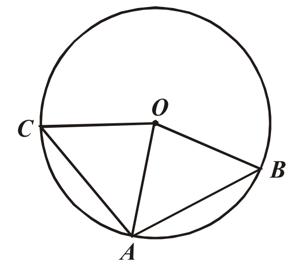


Fig. 12.23

Hence Proved.

Example 4 : If two circles are intersecting each other at two pionts,then prove that line joiniung their centres will be perpendicular bisector of a common chord.

Solution : Given : Fig. 12.24 is having two circles whose centres are O and P respectively, which intersect each other on A and B.

To prove : OP is perpendicular bisector of AB

Construction : Join OA, OB , PA and PB

Proof : In ΔOAP and ΔOBP

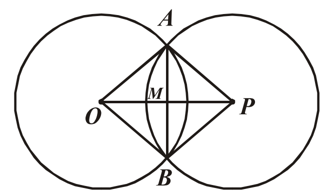


Fig. 12.24

$AO = BO$ (radii of circle)

$PA = PB$ (radii of circle)

$OP = OP$ (common)

$\Delta OAP \cong \Delta OBP$ (From SSS congruence)

Therefore, corresponding angle will be equal of congruence triangles.

$$\angle AOP = \angle BOP$$

or $\angle AOM = \angle BOM$ .....(i)

Now in ΔAOM and ΔBOM

$OA = OB$ (radii of circle)

$\angle AOM = \angle BOM$ (from equation (i))

$OM = OM$ (common)

$\Delta AOM \cong \Delta BOM$ (from SAS congruence)

Therefore, corresponding angle and side will be equal in congruence triangles.

Means $AM = BM$ ....(ii)

$\angle AMO = \angle BMO$ .....(iii)

But $\angle AMO + \angle BMO = 180^\circ$

or $\angle AMO = \angle BMO = 90^\circ$ .....(iv)

From equation (ii) and (iv),

OP is perpendicular bisector of AB **Hence proved.**

Example 5 : In a circle of 10 cm radius two chords $AB = AC = 12$ cm, then find out the length of chord BC.

Solution : In fig. 12.25

ΔABC an insoceles triangle. AD is bisector of $\angle BAC$. Therefore AD is prependicular bisector of BC.

Here, $AC = AB = 12$ cm

$OA = OC = 10$ cm

and $BD = CD$

$\because$ In a ΔADC according to Bodhayan Theorem.

$$CD^2 = AC^2 - AD^2$$

$$CD^2 = 144 - AD^2 \quad .....(i)$$

Similarly in ΔOCD

$$CD^2 = OC^2 - OD^2$$

$$CD^2 = 100 - (OA - AD)^2 = 100 - (10 - AD)^2$$

$$CD^2 = 20\,AD - AD^2 \quad .....(ii)$$

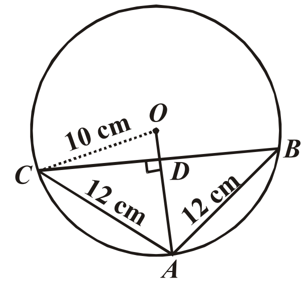


Fig 12.25

From the equation (i) and (ii) by soving

$$AD = 7.2 \text{ cm}$$

Putting the value of AD in equation (i)

$$CD^2 = 144 - (7.2)^2 \text{ or } CD = 9.6 \text{ cm}$$

Hence chord $BC = 2CD = 2\times 9.6 = 19.2$ cm

Example 6 : Prove that out of two chords, longest chord is nearer to centre of circle.

Solution : Given : In fig. 12.26

A circle whose centre is O and chord $CD >$ Chord AB

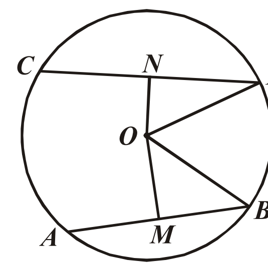


Fig. 12.26

To prove : $ON < OM$

Construction : Join OB and OD

Proof : OM and ON are perpendicular on AB and CD respectively.

So, $$MB = \frac{1}{2}AB \text{ and } ND = \frac{1}{2}CD \quad .....(i)$$

Now in ΔOMB

$$MB^2 = OB^2 - OM^2 \quad .....(ii)$$

and in ΔOND

$$ND^2 = OD^2 - ON^2 \quad .....(iii)$$

Given that AB < CD

or $$\frac{1}{2}AB < \frac{1}{2}CD$$

$$MB < ND \quad \text{(from equation (ii))}$$

or $$MB^2 < ND^2 \quad .....(iv)$$

from equation (ii), (iii) and (iv)

$$(OB^2 - OM^2) < (OD^2 - ON^2)$$

But $OB = OD$ (radius of circle)

So, $$-OM^2 < -ON^2$$

or, $$OM^2 > ON^2$$

or $$ON < OM$$ **Hence Proved.**

Example 7 : In figure 12.27, chord AB = chord CD in a circle, then prove $DQ = BQ$.

Solution : Given : Chord AB = Chord CD

To prove : $DQ = BQ$

Construction : $OL \perp AB$ and $OM \perp CD$ and Join OQ

Proof : $AB = CD$ (given)

or $OL = OM$ .....(i)

Now, in ΔOMQ and ΔOLQ

$OQ = OQ$ (common arm)

$OM = OL$ (from equation (i)

$\angle OMQ = \angle OLQ$ (right angle)

$\angle OMQ \cong \Delta OLQ$ (from RHS)

So the corresponding sides will be equal in congruent triangles

In other words, MQ = LQ .....(ii)

But $MD = \frac{1}{2}CD$ and $LB = \frac{1}{2}AB$

$AB = CD \Rightarrow MD = LB$ .....(iii)

Substracting equation (iii) form (ii)

$MQ - MD = LQ - LB$

Hence $DQ = BQ$ **Hence Proved.**

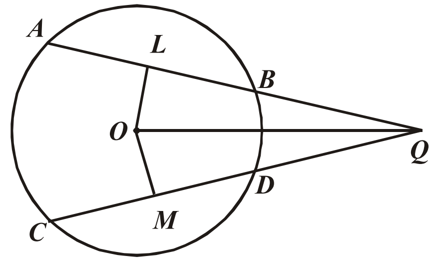


Fig. 12.27

Exercise 12.2

1. Write True / False and if possible then give reason of your answer.
   (i) Chords of a circle AB and CD are 3 cm and 4 cm which make angles at centre are of 70° and 50° respectively.
   (ii) The chords of a circle of length 10 cm and 8 cm. Which apart from the centre are 8 cm and 5 cm respectively.
   (iii) Two chords of a circle AB and CD which are 4 cm distance from the centre then $AB = CD$.
   (iv) Two congruence circles of centres O and O' intersects each other at point A and B then $\angle AOB = \angle AO'B$
   (v) One circle can be drawn by three point which are in a line.
   (vi) A circle of radius 4 cm can be drawn passing two points A and B, if $AB = 8$ cm.
2. If the radius of a circle is 13 cm and a length of its one chord is 10 cm, then find out the distance of this chord form the centre of circle.
3. Two chords of a circle AB and CD whose lengths are 6 cm and 12 cm respectively are parallel to each other and also situated on one side of centre of circle. If the distance between AB and CD is 3 cm then find out the radius of the circle.
4. In fig. 12.28, AB and CD intersect each other at point E. Then prove that arc DA = arc CD.

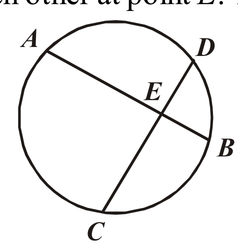


Fig. 12.28

5. In fig. 12.29, AB and CD are two equal chords of a circle, O is the centre of circle. If $OM \perp AB$ and

$ON \perp CD$, then prove that $\angle OMN = \angle ONM$.

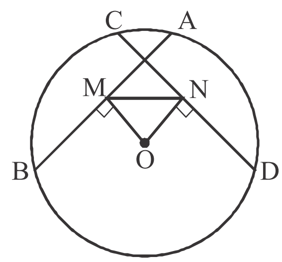


Fig. 12.29

6. In fig. 12.30, O and O' are the centre of circle. $AB || OO'$ then prove that $AB = 2OO'$.

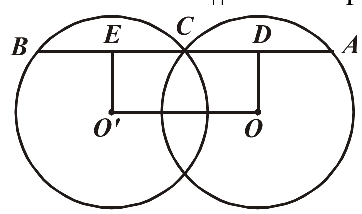


Fig. 12.30

7. AB and CD are two chords of a circle that $AB = 10$ cm and $CD = 24$ cm and $AB || CD$. The distance between AB and CD is 17 cm. Calculate the radius of circle.
8. In a circle of 10 cm radius, the length of two parallel chords are 12 cm and 16 cm. Find out the distance between these if the chords are

   (a) Same side of centre (b) Opposite side of centre.
9. Vertices of a cyclic quadrilateral are situated in such a way that $AB = CD$, then prove that AC = BD.
10. If two equal chords of a circle intertsect each other, then prove their successive parts will be equal to their corresponding parts.
11. Prove that bisector of two parallel chords passes through the centre of circle.

12.07. The Angle Subtended by on arc of a circle

You have learned in previous section that the end points of an arc divide the circle in two parts. If you take two equal chords then we can say about their arc. Is the arc made by a chord equal to arc made by other chord?

Let us solve this puzzle.

Activity :

Draw a circle on a paper. Draw two equal chords AB and CD in it then you will obtain arc AB and CD. (see fig. 12.31 (i))

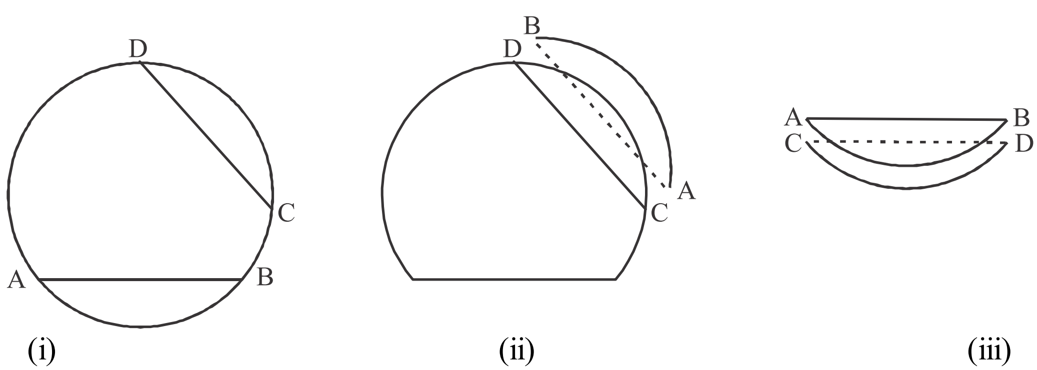


Fig. 12.31

According to fig. 12.31 (ii), chord AB and chord CD cut in the same direction and try to put AB over CD and try to cover it. What have you noticed ?

. Now cut in same direction of chord CD and according to fig. 12.31 (iii) Try to cover each other. Now you will see that arc AB and CD completely cover each others. In other words, equal chords make congurent arcs.

So, If the two chords of a circle are equal then their arc are congurent and opposite of this if two arcs are congurent then their corresponding chords must be equal.

Here the angle made by arc also may be defined in content to angle made an centre by corresponding chord in other words short arc AB make an angle $\angle AOB$. Let us see fig. 12.32 according this defination and theorem 12.1, we can say that " In any circle congurent arcs (or equal arcs) make equal angle on the centre of circle".

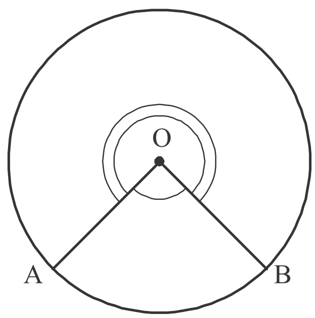
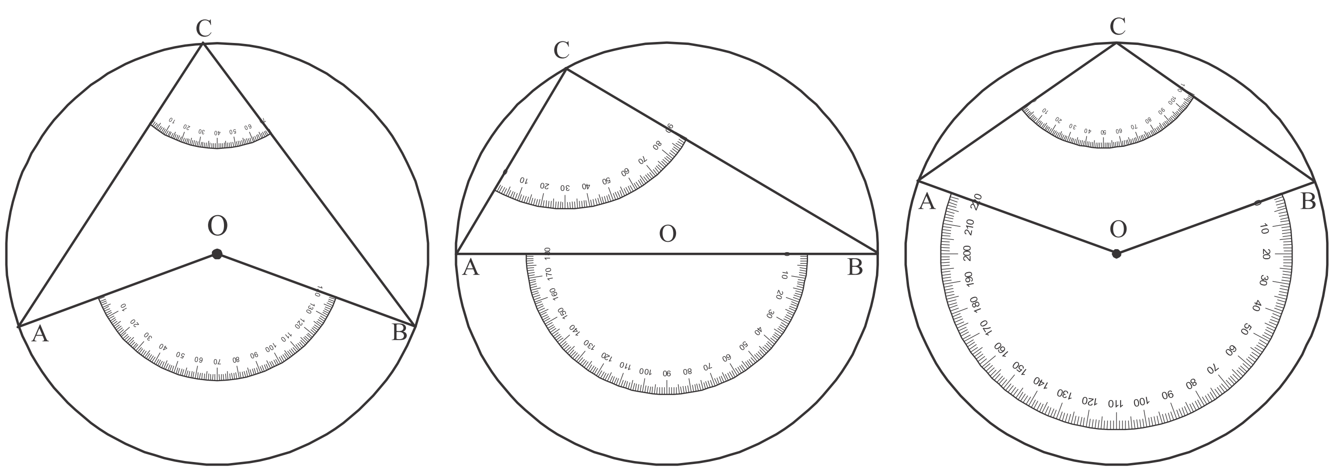


Fig. 12.32

Let us understand the relation between the angle made by an arc and angle of any point in the circle with an activity. In fig 12.33 (i) , (ii) and (iii). you have short arc AB and AB half circle, Big arc AB, made the various angle on the centre of circle. The remaining angles of circle appearing to be measure with the help of photocopy image of protector.



(i) (ii) (iiii)

Fig. 12.33

You have to observe each image carefully, in all angle made by arc AB on the centre $\angle AOB$ and remaining part of circle ACB made angle $\angle ACB$, what relation between them you are able to see

Read the measurement of angles with the help of protector.

In fig. 12.33 (i) $\angle AOB = 140^o$ and $\angle ACB = 70^o$

In (ii) $\angle AOB = 180^\circ$ and $\angle ACB = 90^\circ$

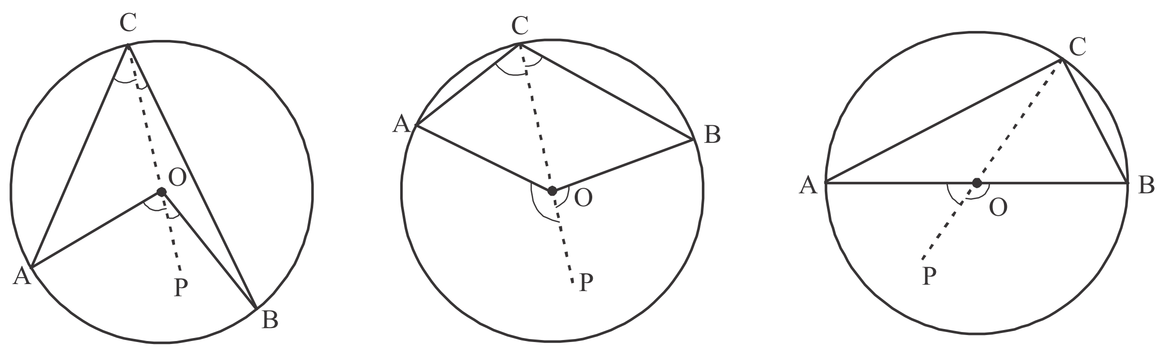
In (iii) obtuse angle $\angle AOB = 220^\circ$ and $\angle ACD = 110^\circ$

It is clear form all the figures the measurement of angle at centre is two times of angle in remaining part of circle is two times of angle in remaining part of circle.

Revise this activity by taking another values of angles.

Let us prove this result with the help of theorem.

Theorem 12.8 : The angle made by an arc on centre is two times of angle made at any point of remaining circle.



(i) (ii) (iii)

Fig. 12.34

Given : Angle subtended by arc AB to centre O is $\angle AOB$ and angle on remaining part is $\angle ACB$.

To prove : $\angle AOB = 2\angle ACB$

Construction : Produce C to P through O.

Proof : ΔAOC is an isoceles triangle ($\because$ OA = OC are the radius of circle)

So $\angle ACO = \angle OAC$ (opposite angles of equal sides are equal) . . . (i)

In ΔAOC angle $\angle AOP$ is exterior angle

$\angle AOP = \angle ACO + \angle OAC$ $(\angle ACO = \angle OAC$ from eq. (i))

$\angle AOP = \angle ACO + \angle ACO$

$\angle \text{AOP} = 2\angle \text{ACO}$ . . . (ii)

Similarly $\angle BOP = 2\angle BCO$ . . . (iii)

$\angle \text{AOP} + \angle \text{BOP} = 2\angle \text{ACO} + 2\angle \text{BCO}$

$\angle \text{AOP} + \angle \text{BOP} = 2(\angle \text{ACO} + \angle \text{BCO})$

$\angle AOB = 2\angle ACB$

from fig 12.34 (i), (ii) and (iii) **Hence proved**

In fig. 12.34. (iii) $\angle ACB$ is angle made on half circle.

Here $\angle AOB = 180^\circ$; so $\angle ACB = 90^\circ$

Sub theorem : Semi circle subtends right angle.

Let us now discuss on the angles subtended by same arc on remaining parts of circle with an activity.

Activity :

Draw a circle and also draw two chords AC and AD . By drawing the same angle image of protector part is according to fig. 12.35 taking CA and DA as a photocopy image of protector and C and D point.

Now you will see that the angle made in the photocopy of protector. By increasing the other sides . they will meet only at point B.

In fig. $\angle ADB = \angle ACB = 60^\circ$

Hence, angle made by an arc in all remaining part of the circle will be same

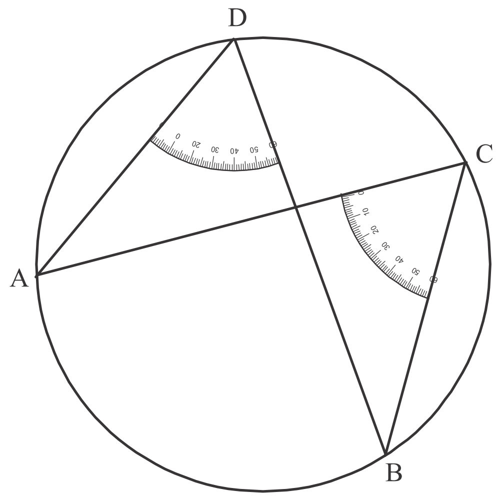


Fig. 12.35

Illustarative Example

Example 1. In Fig. 12.36, AB is diameter and ? DAB? 40° than find the value of $\angle DCA$.

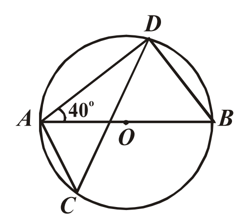


Fig. 12.36

Solution : AB is diameter of circle , so $\angle ADB = 90^\circ$

Now $\angle DBA = 180^\circ - (90^\circ + 40^\circ)$

$\angle DBA = 50^\circ$

$\therefore$ $\angle DBA$ and $\angle DCA$ are the angle of same sector.

therefore $\angle DCA = \angle DBA = 50^\circ$

$\Rightarrow$ $\angle DCA = 50^\circ$

Example 2. In fig. 12.37, arc AB and AC makes angle on O. centre of circle arc 80^o and 120^o . Find the value of $\angle BAC$ and $\angle BOC$.

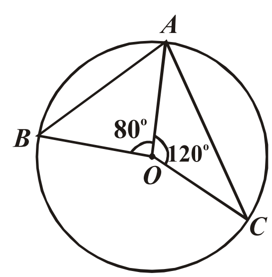


Fig. 12.37

Solution : $\angle BOC = 360^\circ - (120^\circ + 80^\circ)$

So, $\angle BOC = 160^\circ$

$\angle BAC = \frac{1}{2}\angle BOC = \frac{1}{2}\times 160^\circ$

therefore $\angle BAC = 80^\circ$

Example 3. In a quadrilateral ABCD, AB=AC=AD, then prove that $\angle BAD = 2(\angle BDC + \angle CBD)$.

Solution : Given : $AB = AC = AD$. In other words, point A is equidistant form B, C and D. So centre of circle is A.

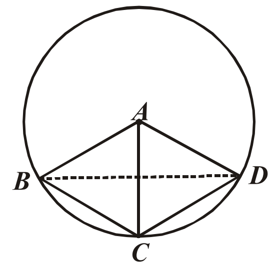


Fig. 12.38

Now arc BC at centre makes $\angle BAC$ and makes $\angle BDC$ on the remaining part of circle.

$\therefore \quad \angle BAC = 2\angle BDC$ . . . (i)

Similarly arc CD makes $\angle DAC$ on centre and $\angle CBD$ on remaining part of circle.

$\therefore \quad \angle CAD = 2\angle CBD$ . . . (ii)

Adding equation (i) and (ii)

$$\angle BAC + \angle CAD = 2(\angle BDC + \angle CBD)$$

$$\angle BAD = 2(\angle BDC + \angle CBD)$$ **Hence proved.**

Example 4. Taking equal side of an triangle isoceles as a diameter, if we draw a circle, prove that it bisects the third unequal side.

Solution : Given : In fig. 12.39 an isoceles ΔABC in which $AB=AC$, taking AC as diameter the circle intersect the unequal side BC at D.

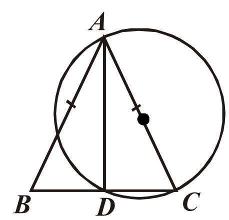


Fig. 12.39

To prove : $BD = DC$

Proof : The circle draw taking AC as diameter the $\angle ADC$ is an angle of half circle.

So $\angle ADC = 90^\circ$

Now in ΔABD and ΔACD

$AB = AC$ (given)

$AD = AD$ (common arm)

$\angle ADB = \angle ADC$ (right angle)

? ABD ? ? ACD (from RHS)

Hence congruent triangle will have equal corresponding sides

Hence $BD = CD$ **Hence Proved**

Example 5. Prove that the angle of major segment is an acute angle.

Solution : Given : In the fig. there is a circle whose centre is O, then segment is ACB.

To prove : $\angle ACB < 90^\circ$

Construction : Join OA, OB and AB

Proof : Angle subtended by arc AB at centre is $\angle AOB$ and angle on remaining part is $\angle ACB$.

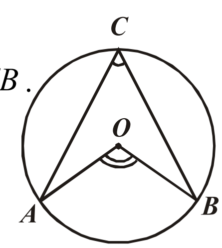


Fig. 12.40

$$\angle ACB = \frac{1}{2}\angle AOB \quad \text{. . . (i)}$$

But $\angle AOB < 180^\circ$ (one angle as ΔAOB)

$\therefore \qquad \frac{1}{2}\angle AOB < \frac{1}{2}\times 180^\circ$

Hence $\qquad \frac{1}{2}\angle AOB < 90^\circ \qquad \text{. . . (ii)}$

from equation (i) and (ii)

$\angle ACB < 90^\circ$ **Hence Proved**

Example 6. AOC is diamtere of a circle and arc $AXB = \frac{1}{2}$ of arc BYC. then find the value of $\angle BOC$.

Solution : arc $AXB = \frac{1}{2}$ arc BYC

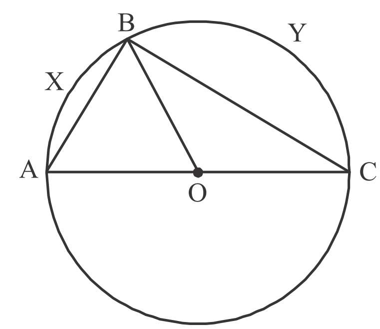


Fig. 12.41

$\therefore \qquad \angle AOB = \frac{1}{2}\angle BOC$

and $\quad \angle AOB + \angle BOC = 180^\circ$

$\frac{1}{2}\angle BOC + \angle BOC = 180^\circ$

$\angle BOC = \frac{2}{3}\times 180^\circ = 120^\circ$

Example 7. Find the value of x in fig. 12.42.

Solution : $\angle DAC = \angle DBC = 30^\circ$ (Angles of same segment) . . . (i)

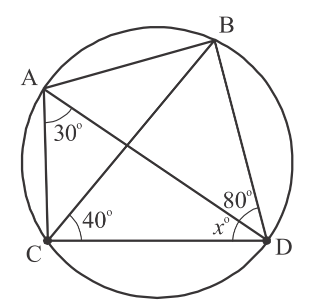


Fig. 12.42

In Δ DBC

$\angle DBC + \angle DCB + \angle BDC = 180^\circ$

$30^\circ + 40^\circ + \left(x + 80^\circ\right) = 180^\circ$ (from fig. and (i))

$x + 80 = 180 - 70$

$x = 110 - 80$

$x = 30^\circ$

Example 8. In fig. 12.43, ΔABC is an equilateral triangle, and 'O' is the centre of the circle. Also AO is increased and then it meets the circle on D. Prove that ΔOBD is an equialteral triangle.

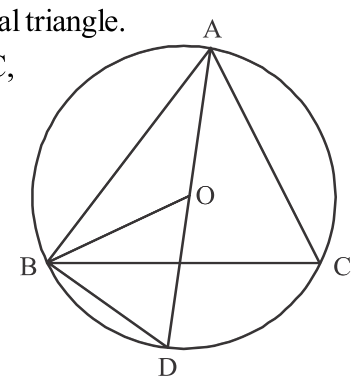


Fig. 12.43

Solution : Given : Δ ABC is an equilateral triangle, O is centre of triangle ABC, on producing A it meets with centre at 'D'.

To prove : Δ OBD is an equilateral triangle.

Proof : OB and OD (radius of same circle)

So $\quad \angle OBD = \angle ODB \qquad \text{. . . (i)}$

∵ Δ ABC is an equilateral triangle

So $\quad \angle C = 60^\circ \qquad \text{. . . (ii)}$

$\angle ADB = \angle C$ (from equation (ii) angle made on circle)

So $\angle ADB = 60^o$ (from equation (i))

but $\angle ADB$ and $\angle ODB$ (both are same)

but $\angle ODB = 60^o$

$\therefore$ $\angle OBD = 60^o$ (from equation (i))

But the sum of all the angle of triangle is 180º, so Δ OBD's third angle will also be 60º.

Hence Δ OBD is an equilateral triangle. **Hence proved**

Exercise 12.3

1. State true or false for every statement and also write the justification of your answers.
   (i) The angle subtended by any chord on any two points of circle are equal.
   (ii) In fig. 12.44, AB is diameter of circle and C is any point on the circle then $AC^2 + BC^2 = AB^2$
   (iii) In fig. 12.44, If $\angle ADE = 120^o$, then $\angle EAB = 60°$
   (iv) In fig. 12.44, $\angle CAD = \angle CED$

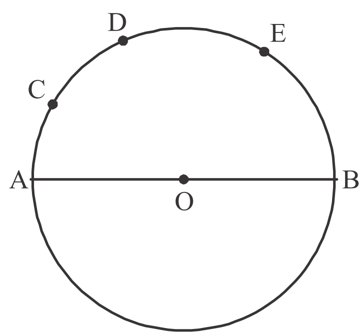


Fig. 12.44

2. In fig. 12.45 ; $\angle ABC = 45^o$, then prove that $OA \perp OC$

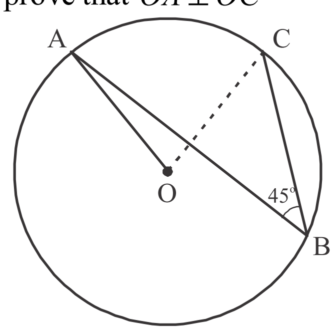


Fig. 12.45

3. O is the centre of a circle ΔABC and D is mid point of base BC then prove that $\angle BOD = \angle A$
4. On common diagonal AB two right triangles ACB and ADB are drawn in such a way that they are situated on opposite side, then prove that $\angle BAC = \angle BDC$
5. Two chords of AB and AC of a circle subtends angle of 90º and 150º at the centre. Find the angle $\angle BAC$ if AB and AC in opposite side of centre.
6. If O is the circumcentre of ΔABC, then prove that $\angle OBC + \angle BAC = 90^o$
7. If the length of a chord is equal to its radius, then find out the angle on the major segment in a circle.

8. In fig. 12.46, $\angle ADC = 130^\circ$ and chord BC = chord BE. then find the value of $\angle CBE$.

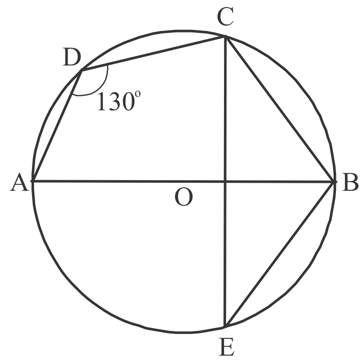


Fig. 12.46

9. In fig. 12.47, angle $\angle ACB = 40^\circ$, then find the value of $\angle OAB$.

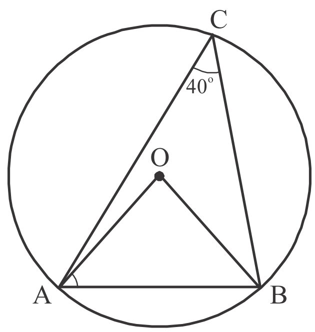


Fig. 12.47

10. In fig. 12.48, AOB is the diameter of a circle and C, D and E are three points situated on semi circle. Find the value of $\angle ACD + \angle BED$

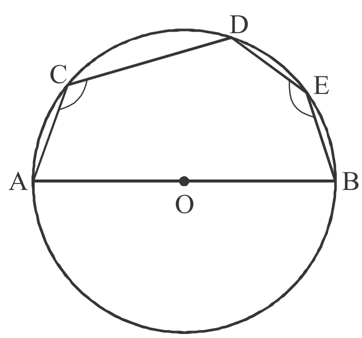


Fig. 12.48

12.8 Cyclic Quadrilateral

Such quadrilateral whose all four vertices are situated on the circle is known as Cyclic Quadrilateral. See fig. 12.49 and 12.50. These Quadrilateral have a special property for that observe carefully on an activity.

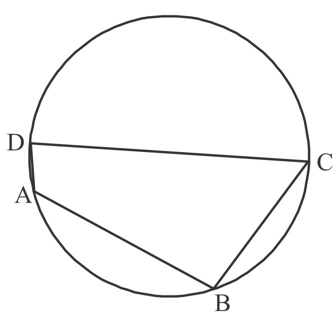


Fig. 12.49

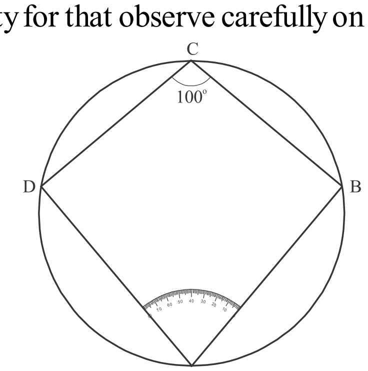


Fig. 12.50

Activity :

1. Draw a circle in your note book.
2. Paste a photocopy of protector at any point of circle, chosen arbitary by you. Here 80^o angle is pasted as shown in fig. 12.50 in $\angle A$.
3. Increase the sides of this angle so that these can touch the circle at any point. Thus you will get $\angle BAD$.
4. Expecting arc DAB, take any point C on the remaining part of circle and complete the quadrilateral.
5. $\angle A$ is opposite to $\angle C$. Find the value of $\angle C$. with the help of protector measure its value. Here you will get the value of $\angle BCD = 100^\circ$ in other words $\angle A + \angle C = 180^\circ$.

Sum of remaining two angles will also be 180^o because sum of all angle of quadrilateral is 360^o.

In other words, the opposite angles of a quadrilateral are supplementary. This can prove by theorem.

Theorem 12.8

Opposite angles of a quadrilateral are supplementary or their total is 180^o

Given : $ABCD$ is a cyclic quadrilateral.

To prove : $\angle A + \angle C = 180^\circ$

$\angle B + \angle D = 180^\circ$

Construction : Join O to B and D

Proof : By arc DAB angle made on centre of circle is x^o and on remaining part angle made is $\angle C$

So, $\angle C = \frac{1}{2} x^o$ . . . (i)

Similarly by arc DCB angle made on centre is y^o and on remaining part angle made is $\angle A$

So, $\angle A = \frac{1}{2} y^o$ . . . (ii)

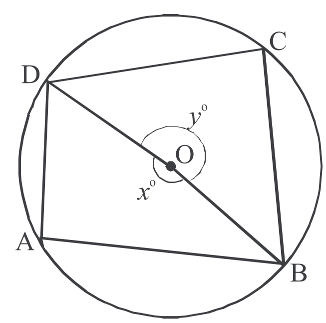


Fig. 12.51

On adding equation (i) and (ii)

$$\angle C + \angle A = \frac{1}{2}(x^o + y^o)$$

$$\angle C + \angle A = \frac{1}{2} \times 360^o = 180^o \quad . . . (iii)$$

Since sum of all angles of quadirlateral is 360^o

So, $\angle B + \angle D = 360^o - (\angle A + \angle C)$

$\angle B + \angle D = 360° - 180°$ (from (iii))

$\angle B + \angle D = 180°$

Hence proved

Opposite of this theorem is also true as follows :

Theorem 12.9 : (Converse of theorem 12.8)

Opposite angles of a quadrilateral are supplementary, then it is a cyclic quadrilateral.

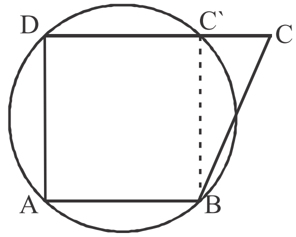


Fig. 12.52

Given : ABCD is a quadirlateral in which

$\angle BAD + \angle BCD = 180^\circ$

and $\angle ABC + \angle ADC = 180^\circ$

To prove : $ABCD$ is a cyclic quadrilateral.

Proof : Let a circle passes through A, B and D but in place of C it passes through C', then joining $C'B$, $ABC'D$ becomes a cyclic quadrilateral.

So $\angle BAD + \angle BC'D = 180^\circ$ (opposite angles of cyclic quadrilateral are supplemantary) . . . (i)

But $\angle BAD + \angle BCD = 180^\circ$ (given) . . . (ii)

from equation (i) and (ii)

$\angle BAD + \angle BC'D = \angle BAD + \angle BCD$

$\Rightarrow$ $\angle BCD = \angle BC'D$ . . . (iii)

But $\angle BC'D$ is an exterior angle of $\Delta BCC'$, $\angle BC'D = \angle BCD + \angle CBC'$ (Exterior angles is sum of interior opposite angles)

$\Rightarrow$ $\angle BC'D > \angle BCD$ . . . (iv)

from equation (iii) and (iv). It is clear $\angle BC'D > \angle BCD$ that is only possible when BC and BC' are coincide or point C and C' are coincide.

So, ABCD is a cyclic quadrilateral

Hence, ABCD is a cyclic quadrilateral **Hence proved**

12.9. Interior Oppostie Angles of Cyclic a Quadrialteral

By increasing a side of a cyclic quadrialteral, the angle made is known as exterior angle of cyclic quadrilateral. (see fig. 12.53) $\angle CBE$ is exterior angle of cyclic quadrilaterel . Similarly you can draw an exterior angle on every vertex of cyclic quadrilateral.

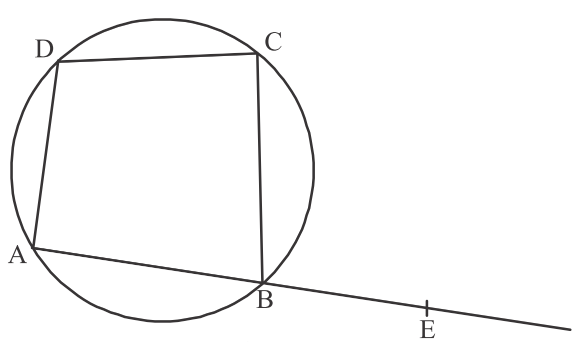


Fig. 12.53

$\angle ABC$ and $\angle ADC$ are known as corresponding and interior opposite angles of exterior angle of $\angle CBE$.

Let us understand what is the relation between exterior and interior opposite angles. Let us try an activity for this.

Activity :

1. Draw a circle.
2. As in fig. 12.53 take any two points A and B and increase them by joining them up to E.

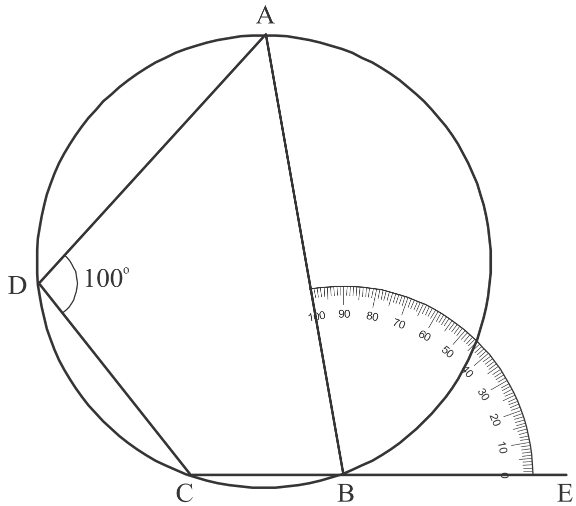


Fig. 12.54

3. From the photocopy of protector cut a portion of a sutaible a measurment (Here angles 100°and paste it in at point B such a way that coincide with base BE)
4. Increase the side EB so that it will meet at a point C of a circle.
5. Arc of circle ABC expect take a point D on the remaining portion of circle complete the cyclic quadrilateral.
6. Measure $\angle ADC$ with the help of protector,you will find that the measurement of angle $\angle ADC = 100^{\circ}$ i.e. the measurment of exterior angle of cyclic quadrilateral is equal to interior opposite angle of cyclic quadrilateral.

   Let us prove this with the help of theorem.

Theorem 12.10 : By increasing a side of cyclic quadrilateral the make exterior angle is equal to interior opposite angle.

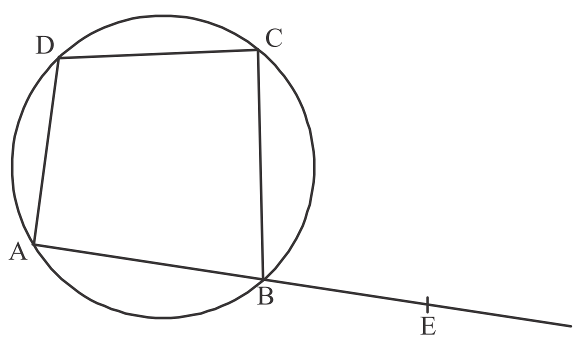


Fig. 12.55

Given : $ABCD$ is a cyclic quadrilateral whose AB is increased up to E point.

To prove : $\angle CBE = \angle ADC$

Proof : $ABCD$ is a cyclic quadrilateral

So, $\angle ABC + \angle ADC = 180^{\circ}$ . . . (i)

$\angle ABC + \angle CBE = 180^{\circ}$ (from linear pair) . . . (ii)

From equation (i) and (ii)

$$\angle ABC + \angle ADC = \angle ABC + \angle CBE$$

or $\angle ADC = \angle CBE$ **Hence proved**

Can you prove the opposite of this theorem if the exterior angle and interior opposite angles are equal then it is a cyclic quadrilateral.

Theorem 12.11 :

In a quadrilateral, the exterior angle made by increasing a side of quadrilateral, is equal to interior opposite to angle, then it is a cyclic quadrilateral.

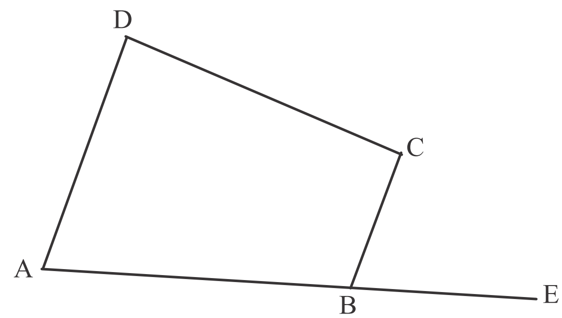


Fig. 12.56

Given : The $\angle CBE$ is exterior angle of quadrialteral $ABCD$,

$$\angle ADC = \angle CBE$$

To prove : ABCD is a cyclic quadrilateral.

Proof : $\angle ADC = \angle CBE$ (given)

Adding $\angle ABC$ on both side

$$\angle ABC + \angle ADC = \angle ABC + \angle CBE$$

$$\angle ABC + \angle CBE = 180^\circ \quad \text{(From linear angle pair)}$$

So, $\angle ABC + \angle ADC = 180^\circ$

Since $\angle ABC$ and $\angle ADC$ are the opposite angles of quadrialteral $ABCD$.

Hence $ABCD$ is a cyclic quadrilateral

Illustrative Examples

Example 1. In fig 12.57, ABCD is a cyclic quadrilateral if $\angle AOC = 136^\circ$ then find the value of $\angle ABC$

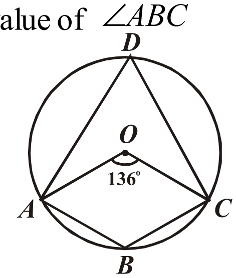


Fig. 12.57

Solution : The arc ABC makes an angle on centre O and remaining part arc

$\angle AOC$ and $\angle ADC$.

So $\angle ADC = \dfrac{1}{2}\angle AOC = \dfrac{1}{2} \times 136^\circ$

$\angle ADC = 68^\circ$

$\therefore$ $ABCD$ is a quadrilateral there fore sum of oppostie angle will be 180°

$\therefore$ $\angle ADC + \angle ABC = 180^\circ$

$\angle ABC = 180^\circ - 68^\circ$

or $\angle ABC = 112^\circ$

Example 2. In fig. 12.58, $ABCD$ is a cyclic quadrilateral, find the value of x and y.

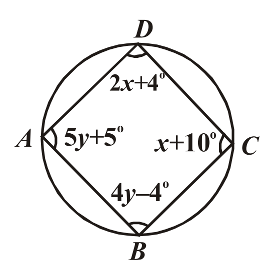


Fig. 12.58

Solution : Opposite angles of a cyclic.

$$(2x + 4^\circ) + (4y - 4^\circ) = 180^\circ$$

or $2x + 4y = 180^\circ$

or $x + 2y = 90^\circ \quad \ldots (1)$

Similarly $(x + 10^\circ) + (5y + 5^\circ) = 180^\circ$

or $x + 5y = 165^\circ \quad \ldots (2)$

By solving equation (i) and (ii)

$$x = 40, \quad y = 25$$

So $\quad x = 40 \quad y = 25$

Example 3. In fig. 12.59, the centre of circle is O and arc BCD subtended an angle 140° at the centre of circle.Find the value of $\angle BAD$ and $\angle DCE$.

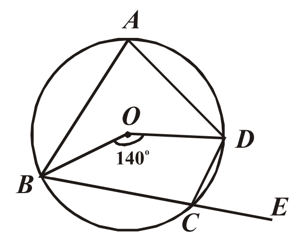


Fig. 12.59

Solution : By arc BCD the angle made on centre and remaining part are $\angle BOD$ and $\angle BAD$.

So, $\quad \angle BAD = \frac{1}{2} \times \angle BOD = \frac{1}{2} \times 140^\circ$

So, $\quad \angle BAD = 70^\circ$

But $\angle DCE$, is exterior angle of cyclic quadrilateral

Therefore, exterior angle will be equal to interior opposite angles.

$$\angle DCE = \angle BAD$$

or $\quad \angle DCE = 70^\circ$

Example 4. In fig 12.60, ABCD is a cyclic quadrilateral drawn a line AE parallel to CD and BA is increased upto F. If $\angle ABC = 92^\circ$ and $\angle FAE = 20^\circ$, then find the value of $\angle BCD$.

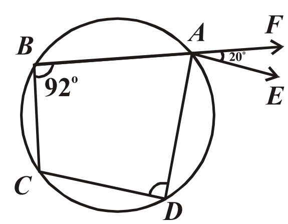


Fig. 12.60

Solution : $ABCD$ is a cyclic quadirlateral

$$\angle ABC + \angle CDA = 180^\circ$$

$\Rightarrow \quad \angle CDA = 180^\circ - 92^\circ$

$\Rightarrow \quad \angle CDA = 88^\circ$

But $\quad CD \parallel AE$

$$\angle DAE = \angle CDA$$

$$\angle DAE = 88^\circ$$

$$\angle DAF = \angle FAE + \angle DAE = 20^\circ + 88^\circ$$

$$\angle DAF = 108^\circ$$

$$\angle DAB = 180^\circ - 108^\circ = 72^\circ$$

Now, $\quad \angle BCD + \angle DAB = 180^\circ$

$$\angle BCD = 180^\circ - \angle DAB = 180^\circ - 72^\circ$$

$$\angle BCD = 108^\circ$$

Example 5. Prove that, cyclic parallelogram is a rectangle.

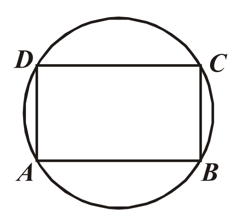


Fig. 12.61

Solution : Given : ABCD is a cyclic parallelogram ,

So, $\quad \angle B + \angle D = 180^\circ \quad \ldots$ (i)

$\quad \angle B = \angle D \quad \ldots$ (ii)

From equation (i) and (ii)

$$\angle B = \angle D = 90^\circ$$

Similarly, $\angle A = \angle C = 90^\circ$

Hence, $ABCD$ is a rectangle. **Hence Proved.**

Example 6. If two sides are parallel of a cyclic quadrilateral then prove that remaining sides and diagonal are equal.

Solution : Given : In cyclic quadrilateral ABCD

$AB \parallel DC$

To prove : (i) $AD = BC$

(ii) $AC = BD$

Proof : $AB \parallel DC$ and BC is a transverasl line.

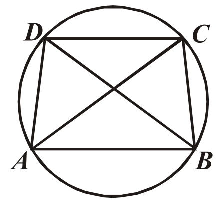


Fig. 12.62

So, $\angle ABC + \angle DCB = 180^\circ$ . . . (i)

But, ABCD is cyclic quadrilateral

So, $\angle ABC + \angle ADC = 180^\circ$ . . . (ii)

From equation (i) and (ii)

$\angle DCB = \angle ADC$ . . . (iii)

Now, In ΔADC and ΔBCD

$\angle ADC = \angle DCB$ (from equation (iii))

$\angle DAC = \angle DBC$ (angle of same sector)

and $DC = DC$ (common)

$\therefore$ $\Delta ADC \cong \Delta BCD$ (ASA congurence)

So, the corresponding sides are equal of congurence trianlge.

hence $AD = BC$

and $AC = BD$ **Hence proved**

Example 7. In fig. 12.63, $ABCD$ is a quadrilateral, in which $AD = BC$ and $\angle ADC = \angle BCD$, then prove that $ABCD$ is a cyclic quadrilateral.

Solution : Given : In a quadrilateral ABCD, AD = BC and $\angle ADC = \angle BCD$.

To prove : $ABCD$ is a cyclic quadrilateral

Construction : $DN \perp AB$ and $CM \perp AB$ drawn

Proof :

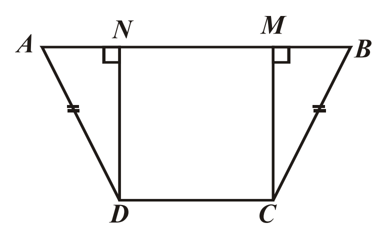


Fig. 12.63

$\angle ADC = \angle BCD$ (given) . . . (i)

$\therefore$ $\angle ADN = \angle ADC - 90^\circ$

$= \angle BCD - 90^\circ$ [from equation (i)]

$\angle ADN = \angle BCM$ . . . (ii)

Now, In ΔAND and ΔBMC

$\angle AND = \angle BMC$ (right angle)

$\angle ADN = \angle BCM$ [from equation (ii)]

and $AD = BC$ (given)

$\therefore$ $\Delta AND \cong \Delta BMC$ (AAS congurence)

Therefore the corresponding angle will be equal of congurence triangles.

Hence, $\angle A = \angle B$ . . . (iii)

Similarly $\angle C = \angle D$ . . . (iv)

But $\angle A+\angle B+\angle C+\angle D=360^\circ$

From equation (iii) and (iv),

$$2\angle B+2\angle D=360^\circ$$

$\therefore$ $\angle B+\angle D=180^\circ$

$\therefore$ $ABCD$ is a cyclic quadrilateral **Hence proved**

Exercise 12.4

1. One angle of cyclic quadrilateral is given, then find the opposite angles.

   (i) 70^o (ii) 135^o (iii) $112\frac{1}{2}^\circ$ (iv) $\frac{3}{5}$ right angle (v) 165^o

2. Find the opposite angle of cyclic qudrilateral in which one angle is

   (i) $\frac{2}{7}$ of other angle (ii) $\frac{11}{4}$ of other angle

3. In fig. 12.64, find the all four angles of a cyclic quadrilateral $ABCD$.

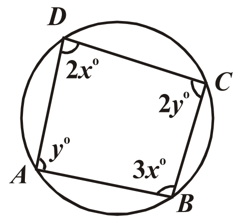


Fig. 12.64

4. In fig. 12.65, few angles are denoted by a, b, c and d. Find the measurement of these angle.

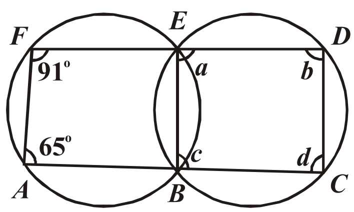


Fig. 12.65

5. If in a cyclic quadrilateral $ABCD$, $AD \parallel BC$, then prove $\angle A=\angle D$.
6. $ABCD$ is a cyclic quadrilateral, on increase AB and DC, meet at point E. Prove that ΔEBC and ΔEDA are similar.
7. Prove that a quadrilateral made from bisector of angles of cyclic quadrilateral is also a cyclic quadrilateral.

Miscellaneous Exercise - 12

1. In a circle of a radius 10 cm the length of chord which is 6 cm apart form the centre is.

   (a) 16 cm (b) 8 cm (c)) 4 cm (d) 5 cm

2. In a circle of radiud 13 cm a 24 cm long chord has been drawn, then the distance of chord from the centre is

   (a) 12 cm (b) 5 cm (c) 6.5 cm (d) 12 cm

3. The measurment of short arc is :

   (a) less the 180^0 (b) more than 180^0 (c) 360^0 (d) 270^0

4. Measurment of big arc is :

   (a) less then 180^0 (b) more than 180^0 (c) 360^0 (d) 90^0

5. In a circle the chords are at same distance from the centre, than one chord is .... of the other.
(a) double (b) triple (c) half (d) equal

6. Measurment of an arc is 180^{0}, then the arc is :
(a) big arc (b) short arc (c) circle (d) half circle

7. Number of circles which passes through three collinear points is.
(a) one (b) two (c) zero (d) infinite

8. If in any circle arc AB = arc BA, then arc is :
(a) big arc (b) short arc (c) half circle (d) circle

9. If the diameter of a circle bisect the two chords, then these chord will be :
(a) parallel (b) perpendicular (c) intersecting (d) none of the above

10. If the arc in two congurent circles are equal, then their corresponding chords will be :
(a) parallel (b) equal (c) perpendicular (d) intersecting

11. If AD is a diameter of any circle and AB is chord. If AD = 34 cm, AB = 30 cm, then the distance of AB from centre is :
(a) 17 cm (b) 15 cm (c) 4 cm (d) 8 cm

12. In fig. 12.66, if OA = 5 cm, AB = 8 cm and chord OD is perpendicualr AB, then CD is equal to :

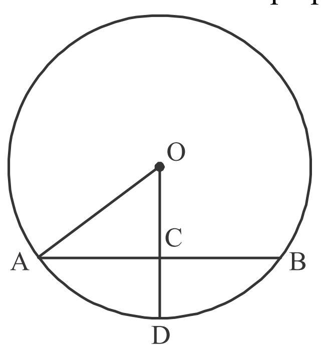


Fig. 12.66

(a) 2 cm (b) 3 cm (c) 4 cm (d) 5 cm

13. If AB = 12 cm, BC = 16 cm and line segment AB is perpendicular on BC, then the radius of circle passing through A, B and C is :
(a) 6 cm (b) 8 cm (c) 10 cm (d) 12 cm

14. In fig. 12.67, if $\angle ABC = 20^{o}$, then $\angle AOC$ is equal to :

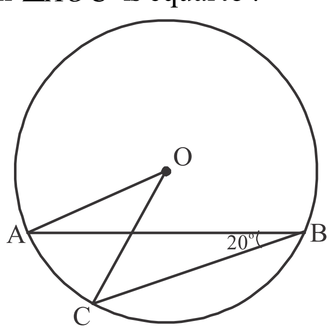


Fig. 12.67

(a) 20^{0} (b) 40^{0} (c) 60^{0} (d) 10^{0}

15. In fig. 12.68, if AOB is a diameter of circle and AC = BC. then $\angle CAB$ is equal to -

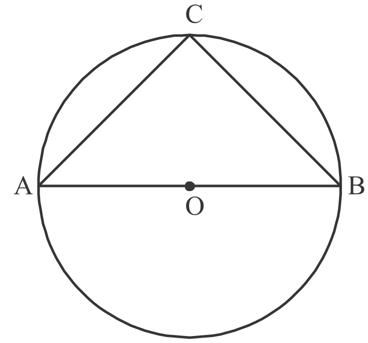


Fig. 12.68

(a) 30° (b) 60° (c) 90° (d) 45°

16. In fig. 12.69, if $\angle OAB = 40^\circ$, then $\angle ACB$ is equal to :

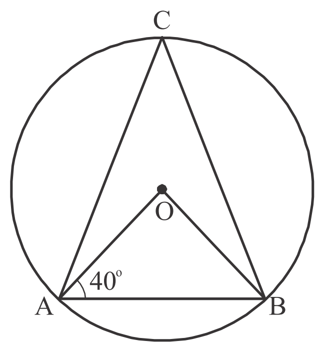


Fig. 12.69

(a) 50° (b) 40° (c) 60° (d) 70°

17. In fig. 12.70, if $\angle DAB = 60^\circ$ and $\angle ABD = 50^\circ$, then $\angle ACB$ is equal to ;

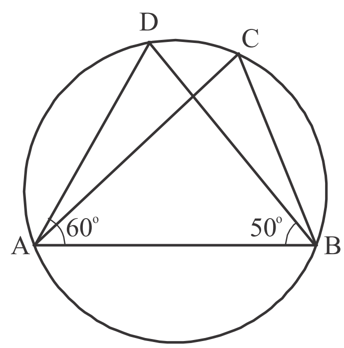


Fig. 12.70

(a) 60° (b) 50° (c) 70° (d) 80°

18. Side AB of a quadrilateral is diameter of its outer circle and $\angle ADC = 140^\circ$ then $\angle BAC$ is equal to :

(a) 80° (b) 50° (c) 40° (d) 30°

19. In fig. 12.75, BC is the diameter and $\angle BAO = 60^\circ$ then $\angle ADC$ is equal to :

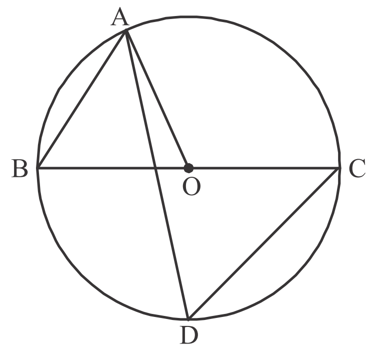


Fig. 12.71

(a) 30° (b) 45° (c) 60° (d) 120°

20. In fig. 12.72. $\angle AOB = 90^\circ$ and $\angle ABC = 30^\circ$ then $\angle CAO$ is equal to :

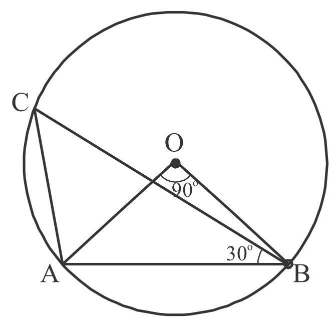


Fig. 12.72

(a) 30° (b) 45° (c) 90° (d) 60°

21. If two equal chords of a circle intersect each other, then prove that two parts of a chord are equal to other are equal to other corresponding both part of chord.
22. If *P, Q* and *R* are the mid points of sides *BC*, *CA* and *AB* of a triangle and *AD* is perpendicular on *BC* form vertex *A*, then prove that *P, Q, R* and *D* are cyclic.
23. *ABCD* is a parallelogram, a circle is drawn passing through *A* and *B* in such a way that it intersect *AD* at *P* and *BC* at *Q*, then prove that *P*, *Q*, *C* and *D* are cyclic
24. Prove that the bisector of any angle of triangle and perpendicular bisector of its opposite side if intersect each other then it intersect on circumcircle.
25. In any circle *AYDZBWCX,* if two chord AB and CD intersect each other at right angle , see fig. 12.73, then prove arc CXA + arc DZB = arc AYD + arc BWC = Semi circle.

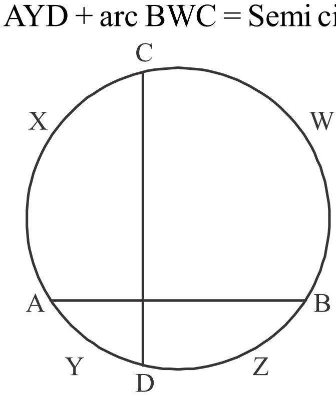


Fig. 12.73

26. If ABC is equilateral triangle in a circle and P is any point on short arc BC, which is not coincidence of B and C. Then prove that PA is bisector of $\angle BPC$.

27. In fig. 12.74, AB and CD are two chords of a circle which intersect each other at point E, then prove that

$\angle AEC = \frac{1}{2}$ (Angle made by arc CXA on centre + Angle made by arc DYB on centre)

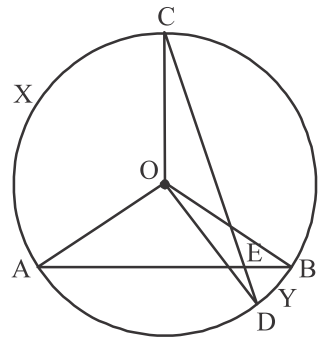


Fig. 12.74

28. If bisectors of oppostite angles in a cyclic quadrilateral *ABCD*, intersect the circular at *P* and *Q* points. then prove that *PQ* is the diameter of circle.

29. The radius of a circle is $\sqrt{2}$ cm . This circle is divided into two parts by 2 cm long chord. Then prove that an angle of 45° is subtends at any point of major segment by this arc.

30. *AB* and *AC* are two chords having radius *r* of a circle in such a way that AB = 2AC. If the distance of *AB* and *AC* from the centre is p and q respectively then prove that $4q^2 = p^2 + 3r^2$

31. In fig. 12.75 , O is the centre of circle $\angle BCO = 30^{\circ}$, then find the value of x and y.

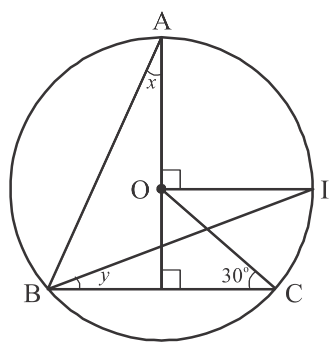


Fig. 12.75

32. In fig. 12.76, O is the centre of circle, Where BD = OD and $CD \perp AB$, then find the value of $\angle CAB$.

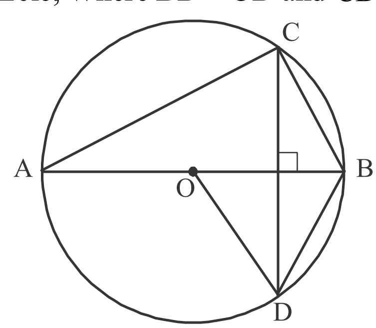


Fig. 12.76

33. Prove that out of all chords which passes through any point of circle, that chord will be smallest which is perpendicular on diameter which passes through that point.

Important Points

1. A circle is a set of all those points lying in the plane which are at a constant distance form a fixed point lying in that plane.
2. Equal chords of a circle (or congurent circles) subetend equal angle at the centre of circle (or corresponding centres).
3. If two chords of any circle (or congurent circle) subtend equal angles at a centre (or on corresponding centres) then chords are equal.
4. The perpendicular drawn from centre of a circle to any chord, it bisects the chord.
5. Any chord passing through centre of circle and bisects any other chord, then it is perpendicular on the chord.
6. Only one circle can be drawn from three non-collinear points .
7. Chords of a circle (or congurent circle) are at equal distance form centre of circle (or corresponding centres of circle).
8. Chords equidistant from centre of a circle, are equal in size.
9. If two arcs of any circle are congurent, then corresponding chord will also be equal and opposite, if two chords of circle are equal then their corresponding arc (long, short) are aslo congurent.
10. Congurent arc of any circle make equal angle at the centre of circle.
11. Angle subtend by any arc at the centre is double of that subtend by same arc in remaining portion of circle at any point.
12. Angles are equal in same segment.
13. The angle in a semi circle subtendended by its base is right angle.
14. If the line segment joining the two points and their inclined line subtend same angle on two other points then all four points will be on the circle subtend equal angle on two other points then all four points will be on the circle.
15. Sum of two opposite angles of cyclic quadrilateral is 180^o.
16. If the sum of two opposite angles of quadrilateral is 180^o, then it will be cyclic quadrilateral.
17. By producing the one side of cyclic quadrilateral the exterior angle so formed is equal to the opposite interior angle.

Answer Sheet

Exercise 12.1

1. (i) Interior (ii) Exterior (iii) Diameter (iv) Semi Circle (v) Three
2. (i) True (ii) False (iii) False (iv) True (v) True (vi) False (vii) False

Exercise 12.2

1. (i) False : Because larger chord subtend larger angle as compared to small chord.
   (ii) False : Because larger chord situated at small distance form the circle.
   (iii) True : Because both chords are at equal distance form the centre.
   (iv) True : Because equal chords of congruent circles subtends equal angle at corresponding centres of circles.
   (v) False : Because a circle which passes through two points can not pass through the third point of line.
   (vi) True : Because AB is diameter.

2. 12 cm 3. $3\sqrt{5}$ cm 7. 13 cm 8. (i) 2 cm (ii) 14 cm

Exercise 12.3

1. (i) False : If both points are situated on one side (major or minor segment), then become equal otherwise not.
(ii) False : Because $\angle C$ is right angle and $AB^2 = AC^2 + BC^2$
(iii) True : After joining AD, DE, DB and EB , $\angle ADB = 90^\circ$ then $\angle BDE = 120 - 90 = 30^\circ$, here $\angle BDE = \angle EAB = 30^\circ$ as subtended on the same arc segment.
(iv) True : Since chords of congruent circles subtend same angle at the corresponding centres.
(v) False : Since, circle passing through two out of three collinear points can never pass through third collinear point.
(vi) True : $\angle CAD = \angle CED$ as these are angles subtend on the same arc segment after joining AC, CD, AD, DE and CE.

2. 120° 6. 60° 9. 100° 10. 50° 11. 270°

Exercise 12.4

1. (i) 110° (ii) 45° (iii) $67\frac{1}{2}^\circ$ (iv) 126° (v) 15°
2. (i) $45^\circ, 40^\circ$ (ii) $132^\circ, 48^\circ$
3. $\angle A = 60^\circ$, $\angle B = 108^\circ$, $\angle C = 120^\circ$, $\angle D = 72^\circ$
5. $a = 65^\circ, b = 89^\circ, c = 91^\circ, d = 115^\circ$

Miscellaneous Exercise 12

1. (a) 2. (b) 3. (a) 4. (b) 5. (d) 6. (d) 7. (a) 8. (c) 9. (a) 10. (b) 11. (d) 12. (a) 13. (c) 14. (b) 15. (d) 6. (a) 17. (c) 18. (b) 19. (c) 20. (d)

13

Circle and Tangent

13.01. Introduction

In the previous chapter we have studied various concepts of circles such as chords, angle made by arcs, cyclic quadrilateral etc. In this chapter we will study a line and a circle of their various positions on a plane and their corresponding properties which appears ?

13.02. Secant and Tangent Line

On a white paper draw a circle and a line. Now compare this figure with the following figures. Definately one shape will appear similar to the figures given below means that a line and a circle drawing together is possible as there figures below given figure 13.01.

Let us consider the three figures :

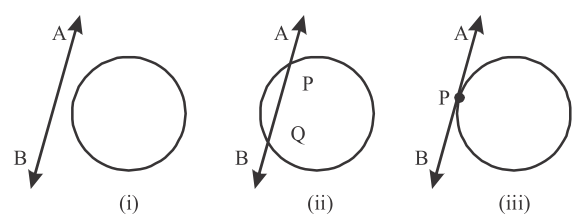


Fig. 13.01

1. In the fig. 13.01 (i). The line is outside the circle so means line and circle are separate figures on the same plane. There is no relation between them.
2. fig. 13.01 (ii) AB is secant for the circle. If a line intersects a circle at two points then it is called secant line.
3. In fig. 13.01 (iii) AB line is tangent for the circle . Here the line AB is passing by touching the circle at point P or we can say in other words that AB line which is intersecting a circle at one point only. Here point P will be the tangent point of the circle and line AB. *It means the line which intersects the circle at a point only is known as a tangemt line.*

To understand the concept of tangent, let us perform the following activity :

Activity :

On a drawing board or on a wood table, put a plain paper and fix two pins A and B. Now keeping normal stress tie up a black colour thread on these two points and draw a circle on the other side of another paper. See fig. 13.02 (i).

Now shift the circle drawn on paper in such away that the thread appear like it is dividing it in two parts to the circle. Name these points as P and Q. Keeping the paper on which circle is drawn stable and fix a pin on point P. In this way the paper can move with respect to point P. See figure 13.02 (ii). Now move the paper on which circle is drawn and observe this process. We observe the following :

(i) The distance between P and Q is reducing with moving means in every situation the chord length decreases form its initial length. See fig. 13.02 (iii).

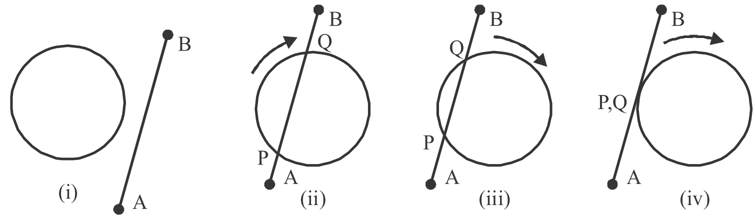


Fig. 13.02

(ii) When point Q approaches to point P or when both coincides then the length of chord becomes zero. The line looks like intersecting at one point. See fig. 13.02 (iv). In this setuation line AB touches the circle at point P.

(iii) If circle is moved more in the same direction then we observe that the length of the chord increases up to a certain limit and after that it decreases and again we get the results as described in (i) and (ii).

Repeat the same process in opposite direction you will get the same result. After this activity we can say that

The secant which is the chord of a circle, whose both piercing ends coincided in special conditions then it is converted into tangent line or we can say that at a point of circle one and only one tangent can exists.

Activity :

On a plain paper that draw a circle and its secant PQ. Now draw parallel lines to PQ. You will observe after the few secants the length of chords is reducing continuously. In one condition the measurment of chord cut by secant tends zero.

It means piercing lines P_1Q_1 and P_2Q_2 on both side becomes tangent . See fig. 13.03. It is clear from this experiment that.

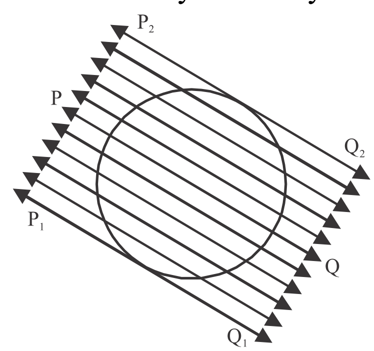


Fig. 13.03

There can not be more then two tangents parallel to the secant or there may exists only two parallel tangents on any circle.

Activity :

Draw a circle with the help of the compass . Draw many radii in the circle with the help of scale and paste it on the card board and cut it along the circumference of the circle.Thus a circular wheel is ready . Now fin a pin on the centre of this wheel and roll this wheel on the ground with respect to the centre of the circle. What will you observe? You will observe that ***at the line of rolling wheel are radii of circle appear perpendicular with respect to the horizontal.*** See fig. 13.04.

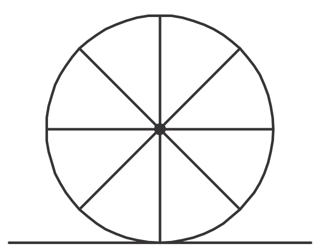


Fig. 13.04

Theorem 13.1

Tangent drawn form any point to a circle, is perpendicular on the line (radius) which joins the centre.

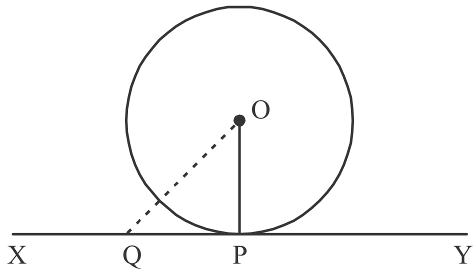


Fig. 13.05

Given : *O* centre of the circle. Tangent *XY* touches the circle at point *P* and *OP* is radius.

To prove : $OP \perp XY$

Construction : Take a point *Q* on *XY* and join *OQ*.

Proof : $\because$ Every point of tangent will be outside the circle except the point of contact.

$\therefore$ OP < OQ (The distance of point outside the circle is more than the radius). It means OP (radius) will be the smallest in distance form the points situated on XY. But as we know that perpendicular is the smallest in all distances of a straight line.

Hence $OP \perp XY$ **Hence Proved.**

Theorem 13.2 (Converse of Theorem 13.1)

If as a line drawn form any point situated on a circle is perpendiuclar to the radius then it is tangent.

Given : *O* is the centre of circle and *OP* is radius and $OP \perp XY$

To prove : *XY* is a tangent on point *P*.

Construction : Join *O* and *Q* which is a point on *XY*.

Proof : $\because$ $OP \perp XY$

$\therefore$ $OP < OQ$

(The perpendicular drawn form any point to a line is smallest in all line segments that join this line.) Since all points including *Q* lying on *XY* are outside the circle. Hence *XY* is a tangent.

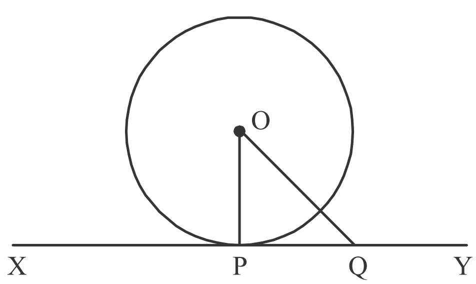


Fig. 13.06

So, perpendicular drawn from a point on any line is the smallest segment among all the line segments drawn from that point to all points on the line.

Hence Proved.

13.03. Number of tangents which can be drawn from any point on a circle

In the preceding section we have studied about secant and tangent. How many tangents can be drawn from inside and outside points of circle and what is the relation between these tangent lines? Let us solve this problem with the help of the following figures.

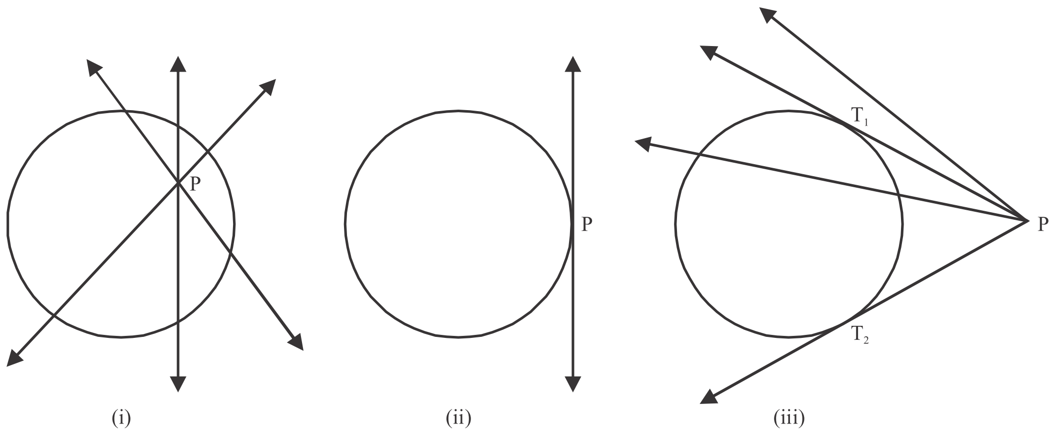


Fig. 13.07

A circle is situated in a plane and we want to choose a point then we will choose it from inside the circle, on the circle and outside the circle. Out of there three situations, we will choose any one situation. Now, we will think on all points one by one.

(i) When point P is situated inside the circle? Then all the lines passing through the points P, are secant lines and number of tangents is zero. See fig. 13.07 (i) It means in such condition the number of tangent will be zero.

(ii) When point P is situated on the circle then we have learnt in the last section that only one tangent can be drawn form a point which is on the circle. See fig, 13.07 (ii).

(iii) When point P is situated outside the circle, only two tangent lines can be drawn. Remaining lines may be either secant or outside the circle. See fig. 13.07 (iii). In the fig. two tangent PT_1 and PT_2 are appearing from point P. Can you tell what is the relation between them? Are these tangents really equal? Let us prove this concept with the help of following theorems.

Theorem 13.3 :

The tangents drawn form an exterior point to a circle are equal.

Given : Two tangents PT_1 and PT_2 are drawn form a point P to a circle with centre O.

To prove : $PT_1 = PT_2$

Construction : Join O to T_1, T_2 and P

Proof : In ΔOPT_1 and ΔOPT_2

$$\angle OT_1P = \angle OT_2P = 90°$$

(Tangent and radius are perpendicular by theorem 13.1)

$OT_1 = OT_2$ (radius of same circle)

$OP = OP$ (common)

According to R.H.S. congruence

$$\Delta OPT_1 \cong \Delta OPT_2$$

Hence, $PT_1 = PT_2$ **Hence Proved**

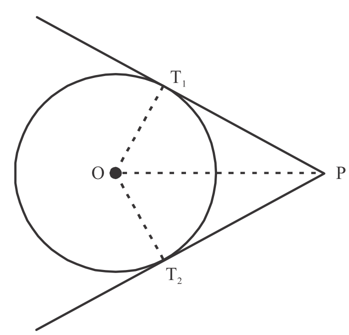


Fig. 13.08

Illustrative Examples

Example 1. Find the length of the tangent, if the distance between tangent point and centre is 13 cm and radius of the circle is 5 cm.

Solution : Since $OQ^2 = OP^2 + PQ^2$ (in right Δ OPQ)

(In right ΔOPQ)

$\Rightarrow$ $PQ^2 = OQ^2 - OP^2$

$= 13^2 - 5^2 = 169 - 25 = 144$

$\Rightarrow$ $PQ = \sqrt{144} = 12$

So, length of tangent is 12 cm

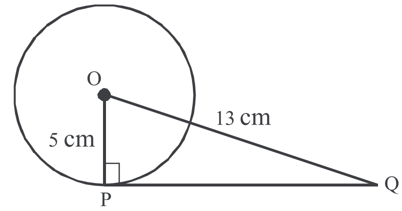


Fig. 13.09

Example 2. In two concentric circles if the chord of larger circle touches the smaller one then prove that the point of contact bisects the chord.

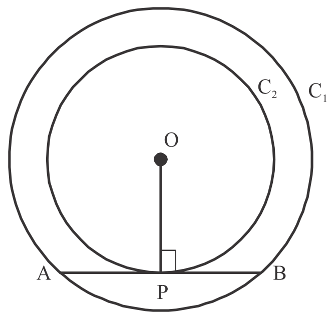


Fig. 13.10

Solution : Given : AB is a chord of larger circle C_1 which touches the smaller circle at point P.

To prove : $AP = PB$

Proof : AB touches the circle C_2 at point P.

So, $OP \perp AB$ (According to theorm 13.1)

Since O is also centre of circle C_1 and AB is chord of circle C_1, so (as per class IX theorem) perpendicular drawn form centre of the circle to the chord, then it bisects the chord.

Hence,

$AP = PB$ **Hence proved**

Example 3. A circle touches the side BC of ΔABC at P externally and AB and AC are produced they meet to the circle at Q and R respectively then prove that $AQ = \frac{1}{2}$ (Perimeter of ΔABC)

Solution : Given : ΔABC, side BC touches the circle at point P and by producing side AB and AC they touches the circle at point Q and R, respectively.

To prove : $AQ = \frac{1}{2}$ (Perimeter of ΔABC)

Proof : AQ = AR (According to theorem 13.2) . . . (1)

Similarly BQ = BP . . . (2)

CP = CR . . . (3)

AQ + AR = [AB + BQ] + [AC + CR]

$\Rightarrow$ = [AB + BP] + [AC + CP]

= AB + (BP + CP) + AC

$\Rightarrow$ 2AQ = AB + BC + AC from equation (1)

$\Rightarrow$ $AQ = \frac{1}{2}[AB + BC + AC]$

$\Rightarrow$ $AQ = \frac{1}{2}$ (Perimeter of ΔABC) **Hence Proved**

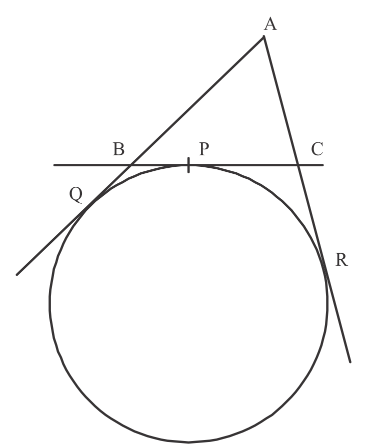


Fig. 13.11

Example 4. The side AB, BC and CA of ΔABC touches a circle of radius 4 cm at points L, M and N respectively. If $AN = 6$ cm, $CN = 8$ cm then find the perimeter of ABC.

Solution : Let 'O' be the centre of the circle inscribed in the ΔABC.

means $OL = OM = ON = 4$ cm

Let $BL = x$ cm

$\Rightarrow$ $BL = BM = x$ (See fig. 13.12)

$\because$ $AN = AL = 6$ cm

Similarly, $CN = CM = 8$ cm

$BC = (x+8)$ cm $= a$ and $AB = (x+6)$ cm $= c$

and $AC = 6 + 8 = 14$ cm $= b$

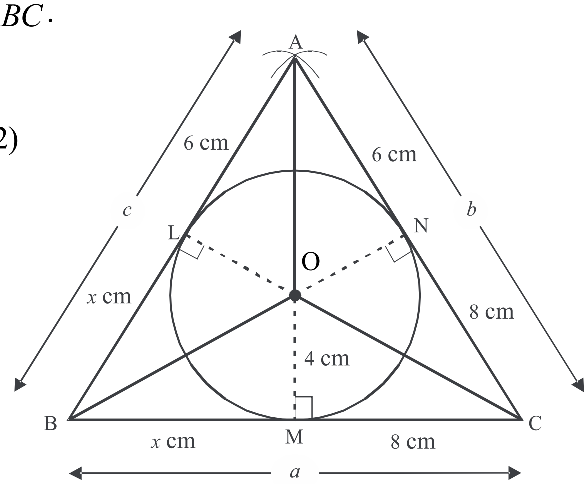


Fig. 13.12

According to Hiron's formula.

$$2s = a + b + c$$

$\Rightarrow$ $2s = x + 8 + 14 + x + 6$

$\Rightarrow$ $2s = 2x + 28$

$\Rightarrow$ $s = x + 14$

Area of ΔABC $= \sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{(x+14)(x+14-x-8)(x+14-14)(x+14-x-6)}$$

$$\sqrt{(x+14)\times 6\times x\times 8} = \sqrt{48x(x+14)} \quad \ldots(1)$$

and Area of ΔABC = Area of ΔAOB + Area of ΔBOC + Area of ΔAOC

$$= \frac{1}{2}AB\times OL + \frac{1}{2}BC\times OM + \frac{1}{2}AC\times ON$$

$$= \frac{1}{2}(x+6)\times 4 + \frac{1}{2}(x+8)\times 4 + \frac{1}{2}\times 14\times 4$$

$$= 2(x+6) + 2(x+8) + 28$$

$$= 2x + 12 + 2x + 16 + 28$$

$$= 4x + 56 \quad \ldots(2)$$

From equation (1) and (2)

$$\sqrt{48x(x+14)} = 4x + 56$$

$$4\sqrt{3x(x+14)} = 4(x+14)$$

$\Rightarrow$ $\sqrt{3x(x+14)} = (x+14)$

Squaring both sides

$$3x(x+14) = (x+14)^2$$

$$3x = x + 14$$

$$3x - x = 14$$

$$x = 7$$

So, $AB = 6 + 7 = 13$ cm

$BC = 7 + 8 = 15$ cm

Hence, perimeter of $\Delta ABC = (13 + 15 + 14) = 42$ cm.

Exercise 13.1

1. Write true or false. Also write the reason of your answer.

 (i) Tangent of a circle is the line which intersect the circle at two points.

 (ii) A tangent XY, touches a circle at point P whose center is O and Q is any other point on the tangent, then $OP = OQ$.

 (iii) LM and XY two tangents at the points P and Q on the circle if PQ is diameter then $LM \| XY$.

 (iv) $'O'$ is the centre of circle situated on the other circle whose centre is A. If circle having centre $'O'$ passes form A and B such that AOB are in one line then the tangents drawn from B will pass through the intersecting points of both circles.

2. Fill in the blanks :

 (i) .... tangents can be drawn form a point situated on circle.

 (ii) A line which intersects the circle at two points is known as ....

 (iii) A circle can have .... parallel tangents.

 (iv) The common point of tangent and circle is known as ....

3. Two concentric circles have radius 5 cm and 3 cm respectively. Find the length of the chord of circle which touches the smaller circle.

4. The length of tangent is 4 cm drawn form any point, 10 cm away form the centre of the circle then what will be the radius of that circle?

5. A circle with centre at $'O'$ touches the four side of a quadrilateral $ABCD$ internally in such a way that it divides AB in 3 : 1 and $AB = 8$ cm then find the radius of the circle where $OA = 10$ cm.

6. A circle touches the all sides of a quadrilateral. Prove that the angle made by opposite sides at the centre are supplementry.

7. In fig. 13.13 centre of a circle is O and the tangents drawn form a point P are PA and PB which touches the circle at A and B respectively then prove that OP is the bisector of line AB.

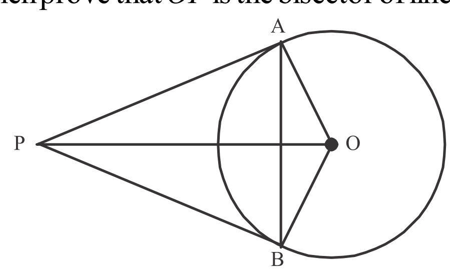


Fig. 13.13

8. In figure 13.13, O is the centre of the circle and from a exterior point $'P'$ two tangents PA and PB are drawn to the circle at A and B respectively then prove that $PAOB$ is a cyclic quadrilateral.

So far, we have learnt about tangents to the circles and solved many problems related to tangents and the circle. If a chord is drawn from a point of contact of the tangent to the circle then we will discuss segments made by the chord. By doing so we may come across some more informations. Let us try to understand these problems.

13.04. Angles of Alternate Segment of Circle

In fig. 13.14, chord AB of any circle, is drawn form tangent point A on the tangent PAQ which makes $\angle BAP$ and $\angle BAQ$ with PAQ.

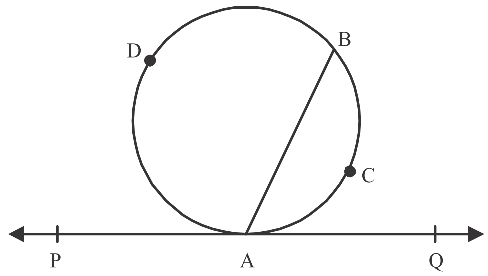


Fig. 13.14

The chord AB divide the circle into two segments ADB and ACB. The segments ADB and ACB of the circle are alternate segment of $\angle BAQ$ and $\angle BAP$.

Theorem 13.4

If a chord is drawn form a point of contact A, of the tangent PQ, of the circle then angle made by this chord with the tangent are equal to the respective alternate angles made by segments with this chord .

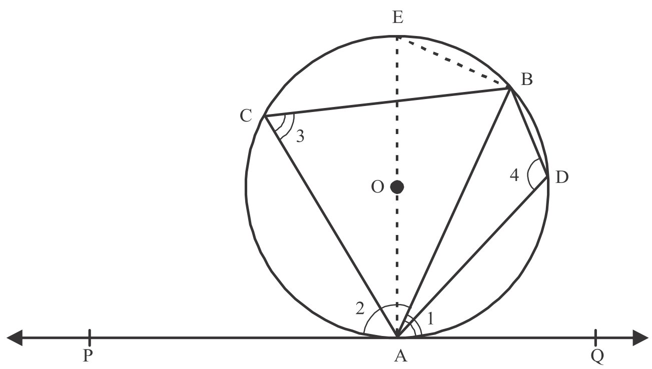


Fig. 13.15

Given : PQ is a tangent to the circle at point A, chord AB make $\angle 1$ and $\angle 2$ respectively with this tangent. $\angle 3$ and $\angle 4$ respective are alternate angles of $\angle 1$ and $\angle 2$ made in alterante segments at point C and D.

To be prove : $\angle 1 = \angle 3$ and $\angle 2 = \angle 4$

Construction : Draw diameter AOE and join EB

Proof : In $\Delta\, AEB$

$\angle ABE = 90°$ (angle made by semi circle)

$\therefore \quad \angle AEB + \angle EAB = 90°$ ... (1)

$\because \quad \angle EAP = 90°$ (diameter is perpendicular to the tangent)

$\therefore \quad \angle EAB + \angle 1 = 90°$ ... (2)

From equation (1) and (2)

$\angle EAB + \angle 1 = \angle AEB + \angle EAB$

$\Rightarrow \quad \angle 1 = \angle AEB$ ... (3)

$\because \quad \angle AEB = \angle 3$ (angles are equal in same circle segment) ... (4)

From eqaution (3) and (4)

$\angle 1 = \angle 3$ ... (5)

Again $\quad \angle 1 + \angle 2 = 180°$ (linear pair)

$\angle 3 + \angle 4 = 180°$

(opposite angles of cyclic quadrilateral are supplementary)

$\angle 1 + \angle 2 = \angle 3 + \angle 4$

But $\angle 1 = \angle 3$ (from equation 5)

$\angle 2 = \angle 4$ **Hence Proved**

Theorem 13.5 (Converse of 13.4)

If a line is drawn at one end of a chord of a circle in such a way that angle made with the chord are equal to alternate angle made by the chord in segment then this line is tangent to the circle.

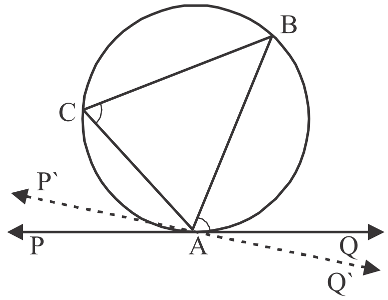


Fig. 13.16

Given : AB is a chord of any circle and PAQ is a line such that $\angle BAQ = \angle ACB$

Where, C is any point in alternate segment.

To prove : PAQ is a tangent.

Solution : $\angle BAQ = \angle ACB$ given. . . (1)

Let instead of PAQ line $P'AQ'$ touches the circle at point A.

So, $\angle BAQ' = \angle ACB$ (by theorem) . . . (2)

From equation (1) and (2)

$\angle BAQ = \angle BAQ'$ . . . (3)

According to the figure $\angle BAQ' = \angle BAQ + \angle QAQ'$ . . . (4)

i.e. $\angle BAQ = \angle BAQ + \angle QAQ'$

hence, $\angle QAQ' = 0$

This is only possible when PAQ and $P'AQ'$ coincides with each other

It means PAQ is a tangent to the circle at point A. **Hence Proved**

Illustrative Examples

Example 1. Write true or false and give reason of your answer.

(i) According to fig. 13.17, $\angle A = 70°$, where PQ touches to the circle at point C.

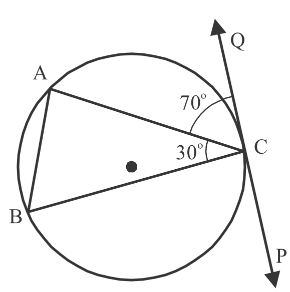


Fig. 13.17

Solution : False, since $\angle A$ is the alternate segment angle of $\angle PCB's$.

So, $\angle A = \angle PCB$

$\angle PCB = 180° - (70° + 30°) = 80°$

$\angle A = 80°$

Example 2. In figure 13.18, PQ is a tangent of a circle whose centre is O which touches the circle at point R. If $\angle TRQ = 30°$ then find $\angle SOR$ and $\angle RTO$.

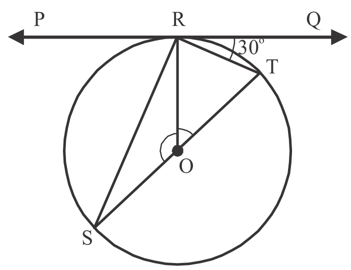


Fig. 13.18

Solution: Since diameter of the circle is SOT

$\therefore$ $\angle SRT = 90°$

and by chord RT, $\angle$ RST is alternate segment of $\angle$ TRQ

So $\angle RST = \angle TRQ = 30°$

But ΔORS is an Isosceles triangle. Radii of circle are OS = OR

$\angle RST = \angle SRO = 30°$

$\therefore$ $\angle SOR = 180° - (30° + 30°) = 180° - 60° = 120°$

and $\angle ORT = \angle SRT - \angle SRO$

$= 90° - 30° = 60°$

Now, in ΔORT

OR = OT (radii of a circle)

$\angle RTO = \angle ORT = 60°$

Hence, $\angle SOR = 120°$

Example 3. In figure 13.19 PQ and RS are tangents at point A and C respectively if $\angle ABC = 60°$ and $\angle BAP = 40°$ then find the value of $\angle BCR$.

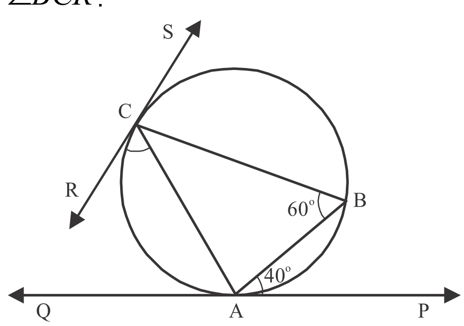


Fig. 13.29

Solution: Tangent PQ and chord AB passes through point A

$\therefore$ $\angle ACB = \angle BAP = 40°$ (By thorem 13.4) . . . (1)

Similarly tangent line CR and chord AC passes through point C

$\therefore \qquad \angle ACR = \angle ABC = 60°$ ... (2)

Adding equation (1) and (2)

$$\angle ACB + \angle ACR = 40 + 60 = 100°$$

or $\qquad \angle BCR = 100°$

Example 4. In figure 13.20, M is middle point of the segment AB, taking AM, MB and AB as diameter of semi circles has been drawn in one side. Taking O as centre and r radius of a circle is drawn in such a way that it touches the all the three circles, then porve that $r = \frac{1}{6}AB$.

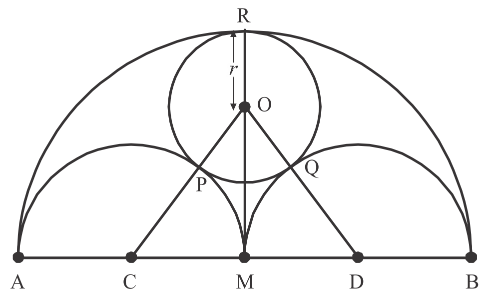


Fig. 13.20

Solution: Given: In figure 13.20, according to the question semicircles are drawn by taking C, M, D and O as centre.

To prove : $r = \frac{1}{6}AB$

Proof : Let AB = a then $AM = \frac{a}{2}$

But $\qquad$ AC = CM = MD = DM = CP = DQ are radii of equal semicircles

$\therefore \qquad CM = MD = CP = DQ = \frac{a}{4}$ ... (1)

Now, $\qquad OC = OD = \left(\frac{a}{4} + r\right)$ ... (2)

$$OM = (MR - OR) = \left(\frac{a}{2} - r\right) \qquad ... (3)$$

Since ΔOCD is an isoscles triangle whose side OC = OD and M is the mid point of CB

$\therefore \qquad OM \perp CD$

Now, in right triangle OMC, $\quad OC^2 = CM^2 + OM^2$

So, from equation (1), (2) and (3)

$$\left(\frac{a}{4} + r\right)^2 = \left(\frac{a}{4}\right)^2 + \left(\frac{a}{2} - r\right)^2$$

$$\Rightarrow \qquad \frac{a^2}{16} + r^2 + \frac{1}{2}ra = \frac{a^2}{16} + \frac{a^2}{4} + r^2 - ra$$

$$\Rightarrow \qquad \frac{1}{2}ra + ra = \frac{a^2}{4}$$

$\Rightarrow \qquad \frac{3}{2}ra = \frac{a^2}{4}$

$\Rightarrow \qquad a(6r - a) = 0$ but $a \neq 0$

$\therefore \qquad 6r = a$, $r = \frac{1}{6}a$ or $r = \frac{1}{6}AB$ Hence proved

Exercise 13.2

1. According to figure 13.21 answer the following questions:

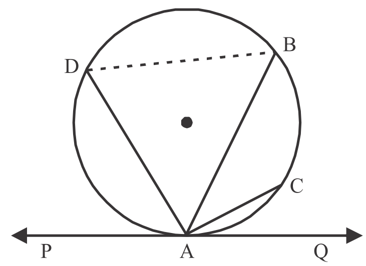


Fig. 13.21

(i) $\angle BAQ$ is an alternate segment of circle.

(ii) $\angle DAP$ is an alternate segment of circle.

(iii) If B is joined with C then $\angle ACB$ is equal to which angle?

(iv) $\angle ABD$ and $\angle ADB$ is equal to which angles.

2. According to figure 13.22 if $\angle BAC = 80^\circ$ then find the value of $\angle BCP$

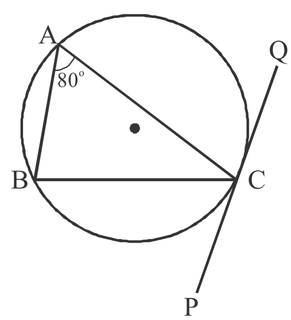


Fig. 13.22

3. According to figure 13.23, PQ and XY are parallel tangents. If $\angle QRT = 30^\circ$ then find the value of $\angle TSY$.

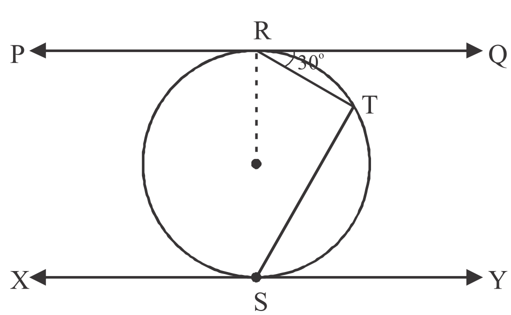


Fig. 13.23

4. Figure 13.24, in a cyclic quadrilateral ABCD diagonal AC bisects the angle C. Then prove that diagonal BD is parallel to the tangent PQ of a circle which passes through the points A, B, C and D.

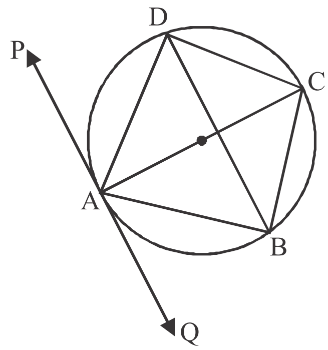


Fig. 13.24

Answer

Exercise 13.1

1. (i) **False :** Tangent intersects the cirlce at one and only one point .
   (ii) **False :** Because OP is perpendicular to the tangent and perpendicular is the smallest among the distances.
   (iii) **True :** Tangent is perpendicular to its diameter.
   (iv) **True :** Because AOB is a diameter and angle made on semi circle is always a right angle.
2. (i) one (ii) Secent (iii) Two (iv) Tangent
3. 8 cm 4. $2\sqrt{21}$ cm 5. 8 cm

Exercise 13.2

1. (i) ADB, (ii) ACBD, (iii) $\angle BAP$, (iv) $\angle DAP$ and $\angle BAQ$
2. 80° 3. 60°

14

Constructions

14.01 Introduction

In previous chapter we have understood and learnt the properties related to circle, secant line, tangent and alternate segment. Here we will learn how to construct the constructions related to theorems studied in previous chapters. We have discussed about congruent lines and points in the chapter of Locus, in whcih we have studied about incentre and circumcentre.

Now, we will try to understand these principles through construction by using basic principles and theorems.

Construction : In geometry any geometrical construction related to the puzzle is called 'construction'.

14.02. Internal Division of a Line Segment in a Given Ratio

Construction 1 : Draw a line segment of length 7.4 cm and divide it into 3 : 5 internally.

Steps of Construction

1. Draw a line segment AB = 7.4 cm.
2. Make an actue angle BAX.
3. Taking any suitable radius in the compass, mark eight (3+5) arcs $P_1, P_2, P_3, ..., P_8$ such that $AP_1 = P_2P_3 = ... = P_7P_8$
4. Join B to P_8.
5. Draw $P_3C \parallel P_8B$ from P_3 (for it draw $\angle AP_3C = \angle AP_8B$ which meets AB at C)

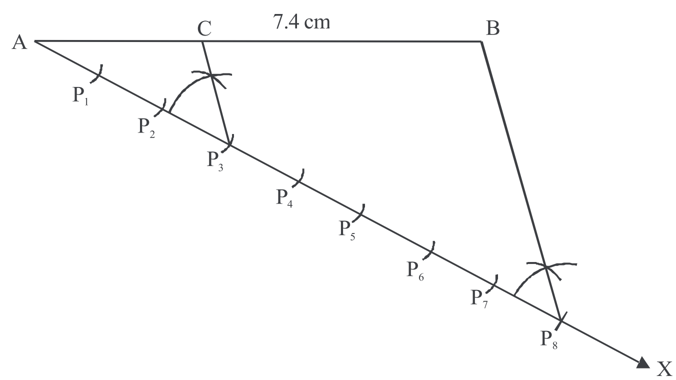


Fig. 14.01

Hence, the line segment AB is divided at C in the ratio of 3 : 5.

Construction 2 : Draw a line segment ML of length 9.7 cm on which find a point N such that

$$MN = \frac{4}{5}ML$$

Steps of Construction

1. Draw a line segment ML = 9.7 cm.
2. Construct on actue angle LMX.
3. Taking any suitable radius in the compass mark five arcs A_1, A_2, A_3, A_4 and A_5 such that

$MA_1 = A_1A_2 = ... = A_4A_5$

4. Join A_5 to L.

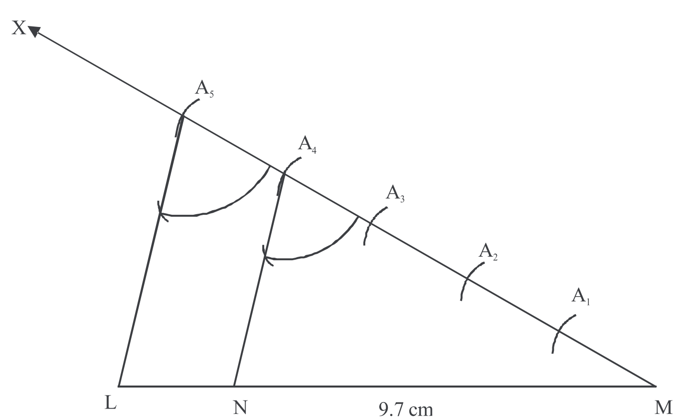


Fig. 14.02

5. Draw $A_4N \parallel A_5L$ from A_4 (for it construct $\angle MA_4N = \angle MA_5L$)

6. Point N divides the line segment ML such that $MN = \frac{4}{5}M$

Verification : In the triangle $MLA_5, NA_4 \parallel LA_5$

$$\therefore \quad \frac{LN}{NM} = \frac{A_5A_4}{MA_4} \text{ (fundamental proportional theorem)}$$

$$\Rightarrow \quad \frac{LN}{NM} + 1 = \frac{A_5A_4}{MA_4} + 1$$

$$\Rightarrow \quad \frac{LN + NM}{NM} = \frac{A_5A_4 + MA_4}{MA_4}$$

$$\Rightarrow \quad \frac{ML}{NM} = \frac{MA_5}{MA_4} = \frac{5}{4}$$

$$\Rightarrow \quad \frac{ML}{NM} = \frac{4}{5}$$

Hence, N is the point on the line segment ML such that $MN = \frac{4}{5}ML$

14.03 Construction of a Tangent to a Point on the Circle

Construction : Draw a tangent on a point at the circle whose radius is given.

Steps of construction:

(i) Take O as centre and draw a circle of radius r.

(ii) Mark a point P at the circumference of the circle and join O to P.

(iii) Draw a perpendicular AB at the point B.

Here AB is the required tangent.

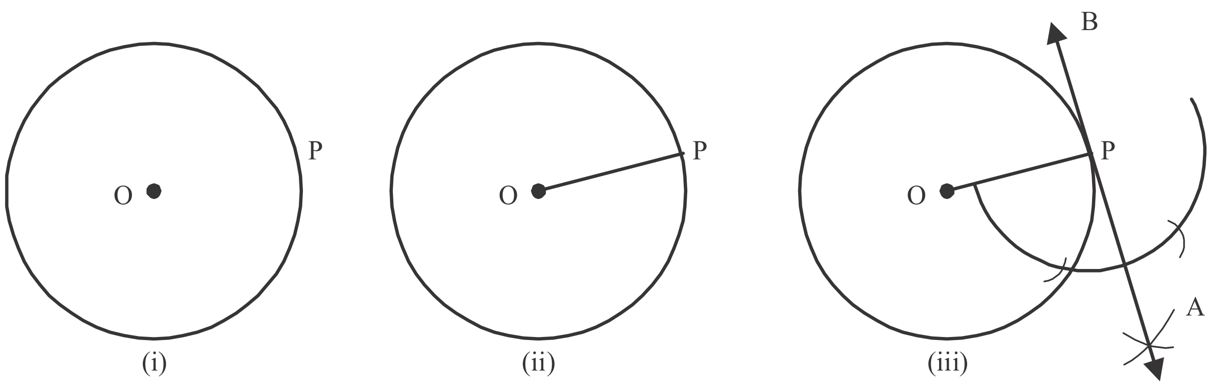


Fig. 14.03

Wrong method :

If we draw a circle and choose any point on it and then draw a line passing through this point, then this will not be a perfect method.

14.04 Construction of a Tangent from a Point Outside the Circle

Construction 4. To construct a tangent to a point outside the circle when centre is given.

Steps of construction

(i) Take a point *P* outside the circle whose centre is O.

(ii) Meet *P* to *O*.

(iii) Draw a perpendicular bisector of *OP* that intersects *OP* at *M*.

(iv) Taking *M* as the centre and radius *MO* draw a circle which intersects the first circle at A and B.

(v) Join A to P and B to P.

Hence AP and BP are the required tangents.

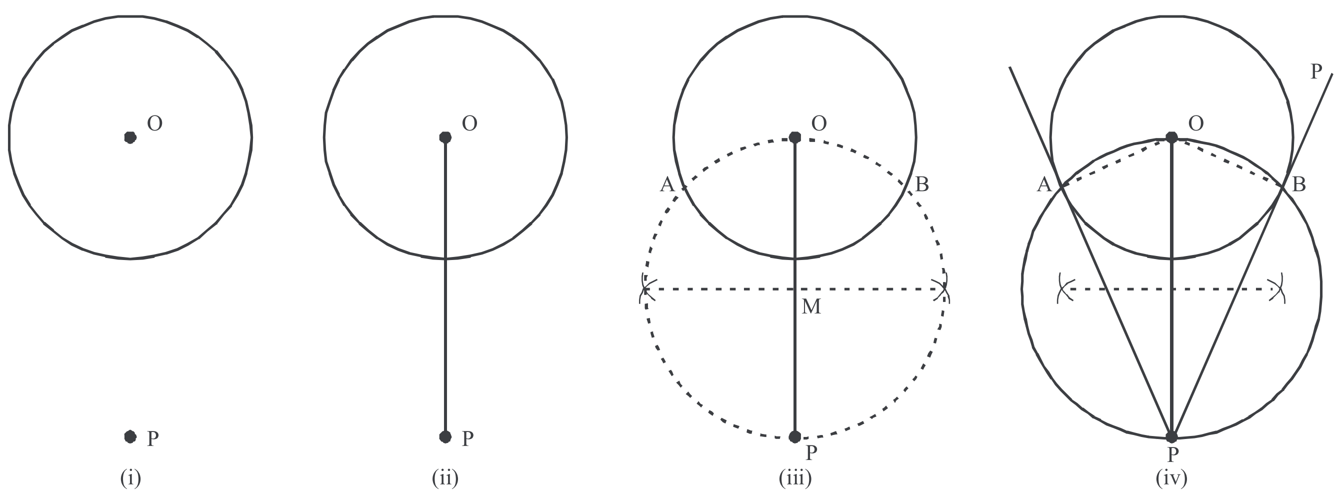
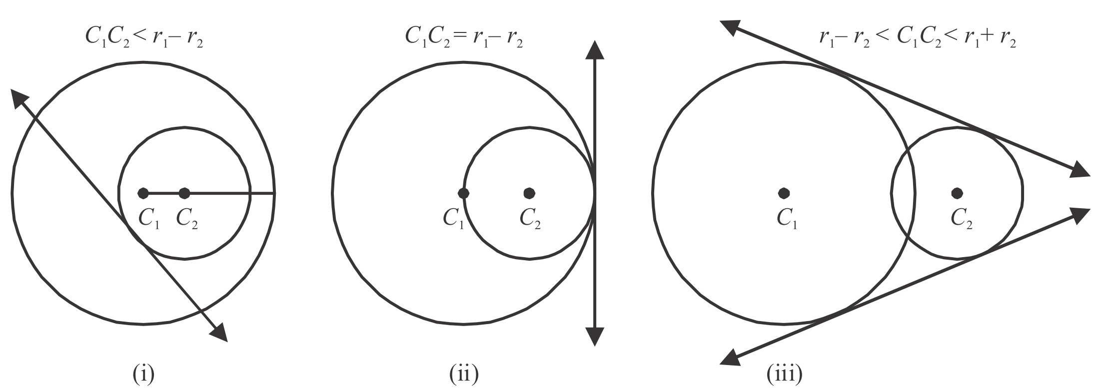


Fig. 14.04

14.05. Common Tangents

To find the maximum number of tangents to a circle or more than one circle we have to consider the different conditions. See the figures given below:



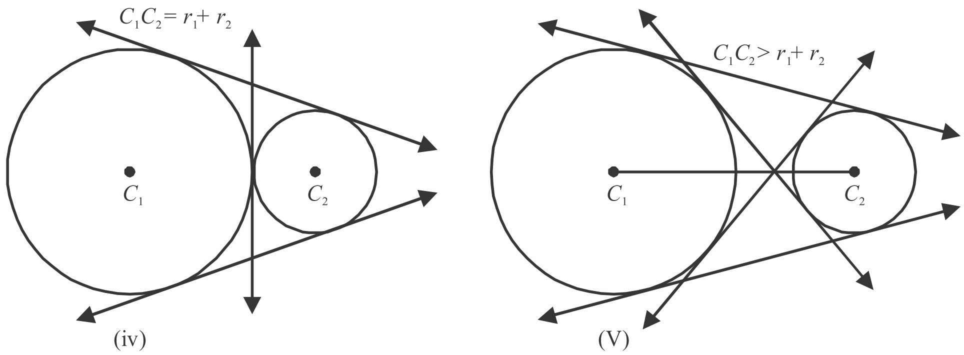


Fig. 14.05

I. In figure 14.05(i), if a tangent is drawn to the smaller ercircle, it intersects the larger circle. So, if $C_1C_2 < r_1 - r_2$ then the number of common tangents is zero.

II. In figure 14.05(ii), there is only one common tangent to both the circles and these circles are in the same side of the tangent.This is called a direct common tangent. Now when $C_1C_2 = r_1 - r_2$, then the number of common tangent is only one.

III. In the figure 14.05(iii), both the circles intersect each other so there are two tangents to both the circles in two opposite sides.Since two circles are in the middle of the tangents i.e., in the same side of each tangent. So both the tangents are considered as direct common tangents. When $r_1-r_2 < C_1C_2 < r_1+r_2$, then the number of direct common tangents will be 2.

IV. In the figure 14.05(iv), both the circles touch each other externally, so the number of common tangents is three. Here one common tangent is situated at the common touch point of the two circles, which is called the transversal common tangent, as two circle are situated at the two different sides of this tangent.

Thus if $C_1C_2 = r_1 + r_2$, then the total number of common tangents will be 3 (2 direct common tangents and one transversal common tangent).

V. In the figure 14.05(v), two circles do not touch each other, so the number of common tangents to the two circles is 4. (2 direct common tangents and two transversal common tangents).

Thus, when $C_1C_2 > r_1 + r_2$, then the total number of common tangents is 4 in which there 'two' direct and two transversal tangents.

Construction 5 : To construct a common tangents of both circles where distance between centres of both the circles having different radius is known.

Steps of construction

(i) Draw a line segment $C_1C_2 = 5.5$ cm, with centres C_1 and C_2 and radii $r_1 = 3.5$ cm and $r_2 = 2.5$ cm, draw two circles respectively.

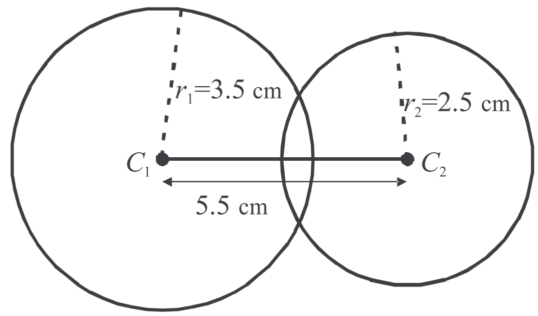


Fig. 14.06

(ii) With the centre of a larger circle C_1 and radius $r_1 - r_2 = 3.5 - 2.5 = 1$ cm, draw another circle. Taking C_1C_2 as diameter draw a dotted circle which intersects the smallest circle at P.

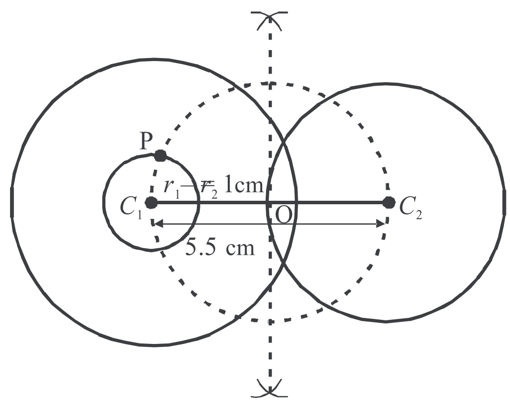


Fig. 14.07

(iii) Join P to C_2 and from C_2 draw a tangent to the circle of radius $r_1 - r_2$. Join C_1 to P and extend it to meet the largest circle at Q of radius r_1. Taking Q as center and radius PC_2 draw an arc which intersects the circle of radius r_2 at R. Join Q to R.

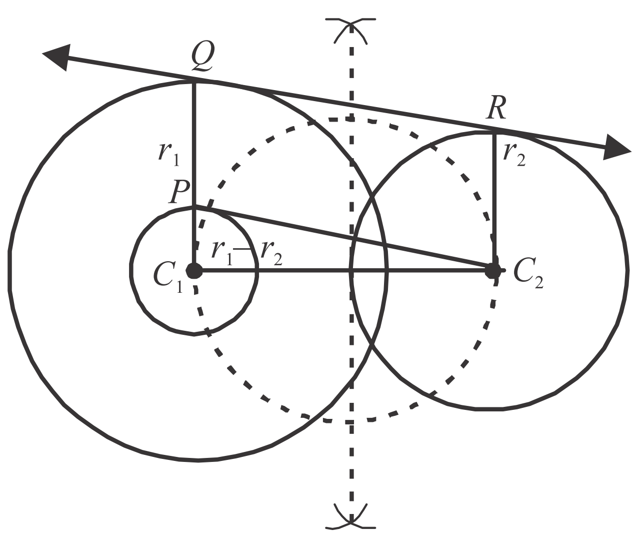


Fig. 14.08

Note: In this way we get a rectangle $PQRC_2$. Therefore C_1Q and C_2R are the perpendiculars at the line QR. So QR is called the direct common tangent to both the circles. Similarly, another tangent line can also be drawn in opposite side.

Important note : To construct a direct common tangent line.

(i) Find out $r_1 - r_2$.

(ii) Draw circle with radius $r_1 - r_2$ must be co-centred of the larger circle.

(iii) If twocircles have equal radii, then join C_1 to C_2 and draw perpendiculars at both ends C_1C_2, and join these intersecting points what is obtained is the required tangent.

Construction 6 : To construct a common transversal tangent to two circles (radii r_1 and r_2 and centres C_1 and C_2 are given).

Example : The centres of two circles with radii 2.5 cm and 1.0 cm are at a distance of 7 cm to each other. Draw a common transversal tangent line.

Steps of Constructions :

(i) Draw a line segment C_1C_2 = 7 cm with the centres C_1 and C_2 and radii 2.5 cm and 1.0 cm, draw two circles at both end points.

(ii) With the radius equal to the sum of the radii of both the circles *i.e.*, $r_1 + r_2 = 2.5 + 1.00 = 3.5$ cm and from centre C_2 of smaller circle, draw a dotted circle of radius 3.5 cm. Find the bisector of the diameter C_1C_2 and taking the bisecting point O as centre draw a dotted circle taking radius as $OC_1 = OC_2$. Which intersects circle with radius 3.5 cm at R. Join C_1 to R, and draw the tangent to the circle of radius $r_1 + r_2$ from C_1.

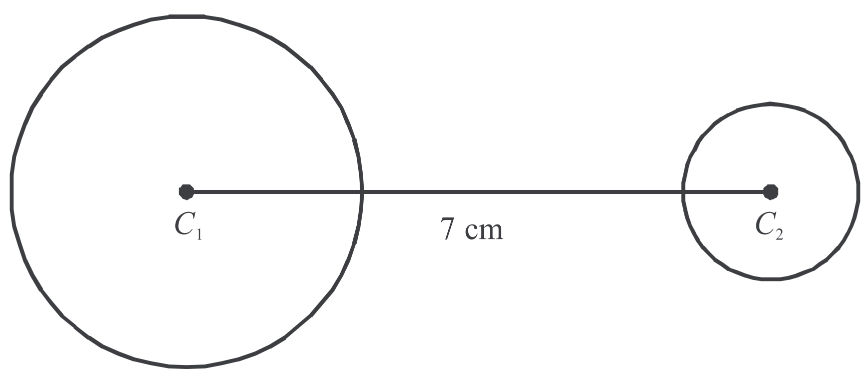


Fig. 14.09

(iii) Join R to C_2 which touches the smaller circle at Q. Taking Q as centre and RC_1 as radius draw an arc whcih intersect the larger circle at P.

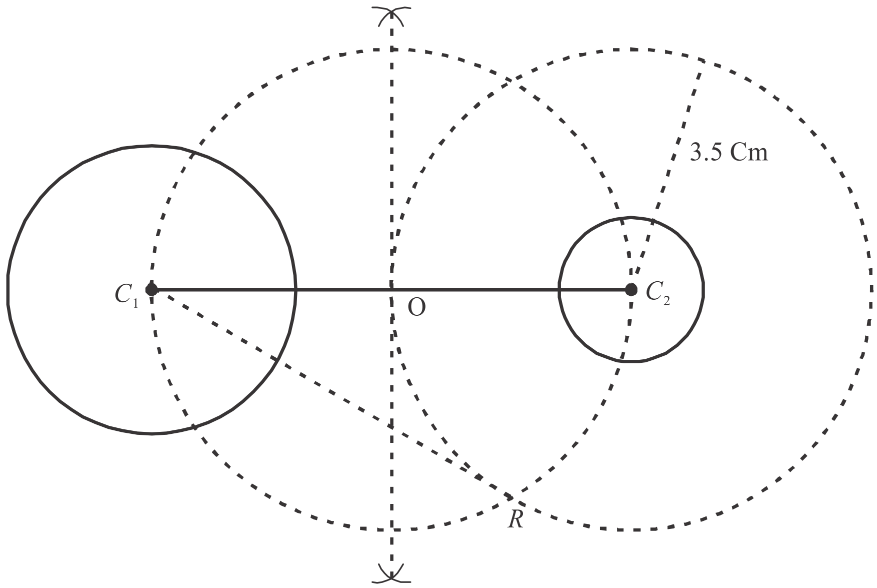


Fig. 14.10

Joint P to Q. Hence PQ, is the required tangent.

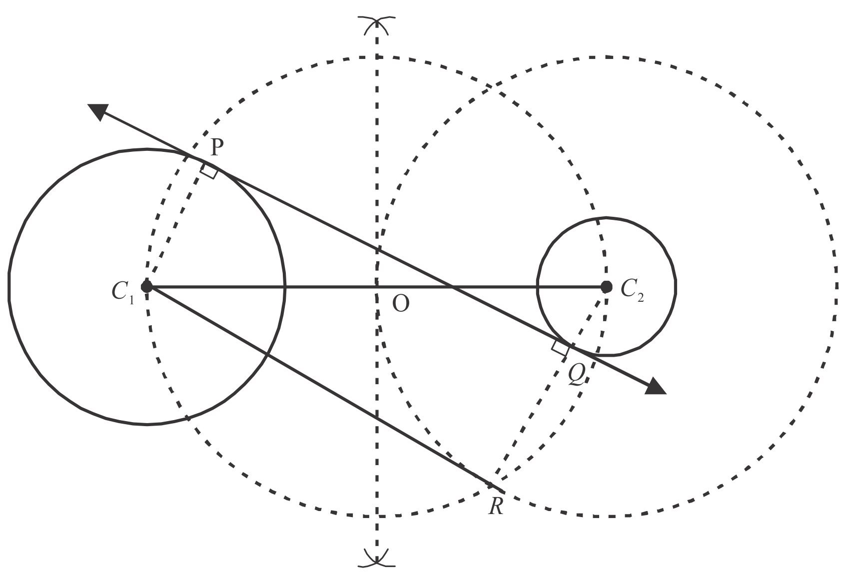


Fig. 14.11

Illustrative Examples

Example 1. Draw a tangent to a circle of radius 2.5 cm.

Solution :

Steps of Construction

(i) Draw a circle with the centre O and radius 2.5cm.

(ii) Take a point P on the circle and join O to P.

(iii) Draw a perpendicular at P i.e., draw $\angle OPA = 90°$ Extend the perpendicular to both sides upto A and B.

Hence, APB is the required tangent.

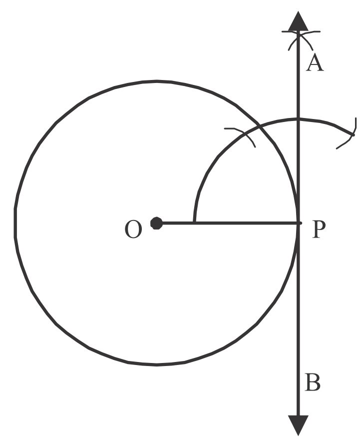


Fig. 14.12

Example 2. Construct a circle with the centre O and radius 2.8 cm. Take a point P at a distance of 4.3 cm from the centre and draw a pair of tangents to the circle from it. Also measure the tangents and verify that they are equal.

Steps of Construction:

(i) With the centre O and radius 2.8 cm draw a circle.

(ii) Take a point P outside the circle at a distance of 4.3cm from its centre and join O to P.

(iii) Draw the perpendicular bisector of OP, find its mid point M.

(iv) Take M as centre and radius $OM = PM$ and draw a circle which intersects the given circle at the points A and B.

(v) Join A to P and B to P.

Now PA and PB are the required tangents.

By measuring PA and PB, we find that PA = PB = 3.2 cm.

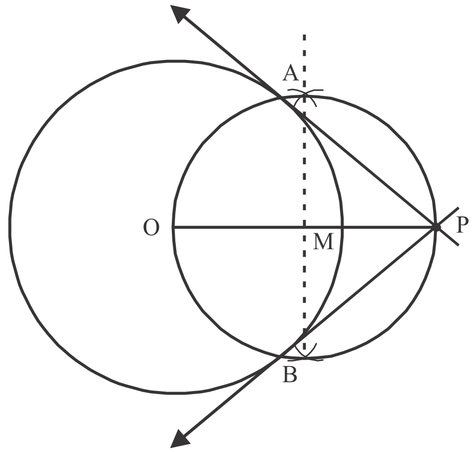


Fig. 14.13

Example 3. Draw a circle with center O and radius 3 cm. Draw its two radii OA and OB such that the angle between them is 180^{o} and draw a pair of tangents at A and B.

Steps of Construction :

(i) Draw a circle with centre O and radius 3 cm.

(ii) Draw two radii OA and OB such that angle between them be 120^{o}.

(iii) Draw the perpendicular at A and B on two radii which intersect each other at P.

(iv) Join P to A and P to B.

Here PA and PB are the required pair of tangents.

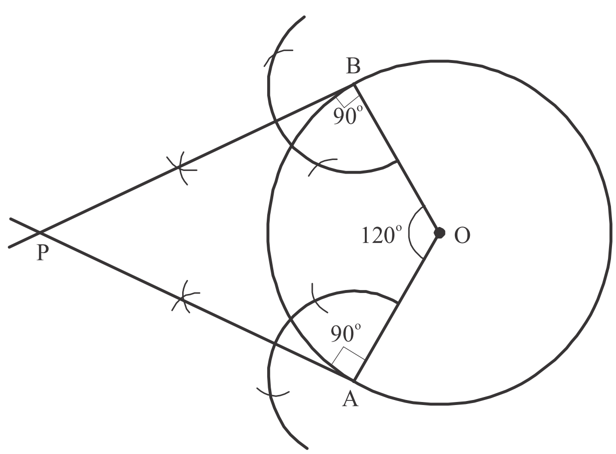


Fig. 14.14

Example 4. Draw two tangents to the circle of radius 4 cm such that the measure of the angle between them is 80^{o}.

Solution : We have $\angle APB = 80°$

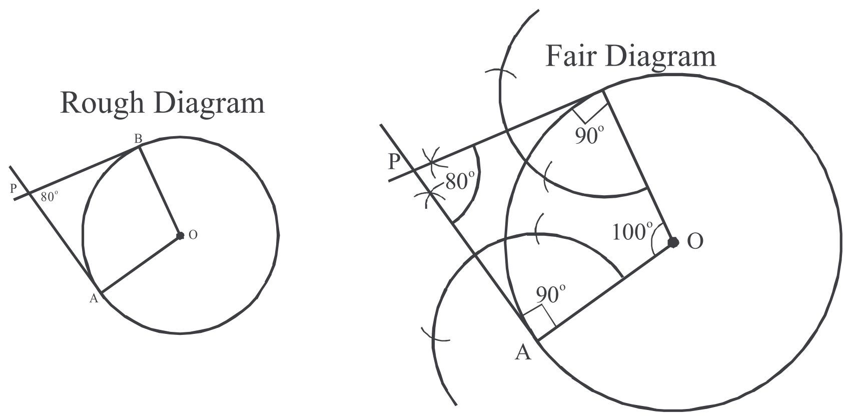
and we know that $\angle A = \angle B = 90°$

$\therefore$ In a quadrilateral $AOBP$

fourth angle $\angle AOB = 360 - (80 + 90 + 90) = 360 - 260 = 100°$

$\therefore$ the angle between two radii OA and OB

$= \angle AOB = 100°$



(i) (ii)

Fig. 14.15

Steps of Construction :

(i) Draw a circle of radius 4 cm with centre O.

(ii) Make an angle AOB of 100^{o} at the center O.

(iii) Draw the perpendiculars at A and B respectively, which intersects each other at P.

(iv) Join A to P and B to P.

Here AP and AB are the required tangents including the angle APB = 80^{o}.

Example 5. The distance between the centres of two circles whose radii are 4 cm and 3 cm respectively is 6.5 cm, draw any one direct common tangent to these circles.

Steps of Construction :

Let C_1 and C_2 are the centre of both the circles respectively where C_1C_2 = 6.5 cm.

(i) Draw a line segment C_1C_2 = 6.5 cm and aslo draw two circles with C_1 as centre and radius 4 cm and C_2 as centre and radius 3 cm.

(ii) Taking the centre C_1 of the larger circle and radius $r_1 - r_2 = 4 - 3 = 1$cm, draw a circle.

(iii) Bisect C_1C_2 at M.

(iv) Take M as centre and $C_1M = C_2M$ as radius, draw another circle, which intersects the circle of radius 1 cm at P.

(v) Join C_2 to P and join C_1 to P and produce it to Q.

, (vi) Join Q to R.

Here QR is the required tangent.

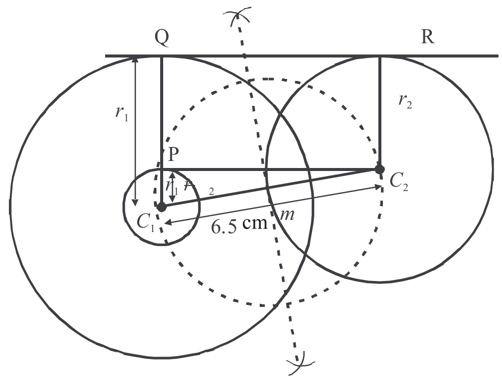


Fig. 14.16

Example 6. The distance between the centres of the two circles of 8 cm. Radius of one circle is 2.5 cm and that of other is 1.5 cm. Draw transversal common tangent line to the circles.

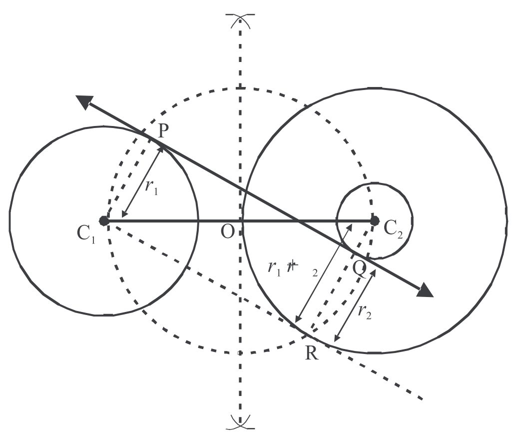


Fig. 14.17

Steps of Construction :

Let C_1, C_2 and r_1, r_2 are centre and radii of both the circles respectively where r_1 = 2.5 cm, r_2 = 1.5 cm and C_1C_2 = 8 cm.

(i) Draw a line segment C_1C_2 = 8 cm.

(ii) Draw two circles with centre C_1 and C_2 radius 2.5 cm and 1.5 cm respectively at both the ends.

(iii) Draw a circle with C_2 as centre and radius 2.5 + 1.5 = 4.0 cm.

(iv) Take C_1C_2 as diameter. Draw another circle which intersects the circle of radius 4 cm at *R*. Join C_1 at *R*. Join C_2 to *R* which intersects the circle of radius 1.5 cm at *Q*.

(v) Take *Q* as centre and radius equal to RC_1. Draw an arc which intersect the circle with centre C_1 at *P*. Join *Q* to *P*.

Here *PQ* is the required transversal tangent.

Exercise 14.1

1. Draw a line segment of length 6.7 cm and divide it into the ratio of 2 : 3 internally.
2. Draw a line segment AB of length 8.3 cm. Find a point C at AB such that $AC = \frac{1}{3}AB$ and also verify it.
3. Take a point P at the circle of radius 2.8 cm and draw a tangent to the circle at P.
4. Draw the tangents to both the ends of the diameter of circle of radius 3 cm. Will the tangents intersects each other? If yes, justify your answer.
5. Draw a chord of lengh 2.3 cm in a circle of radius 13.1 cm and draw the tangent to its both ends.
6. Draw a tangent to a circle of radius 2.7 cm.
7. Draw a circle with centre O and radius 2.4 cm. Draw two radii OA and OB such that the angle between them is 60^o. Draw the tangents to A and B to intersect each other at T. Measure the angle ATB.
8. Draw two tangents to the circle of radius 13.2 cm, such that the angle between them is 70^o.
9. Draw a circle of radius 3cm. Construct a pair of tangents from an exterior point 5 cm away from its centre.
10. The distance between the centres of two circles is 8 cm. Radius of one circle is 3 cm and that of other is 4 cm. How many common tangents can be drawn on these circles and construct two direct common tangents.
11. Construct a common transversal tangents of the circles whose radii are 1.7 cm and 2.8 cm respectively and centres are 6cm apart.

14.06. Constuction of Circumcircle and Incircle of a Triangle

(A) Construction of circumcircle of a triangle

The *circumcircle* of a triangle is a circle passing through the vertices of the triangle. The centre of the circle is called the circumcentre of the triangle. The circumcentre of a triangle is a point where the perpendicular bisectors of three sides of triangle intersect each other.

To construct a circumcircle of triangle, the following steps are to be taken :

(i) To find the circumcircle of a triangle, perpendicular bisectors of any of the two sides are drawn. The intersection points of these bisectors is the circumcentre of the triangle.

(ii) Take circumcentre as centre and radius between the centre and any of the vertex.

(iii) Draw a circle which passes through three vertices of the triangle. It is the required circumcircle.

Example 7. Construct a ΔABC in which BC = 3.8 cm, $\angle B = 60°$ and $\angle C = 55°$. Draw a circumcircle of this triangle and check the position of circumcentre.

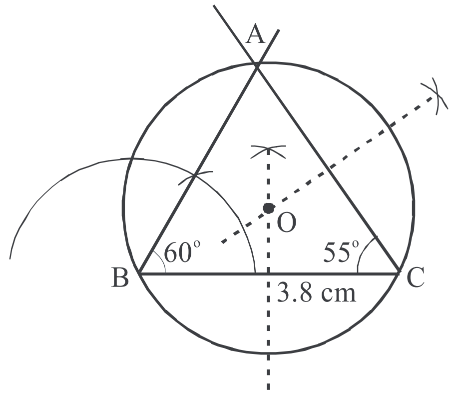


Fig. 14.18

Steps of Construction :

(i) Construct ΔABC according the given measurements.

(ii) Draw the perpendicular bisectors of the sides BC and AC that intersect each other at the point O.

(iii) Take O as centre and radii $OA = OB = OC$, draw a circle.

This is the required circumcircle

Note : *The circumcentre of an acute angled triangle always lies inside it.*

Example 8. Construct ΔABC in which side BC = 4 cm, $\angle B = 40^\circ$ and $\angle A = 90^\circ$. Construct the circumcircle of the triangle. Also find the position of its circumcentre.

Steps of Construction :

Since, in a triangle ABC

$\angle A + \angle B + \angle C = 180^\circ$

$\therefore$ $90^\circ + 40^\circ + \angle C = 180^\circ$

$\Rightarrow$ $\angle C = 180^\circ - 135^\circ = 50^\circ$

(i) Draw a line segment BC = 4 cm, $\angle B = 40^\circ$ and $\angle C = 50^\circ$, in this way we get $\angle A = 90^\circ$.

(ii) Draw the perpendicular bisectors of the sides AB and AC to intersect each other at O.

(iii) Take O as centre and radius $OA = OB = OC$ draw a circle, which passes through the vertices of ΔABC.

This is the required circumcircle of ΔABC.

Note : The circumcentre of a right angled triangle always falls at the hypotenuse.

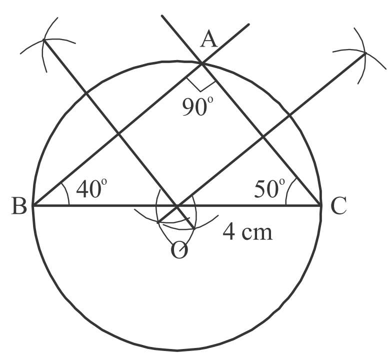


Fig. 14.19

Example 9. Construct a circumcircle of a triangle with base BC= 6 cm, $\angle B = 30^{\circ}$ and $\angle C = 25^{\circ}$. Also find the position of the circumcentre of the triangle.

Steps of Construction :

(i) Construct the ΔABC according to the given measures.

(ii) Draw the perpendicular bisector of sides AB and AC which intersect each other at the point O. It is the circumcentre of the triangle.

(iii) Take O as centre and radii = OA = OB = OC, draw a circle, which passes through the three vertices A, B and C of given triangle.

Now the constructed circle is the required circle.

Note : The circumcentre of an obtuse angled triangle always falls in somewhere outside the triangle. According to the nature of the triangles the position of the circumcentre is always different.

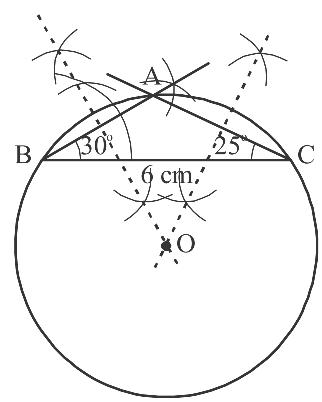


Fig. 14.20

(a) In an acute angled triangle, the position of its circumcentre is inside the triangle.

(b) In an right angled triangle, the position of its circumcentre is always at its hypotenuse.

(c) In an obtuse angled triangle the position of its circumcentre is always at an external point of the triangle.

(B) Construction of an incircle of triangle

The incircle of a triangle is a circle drawn inside a triangle touching all the sides of the triangle. The centre of this circle is called the incentre of the triangle. The intersecting point of the bisectors of angles of a triangle is called the incentre of the triangle. We use the following steps to understand the construction of incircle.

(i) Draw bisector of any two angles of the given triangle.

(ii) Draw a perpendicular to any one side of the triangle form the intersecting point of bisetors (bisector of any angle is at the same distance from the two sides of the angle).

(iii) Take radius equal to the length of the perpendicular drawn and intersecting point as centre ad draw a circle. It will be the incircle of the given triangle.

Example 10. Construct an incircle of the ΔABC in which base BC = 5.8 cm, AB = 5 cm and $\angle B = 55^{\circ}$.

Steps of Construction :

(i) Draw a triangle ABC accord to the given measurements in the question.

(ii) Draw the bisector of the $\angle B$ and $\angle C$, which intersect each other at *O*. *O* is the incenntre.

(iii) Draw $OM \perp BC$ form *O*.

(iv) Take *O* as centre and *OM* as radius. Draw a circle which touches the sides of ΔABC at *P, M* and *N* respectively.

This is the required incircle of ΔABC

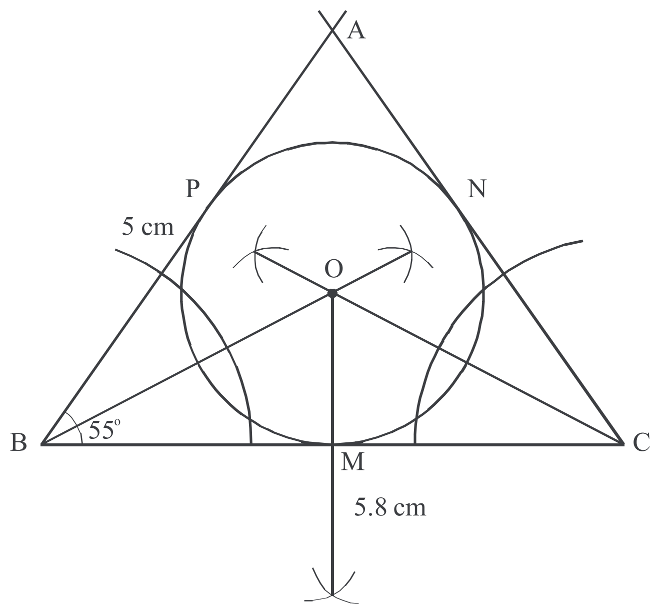


Fig. 14.21

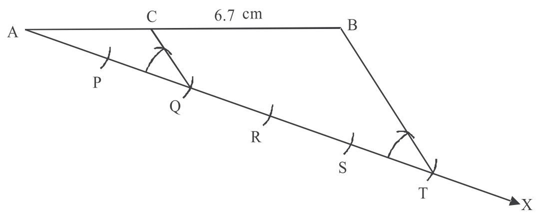
Exercise 14.2

1. Find the true or false in the following statements and justify your answer if possible.
   (i) Circumcircle and incircle of an equilateral triangle can be drawn form the same centre.
   (ii) All three sides of a triangle touch its incircle.
   (iii) If the triangle is obtuse angled, its circumcentre will fall at one of its sides.
   (iv) The circumference of the triangle lies inside, if the triangle is an acute triangle.
   (v) The construction of incircle is being done by obtaining the point of intersection of two perpendicular of sides and bisector of two angles.
2. Construct an incircle of an equilateral triangle with side 4.6 cm, Is its incentre and circumcentre are coincidence? Justify your answer.
3. Construct an incircle of a triangle with AB = 4.6 cm, AC = 4.2 cm and $\angle A = 90^{o}$.
4. Draw a circumcircle of a triangle with sides respectively 5, 12 and 13 cm. Why does its circumcentre falls at the side of length 13 cm?
5. Draw a circumcircle of a triangle whose sides are 5 cm, 4.5 cm and 7 cm. Where and why does its circumcentre lie?
6. Construct a ΔABC in which AB = 6 cm, BC = 4 cm and $\angle B = 120^{o}$ also construct incircle of this triangle.

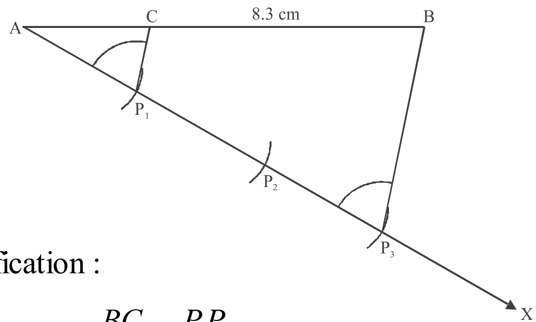
Answers

Exercise 14.1

1.



2.



Verification :

$$\therefore \qquad \frac{BC}{AC} = \frac{P_1P_3}{AP_1} \qquad \text{(Fundamental Proportional Theorem)}$$

or $$\frac{BC}{AC} + 1 = \frac{P_1P_3}{AP_1} + 1$$

or $$\frac{BC + AC}{AC} = \frac{P_1P_3 + AP_1}{AP_1}$$

or $$\frac{AB}{AC} = \frac{AP_1 + P_1P_3}{AP_1} = \frac{3}{1}$$

or $$\frac{AC}{AB} = \frac{1}{3}$$

Hence, C is such a point on AB that $AC = \frac{1}{3}AB$.

Do other construction yourself with the help of your teacher.

Exercise 14.2

(1) (i) True, since circumcentre, incentre and orthocentre of an equilateral triangle are coincides.

(ii) True, since an incircle is drawn with the radius equal to length of perpendicular to a side from incentre.

(iii) False, a right angled triangle has its circumcentre on its hypotenuse.

(iv) True,

(v) False, to construct an incircle of a triangle by bisecting its angle not its sides.

4. Since, the side of length 13 cm is the hypotenuse of the given triangle and circumcentre of a right angled triangle always falls on its hypotenuse.

15

Circumference of a Circle and Area

15.01 Introduction

In previous chapters, we have learnt about circle related definations and properties. In this chapter we shall study about the circumference and area of the some circular figures.

A circle is a locus of a point which moves in a plane is such a manner that its distance form a fixed point in the same plane remains constant.

15.02 Circumference of the Circle

To find the circumference (approximately) mark a point C at the rim of a circular disc and put it in a plane such that the point C touches a point A in the plane. Now rotate the circular disc carefully.

That the point C may again touch the plane at the point B (See the fig. 15.01)

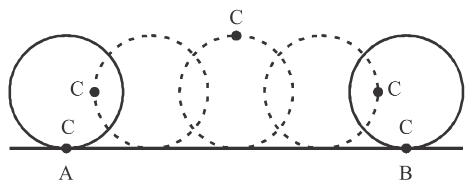


Fig. 15.01

Now measure the line segment AB. The length of the line segment AB is equal to the circumference of circular disc. So we can say that the distance covered by a wheel in one complete rotation is called its circumference. It is also known as the perimeter.

Circumfrance of a circle $= 2\ \pi r$ or $\pi \times d$

where r stands for radius and d stands for diameter of the circle

So, $\quad d = 2r$

or $\quad \dfrac{\text{Circumference}}{\text{diameter}} = \pi$

Note : Words "the circle and the circumference" of the circle are completely different. A circle is a figure in the plane while circumference is a length measured. The ratio of the circumference of a circle with its diameter is a constant which is denoted by the Greek letter π (Pie)

The value of π is calculated up to 5,00,000 (half million) decimal places by computer. We will take the value of $\pi = \dfrac{22}{7}$. A more accurate value of π is $\dfrac{62832}{200000}$ approximately 3.1416 which was given by the great Indian astronomer Aryabhatta (499 AD). π is a irrational number. The value of $\pi = 3.1416$ is correct to four decimal places.

The circumferences of the circle $C\,(0, r)$ is $2\pi r$ see (fig. 15.02).

$$\frac{\text{Circumference}}{\text{diameter}} = \pi$$

or $\quad \text{Circumference} = \pi \times \text{diameter}$

If C represents the circumference and D represents the diameter, then

$$C = \pi \times D = \pi \times 2r \quad \text{since diameter} = 2 \times \text{radius}$$

$$= 2\pi r$$

15.03. Area of a Circle

If a circle is drawn on a graph paper, we can obtain the area of the circle by counting the square surrounded by it. In this way, we get

$$\frac{\text{Area of the circle}}{(\text{radius})^2} = \pi$$

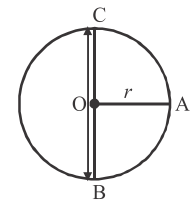


Fig. 15.02

or $\quad \frac{A}{r^2} = \pi$ where A is the area and r represents the radius of the circle.

or $\quad A = \pi r^2$

15.04. Area between Two Concentric Circles

Circles having the same centre but with different radii are called concentric circles (fig. 15.03). If r_1 and r_2 are the radii of two concentric circles where $(r_1 > r_2)$, then the area between two circles.

$$= \pi r_1^2 - \pi r_2^2$$

$$= \pi\left(r_1^2 - r_2^2\right)$$

Following results can be obtained from the above given facts :

(i) The distance covered by a wheel in one complete rotations is equal to circumference of the wheel.

(ii) Number of rotations made by a wheel in one minute

$$= \frac{\text{distance covered in per minute}}{\text{circumference}}$$

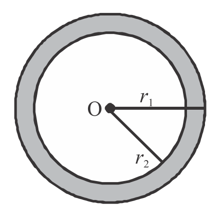


Fig. 15.03

(iii) Area of the circle is πr^2

Example 1. Find the radii of the circles given below, when

(i) The circumference of circle is 132 cm.

(ii) The circumference of circle is 176 cm.

Solution : (i) We have, circumference of circle = 132 cm.

or $\quad 2\pi r = 132$ (where r is the radius of circle)

or $\quad 2 \times \frac{22}{7} \times r = 132$

or $\quad r = \frac{132 \times 7}{2 \times 22} = \frac{42}{2}$

or radius $= 21$ cm.

(ii) We have, circumference of the circle = 176 cm

or $2\pi r = 176$ (where, r = radius of the circle)

or $2 \times \frac{22}{7} \times r = 176$

$$r = \frac{176 \times 7}{2 \times 22} = \frac{56}{2} = 28$$

hence radius of the circle = 28 cm.

Example 2. Find the area of a circle whose radius is 7 cm.

Solution : Given, radius of the circle =7 cm

$\therefore$ Area of the circle $= \pi r^2$

$$= \frac{22}{7} \times 7 \times 7 = 154$$

Hence, area of the circle = 154 cm^2

Example 3. A wheel of a bicycle makes 5000 rotations to complete a distance of 11 km. Find the diameter of the wheel.

Solution : We have, number of rotations = 5000

Distance covered = 11 km

Distance covered in one rotation by wheel

$$= \frac{\text{Total distance covered}}{\text{Number of rotations}}$$

$$= \frac{11}{5000} \text{km}$$

$$= \frac{11}{5000} \times 1000 \times 100 cm$$

$$= 220 \text{ cm}$$

Let radius of the wheel be r

Circumference = 220 cm

$$2\pi r = 220 \text{ cm}$$

$$2 \times \frac{22}{7} \times r = 220 \text{ cm}$$

$$r = \frac{220 \times 7}{2 \times 22} \text{cm} = 35 \text{cm}$$

diameter $= 2r = 2 \times 35 = 70$ cm

Exercise 15.1

1. Radius of a circle is 3.5 cm. Find its circumference and area.
2. The circumference of a circle is 44 m. Find the area of the circle.
3. The radius of a semi-circle shaped plot is 21 metre. Find the radius of the wheel.
4. A wheel covers a distance of 88 metres in 100 rotation. Find the radius of the wheel.
5. The area of a circular plates is 154 cm^2. Find its circumference.
6. The circumference of a circle is equal to the perimeter of a square. If area of the square is 484 m^2, then find area of the circle.
7. The cost of constructing a boundry wall of a circular field is ₹ 5280 at the rate of ₹ 24 per metre. Find the cost of ploughing this field at rate of ₹ 0.50 per metre ^{2}.
8. The radius of a circular grass land is 35 m. There is a foot-path of width 7 m. around in it. Find the area of the foot path.
9. The area between two concentric circles wil be

   (a) πR^2 (b) $\pi(R+r)(R-r)$ (c) $\pi(R^2-r)$ (d) None of these.
10. The radii of two concentric circles are 4 cm and 3 cm. The area between these two circles will be :

   (a) 22 cm^2 (b) 12 cm^2 (c) 32 cm^2 (d) 18 cm^2

15.05. Area of Sector and Segment of a Circle

The portion (or part) of the circular region enclosed by two radii and the corresponding arc is called a sector of the circle.

In the fig. 15.04., take a sector *AOB* in the circle (*O*,*r*). Let $\angle AOB = \theta$ and $\theta < 180^\circ$. When angle at the centre is increased, then the length of the arc *AB* is also increased in the same ratio. When an arc subtends an angle of at the centre then the length of arc = length of the arc of semi circle = πr.

∵ The length of the arc that subtends an angle of 180° at the centre is = πr.

∴ The length of the arc that subtends an angle of $\theta = \frac{\pi r\theta}{180^\circ} = 2\pi r \times \frac{\theta}{360^\circ}$

or $$L = 2\pi r \times \frac{\theta}{360^\circ} \qquad \text{. . . (i)}$$

Similarly, when angle at the centre is 180°, then the area of its corresponding sector is $=\frac{\pi r^2}{2}$

When the angle at the centre is θ, then area of the corresponding sector

$$A = \frac{\pi r^2\theta}{2\times180} = \frac{\pi r^2\theta}{360}$$

or $$A = \pi r^2 \times \frac{\theta}{360} \qquad \text{. . . (ii)}$$

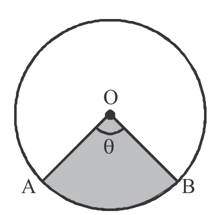


Fig. 15.04

From (i) and (ii) we get

$$A = \frac{1}{2} L \times r$$

Note : Here angle θ is taken in degree.

Some Important Results

(i) The minute hand of a watch moves round 6° during one minute.

(ii) The hour hand of a watch moves round $\left(\frac{1}{2}\right)^{\circ}$ during one minute.

Illustrative examples

Example 1. The length of an arc of a circle is 4 cm and its radius is 6 cm. Find the area of this sector of the circle.

Solution : We have, length of the arc of the circle = 4 cm.

and radius = 6 cm.

We know that the length of the sector $= \frac{\pi r\theta}{180°}$

or $4 = \frac{\pi \times 6 \times \theta}{180}$

or $\theta = \frac{4 \times 180°}{\pi \times 6}$

or $\theta = \frac{4 \times 30°}{\pi} = \frac{120°}{\pi}$

$\therefore$ Area of the sector $= \frac{\pi r^2\theta}{360°} = \frac{\pi \times (6)^2 \times \left(\frac{120°}{\pi}\right)}{360°}$

or $\frac{6 \times 6 \times 120°}{360°} = \frac{6 \times 6}{3} = \frac{36}{3} = 12$

Hence, area of the given sector = 12 cm²

This problem can also be solved with the help of formula $A = \frac{1}{2} L \times r$

Example 2. The angle subtended at the centre by an arc of a circle is 50°. If the length of the arc is 5π cm, find the area of the mirror sector made by this arc of the circle. (Take $\pi = 3.14$)

Solution : Given, the length of the arc $l = 5\pi$ cm

and angle of the sector $\theta = 50°$

Length of the arc $L = \frac{\pi\theta}{180°}$

$r = \frac{5\pi \times 180°}{50\pi} \text{cm} = 18\text{cm}$

Area of the sector $= \frac{1}{2} l \times r$

$$= \frac{1}{2} \times 5\pi \times 18 \text{ cm}^2 = 45\pi \text{ cm}^2$$

$$= 45 \times 3.14 \text{ cm}^2 = 140.3 \text{ cm}^2$$

Example 3. Radius of a circle is 7 cm and the angle of the sector is 90°. Find the length of the arc and the area of the minor sector of the circle. $(Take\pi = \frac{22}{7})$.

Solution : Given, radius of the circle = 7 cm

and angle of sector $= 90^\circ$

Length of the arc of the sector $L = \frac{\pi r \theta}{180^\circ} = \frac{22}{7} \times \frac{7 \times 90^\circ}{180^\circ}$

or $l = 11\text{cm}$

Area of the sector $= \frac{1}{2} \times Lr\times$

$$= \frac{1}{2} \times 11 \times 7 = \frac{77}{2} = 38.5 \text{ cm}^2$$

Example 4. Given figure is an equilateral triangle, whose length of one side is 20 cm. Form each of the vertex of the triangle three arcs are drawn with a radius 10 cm. Find out the area of the shaded portion.(Take $\pi = 3.14$ and $\sqrt{3} = 1.73$)

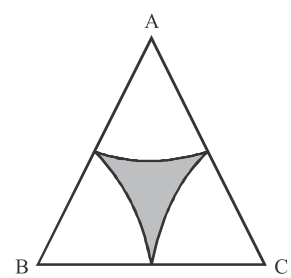


Fig.15.05

Solution : Given, length of the side of equilateral triangle = 20 cm.

$\therefore$ area of the triangle $= \frac{\sqrt{3}}{4} a^2 = \frac{\sqrt{3}}{4} \times (20)^2 \text{ cm}^2$

$$= \frac{\sqrt{3}}{4} \times 20 \times 20 \text{ cm}^2$$

$$= 1.73 \times 100 \text{ cm}^2 = 173 \text{ cm}^2$$

We know that, measured degree of each of the angle of an equilateral triangle = 60°
There for, area of three sectors will be equal.

$\therefore$ Area of three sectors $= 3 \times \frac{\pi r^2 \theta}{360^\circ}$

$$= \frac{3 \times 3.14 \times 10^2 \times 60^\circ}{360^\circ} \text{ cm}^2$$

$$= 157 \text{ cm}^2$$

Hence, area of the shaded portion $= (173 - 157)\text{cm}^2 = 16\text{cm}^2$

Example 5. The length of an hour hand of a clock is 6 cm. Find the area of the sector swept by this hour hand with in 90° minutes.

Solution : Given, length of hour hand $= 6$ cm

$\therefore$ The radius of the sector $= 6$ cm

The angle subtended by this hand in 12 hours $= 360^{\circ}$

The angle subtended by this hand in 1 hour $= \dfrac{360}{12} = 30^{\circ}$

The angle subtended by this hand in 1 minutes $= \dfrac{30^{\circ}}{60^{\circ}} = \left(\dfrac{1}{2}\right)$

$\therefore$ The angle subtended by this hand in 90 minutes $= \dfrac{1}{2} \times 90 = 45^{\circ}$

The area swept by this hand = Area of the sector

$$= \frac{\pi r^2 \theta}{360^{\circ}}$$

$$= \frac{\frac{22}{7} \times 6^2 \times 45^{\circ}}{360^{\circ}} \text{cm}^2 = \frac{22 \times 6 \times 6 \times 45^{\circ}}{7 \times 360^{\circ}} \text{cm}^2$$

$$= \frac{22 \times 36}{7 \times 8} \text{cm}^2 = \frac{792}{56} \text{cm}^2 = 14.14 \text{cm}^2$$

Hence, the area swept by the hour hand of the clock is 90 minutes = 14.14 cm^2

15.06. Area of the Segment of Circle

Every chord of a circle divides the circle into two parts and each of the part is called the segment. The greater part is called major segment and smaller part is known as minor segment of the circle.

The centre of the circle is *O* and radius is r (see fig 15.06). Let the chord *PQ* divides the circle into two segments. We are to find the area of the minor segment (*PQR*) of the circle.

Let $\angle POQ = \theta^{\circ}$

Then $\angle POM = \angle QOM = \dfrac{\theta}{2}$

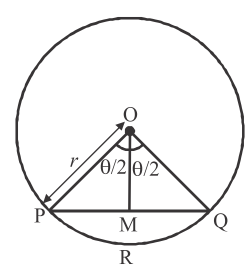


Fig. 15.06

Area of sector *OPRQ* = area of sector *PRQ* + area of Δ *POQ*

$\therefore$ area of segment *PRQ* = area of sector *OPRQ* – area of Δ *POQ*

$$= \frac{\pi r^2 \theta}{360^{\circ}} - \frac{1}{2} PQ \times OM$$

$$= \frac{\pi r^2 \theta}{360^{\circ}} - \frac{1}{2} \times 2PM \times OM = \frac{\pi r^2 \theta}{360^{\circ}} - r \sin\frac{\theta}{2} \times r \cos\frac{\theta}{2}$$

$\Rightarrow$ Area segment *PRQ* $= \dfrac{\pi r^2 \theta}{360^{\circ}} - \dfrac{r^2}{2} \sin\theta$ $\quad \left[\sin\theta = 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}\right]$

Example 6. A chord of the circle with radius 5 cm subtends a right angled at the centre. Find the area of the minor segment by this chord.

Solution : Given, radius of the circle = 5 cm and the angle subtended at the centre by given chord = 90^o

$$\text{Area of sector } OPRQ = \frac{\pi r^2\theta}{360°}$$

$$= \frac{\frac{22}{7}\times 5^2 \times 90°}{360°}$$

$$= \frac{22\times 25\times 90°}{7\times 360°} = \frac{550}{28} = 19.64$$

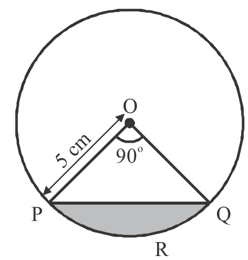


Fig. 15.07

$$\text{Area of } \Delta\ POQ = \frac{1}{2}\times OP\times OQ$$

$$= \frac{1}{2}\times 5\times 5 = \frac{25}{2} = 12.50$$

$\therefore$ Area of minor segment of the circle = Area of sector OPRQ - area of Δ POQ

$$= 19.64 - 12.50 = 7.14$$

Hence, area of minor segment of the circle = 7.14 cm^2

This result also can be obtained with formula area of minor segment of circle

$$= \frac{\pi r^2\theta}{360°} - \frac{1}{2}r^2\sin\theta.$$

Example 7. A chord of a circle radius 14 cm subtends an angle of 30^o at the centre. Find the areas of both, minor segments and major segment of the circle. $(Take\ \pi = \frac{22}{7})$.

Solution : Given, radius of the circle r = 14 cm.

Angle subtended at the centre by chord $\theta = 30°$

We know that area of minor segment of the circle

$$= \frac{\pi r^2\theta}{360°} - \frac{1}{2}r^2\sin\theta$$

$$= \frac{\frac{22}{7}\times 14\times 14\times 30°}{360^o} - \frac{1}{2}\times 14\times 14\sin 30°$$

$$= \frac{22\times 2\times 14}{12} - \frac{1}{2}\times 196\times\frac{1}{2}$$

$$= \frac{616}{12} - 49 = 51.33 - 49$$

$$= 2.33$$

Area of the major segment of the circle

= area of the circle - area of the minor segment of the circle

$$= \pi r^2 - 2.33 = \frac{22}{7} \times (14)^2 - 2.33$$

$$= \frac{22 \times 14 \times 14}{7} - 2.33$$

$$= 22 \times 2 \times 14 - 2.33$$

$$= 616 - 2.33$$

$$= 613.67 \text{ cm}^2$$

Example 8. A chord in a circle radius 12 cm substeds an angle of 120° at the centre. Find the area of corresponding segment of the circle. (Take $\pi = 3.14$ and $\sqrt{3} = 1.73$)

Solution : Here. r = 12 cm and $\theta = 120^\circ$

Area of the corresponding segment = area of the minor segment

= area of sector OAB – area of Δ OAB. . . . (i)

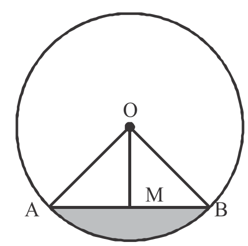


Fig. 15.08

Area sector $OAB = \frac{\theta}{360} \times \pi r^2 = \frac{120}{360} \times 3.14 \times 12 \times 12 \text{ cm}^2$

$$= 3.14 \times 48 \text{ cm}^2 \quad \text{. . . (ii)}$$

To find th area of Δ OAB, draw $OM \perp AB$

AM = BM and $\angle AOM = \angle BOM = 60^\circ$

Not $\frac{OM}{OA} = \cos 60^\circ$ $\therefore$ $OM = OA \cos 60^\circ = 12 \times \frac{1}{2} = 6$

$$\frac{AM}{OA} = \sin 60^\circ$$

$\therefore$ $AM = OA \sin 60^\circ = 12 \times \frac{\sqrt{3}}{2} = 6\sqrt{3}$

$$AB = 2 \times 6\sqrt{3} \text{ cm} = 12\sqrt{3} \text{ cm}$$

Area of Δ OAB $= \frac{1}{2} \times AB \times OM = \frac{1}{2} \times 12\sqrt{3} \times 6 = 36\sqrt{3}$ . . . (iii)

So form equation (i), (ii) and (iii), we get

Area of corresponding segment $= \left(3.14 \times 48 - 36\sqrt{3}\right) \text{cm}^2$

$$= (3.14 \times 48 - 36 \times 1.73) \text{ cm}^2$$

$$= 12(12.56 - 3 \times 1.73) \text{ cm}^2$$

$$= 12(12.56 - 5.19) \text{ cm}^2 = 88.44 \text{ cm}^2$$

Exercise 15.2

1. Radius of a circle is 7 cm and the angle subtended at the centre by an arc is 60^o. Find the length of the arc.
2. Radius of a circle is 10.5 cm and angle of the sector is 45^o. Find the area of the minor sector of the circle. $\left(\pi = \frac{22}{7}\right)$
3. The length of an arc is 12 cm and radius of a circle is 7 cm respectively. Find the area of the minor sector of circle.
4. An arc of a circle with radius 21 cm subtends an angle of 60^o at the centre. Find :
   (i) Length of the arc
   (ii) Area of the sector formed by this arc
   (iii) Area of the segment formed by corresponding chord.
5. The lenght of the minute hand of a clock is 10.5 cm. Find the area of the sector swept by the minute hand in 10 minutes $\left(\pi = \frac{22}{7}\right)$
6. Radius of a circle is 3.5 cm and the angle subtended by a chord at the centre is 90^o. Find the area of the minor segment of the circle formed by this chord. $\left(\pi = \frac{22}{7}\right)$
7. The circumference of a circle is 22 cm. Find the area of its one quadrant.
8. Minute hand of a clock is 5 cm long. Find the area of the sector formed by this hand in 7 minutes.
9. Given figure 15.09 *ABCD* is a rectangle, in which side *AB* = 10 cm and *BC* = 7 cm. From the each vertex of the rectangle four circles with radii 3.5 cm each are drawn. Find the area of shaded portion. $\left(\pi = \frac{22}{7}\right)$

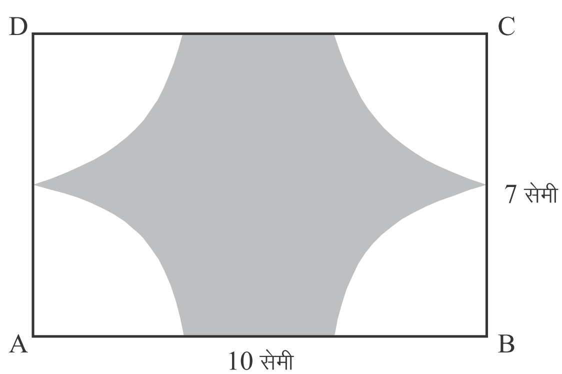


Fig.15.09

15.07. Area of Combination of Plane Figures

Combination means forming a new shape by joining two or more plane figures. Now, we will calculate the area of different type of figures related to the circles. Now we shall try to calculate the areas of some other figures. We come across these types of figures in our daily life and also in the form of various interesting designs. Flower beds,drain covers, window designs on table covers are some of such examples. To find the areas of these figures several examples are given below.

Example 9. There are two circular portions on two sides of a square lawn of side 58 *m* .The centre of each circular portion is the point of intersection of the diagonals of the square lawn. Find the area of the lawn in whole.

Solution : We have, side of the square = 58 m

$\therefore$ length of its diagonal $= \sqrt{58^2 + 58^2}$

$$= 58\sqrt{2}\ \text{m}^2$$

So the radius of the circle whose centre is the intersecting point of its diagonals

$$= \frac{58\sqrt{2}}{2} = 29\sqrt{2}$$

The area of one circular end = area of the sector of the circle with radius $29\sqrt{2}$ and centre angle is 90°

$$= \left[\frac{\pi r^2\theta}{360^\circ} - r^2 \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right] = \left[\frac{\pi\theta}{360^\circ} - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right] r^2$$

$$= \left[\frac{22}{7}\times\frac{90^\circ}{360^\circ} - \sin 45^\circ \cos 45^\circ\right](29\sqrt{2})^2\text{m}^2$$

$$= \left[\frac{11}{14} - \frac{1}{2}\right]\times 29\times 29\times 2\ \ \text{m}^2$$

$$= 29\times 29\times 2\times\frac{4}{14}\ \text{m}^2$$

$$= \frac{3364}{7}\ \ \text{m}^2$$

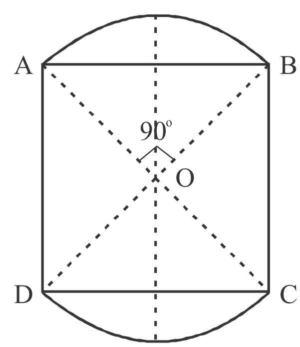


Fig. 15.10

Area of the lawn with two circular portion.

= area of the square + 2 (area of the one end)

$$= \left[58\times 58 + 2\times\frac{3364}{7}\right]\ \text{m}^2$$

$$= 3364\left[1 + \frac{2}{7}\right]\ \text{m}^2$$

$$= 3364\times\frac{9}{7}\ \text{m}^2$$

$$= 4325.14\ \text{m}^2$$

Example 10. Diameter of a circular grass land is 42 m. It has 3.5 metre wide path around it on the outside. Find the cost of gravelling the path at the rate of a ₹ 4 per metre^2.

Solution : Given, diameter of the grass land = 42 m

$\therefore$ Radius of the grass land = 21 m

Radius of the grass land with path $= (21+3.5) = 24.5$ m

Area of the path $= \left[\pi(24.5)^2 - \pi(21)^2\right]$ m^2

$= \pi\left[(24.5)^2 - (21)^2\right]$ m^2

$= \pi\left[(24.5+21)(24.5-21)\right]$ m^2

$= \pi[45.5\times3.5]$ m^2

$= \frac{22}{7}\times45.5\times3.5$ m^2 = 500.5 m^2

Cost of the gravelling the path = ₹ 500.5×4 = ₹ 2002

Example 11. In the fig. 15.11 is a square *ABCD* with side 14 cm. Find the area of shaded portion.

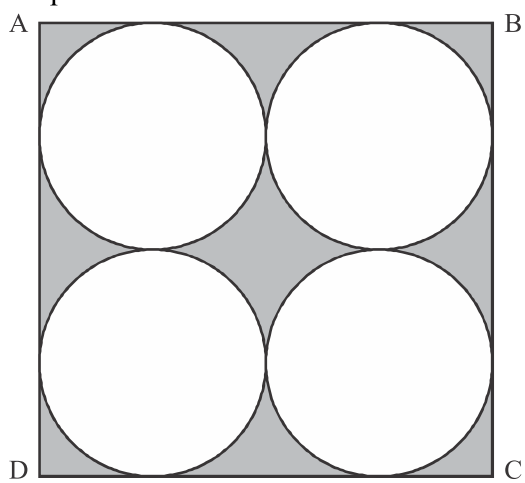


Fig. 15.11

Solution : We have side of the square *ABCD* = 14 cm

$\therefore$ Area = (side)^2 $= 14\times14 = 196\,\text{cm}^2$

Diameter of every circle $= \frac{14}{2}$ cm = 7 cm

$\therefore$ Radius of every circle $= \frac{7}{2}$ cm

So, area of one circle $= \pi r^2$

$= \frac{22}{7}\times\frac{7}{2}\times\frac{7}{2}$ cm^2

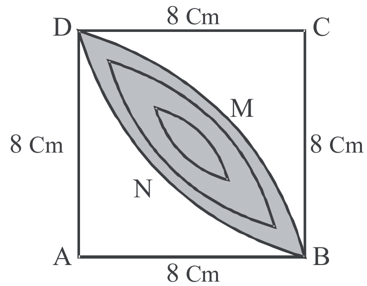
$= \frac{154}{4} = \frac{77}{2}$ cm^2

Area of all four circles $= 4\pi r^2$

$$= 4 \times \frac{77}{2} \text{cm}^2 = 154 \text{cm}^2$$

Area of shaded portion $= (196 - 154)\text{cm}^2 = 42 \text{cm}^2$

Example 12. Find the area of shaded portion in following figure which is the region between the two quadrants of the circle with radii 8 cm each.



Solution : Given, radii of quadrants $ABMD$ and $BNDC$ = 8 cm each.

The sum of their areas $= 2 \times \frac{1}{4}\pi r^2 = \frac{1}{2}\pi r^2$

$$= \left[\frac{1}{2} \times \frac{22}{7} \times 64\right]$$

$$= \frac{704}{7} \text{cm}^2$$

Area of the square $ABCD = (8 \times 8)cm^2 = 64 cm^2$

Area of the shaded region

= sum of the area of two quadrants - area of the square $ABCD$

$$= \left[\frac{704}{7} - 64\right]cm^2 = \left[\frac{704 - 448}{7}\right]cm^2 = \frac{256}{7} \text{cm}^2$$

Exercise 15.3

1. Find the circumference of the incircle of square with side 14 cm.
2. The difference between the circumference and the radius of a circle is 74 cm, find the area of the circle.
3. In the given fig. 15.14 O is the centre of a circle. $\angle AOB = 90°$ and OA = 3 cm, then find the area of the shaded region.

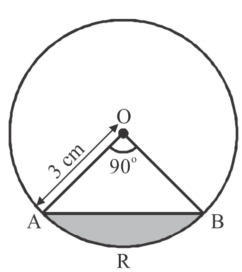


Fig. 15.14

4. If perimeter of a circle is equal to the perimeter of a square, then find the ratio of their areas.
5. The radius of a circular park 3.5 metre. There is a footpath of width 1.4 m around the park. Find the area of the footpath (see fig. 15.15).

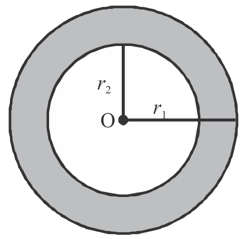


Fig. 15.15

6. Find the area of the square inscribed in a circle of radius 8 cm.
7. In the fig. 15.16, there is a quadrant *ABMC* of a circle with radius 14 cm and with *BC* as diameter a semicircle is drawn. Find the area of shaded port.

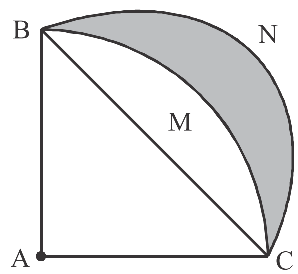


Fig. 15.16

8. In the given figure 15.17 *AB* is the diameter of a circle, *AC* = 6 cm and *BC* =8 cm. Find the area of the shaded portion.

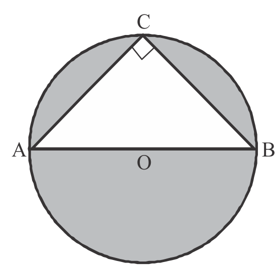


Fig. 15.17

9. Find the area of the shaded design of the given figure 15.18 where *ABCD* is a square with side 10 cm and taking every side of the square as the diameter four semi-circles are drawn. $(\pi = 3.14)$

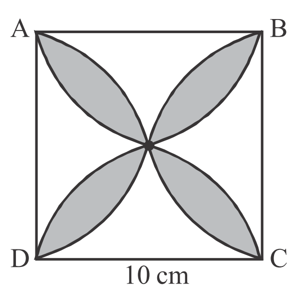


Fig. 15.18

10. In the given figure 15.19 radius of the semicircle is 7 cm. Find area of the circle formed in side the semi-circle.

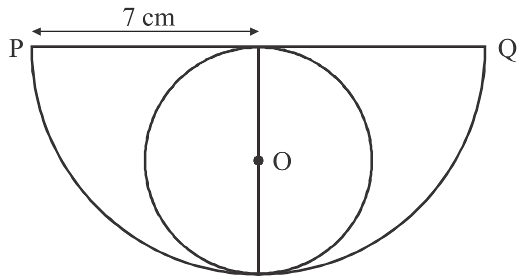


Fig. 15.19

11. The sum of the circumferences of two circles with radii R_1 and R_2 is equal to the perimeter of the circle with radius R. Choose the correct option.

(A) $R_1 + R_2 = R$ (B) $R_1 + R_2 > R$

(C) $R_1 + R_2 < R$ (D) Nothing can be said

12. The circumference of the incircle of a square with side 14 cm is :

(A) 22 cm (B) 44 cm (C) 33 cm (D) 55 cm

Important Points

1. Circumference of the circle $= 2\pi r = \pi d$ (r = radius, d = diameter)
2. Area of the circle $= \pi r^2$
3. Area of the portion between two concentric circles $= \pi\left(r_1^2 - r_2^2\right)$ where $r_1 > r_2$
4. Area of a sector of the circle $A = \dfrac{\pi r^2 \theta}{360°}$
5. Length of the arc of the sector of the circle $= L = \dfrac{2\pi r\theta}{360°}$
6. Area of a sector of the circle $= \dfrac{1}{2} Lr$
7. Area of a segment of a circle $= \dfrac{\pi r^2 \theta}{360°} - \dfrac{1}{2} r^2 \sin\theta$
8. Area of the major segment of a circle

= area of the circle - area of the minor segment of the circle.

Answer Sheet

Exercise 15.1

1. 22 cm, 38.5 cm^2 2. 154 sq.m. 3. 693 sq. metre, 108 metre
4. 14 cm, 5. 144 cm 6. 616 sq. metre
7. 1925 8. 1386 sq. metre 9. B
10. A

Exercise 15.2

1. 7.3 cm 2. 43.31 cm^2 3. 42 cm^2 4. (i) 22 cm (ii) 231 cm^2
(iii) 40.047cm^2 5. 57.75 cm^2 6. 3.5 cm^2 7. 9.625cm^2
8. 7.64 cm^2 9. 31.5 cm^2

Exercise 15.3

1. 44 cm 2. 616 cm^2 3. 2.57 cm^2 4. 14 : 11 5. 36.96m^2
6. 128 cm^2 7. 98 cm^2 8. 54.57 cm^2 9. 57 cm^2 10. 38.5 cm^2
11. A 12. B

16

Surface Area and Volume

16.01. Introduction

In the previous chapter we have studied the methods to find area of plane figures *e.g.* ,triangle, rectangle, circle, etc. In this chapter we will study about solid figures. The bricks, match-box, plane walls of house, water tanks, cricket ball etc. are different shapes of solid figures. Main difference between plane figure and solid figure is that plane figures are completely lie in plane whereas solid figures do not lie in a plane. Solid figures lie in space. Solid (cubical figures) are three dimensional.

The surface area of solid figures means sum of the area of all the surfaces and volume is the measure of the amount of space inside a solid figure. Area is measured is square units and volume in cubic units.

16.02. Surface Area and Volume of Cube and Cuboid

A match box, room, chalk box, brick etc., are examples of cuboid. In the figure 16.01 cuboid has shown, which has 6 faces, each face is rectangular in a plane, opposite faces are parallel and congruent and contains three pair of parallel faces. Two adjoining faces meets at a same line segment which is called edges. A cuboid has 12 edges.

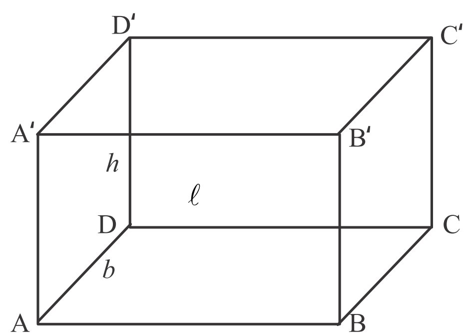


Fig. 16.01

Let $AB = A'B' = D'C' = DC = \ell$

$AD = A'D' = B'C' = BC = b$

$AA' = DD' = BB' = CC' = h$

Let we assume ℓ , b , h as length, breadth and height of the solid figurethen these perdendicular edges form a cuboid.

'If in six parallel faces, all are rectamgular, then it is called cuboid.

Cuboid is also called rectangular face solid *e.g.,* brick, box, room etc.

In cuboid, area of face $ABCD$ = Area of face $A'B'C'D' = \ell \times b$

In cuboid, area of face $ADD'A'$ = Area of face $BCC'B' = b \times h$

In cuboid, area of face $ABB'A'$ = Area of face $DCC'D' = h \times \ell$

Thus, total surface area of the cuboid

= 2 (Area of $ABCD$ + Area of $ADD'A$ + Area of $ABB'A'$)

$$= 2(\ell \times b + b \times h + h \times \ell)$$

$$= 2(\ell b + bh + h\ell) \text{ square units}$$

If height of cuboid is zero then it will take the form of rectangle.

16.03. Cube

When length, breadth and height of cuboid are same *i.e.*, $\ell = b = h \neq 0$ then it is called a cube. All the faces of cube are square and area of each face is same. Figure 16.02 shows a cube. If ℓ is the length of side of a cube then its total surface area

$$= 2(\ell \times \ell + \ell \times \ell + \ell \times \ell)$$

$$= 2(\ell^2 + \ell^2 + \ell^2)$$

$$= 2 \times 3\ell^2$$

$$= 6\ell^2 \text{ square units}$$

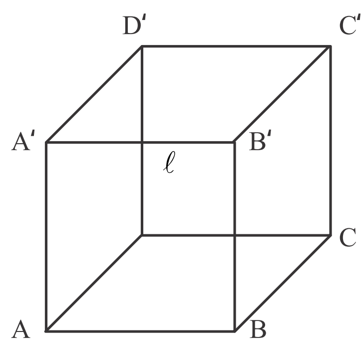


Figure 16.02

Now we will develop the method to find the volume of cube or cuboid.

According to fig. 16.03 take a cube of edge 1unit such cube is called unit cube. 1 cubic cm can be written as 1 cm^3.

Similarly volume of cuboid is measured in cube unit since length, breadth and height of cuboid and different so volume of cuboid will be their product.

Volume of cuboid *(V)* = $\ell \times b \times h$ cube unit

Where ℓ = length, b = breadth, h = height

Volume *(V)* = $\ell \times b \times h$ cube unit

$\because$ length, breadth and height of cube are same

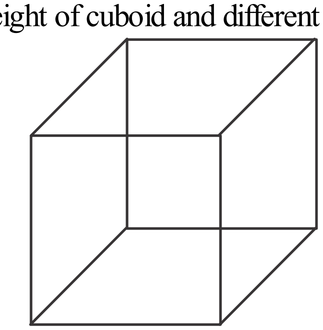


Figure 16.03

Volume of cube = $\ell \times \ell \times \ell = \ell^3$ cube unit

16.04. Diagonal of Cube and Cuboid

The line of adjoining the two opposite vertices of parallel face of cube or cuboid is called diagonal. So there are 4 diagonals.

Note : If ℓ, b, h are length, breadth and height of cuboid then,

Diagonal of cuboid = $\sqrt{\ell^2 + b^2 + h^2}$ units,

For cube, $\ell = b = h$

$\therefore$ diagonal of cube $= \ell\sqrt{3}$ units,

Units related to volume

(i) 1 litre = 1000 cubic cm.

(ii) 1 cubic cm = 10 × 10 × 10 cubic mm. = 1000 cubic mm

(iii) 1 cubic m = 100 × 100 × 100 cubic cm = 100,00,00 cubic cm

(iv) 1 cubic m = 1000 litre = 1 kilo litre

Example 1. The length, breadth and height of a closed wooden box are 90 cm, 50 cm, and 30 cm, respectively. Find the outer surface area of the box.

Solution : Length of box = 90 cm

Breadth of box = 50 cm

Height of box= 30 cm

Total outer surface area of box = 2 (ℓ ×b + b × h + h × ℓ)

$$= 2(90 \times 50 + 50 \times 30 + 30 \times 90) \text{ sq. cm.}$$

$$= 2(4500 + 1500 + 2700) \text{ sq. cm.}$$

$$= 2(8700) \text{ sq. cm.}$$

$$= \frac{17400}{10000} = 1.74 \text{ sq. cm.}$$

Example 2. Total surface area of a cube is 1014 m^2. Find the side of the cube.

Solution : Total surface area of a cube = 1014 sq.m.

Let side of cube $= x$ m

Total surface area of cube = 6 $(\text{side})^2$

$\therefore$ $6x^2 = 1014$

or $x^2 = \frac{1014}{6} = 169$

or $x = \sqrt{169}$ m $= 13$ m

Example 3. If length of cuboid is 12 m, breadth 9 m and height is 8 m then find its diagonal length.

Solution : Length of cuboid = 12 m

Breadth of cuboid = 9 m

Height of cuboid = 8 m

we know that diagonal of cuboid= $\sqrt{(\ell)^2+(b)^2+(h)^2}$

$$=\sqrt{(12)^2+(9)^2+(8)^2}\text{ m}$$

$$=\sqrt{144+81+64}\text{ m}$$

$$=\sqrt{289}=17\text{ m}$$

Example 4. Length of a room is 5 m, breadth 3.5 m and height is 4 m. Find the cost of whitewash at four walls at the rate ₹20 per square meter.

Solution : Length of room = 5 m

Breadth = 3.5 m

Length = 4 m

Area of 4 walls of room = $2(\ell+b)h$

$$=2(5+3.5)\times 4\text{ sq. m}$$

$$=2\times 8.5\times 4\text{ sq.m}$$

$$= 68\text{ sq.m.}$$

Cost of whitewash at four walls = ₹ 68×20

or cost = ₹1360

Example 5. The perimeter of a surface of cube is 28 cm then find volume of the cube.

Solution : Preimeter of a face of cube = 28 cm.

∵ All sides of cube are equal

∴ Preimeter of 1 face = 4 × side

or 28 = 4 × side

or side = $\frac{28}{4}cm=7$ cm.

Volume of cube = $(\text{face})^3=(7)^3$

$$=7\times 7\times 7=343\text{ cube cm.}$$

Example 6. If ratio of length, breadth and height of a cuboid is 6 : 5 : 4 and its total surface area is 33300 sq. cm. then find the volume of the cuboid.

Solution : Let length, breadth and height of cuboid are $6x$, $5x$, $4x$ respectively ,

Area of cuboid = 33300 sq. cm.

∴ $2(\ell b+bh+h\ell)=33300$

or $2(6x\times 5x+5x\times 4x+4x\times 6x)=33300$

or $2(30x^2+20x^2+24x^2)=33300$

or $2\times 74x^2=33300$

or $x^2=\frac{33300}{2\times 74}$

or $x^2 = 225$

$\therefore$ $x = \sqrt{225} = 15$

or $x = 15$ cm

$\because$ ℓ ? 6? 15 cm = 90 cm, $b = 5 \times 15$ cm = 75 cm, $h = 4 \times 15$ cm = 60 cm.

Required volume $= 90 \times 75 \times 60$ cm^3

= 405000 cm^3

Example 7. The dimensions of a box are 3 m × 2 m × 1.80 m. Find the cost of varnishing at all the outer surfaces of the box at the rate of ₹ 12 per m^2.

Solution : Dimensions of box

ℓ = 3 m, b = 2 m, h = 1.80 m.

Total surface area of box = 2 [$\ell \times b + b \times h + h \times \ell$]

= 2 [3 × 2 + 2 × 1.80 + 1.80 × 3] m^2

= 2 [6 + 3.60 + 5.40] m^2

= 2 [6 + 9] m^2 = 2 × 15 m^2 = 30 m^2

Cost of varnishing at 30 m^2 area = ₹ 30 × 12 = ₹ 360

Example 8. Three metalic cubes are of edges 3 cm, 4 cm, and 5 cm. By melting these cubes a new cube is formed. Find the edge of this new cube.

Solution : Volume of cube of side 3 cm = (side)^3 = 3^3 cm^3 = 27 cm^3

Volume of cube of side 4 cm = (side)^3 = $(4)^3$ cm^3= 64 cm^3

Volume of cube of side 5 cm = (side)^3 = $(5)^3$ cm^3= 125 cm^3

Total volume of these cubes = (27 + 64 + 125) cm^3 = 216 cm^3

By melting these cubes, new cube is formed

$\therefore$ Volume of new cube = 216 cm^3

(side)^3 = 216

side $= \sqrt[3]{216} = (6 \times 6 \times 6)^{1/3} = 6$

$\therefore$ Edge of new cube is 6 cm.

Exercise 16.1

1. The length of a cube is 12 cm, breadth is 2 cm and height is 5 cm. Find the total surface area of cuboid and volume of the cuboid.
2. The edges of three cubes are 8 cm, 6 cm, and 1 cm respectively. After melting these cubes a new cube is formed. Find the total surface area of the new cube.
3. The dimensions of a box are 50 cm × 36 cm × 25 cm. How much sq. cm. cloth will be required for making cover of this box ?
4. Each surface area of a cube is 100 cm^2. The cube is cut into two equal parts by a plane which is parallel to the base then find total surface area of equal parts. Then find total surface area of equal part.
5. A box without lid is made by wood of thickness 3 cm. Its outer length is 146 cm, breadth is 116 cm and height is 83 cm. Find the cost of painting the internal surface of the box at the rate ₹ 2 per 1000 sq. cm.
6. The sum of length, breadth and height of cuboid is 19 cm and length of its diagonal is 11 cm. Find the total surface area of the cuboid.
7. A room with square floor of side 6 m contains 180 m^3 air. Find the height of the room.

8. How many bricks are required to make a wall of dimensions 44 m long, 1.5 m height and 35 cm broad. If dimension of 1 brick is 22cm × 10 cm × 7 cm.
9. Find the maximum length of rod which can be kept in a room of size 10 m × 8 m × 6 m.
10. The volume of a cube is 512 cubic metre. Find its edge.
11. 5 m, 30 cm and 3 m are length, breadth and height of a wall respectively. How many bricks of dimensions 20 cm × 10 cm × 7.5 cm will be required to make the wall.
12. The ratio of length, breadth and height of a cubid is 5 : 3 : 2. If total surface area of cuboid is 558 cm^2, then find their edges.

16.05. Surface Area and Volume of Cylinder

You have seen the measuring jar, test tube, round pipe. etc. The objects which have one lateral curved surface and congruent circular cross section is called circular cylinder. The line, joining the centres of circular cross-sections is called axes of cylinder. If the axis of cylinder is perpendicular to circular cross-section then cylinder is called right circular cylinder. In this chapter , whenever the word cylinder is used it means right circular cylinder.

If $ABCO$ is a rectangular area it rotates about line OA then we get a solid cylinder of radius r and height h as shown in fig. 16.04.

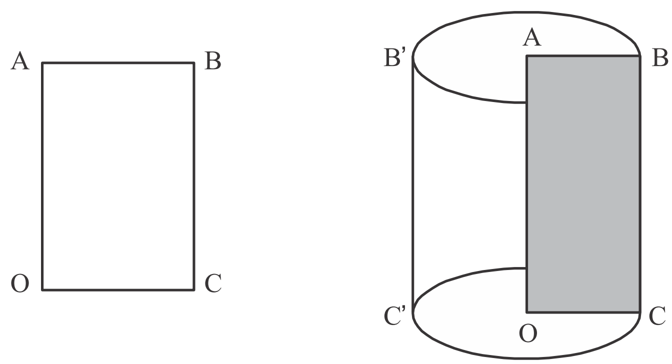


Fig. 16.04

The lines which are parallel to OA and lies on the curved surface of the cylinder are called generators. Here lines BC, B'C' are generators. On placing the cylinder vertically then the downward circular end is called base of the cylinder. In the figure 16.04 CB is the height of the cylinder. The radius of the circular ends is called radius of the cylinder.

In hollow cylinder both the ends are open whereas in solid cylinder they are closed. To find curved surface area of cylinder of radius r and height h (fig. 16.05) , if we cut curved surface along the line segment BC and spread it then it will take shape of a rectangle $BCDE$

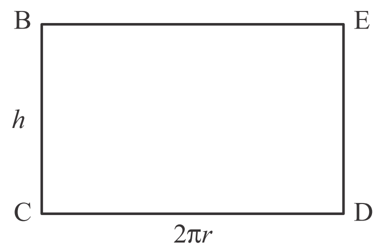


Fig. 16.05

When sheet is in the form of curved surface of cylinder then line segment *BC* and *ED* coincides so

$\therefore$ Area of curved surface of cylinder

$$= \text{Area of rectangle BCDE}$$
$$= 2\pi r \times h$$
$$= 2\pi rh$$

Area of base if cylinder $= \pi r^2$

Total surface area of cylinder = Curved surface+ 2 Area of base

$$= 2\pi rh + 2\pi r^2$$
$$= 2\pi r(h + r)$$

The space occupied by cylinder is called its volume. If radius of cylinder is *r* and height is *h* then

Volume of cylinder = Area of base × height

$$= \pi r^2 \times h$$
$$= \pi r^2 h$$

If outer radius of a hollow cylinder is r_1 and inner radius is r_2 also height is h then in fig. 16.06, volume of hollow cylinder.

$$= \pi r_1^2 h - \pi r_2^2 h$$
$$= \pi\left(r_1^2 - r_2^2\right)h$$

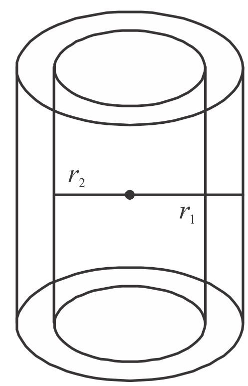


Fig. 16.06

Total surface area of hollow cylinder

$$= 2\pi r_1 h + 2\pi r_2 h + 2\pi r_1^2 - 2\pi r_2^2$$
$$= 2\pi h(r_1 + r_2) + 2\pi\left(r_1^2 - r_2^2\right)$$
$$= 2\pi h(r_1 + r_2) + 2\pi(r_1 + r_2)(r_1 - r_2)$$
$$= 2\pi(r_1 + r_2)(h + r_1 - r_2)$$

Illustrative Examples

Example 1. The area of base of a cylinder is 154 cm^2 and height is 21 cm. Find its volume and curved surface area.

Solution : Area of base of cylinder = 154 cm^2

Height of cylinder (h) = 21 cm

Volume of cylinder = Area of base × height

= 154 × 21 cm^3

= 3234 cm^3

$\therefore$ Area of base $= \pi r^2$

or $154 = \dfrac{22}{7} \times r^2$

or $154 \times 7 = 22 \times r^2$

or $$r^2 = \frac{154\times7}{22} = 49$$

or $$r = \sqrt{49}\text{ cm} = 7\text{ cm}$$

Curved surface area of cylinder $= 2\pi rh$

$$= 2\times\frac{22}{7}\times7\times21\text{ cm}^2$$

$$= 2\times22\times21\text{ cm} = 924\text{ sq. cm.}$$

Example 2. The height of a cylinder is 11 cm amd its curved surface area is 968 cm^2. Find the radius of cylinder.

Solution : Height of cylinder $= 11$ cm

Let radius of cylinder $= r$

Curved surface area of cylinder $= 2\pi rh = 968$

or $$2\times\frac{22}{7}\times r\times11 = 968$$

$$r = \frac{968\times7}{2\times22\times11}\text{ cm} = 14\text{ cm.}$$

Example 3. If volume of cylinder is 448 π cm^3 and height is 7 cm then find curved surface area of cylinder.

Solution : Volume of cylinder $= \pi r^2 h$

∴ $$448\pi = \pi\times r^2\times7$$

or $$448 = 7r^2$$

or $$r^2 = \frac{448}{7} = 64$$

or $$r = \sqrt{64} = 8\text{ cm}$$

Curved surface area of cylinder $= 2\pi rh$

$$= 2\times\frac{22}{7}\times8\times7\text{ sq.}$$

$$= 44\times8 = 352\text{ sq. cm.}$$

Total surface area of cylinder $= 2\pi r(h+r)$

$$= 2\times\frac{22}{7}\times8(7+8)\text{ sq.}$$

$$= 2\times\frac{22}{7}\times15\times8\text{ sq.cm} = \frac{5280}{7}\text{ sq. cm} = 754.28\text{ sq. cm.}$$

Example 4. A hollow cylinder is of height 21 decimetre. Its outer and inner diameter are 10 cm and 6 cm respectively. Find its volume.

Solution : Height of hollow cylinder = 21 dm. [$\because$ 10 cm = 1 dm]

$$= 21 \times 10 = 210 \text{ cm.}$$

Outer diameter of hollow cylinder = 10 cm.

$\therefore$ Outer radius $(r_1) = \frac{\text{diameter}}{2} = \frac{10}{2}\text{cm} = 5\text{cm.}$

Inner diameter of hollow cylinder = 6 cm.

Inner radius $(r_2) = \frac{6}{2}\text{cm} = 3\text{cm.}$

Volume of hollow cylinder $= \pi\left(r_1^2 - r_2^2\right)h$

$$= \frac{22}{7}\left[(5)^2 - (3)^2\right] \times 210 \text{ cm}^3$$

$$= \frac{22}{7}[25 - 9] \times 210 \text{ cm}^3$$

$$= \frac{22}{7} \times 16 \times 210 \text{ cm}^3 = 10560 \text{ cm}^3$$

Example 5. The ratio of radius and height of a cylinder is 1 : 3. If volume of cylinder is 3234 cm^3 then find total surface area of cylinder.

Solution : Let r is radius of cylinder and height is 3 r

Volume of cylinder $= \pi r^2 h$

$$= \frac{22}{7} \times r^2 \times 3r = 3234$$

$$r^3 = \frac{3234 \times 7}{22 \times 3}$$

or $r^3 = 343$

or $r^3 = (7)^3$

or $r = 7$

Thus, height of cylinder $3 \times 7\text{cm} = 21\text{cm.}$

Total surface area of cylinder $= 2\pi r(h + r)$

$$= 2 \times \frac{22}{7} \times 7(21 + 7) \text{ cm}^2$$

$$= 2 \times 22 \times 28 \text{ cm}^2$$

$$= 1232 \text{ cm}^2.$$

Example 6. Length of a roller is 2 m and its diameter is 1.4 m. Find, how much field will be flat by roller in 5 revolutions ?

Solution : Length of roller (h) = 2 m

diameter of roller = 1.4 m

Radius of roller $= \frac{1.4}{2} = 0.7$ m

Area of flat surface by roller in 1 revolution = Curved surface area

$$\therefore \quad 2\pi rh = 2\times\frac{22}{7}\times 0.7\times 2 \text{ m}^2$$

$$= 2\times 22\times\frac{1}{10}\times 2 \text{ m}^2$$

$$= 8.8 \text{ m}^2$$

$\therefore$ In 5 revolution required flat area = 8.8 × 5 = 44 m^2

Exercise 16.2

1. The diameter of a cylinder is 14 cm and height is 15cm. Find the total surface area of cylinder and its volume.
2. Find the curved surface area, total surface area and volume of a right circular cylinder having radius of the base 3 cm and height 7 cm.
3. Find the curved surface area and volume of cylinder whose height is 21 cm and area of its one end is 154 cm^2 .
4. Find the ratio of curved surface area and volume of two right circular cylinder whose radius are in ratio 2: 3 and heights are in the ratio 5 :4.
5. The total surface area of a solid cylinder is 462 cm^2 . Its curved surface area is one third of total surface area. Find the volume of cylinder.
6. Find the volume of cylinder whose curved surface area is 660 cm^2 and height is 15 cm.
7. If the volume of a cylinder is $30\pi cm^3$ and area of base is $6\pi cm^2$, then find height of cylinder.
8. The volume and curved surface area of a cylinder are 1660 cm^3, and 660 cm^2 respectively. Find the radius and height of cylinder.
9. Find the ratio of total surface area and curved surface area of a cylinder whose height and radius are 7.5 cm and 3.5 cm respectively.
10. A 20 m deep welll with diameter 7 m is dug and the earth from digging is evenly spread out to form a plateform 22 m $\times$ 14 m. Find the height of plateform.
11. 30800 cm^3 water can be filled in a cylindrical vessel. If its internal radius is 14 cm, then find its internal curved surface area.
12. If width of hollow cylinder is 2 cm. Its internal diameter is 14 cm and height is 26 cm. Both the ends of cylinder are open, then find the total surface area of hollow cylinder.
13. If both ends of a hollow cylinder are open. Its height is 20 cm and internal and external radius are 26 cm and 30 cm respectively. Find the volume of this hollow cylinder.

16.06 Cone

You have seen joker's cap, icecream cone. These objects are like as cone. In the given figure 16.07 , VO is a line segment, other line segment VA makes an angle θ with VO revolves about side VO then form a cone.

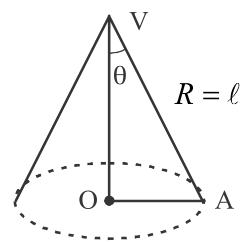


Fig. 16.07

Length of line segment VO is called height of cone and is generally denoted h . Line segment VO is axis of cone. Point V is called vertex of cone and side VA is called slant height of cone which is generally expressed by l. Base of cone is a circle whose centre is O and radius $OA = r$ (say). If line VO is perpendicular to base then cone is called right circular cone. In this chapter we will study about right circular cone.

The solid obtained on revolving a right angled triangle about one of its sides (other than hypotenuse) is called a cone or a right cicular cone.

$$VA^2 = VO^2 + OA^2$$

or $$\ell^2 = h^2 + r^2$$

$$\ell = \sqrt{h^2 + r^2}$$, VA is called slant height of cone

Take sector OAB of circle joins sides OA and OB such that we can obtained a cone, perimeter of its base is equal to length of arc AB and radius of sector is slant height of cone. According to fig. 16.08.

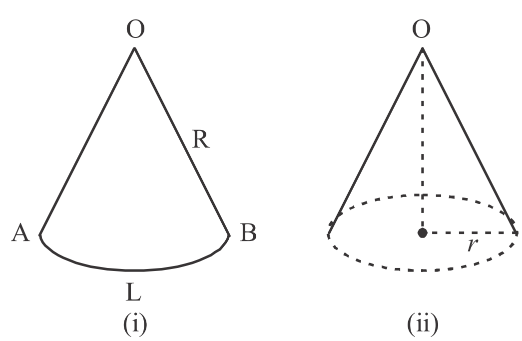


Fig. 16.08

= Curved surface area of cone.

$$= \frac{1}{2} L \times R$$

$$= \frac{1}{2} \text{ Perimeter of base of cone} \times \text{slant height}$$

$$= \frac{1}{2} \times 2\pi r \times \ell$$

Slant surface of cone $= \pi r \ell$

Total surface area of cone = curved surface of cone + area of base of cone

$$= \pi r \ell + \pi r^2$$

$$= \pi r(\ell + r)$$

If r and h are radius and height of cone, then

Slant height of cone $= \ell = \sqrt{r^2 + h^2}$

Total surface area of cone $= \pi r(\ell + r)$

Volume of cone $= \frac{1}{3}\pi r^2 h$

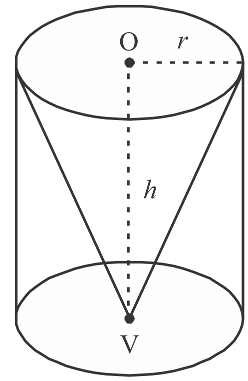


Fig. 16.09

We know that $\pi r^2 h$ is volume of cylinder whose radius is r and height h. If radius and height of cone are r and h respectively then volume of cone will be $\frac{1}{3}\pi r^2 h$. It can be verified by an experiment that volume of cone is one third the volume of cylinder of same radius and same height.

Take a measuring jar of radius r and h. Its volume is $\pi r^2 h$. According to fig, 16.09. Take a cone of same radius and height. Let volume of cone is V. Fill the cone with water and drop in the cylinder you will see that after repeating this process three times then jar will fill with water completely. So it concludes that volume of cone $V = \frac{1}{3}\pi r^2 h$.

Illustrative Examples

Example 7. The diameter of ther base and height of a cone are 12 m and 10 m. Find the total surface area of cone.

Solution : Given : Diameter of base of cone = 12 m

Radius of cone $= \frac{\text{diameter}}{2} = \frac{12}{2}\text{m} = 6\text{m}$

and slant height of cone $(\ell) = 10$ m

Total surface area of cone $= \pi(\ell + r)r$

$$= \frac{22}{7}(10+6)\times 6 = \frac{22}{7}\times 16\times 6$$

$$= \frac{2112}{7} = 301.71$$

Thus, total surface area of cone = 301.71 m^2

Example 8. The curved surface area of a cone is 2035 cm^2 and diameter of its base is 35 cm. Find the slant height of cone.

Solution : Given : Curved surface area of a cone = 2035 cm^2

$\therefore$ Radius of cone $(r) = \frac{\text{diameter}}{2} = \frac{35}{2}\text{cm} = 17.5\text{cm}$

$\therefore$ $2035 = \frac{22}{7} \times 17.5 \times \ell$

or $2035 = 22 \times 2.5 \times \ell$

or $\ell = \frac{2035}{22 \times 2.5} = \frac{2035}{55}$

or $\ell = 37$

Thus, slant height of cone will be 37 cm.

Example 9. The volume of cone of height 9 cm is 16632 cm^3. Find radius of its base.

Solution : Given : volume of cone = 16632 cm^3

height of cone $(h) = 9$ cm

volume of cone $= \frac{1}{3}\pi r^2 h$

$\therefore$ $16632 = \frac{1}{3} \times \frac{22}{7} \times r^2 \times 9$

or $16632 = \frac{22}{7} \times r^2 \times 3$

or $16632 \times 7 = 22 \times r^2 \times 3$

or $r^2 = \frac{16632 \times 7}{22 \times 3} = \frac{756 \times 7}{3}$

or $r^2 = 252 \times 7$

or $r = \sqrt{36 \times 7 \times 7}$

or $r = 6 \times 7 = 42$

Thus radius of cone = 42 cm.

Example 10. The ratio of radius and height of a cone is 5 : 12 and its volume is 2512 cm^3. Find slant height and radius of base of cone (take $\pi = 3.14$)

Solution : Ratio of radius and height of cone = 5 : 12

Radius of cone $(r) = 5x$ cm

Height of cone $(h) = 12x$ cm

Volume of cone = 2512 cm^3

$\therefore$ $\frac{1}{3}\pi r^2 h = 2512$

or $\frac{1}{3} \times 3.14 \times (5x)^2 \times 12x = 2512$

or $\frac{1}{3} \times 3.14 \times 25x^2 \times 12x = 2512$

or $3.14 \times 25x^2 \times 4x = 2512$

or $314x^3 = 2512$

or $x^3 = \frac{2512}{314} = 8$

or $(x)^3 = (2 \times 2 \times 2)$

or $x^3 = 2^3$

or $x = 2$

Thus, radius of cone $= 5 \times 2 = 10$ cm.

Height of cone $= 12 \times 2 = 24$ cm.

Example 11. A conical tent that is 14 metres high and its base area is 346.5 metre^2. This tent is made of 1.5 metre wide canvas, then find out its length.

Solution : Height of conical tent $h = 14$ m.

Radius $= r$ m

Area of base of cone $= \pi r^2$

Area of base $= 346.5$ m^2

$$\frac{22}{7} \times r^2 = 346.5$$

or $r^2 = \frac{346.5 \times 7}{22}$

or $r^2 = 110.25$

or $r = 10.5$ m

Slant height of tent $\ell = \sqrt{r^2 + h^2}$

$= \sqrt{(10.5)^2 + (14)^2}$

$= \sqrt{110.25 + 196}$

$= \sqrt{306.25} = 17.5$ m.

Area of canvas = Curved surface area of tent

$= \pi r \ell$ m^2

$= \frac{22}{7} \times 10.5 \times 17.5$ m^2

$= 577.5$ m^2

$$\text{Length of canvas} = \frac{\text{Area}}{\text{Width}}$$

$$= \frac{577.5}{1.5} \text{ m} = 385 \text{ m}$$

Exercise 16.3

1. Find the curved surface area, total surface area, and volume of a cone whose height is 28 cm and radius of base is 21 cm.
2. Find the slant height of the right circular cone whose volume is 1232 cm^3 and height is 24 cm.
3. Find the total surface area of cone whose diameter of base is 14 m, and slant height is 25 m.
4. Find the curved surface area and total surface area of right circular cone whose radius of base is 14 cm and slant height is 50 cm.
5. Find the volume of right circular cone whose radius of base is 6 cm and height is 8 cm.
6. Find the radius of base of a cone whose curved surface area is 1884.4 m^2 and its slant height is 12 cm.
7. Find the height of the right circular cone of slant height is 25 cm and area of its base is 154 cm^2.
8. The base of two cones are of same diameter. Ratio of their slant height is 5 : 4.If curved surface area of smaller cone is 400 cm^2 then find the curved surface area of bigger one.
9. The ratio of slant height and radius of a cone 7 : 4. If its curved surface area is 792 cm^2 , then find its radius.
10. The circumference of base of a conical tent is 9 m and height is 44m. Find the volume of air inside it.
11. The radius and height of a conical vessel are 10 cm and 18 cm respectively, which is filled with water to the brim. It is poured in a cylindrical vessel of radius 5 cm. Find the height of water level in cylindrical vessel.
12. A cone of maximum height is cut from a cube of edge14 cm. Find the volume of cone.
13. Find the slant height, curved surface area and total surface area and volume of a cone whose base radius and heights are 7 cm and 24cm respectively.
14. The radius of a sector is 12 cm and angle is 120^o. By coinciding its straight sides a cone is formed. Find the volume of that cone.

16.07. Sphere

A sphere or semi-sphere is a solid obtained on revolving full or half respectively a circle about any diameter of it.

We can define sphere as set of all the points in space are called sphere which are at equal distant from a fixed point. Fixed point is called centre of the sphere. Distance from centre to any point is called centre of the sphere. Distance from centre to any point of this set is called radius, cricket ball, football are examples of sphere.

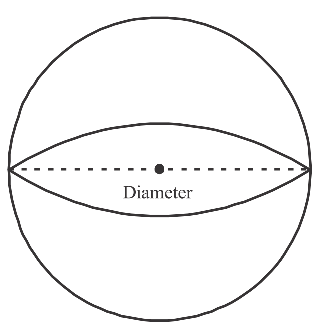


Fig. 16.10

The line segment which passes through the centre of sphere and whose two end points lie on sphere is called diameter of sphere. All the diameter of sphere are of same length. Diameter of sphere is twice of its radius. The space occupied by sphere is called its volume.

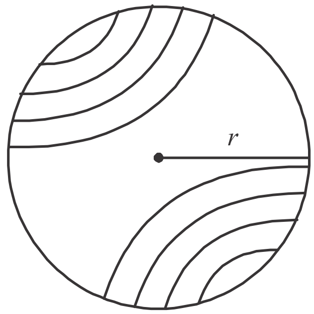


Fig. 16.11

Following formula can be given without proof.

If r is radius of sphere then

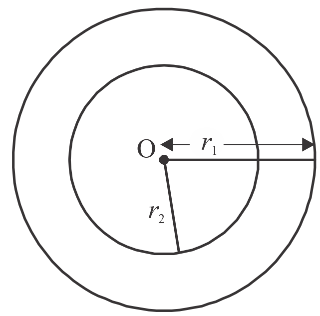


Fig. 16.12

volume of sphere $= \frac{4}{3}\pi r^3$

The surface area of sphere $= 4\pi r^2$

Volume of hemisphere of radius $= \frac{2}{3}\pi r^3$

Total surface area of hemisphere $= 3\pi r^2$

If r_1 and r_2 are external radius of spherical shell

Volume of spherical shell $= \frac{4}{3}\pi\left(r_1^3 - r_2^3\right)$

Total surface area of spherical shell $= 4\pi\left(r_1^2 + r_2^2\right)$

Illustrative Examples

Example 12. Find the surface area of sphere of radius 7 cm.

Solution : Radius of sphere $= 7$ cm.

The surface area of sphere $= 4\pi r^2$

$$= 4 \times \frac{22}{7} \times 7 \times 7 \text{ cm}^2$$

$$= 22 \times 28 \text{ cm}^2$$

$$= 616 \text{ cm}^2$$

Example 13. The radius of a hemisphere is 3.5 cm. Find its volume and total surface area.

Solution : Radius of hemisphere $(r) = 3.5$ cm.

Volume of hemisphere $= \frac{2}{3}\pi r^3$

$$= \frac{2}{3} \times \frac{22}{7} \times (3.5)^3 \text{ cm}^3$$

$$= \frac{2}{3} \times \frac{22}{7} \times 3.5 \times 3.5 \times 3.5 \text{ cm}^3$$

$$= \frac{2}{3} \times 22 \times 0.5 \times 12.25 \text{ cm}^3$$

$$= \frac{269.5}{3} \text{cm}^3 = 89.83 \text{ cm}^3$$

Thus, total surface area of hemishpere $= 3\pi r^2$

$$= 3 \times \frac{22}{7} \times (3.5)^2 \text{ cm}^2$$

$$= 3 \times \frac{22}{7} \times 3.5 \times 3.5 \text{ cm}^2$$

$$= 3 \times 22 \times 0.5 \times 3.5 \text{ cm}^2$$

$$= 115.5 \text{ cm}^2$$

Example 14. The radius of a lead sphere is 5 cm. How many small spheres of 5 mm radius can be formed form this sphere?

Solution : Radius of lead sphere is $(r) = 5$ cm

Volume of big lead sphere $(v) = \frac{4}{3}\pi (5)^3 \text{ cm}^3$

$$= \frac{4}{3}\pi \times 5 \times 5 \times 5 \text{ cm}^3$$

Radius of small lead sphere $(r_1) = 5$ mm $= \frac{5}{10}\text{cm} = 0.5 \text{cm}$

Volume of 1 small lead sphere $= \frac{4}{3} \times \pi \times 0.5 \times 0.5 \times 0.5 \text{ cm}^3$

No. of small lead sphere $= \frac{\text{Volume of big shere}}{\text{Volume of small sphere}}$

$$= \frac{\frac{4}{3} \times \pi \times 5 \times 5 \times 5}{\frac{4}{3} \times \pi \times 0.5 \times 0.5 \times 0.5} = 1000 \text{ small spheres}$$

Example 15. The surface area of a ball is 1386 cm^2. Find the radius of ball.

Solution : Given : surface area of ball (sphere) = 1386 cm^2

or $$4\pi r^2 = 1386$$

or $$4 \times \frac{22}{7} \times r^2 = 1386$$

or $$r^2 = \frac{1386 \times 7}{4 \times 22} = 110.25$$

or $$r = \sqrt{110.25} = 10.5$$

Thus radius of ball will be 10.5 cm.

Example 16. The ratio of surface areas of two sphere is 4 : 9 find the ratio of their surface areas and volumes.

Solution : Let r_1 and r_2 are radius of two spheres and $4\pi r_1^2$ and $4\pi r_2^2$ are their surface areas

Ratio of surface areas of two sphere $\frac{\pi r_1^2}{\pi r_2^2} = \frac{4}{9}$

or $$\frac{r_1^2}{r_2^2} = \frac{4}{9}$$

or $$\frac{r_1}{r_2} = \sqrt{\frac{4}{9}} = \frac{2}{3}$$

Ratio of volumes of two sphere $= \dfrac{\frac{4}{3}\pi r_1^3}{\frac{4}{3} r_2^3} = \dfrac{r_1^3}{r_2^3}$

$$= \left(\frac{r_1^3}{r_2^3}\right) = \left(\frac{2}{3}\right)^3 = 8:27$$

Exercise 16.4

1. Find the surface area and volume of a sphere of radius 1.4 cm.
2. Find the volume of the sphere whose surface area is 616 cm^2.
3. Find the surface area and volume of a hemisphere of radius 4.5 cm.
4. Find the surface area of a sphere whose volume is 38808 cm^3.
5. A cylinder is made of lead whose radius is 4 cm and height is10 cm. By melting this, how many spheres of radius is 2 cm, can be formed?
6. A hollow spherical shell is 2 cm, thick. If its outer radius is 8 cm, then find the volume of metal used in it.
7. How many cones of 3 cm. radius and 6 cm in height are formed by melting a metallic sphere of radius 9 cm?
8. Eight spheres of same radius from a metallic sphere of 10 cm radius, are formed. Find the surface area of each sphere so obtained.

9. Find the volume of sphere if its surface area is 5544 cm^2.
10. The dimensions of a solid rectangular slab of lead is 66 cm, 42 cm and 21 cm respectively. Find by melting this, how many sphere of diameter 4.2 cm can be formed?
11. A sphere of 6 cm diameter is dropped into cylindrical vesselof diameter 12 cm. Find the rise in water in vessel.
12. A hemispherical bowl of internal radius 9 cm is full of liquid. This liquid is to be filled into cylindrical shaped small bottles each of diameter 3 cm and height 4 cm. How many bottles are necessary to empty the bowl?
13. The diameter of a sphere is 0.7 cm. If 3000 spheres, completely filled with water, are drawn out from a water tank then find volume of water drawn out.
14. A hollow hemispherical vessel has external and internal radius as 43 cm and 42 cm. respectively. If cost of colouring it, is 7 paisa per square cm. then find cost of painting the vessel.

Miscellaneous Exercise-16

1. The total surface area of a cube is 486 cm^2 , edge of cube is

   (a) 6 cm (b) 8 cm (c) 9 cm (d) 7 cm
2. The length, breadth and height of cuboid are 3 m, 2 m and 1 m respectively. Total surface area of cuboid will be :

   (a) 12 m^2 (b) 11 m^2 (c) 21 m^2 (d) 22 m^2
3. The diameter of a sphere is 6 cm, the volume of sphere will be

   (a) 16 π cm^3 (b) 20 π cm^3 (c) 36 π cm^3 (d) 30 π cm^3
4. A cylinder has radius of its base 14 cm and 10 cm in height. Curved surface area of cylinder will be :

   (a) 810 cm^2 (b) 880 cm^2 (c) 888 cm^2 (d) 890 cm^2
5. The volume and height of a cone is 308 cm^2 and 6 cm respectively. Radius of its base will be :

   (a) 7 cm (b) 8 cm (c) 6 cm (d) none of these
6. A solid metallic hemisphere has diameter 42 cm. Find the cost of polishing the total surface at the rate 20 paise per cm^2.
7. A cone, a hemisphere and a cylinder are formed by same radius and same height. Write ratio of their volumes.
8. The left part of a solid body is cylindrical and right part is conical. If diameter of cylindrical is 14 cm and length is 40 cm and of diameter cone is 14 cm and height is 12 cm, then find the volume of solid.
9. A metallic sphere of radius 9 cm is melted and then recast into small cones of radius 3 cm and height 6 cm. Find the number of cones thus formed.
10. The population of a village is 4000. There is requirement of 150 *l* water per person daily. There is a water tank of dimensions 20m × 15 m × 6 m in village. For how many days water of this tank will suficient.
11. Three solid metallic sphere of radii 6 cm, 8 cm and 10 cm respectively are melted and then recast into a big sphere. Find the radius of this sphere.
12. A conical vessel has radius 10 cm and height 18 cm and is completely filled with water which is transferred in a cylindrical vessel of radius 5 cm. Find the level of water (height) in cylindrical vessel.
13. A candle of diameter 2.8 cm is formed from a cuboid of dimensions 11 cm × 3.5 cm × 2.5 cm. find the length of candle.
14. The diameter of a metallic sphere is 6 cm. It is melted and drawn into a wire of length 36 m and of uniform circular cross-section.. Find its radius.

Important Points

1. Area of 4 walls of room = 2 height (length + breadth)
   $= 2 \times (l + b) \times h$
2. Total surface area of cuboid $= 2(\ell b + bh + h\ell)$
3. Volume of cuboid $= \ell \times b \times h =$ Area of base × height
4. Area of 4 walls of cube $= 4\ell^2$
5. Total surface area of cube $= 6\ell^2$
6. Volume of cube $= \ell^3$
7. Diagonal of cuboid $= \sqrt{\ell^2 + b^2 + h^2}$
8. Diagonal of cube $= \sqrt{3}\ell$
9. Curved surface area of cylinder $= 2\pi rh$
10. Total surface area of cylinder $= 2\pi r(r + h)$
11. Volume of cylinder $= \pi r^2 h =$ Area of base × height
12. Total surface area of hollow cyclinder $= 2\pi(r_1 + r_2)(h + r_1 - r_2)$
13. Volume of hollow cylinder $= \pi(r_1^2 - r_2^2)h$
14. Curved surface area of cone $= \pi r\ell$
15. Slant height of cone $\ell = \sqrt{h^2 + r^2}$
16. Total surface area of cone $= \pi r(r + \ell)$
17. Volume of cone $= \frac{1}{3}\pi r^2 h = \frac{1}{3} \times$ Area of base × height
18. Surface area of solid sphere $= 4\pi r^2$
19. Volume of soild sphere $= \frac{4}{3}\pi r^3$
20. Curved surface area of hemisphere $= 2\pi r^2$
21. Total surface area of hemisphere $= 3\pi r^2$
22. Volume of hemisphere $= \frac{2}{3}\pi r^3$
23. Total surface area of spherical shell $= 4\pi(r_1^2 + r_2^2)$
24. Volume of spherical shell $= \frac{4}{3}\pi(r_1^3 - r_2^3)$ where $r_1 > r_2$
25. 1 m^3 = 1000 litre = 1 kilo letre
    1 litre = 1000 cm^3
    1 acre = 100 m^2
    1 cm^3 = 1000 mm^3
    1 m^3 = 100,00,00 cm^3

Answer Sheet

Exercise 16.1

1. 426 cm^2, 540 cm^2 2. 486 cm^2 3. 7900 cm^2 4. 400 cm^2
5. ₹ 110.80 6. 240 cm^2 7. 5 m 8. 15000 bricks
9. $10\sqrt{2}$ m 10. 8 m 11. 3000 bricks 12. 15 cm, 9 cm, 6 cm

Exercise 16.2

1. 968 cm^2, 2310 cm^3 2. 132 cm^2, 188.57 cm^3, 198 cm^3
3. 3234 cm^3, 924 cm^2 4. 5 : 6 and 5 : 9 5. 539 cm^3 6. 2310 cm^3
7. 5 cm 8. 5 cm, 21 cm 9. 22 : 15 10. 2.5 m
11. 4400 cm^2 12. 2794 cm^2 13. 3520 cm^3

Exercise 16.3

1. 2310 cm^2, 3696 cm^2, 12936 cm^3 2. 25 cm 3. 704 m^2
4. 2200 cm^2, 2816 cm^2 5. 301.71 cm^3 6. 5 cm approx
7. 24 cm 8. 320 cm^2 9. 12 cm 10. 462 m^3
11. 24 cm 12. 718.67 cm^3 13. 25 cm, 550 cm^2, 704 cm^2, 1232 cm^3
14. 189.61 cm^3

Exercise 16.4

1. 24.64 cm^2, 11.5 cm^3 2. 1437.33 cm^3
3. 190.93 cm^2, 190.93 cm^3 4. 5544 cm^2
5. 15 6. 1240.38 cm^3 7. 54 8. 100 π cm^2
9. 38808 cm^3 10. 1500 11. 1 cm 12. 54
13. 539 cm^3 14. ₹ 397.43

Miscellaneous Exercise- 16

1. (c) 2. (d) 3. (c) 4. (b) 5. (a)
6. ₹ 831.60 7. 1 : 2 : 3 8. 6776 cm^3 9. 54 10. 3 days 11. 12 cm
12. 24 cm 13. 15.6 cm 14. $\frac{1}{10}$ cm = 0.1 cm

❑

17

Measures of Central Tendency

17.01 Introduction :

The collection, classification, tabulation and graphical representation of initial data make the data simple and easy to understand. But when comparitive study of data is to be undertaken or any conclusion form data are to be drawn then it is necessary to make it simple and compact. So that their characterstics can be represented by a single number.

For example, if 300 students of a school are compared to 500 students of another school, it is not possible to come to any conclusion from their series showing marks obtained in different subject. But if a single number is taken from each of these series which is their repersentative, then it will be easy to compare these series. This representative number is taken in the middle of the series, where most of the observations are centred. Such a number is the representative of the whole series and it is called a **measure of** central tendency.

17.02 Measures of Central Tendency and Types of Averages :

Measures of central tendency and averages are ordinarily divided into two parts :

(1) Mathematical Average

(i) Arithmetic Mean or Average [AM]

(ii) Geometric Mean [GM]

(iii) Harmonic Mean [HM]

(2) Average of Position

(i) Median

(ii) Mode

Here at the secondary level, we shall consider only simple questions on arthimetic mean (which is usually called mean), median and mode.

17.03 Arithmetic Mean

To find arithmetic mean from initial data (individual series) : In order to find arithmetic mean from such type of data, we find the sum of all the data and divide it by the number of data. This is also called mean, i.e.

$$\text{Arithmetic Mean} = \frac{\text{Sum of data}}{\text{Number of data}}$$

For example, the marks obtained by student of class X in mathematics are 7, 8, 5, 6, 7, 8, 9, 4, 5 and 6 respectively then average of marks

$$= \frac{\text{Sum of marks (Sum of data)}}{\text{No. of students (No. of data)}}$$

$$= \frac{7+8+5+6+7+8+9+4+5+6}{10}$$

$$= \frac{65}{10} = 6 \cdot 5 \text{ Marks}$$

If values of a variate are $x_1, x_2, \ldots, x_n$ respectively, then

their A.M. $(\bar{x}) = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$

$$= \frac{\sum_{i=1}^{n} x_i}{n}$$

Note : Σ is a letter of Greek alphabet and it is pronounced as 'Sigma' and it is used to denote the operation of addition in Mathematics.

$\sum_{i=1}^{n} x_i$ denotes the sum $x_1 + x_2 + x_3 + \cdots + x_n$.

Thus $\sum_{i=1}^{25} y_i = y_1 + y_2 + y_3 + \cdots + y_{25}$

Illustrative Examples

Example 1. The monthly salaries of 5 employees including Head master in a school are ₹ 8000, ₹ 5000, ₹ 4000, ₹ 2500 and ₹ 1500 respectively. Find the average monthly salary of working employees in school.

Solution : Average monthly salary

$$= \frac{8000 + 5000 + 4000 + 2500 + 1500}{5}$$

$$= \frac{21000}{5} = ₹\, 4200$$ **Answer.**

Example 2. Find the A.M. of the first ten positive odd numbers.

Solution : First ten odd positive numbers are 1, 3, 5, 7, 11, 13, 15, 17, 19 respectively.

So A.M. $(\bar{x}) = \dfrac{1+3+5+7+9+11+13+15+17+19}{10}$

$$= \frac{100}{10} = 10$$ **Answer.**

Example 3. The average of eight consecutive odd numbers is 16, find the numbers.

Solution : Let the first odd number be x, so eight odd numbers are

$x, x+2, x+4, x+6, x+8, x+10, x+12, x+14$

Average of eight odd numbers

$$= \frac{(x)+(x+2)+(x+4)+(x+6)+(x+8)+(x+10)+(x+12)+(x+14)}{8}$$

$$= \frac{8x+2+4+6+8+10+12+14}{8} = \frac{8x+56}{8}$$

Thus $\dfrac{8x+56}{8} = 16$ or $8x + 56 = 128$ or $x = 9$.

Therefore required consecutive numbers are

9, 11, 13, 15, 17, 19, 21, 23 **Answer.**

17.04 Merits, Demerits of Arithmetic Mean :

Merits :

1. It is easy to calculate.
2. It is based upon all the terms.
3. It is also used in other statistical analysis.
4. This mean is fixed and always same.
5. It is possible to test its accuracy.
6. Its value has stability.

Demerits :

1. Sometimes in its calcualtion, such value may occur which is not possible according to nature, e.g. number of members in a family is 3·8 or 5·6
2. It is not possible to calculate if any one value is missing.
3. It is affected very much by exterme values.
4. Determination of this mean is not possible by observation.

Exercise 17.1

1. If marks obtained by ten students of a class in Mathematics are 52, 75, 40, 70, 43, 40, 65, 35, 48 and 52 then find their arithmetic mean.
2. The monthly salaries (in ₹) of subordinate employees of a school are 1720, 1750, 1760 and 1710, then find the airthmetic mean.
3. If the airthmetic mean of the marks 3, 4, 8, 5, x, 3, 2, 1 is 4, then find the value of x.
4. The runs scored in 10 innings by a batsman are 60, 62, 56, 64, 0, 57, 33, 27, 9 and 71 respectively. Find the arithmetic mean of the runs scored by him in these innings.
5. Calculate arithmetic mean of the following marks obtained by 10 students in English in the monthly test :

Roll No.	:	1	2	3	4	5	6	7	8	9	10
Marks obtained	:	30	28	32	12	18	20	25	15	26	14

6. The number of books issued to the students in 10 days from the school library are given below :

300 405 455 489 375 280 418 502 300 476

Find the average number of books issued per day.

7. The average weight of 25 students of section A of a class is 51 kg. whereras the average weight of 35 students of section B is 54 kg. Find the average weight of 60 students of this class.
8. The mean of 5 numbers is 18, If one number is excluded, then mean becomes 16. Find the excluded number.
9. The mean of 13 numbers is 24. If 3 is added to each number, then find their new mean.
10. The monthly salary of 5 employees of a school is ₹ 3000. On the retirement of one employee, the average monthly salary of remaining employees is ₹ 3200. What was the salary of retired employeed at the time of retirement ?

17.05 Arithmetic Average from Discrete Series or Discrete Frequency Distribution

Let frequaency distribution of n values of variate x is as follows :

Value of x	:	x_1	x_2	x_3	$\cdots$	x_k
Frequency f	:	f_1	f_2	f_3	$\cdots$	f_k

It is clear form distribution that out of total n values of x, x_1, is attained f_1 times, x_2, is attained f_2 times. Thus average or arithmetic mean $(\bar{x})$ of variate x will be obtained as follows :

$$\bar{x} = \frac{\overbrace{x_1 + x_1 + \cdots + x_1}^{f_1 \text{ times}} + \overbrace{x_2 + x_2 + \cdots + x_2}^{f_2 \text{ times}} + \cdots + \overbrace{x_n + x_n + \cdots + x_n}^{f_k \text{ times}}}{f_1 + f_2 + \cdots + f_n}$$

$$= \frac{f_1 x_1 + f_2 x_2 + \cdots + f_n x_n}{f_1 + f_2 + \cdots + f_n}$$

$$= \frac{1}{n}\sum_{i=1}^{n} f_i \, x_i, \quad \text{Where } \sum_{i=1}^{n} f_i = n = \text{Number of total values}$$

Working steps :

Step I. First of all make frequaency table form frequency distribution in such a way that I column contains the value of x_i of x and II column contians the frequency f_i of x.

Step II. III column will contain product $f_i x_i$ of x_i and f_i .

Step III. On showing sum of II column by $\sum f_i$ and sum of III column by $\sum f_i x_i$

$$\text{Arithmetic mean } (\bar{x}) = \frac{\sum f_i x_i}{\sum f_i}$$

Therefore for calculation of arithmetic mean table can be prepared as follows :

Calculation of Arithmetic mean

x_i	f_i	$f_i x_i$
x_1	f_1	$f_1 x_1$
x_2	f_2	$f_2 x_2$
x_3	f_3	$f_3 x_3$
$\vdots$	$\vdots$	$\vdots$
x_n	f_n	$f_n x_n$
	$\sum f_i$	$\sum f_i x_i$

$$\text{Mean } \quad \bar{x} = \frac{\sum f_i x_i}{\sum f_i}$$

Note : The value of x is represent by x_i and their corresponding frequencies by f_i . The average value of x is denoted by $\bar{x}$.

Example : Calculate the mean from the following frequency distribution :

x:	5	6	7	8	9	10	11
f:	5	8	9	12	6	6	4

Solution : **Calculation of Arithmetic mean**

x_i	f_i	$f_i x_i$
5	5	25
6	8	48
7	9	63
8	12	96
9	6	54
10	6	60
11	4	44
	$\sum f_i = 50$	$\sum f_i\ x_i = 390$

Thus Arithmetic mean $\bar{x} = \dfrac{\sum f_i x_i}{\sum f_i}$

$= \dfrac{390}{50} = 7 \cdot 8$ **Answer .**

Illustrative Examples

Example 1. Find the arithmetic mean of the following frequency distribution :

x	1	2	3	4	5	6
f	2	5	6	4	2	2

Solution : **Calculation of Arithmetic mean**

x_i	f_i	$f_i x_i$
1	2	2
2	5	10
3	6	18
4	4	16
5	2	10
6	2	12
	$\sum f_i = 21$	$\sum f_i x_i = 68$

Thus Arthmetic mean $(\bar{x}) = \dfrac{\sum f_i x_i}{\sum f_i} = \dfrac{68}{21} = 3\cdot238$ **Answer**

Example 2. The daily salaries of 50 officers of a factory are as follows :

Salary (in₹) x :	**450**	**475**	**500**	**525**	**550**
No. of officers f :	**12**	**13**	**7**	**10**	**8**

Find arithmetic mean of their salaries.

Solution :

x_i	f_i	$f_i x_i$
450	12	5400
475	13	6175
500	7	3500
525	10	5250
550	8	4400
	$\sum f_i = 50$	$\sum f_i x_i = 24725$

Therefore the required A.M. $(\bar{x}) = \dfrac{\sum f_i x_i}{\sum f_i}$

$$= \frac{24725}{50}$$

$= ₹494\cdot5$ **Answer**

Exercise 17.2

Find the mean of the following frequency distribution (Q. 1-4) :

1.

x :	3	5	8	11
f :	2	4	5	3

2.

x :	2	5	7	9	11
f :	1	5	4	7	3

3.

x :	$0\cdot1$	$0\cdot2$	$0\cdot3$	$0\cdot4$	$0\cdot5$	$0\cdot6$
f :	30	60	20	40	10	50

4.

x :	$0\cdot1$	$0\cdot3$	$0\cdot5$	$0\cdot7$	$0\cdot89$
f :	7	8	10	15	10

5. In hundred falmily, the number of childern are :

No. of children	1	2	3	4	5	6
No. of families	45	25	19	8	2	1

Find their arithmetic mean.

6. The weight of students in a class are given in the following table.

Weight (in kgs.)	20	21	22	23	24	25	26	27	28
No. of students	1	2	6	7	4	2	3	2	3

Find their arithmetic mean.

7. If mean of the following distribution is $7\cdot5$, then find the value of P.

x :	3	5	7	9	11	13
f :	6	8	15	P	8	4

8. If mean of the following frequency distribution is $1\cdot46$, then find the unknown frequencies.

x :	0	1	2	3	4	5	*Sum*
f :	46	$\cdots$	$\cdots$	25	10	5	200

17.06 Arithmetic mean from grouped frequency distribution

In such type of frequency distribution, the value of variable is divided in intervals. For example consider the following frequency distribution :

Marks obtained (x)	0-10	10-20	20-30	30-40	40-50
No.of students (f)	5	8	20	14	3

Here, the frequency of a class interval 10-20 is 8, i.e. the number of values of x from 10 to less than 20 is 8. When grouped frequency distribution is prepared from initial data then by seeing distribution, it is impossible to estimate about these data. For example, if values of x are 10, 11, 12, 17, 17, 18, 19, 19·5, or 11, 12, 13, 14, 15, 15, 17, 19 then in each case the class interval will be 10-20 whose frequency is 8.

Hence for the sake of convenience and simplicity it is quite logical to consider the mean of each interval as the mean of x and the frequency of each interval as the frequency x. The mean is calculated by the method used in case of ungroup frequency distribution. For example, for the interval 10-20 $x = \frac{10+20}{2} = 15$ whose frequency is 8.

Thus the ungrouped frequency distribution is obtained from above grouped frequency distribution as follow

Interval (Marks)	0-10	10-20	20-30	30-40	40-50
Marks obtained	5	15	25	35	45
Frequency	5	8	20	14	3

We obain the mean with the help of the method explained earlier as follows:

x_i	f_i	$f_i x_i$
5	5	25
15	8	120
25	20	500
35	14	490
45	3	135
Sum	$\sum f_i = 50$	$\sum f_i x_i = 1270$

Therefore the required A.M. $\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$

$$= \frac{1270}{50}$$

$= 25 \cdot 4$ Marks **Answer** .

Exercise 17.3

Find the arithmetic mean of the following frequency distribution : [1 to 4]

1.

Class	0-10	10-20	20-30	30-40	40-50
Frequency	9	12	15	10	14

2.

Class	0-6	6-12	12-18	18-24	24-30
Frequency	6	8	10	9	7

3.

Marks obtained (x)	100-120	120-140	140-160	160-180	180-200
No. of students(f)	10	20	20	15	5

4.

Class	25 – 35	35 – 45	45 – 55	55 – 65	65 – 75
Frequency	6	10	8	12	4

5. Find the mean of the following frequency distribution :

Weight (in kg.)	40 – 50	50 – 60	60 – 70	70 – 80	80 – 90	90 – 100
No. of students	10	25	28	12	10	15

6. The salaries of the workers of a factory are as follows :

Monthly salary (in ₹)	1000-1200	1200-1400	1400-1600
No. of workers	10	20	20
Monthly salary (in ₹)	1600-1800	1800-2000	
No. of workers	15	5	

Find the arithmetic mean of the salaries.

17.07 Arithmetic mean using assumed mean :

If the values of x in any frequency distribution are very large, when it is difficult to calculate the arithmetic mean and much time is also consumed. In such a situated, it is convenient to calculate the arithmetic mean form short-cut method of assumed mean.

Working Steps :

Step I. We first prepare the frequency table in such a way that first column contains x_i , and the second column contains their frequencies f_i .

Step II. In the third column we write deviation of each x_i from an appropriate value A. Here A is calld the assumed mean.

Step III. In the fourth column we write product of frequency f_i and devaiton d_i, $f_i d_i$.

Step IV. Write sum of column second, $\sum f_i$ and sum of column fourth $\sum f_i d_i$ in the corresponding last column.

Step V. We find the arithmetic mean form the formula

$$\bar{x} = A + \frac{1}{N}\left(\sum f_i d_i\right), \text{ where } N = \sum f_i \ .$$

The above procedure is clear form the following table.

x_i	f_i	$d_i = x_i - A$	$f_i d_i$
x_1	f_1	d_1	$f_1 d_1$
x_2	f_2	d_2	$f_2 d_2$
x_3	f_3	d_3	$f_3 d_3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_k	f_k	d_k	$f_k d_k$
	$N = \sum f_i$		$\sum f_i d_i$

Thus Arithmetic Mean $\left(\bar{x}\right) = A + \dfrac{\sum f_i d_i}{\sum f_i}$

$$= A + \frac{1}{N}\left(\sum f_i \, d_i\right)$$

If in step II, step deviation is found from $u_i = \dfrac{x_i - A}{h}$, where h is class interval then according to step III, in column third, we write $f_i u_i$, i.e. the product of f_i and u_i . The mean is to found by the folowing formula :

$$\bar{x} = A + \frac{\sum f_i u_i}{\sum f_i} \times h$$

Important Remarks :

(i) Generally assumed mean A is taken as mid value of x or value of x having maximum frequency.

(ii) When difference in the value of x is large and the value of x are also larger or frequencies are large then it is convenient to calculate by step deviation $u_i = \dfrac{x_i - A}{h}$.

Calcualtion table for the above formula

x_i	f_i	$u_i = \dfrac{x_i - A}{h}$	$f_i u_i$
x_1	f_1	u_1	$f_1 u_1$
x_2	f_2	u_2	$f_2 u_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_k	f_k	u_k	$f_k u_k$
योग	$\sum f_i$		$\sum f_i u_i$

Thus arithmetic mean $(\overline{x}) = A + \dfrac{\sum f_i u_i}{\sum f_i} \times h$

(Here, by taking A as mid value, the values of u_i are $\cdots -3, -2, -1, \ 0, 1, 2, 3, \cdots$)

The method will be clear form the following examples.

Illustrative Examples

Example 1. Find the arithmetic mean for the following frequency distribution :

x	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**
f	**20**	**43**	**75**	**67**	**72**	**45**	**39**	**9**	**8**	**6**

Solution : First of all we construct the calcualtion table by assuming 25, the corresponding value of maximum frequency 72 as assumed mean. (Here A=25 and h = 5).

Calculation for Arithmetic mean

Variate x_i	Frequency f_i	$u_i = \frac{x_i - 25}{5}$	$f_i u_i$
5	20	−4	−80
10	43	−3	−129
15	75	−2	−150
20	67	−1	−67
25	72	0	0
30	45	1	45
35	39	2	78
40	9	3	27
45	8	4	32
50	6	5	30
Total	$N = \sum f_i = 384$		$\sum f_i u_i = -214$

Thus Arithmetic mean $(\bar{x}) = A + \left(\frac{\sum f_i u_i}{\sum f_i}\right) \times h$

$$= 25 + \left(\frac{-214}{384}\right) \times 5$$

Answer.

$$= 25 - 2 \cdot 786 = 22 \cdot 214$$

Example 2. The following frequency distribution shows weights of 12 students.

Weight (in kg.)	**67**	**70**	**72**	**73**	**75**
No. of students	**4**	**3**	**2**	**2**	**1**

Find mean weight.

Solution : **Calculation table for Arithmetic mean**

Weight (in kg.) x_i	No. of students f_i	$d_i = x_i - 72$	$f_i d_i$
67	4	-5	-20
70	3	-2	-6
72	2	0	0
73	2	1	2
75	1	3	3
Sum	$N = \sum f_i = 12$		$\sum f_i d_i = -21$

Here by taking mid values of $x = 72$ as value of A

$$\text{Mean } (\bar{x}) = A + \frac{1}{N}\left(\sum f_i d_i\right)$$

$$= 72 + \left(\frac{-21}{12}\right)$$

$$= 72 - \frac{7}{4} = 70 \cdot 25 \text{ kg.}$$

Thus mean weight 70·25 kg. **Answer**

Example 3. In the following table, height of villages of some special region from sea level is given. Find the mean height of that region form sea level.

Height (in Metres)	**200**	**600**	**1000**	**1400**	**1800**	**2200**
No. of villages	**142**	**265**	**560**	**271**	**89**	**16**

Solution : Here we shall find the mean by taking A = 1000 and h = 400 and by calculating both types of devaition d_i as well as u_i .

Calculation table for Arithmetic mean

Height (in Metres.) x_i	No. of villages f_i	Deviation $d_i = x_i - 1000$	$f_i d_i$	Devaition $u_i = \frac{x_i - 1000}{400}$	$f_i u_i$
200	142	−800	−113600	−2	−284
600	265	−400	−106000	−1	−265
1000	560	0	0	0	0
1400	271	400	108400	1	271
1800	89	800	71200	2	178
2200	16	1200	19200	3	48
	$\sum f_i = 1343$		$\sum f_i d_i = -20800$		$\sum f_i u_i = -52$

Therefore

(i) Mean by deviation method

$$\bar{x} = A + \frac{\sum f_i d_i}{\sum f_i}$$

$$= 1000 + \frac{-20800}{1343}$$

$$= 1000 - 15 \cdot 488 \text{ approx.}$$

$$= 984 \cdot 512$$ **Answer.**

(ii) Mean by step deviation method

$$\bar{x} = A + \left(\frac{\sum f_i u_i}{\sum f_i}\right) h$$

$$= 1000 + \frac{-52}{1343} \times 400$$

$$= 1000 - 15 \cdot 488 \text{ approx.}$$

$$= 984 \cdot 512$$ **Answer.**

Example 4. Find the mean of the following frequency distribution by step deviation method.

Class interval	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50
Frequency	7	10	15	8	10

Solution : Calculation of Mean

(Here $A = 25$ and $h = 10$)

Class interval	x_i	f_i	$u_i = \dfrac{x_i - 25}{10}$	$f_i u_i$
0 – 10	5	7	–2	–14
10 – 20	15	10	–1	–10
20 – 30	25	15	0	0
30 – 40	35	8	1	8
40 – 50	45	10	2	20
		$\sum f_i = 50$		$\sum f_i u_i = 4$

$$\text{Thus Mean} = A + \left(\frac{\sum f_i u_i}{\sum f_i}\right) \times h$$

$$= 25 + \left(\frac{4}{50}\right) \times 10$$

$$= 25 \cdot 8 \text{ metre}$$ **Answer**

Exercise 17.4

Find the mean of following frequency distribution with the help of assumed mean (Q. 1 – 4):

1.

x	800	820	860	900	920	980	1000
f	7	14	19	25	20	10	5

2.

Weight (in kgs.)	60	61	62	63	64	65
No. of workers	5	8	14	16	10	7

3.

Expenditure (in ₹)	100-150	150-200	200-250	250-300
No. of workers	24	40	33	28
Expenditure (in ₹)	300 – 350	350 – 400	400 – 450	450 – 500
No. of workers	30	22	16	7

4.

Expenditure on water (in ₹)	15-20	20-25	25-30	30-35	35-40	40-45
No. of houses	7	5	7	8	9	11
Expenditure on water (in ₹)	45-50	50-55	55-60	60-65	65-70	
No. of houses	7	5	4	4	3	

5. Find the mean of the following distribution by taking assumed mean as 25.

Class interval	0–10	10–20	20–30	30–40	40–50
f	6	10	13	7	4

6. In the following table, the age distribution of patients of a disease in a particular year is given. Find the average age (in years) per patient.

Age (in years)	5–14	15–24	25–34	35–44	45–54	55–64
No.of patient	6	11	21	23	14	5

7. Find the mean form the following frequency distribution.

Class interval	40-50	50-60	60-70	70-80	80-90	90-100
Frequency	10	25	28	12	10	15

17.10 Median :

If n values of a variable are arranged in ascending or descending order, then the value of the middle term of the arranged series is called the median. If the number of terms is odd then only one term i.e. $\left(\frac{n+1}{2}\right)$th term will be the middle term. But if the number of terms is even then two terms $\left(\frac{n}{2}\text{th}\right)$ and $\left(\frac{n}{2}+1\right)$th will be in the middle and mean of these two terms will be the median. For example, marks obtained by 9 students of the class A are 10, 15, 12, 18, 17, 18, 15, 16, 19 and marks obtained by 8 students of the class B are 19, 15, 18, 14, 17, 16, 15, 15. Arranging these in ascending order :

A : 10 12 15 15 16 17 18 18 19

B : 14 15 15 15 16 17 18 19

Median of A = Middle term (5th term) = 16 marks

Median of B = Mean of middle terms $\left(\frac{4^{th}\text{ term} + 5^{th}\text{ term}}{2}\right)$

$$= \frac{15+16}{2} = 15\cdot 5 \text{ Marks.}$$

17.11 Median from ungrouped or individual series :

Working steps :

Step I. Arranging the n values of variate x in ascending or descending order like

$x_1, x_2, x_3, \ldots, x_n$

Steps II. Now obtain the median by the following formula :

$$\text{Median } (M) = \begin{cases} \dfrac{n+1}{2}\text{th term i.e. } x_{\frac{n+1}{2}}, \text{ if } n \text{ is odd} \\ \text{Mean of the } \frac{n}{2}\text{th and } \frac{n}{2}+1\text{th terms} \\ \text{i.e. } \dfrac{x_{\frac{n}{2}} + x_{\frac{n}{2}+1}}{2}, \text{ if } n \text{ is even} \end{cases}$$

Illustrative Examples

Examples 1. Find the median from following data :

25, 34, 31, 23, 22, 26, 35, 28, 20, 32

Solution : Arranging the given data in ascending order :

S. No.	1	2	3	4	5	6	7	8	9	10
Value of variate (x)	20	22	23	25	26	28	31	32	34	35

Here, number of term $(n) = 10$ (even number)

$$\text{Therefore Median } (M) = \frac{\frac{10}{2}\text{th term} + \left(\frac{10}{2}+1\right)\text{th term}}{2}$$

$$= \frac{x_5 + x_6}{2} = \frac{26+28}{2} = 27 \quad \textbf{Answer}$$

Example 2. Find the median of the following values of variate :

37, 31, 42, 43, 46, 25, 39, 45, 32

Solution : Arranging the given data in ascending order

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
25	31	32	37	39	42	43	45	46

$\because$ 9 values of x in ascending order are $x_1, x_2, \ldots, x_9$

$$\text{Therefore Median } (M) = \left(\frac{9+1}{2}\right) \text{th term} = x_5 = 39$$

Example 3. The values of (x) in ascending order are as follows :

8 11 12 16 $16+x$ 20 25 30

if the median is 18, then find the value of x.

Solution : Here total number of variate is 8 therefore the two terms are 16 and $16+x$ respectively.

$$\text{Therefore median } = \frac{(16)+(16+x)}{2} = 18 \text{ (Given)}$$

$\Rightarrow \quad 32 + x = 36 \quad \Rightarrow x = 4$

Therefore value of $x = 4$ **Answer**

17.12 Median from ungrouped frequency distribution :

The working rule to find the median form ungrouped frequency distribution is as follows :

Working steps :

Steps I. Preparation of cumulative frequency table.

Steps II. Obtain the value of $N/2$, where $N = \sum f_i$

Steps III. Variate value of the cumulative frequency just greater than or equal to $N/2$ will be the median.

Illustrative Examples

Example : Find the median form the following frequency distribution :

x :	1	2	3	4	5	6	7	8	9
f :	8	10	11	16	20	25	15	9	6

Solution : Calculation of median

x_i	f_i	$c.f.$
1	8	8
2	10	18
3	11	29
4	16	45
5	20	65
6	25	90
7	15	105
8	9	114
9	6	120

$N = 120$

Here $\frac{N}{2} = 60.$

The term whose cumulative frequency is just greater than 60, i.e. the term for which the value cumulative frequency 65 is 5. **Answer**

Therefore median = 5

Exercise 17.5

1. Find the median of the following variates :

   25, 34, 33, 13, 20, 26, 36, 28, 19, 34

2. Find median of the following data.

   19, 25, 59, 48, 35, 31, 30, 32, 51.

   If 25 is replaced by 52, then find new median.

3. The marks obtained by students of a class are given below. Find their median.

Marks Obtained	15	20	25	30	35	40	45	50
No. of students	2	8	16	26	20	16	7	4

4. The number of children in 100 families are as follows, find their median :

No. of children	0	1	2	3	4	5	6
No. of families	10	35	27	17	6	3	2

5. Find the median of the following frequency distribution.

x	20	25	30	35	40	45	50	55
f	14	28	33	30	20	15	13	7

17.13 Median from grouped frequency distribution :

Working steps for finding median from grouped frequency distribution are as follows :

Step I. To prepare the cumulative frequency table.

Step II. After calcuating $N/2$, find the class interval of cumulative frequency just greater than it.

Step III. Now to find the median for this class interval with the help of following formula :

$$\text{Median } (M) = \ell + \left(\frac{\frac{N}{2} - C}{f_i}\right) \times h$$

Where ℓ = lower limit of the median class

N = Total frequency $\left(\sum f_i\right)$

C = Cumulative frequency of class preceding the median class

h = interval of the median class

f = frequency of the median class

This method will be clear form the following example.

Example 1. Find median of following frequencydistribution :

Class interval	**10 - 25**	**25 - 40**	**40 - 55**	**55 - 70**	**70 - 85**	**85 - 100**
f_i	**6**	**20**	**44**	**26**	**3**	**1**

Solution : Cumulative frequency table

Class interval	f_i	Cummulative frequency (c)
10-25	6	6
25-40	20	26
40-55	44	70
55-70	26	96
70-85	3	99
85-100	1	100

$N = 100$

Here, $\frac{N}{2} = 50 \Rightarrow$ class interval of median is "$40-55$"

Here, $\ell = 40,\ C = 26,\ h = 15$ and $f = 44$.

$\therefore$ The median $(M) = \ell + \frac{\left(\frac{N}{2} - c\right)}{f} \times h$

$$= 40 + \frac{(50-26)}{44} \times h$$

$$= 40 + \frac{24}{44} \times 15$$

$$= 48 \cdot 18$$

Therefore the median is $48 \cdot 18$, **Answer.**

17.14 Merits and Demerits of Median :

Merits of Median :

(i) It is the best in study of qualitative characterstics.

(ii) Finding median is easy and convenient. Sometimes it can be known by inspection.

(iii) It does not need all data for its calculation.

(iv) Median is always fixed and clear.

(v) Extreme values have no affect on it, while they have more affect on mean.

Demerits of Median :

(i) When values are distruibuted irregularly, median does not represent the representative number and gives absurd conclusion. For example a student obtained 40, 30, 5, 3, 2 marks in 5 subjects. Here the median is 5 which does not properly represent the data.

(ii) When equal importance is to be given to extreme values then the measure of central tendency is not suitable.

(iii) It can not be use in mathematical operations.

Exercise 17.6

1. The marks obtained by 100 students are given in following table. Find median from these.

Marks obtained	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
No. of students	6	20	44	26	3	1

2. The marks of students of a class are given in following frequency distribution. Find the median form these :

Marks obtained	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50
No. of students	4	28	42	20	6

Find the median form following frequency distribution : [Q. 3 to 4]

3.

Class interval	0 – 10	10 – 20	20 – 30	30 – 40
f_i	2	6	10	17
Class interval	40 – 50	50 – 60	60 – 70	70 – 80
f_i	30	15	10	10

4.

Class interval	0–8	8-16	16-24	24-32	32-40	40-48
f_i	42	30	50	22	8	5

17.15 Mode :

The value of a series whose frequency is maximum is called mode. The terms of the series have the maximum tendency to accumulate around the mode.

17.16 Calculation of Mode :

(i) Mode form Individual Series or Discrete Series

From this series we first prepare frequency distribution table. The value with maximum frequency is called mode. This can be easily understood by the following example.

Marks obtained	0	1	2	3	4	5
No. of students	5	8	13	5	3	2

Here it is clear form frequency distribution that frequency of 2 is maximum i.e. 13, thus mode of distribution is 2 marks.

If distribution of frequencies are irregular or value of maximum frequencies are more than one, then it becomes difficult to find mode. In such situation mode is determined by the method of grouping. Here we shall study only regular distributed frequency distribution.

(ii) Mode from ungrouped frequency distribution :

Here in regular frequency distribution, the value whose frequency is maximum, the value of that term is mode.

Example : The marks obtained by some students are as follows, find their mode.

Marks obtained	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
No. of students	**1**	**5**	**15**	**16**	**20**	**19**	**15**	**8**	**7**	**3**	**2**

Solution : Here ferquency of marks 34 is maximum, i.e. 20.

Therefore Mode = 34 Marks

(iii) Mode from grouped frequency distribution :

Working rule to find mode from gruoped frequency distribution is as follows :

Step I. The class of grouped frequency distribution having maximum frequency is called the modal class interval. First we find modal class interval.

Step II. With the help of modal class we find the mode by using the followig formula :

$$\text{Mode} = \ell + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2} \right) \times h$$

where $\ell =$ lower limit of modal class

$f_1 =$ frequency of modal class

$f_0 =$ frequency of the class interval just preceeding the modal class interval

$f_2 =$ frequency of the class interval just after the modal class interval

$h =$ width of the modal class

Illustrative Examples

Example 1. Find mode from the following frequency distribution.

Class	**10-25**	**25-40**	**40-55**	**55-70**	**70-85**	**85-100**
f_i	**6**	**20**	**44**	**26**	**3**	**1**

Solution : Here maximum frequency is 44 of class '40-55'

So Mode class = 40 – 55

Again $\ell = 40,\ f_1 = 44,\ f_0 = 20,\ f_2 = 26$ and $h = 15$

$$\text{Mode according to formula} = \ell + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2} \right) \times h$$

$$= 40 + \left(\frac{44 - 20}{88 - 20 - 26} \right) \times 15$$

$$= 48 \cdot 57$$

Therefore required Mode = 48·57 **Answer**

Exercise 17.7

1. Find the mode of the following distribution :

(i)	2	5	7	5	3	1	5	8	7	5
(ii)	2	4	6	2	6	6	7	8		
(iii)	2·5	2·5	2·1	2·5	2·7	2·8	2·5			

2. Find the Mode of following frequency distribution :

(i)

x	3	4	5	6	7	8
f	2	4	6	3	2	1

(ii)

x	1·1	1·2	1·3	1·4	1·5	1·6
f	20	50	80	60	15	8

3. The number of members of 30 families of a village is given in the following table. Find their Mode.

No. of members	2	3	4	5	6	7	8
No. of families	1	2	4	6	10	3	5

4. The Ages (in years) of 20 students of a class are as follows :

15	16	13	14	14	13	15	14	13	13
14	12	15	14	16	13	14	14	13	15

Find mode by representing these in frequency distribution.

5. The marks obtained by some students are given below, find the mode of marks obtained :

Marks obtained	10	20	30	40	50	60	70	80
No. of students	2	8	16	26	20	16	7	4

Find the mode form following frequency distribution :

6.

Class	10-15	15-20	20-25	25-30	30-35	35-40	40-45
Frequency	3	7	16	12	9	5	3

7.

Marks obtained	0-10	10-20	20-30	30-40	40-50	50-60
No. of students	5	12	14	10	8	6

8.

Marks obtained	20-30	30-40	40-50	50-60	60-70
No. of students	4	28	42	20	6

9.

Height (in cm)	52-55	55-58	58-61	61-64
No. of students	10	20	25	10

Important Points

1. Arithmetic mean $(\bar{x})$:

(i) Individual series : $(\bar{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i$

(ii) Ungrouped distribution : $(\bar{x}) = \frac{\sum f_i u_i}{\sum f_i}$

(iii) By assumed mean : $\bar{x} = A + \frac{\sum f_i d_i}{\sum f_i}$ or $\bar{x} = A + \frac{\sum f_i u_i}{\sum f_i} \times h$

where assumed mean A, $d_i = x_i - A$ and $u_i = \frac{x_i - A}{h}$

2. Median (M)

(i) *Individual series :* Arranging values in ascending or descending order as $x_1, x_2, x_3, \ldots, x_n$

$$\text{Median }(M) = \begin{cases} x_{\frac{n+1}{2}}, & \text{if } n \text{ is odd number} \\ \frac{x_{\frac{n}{2}} + x_{\frac{n}{2}+1}}{2}, & \text{if } n \text{ is even number} \end{cases}$$

(ii) *Ungrouped frequency distribution :* The value from cumulative frequency table cumulative frequency is just greater than $N/2$.

(iii) *Grouped frequency distribution :* The class interval whose frequency is just greater than $N/2$ is median class interval and

$$\text{Median }(M) = \ell + \left(\frac{\frac{N}{2} - C}{f}\right) \times h$$

where ℓ = lower limit of median class interval

$N = \sum f_i$ i.e. Total frequencies

C = C.F. of class interval preceding the median class

h = width of the median class

f = frequency of the median class

3. Mode :

(i) *Individual series* : The term whose frequency is maximum.

(ii) *Ungrouped frequency distribution* : Value of the term with maximum frequency.

(iii) *Grouped frequency distribution* : The class of maximum frequency is called modal class.

and mode $(z) = \ell + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right) \times h$

where ℓ = lower limit of modal class

f_1 = frequency of modal class

f_0 =	frequency of the class interval just preceding the modal class interval
f_2 =	frequency of the class interval just after the modal class interval
h =	width of middle class

Miscellaneous Exercise 17

Four possible options of the following questions are given. Choose the correct option : [1-10]

1. Positional mean is :
   (A) Arthmetic mean (B) Geometric mean
   (C) Harmonic mean (D) Median
2. Mode value of any series is :
   (A) Middle value (B) Value whose frequency is maximum
   (C) Minimum frequency value (D) Limit value
3. The median of following series is :
   520, 20, 340, 190, 35, 800, 1210, 50, 80
   (A) 1210 (B) 520 (C) 190 (D) 35
4. The marks obtained by four students in statics are 53, 75, 42, 70. Mean of their marks is :
   (A) 42 (B) 64 (C) 60 (D) 56
5. A student secured 85, 87 and 83 marks in Mathematics, Physics and Chemistry respectively. The average marks of these subject is :
   (A) 86 (B) 84 (C) 85 (D) 85·5
6. If the arithmetic mean of 5, 7, 9, x is 9, then the value of x is :
   (A) 11 (B) 15 (C) 18 (D) 16
7. The median of the distribution 2, 3, 4, 7, 5, 1 is :
   (A) 3 (B) 4 (C) 2 (D) 20
8. The median of the distribution 1, 3, 2, 5,. 9 is :
   (A) 3 (B) 4 (C) 2 (D) 20
9. The mode of the distribution 3, 5, 7, 4, 2, 1, 4, 3, 4
   (A) 7 (B) 4 (C) 3 (D) 1
10. The no.of students of a school according to their ages are as follows :

Age (in years)	8	9	10	11	12	13	14	15	16	17
No. of students	15	25	40	36	41	37	20	13	5	3

Their mode will be :
(A) 41 (B) 12 (C) 3 (D) 17

Find the airthmetic mean of the following distribution [Q. 11-14]

11.

x	5	6	7	8	9
f	4	8	14	11	3

12.

x	10	15	17	20	22	30	35
f	5	10	2	8	3	6	6

13.

x	19	21	23	25	27	29	31
f	13	15	16	18	16	15	13

14.

x	1	2	3	4	5	6
f	45	25	19	8	2	1

15. Find the arithmetic mean from following frequency distribution :

Weight (in kg)	40-44	44-48	48-52	52-56	56-60	60-64
No. of persons	5	6	5	9	3	2

Find median of the following distribution : [Q. 16-17]

16.

x	0·1	0·2	0·3	0·4	0·5	0·6	0·7
f	30	60	20	40	10	50	35

17.

Measure of shoes	4·5	5·0	5·5	6·0	6·5	7·0	7·5	8·0	8·5	9·0
No. of shoes	1	2	4	5	15	30	60	95	82	75

18. The run scored by players of a cricket tean is as follows :

57, 17, 26, 91, 115, 26, 83, 41, 57, 0, 26.

Find their A.M. median and mode.

Find mode of following distributions : [Q. 19-20]

19.

20.

Class	0 – 20	20 – 40	40 – 60	60 – 80	80 – 100
Frequency	3	15	24	8	5

21. Define Arithmetic mean and given their two demerits.
22. Give main uses of median.
23. Give difference between arithmetic mean and weighted arithmetic mean.

Answers

Exercise 17.1

1. 52 Marks 2. ₹1735 3. 6 4. 43.9 run 5. 22 Marks 6. 400 Books
7. 52.75 kg. 8. 26 9. Mean 24 +3 10. ₹ 2200

Exercise 17.2

1. 7·07 2. 7·55 3. 0·34 4. 0·55 5. 2 6. 23·9 7. 3 8. 76 व 38

Exercise 17.3

1. 26·33 approx 2. 15·45 3. 145·71 4. 49·5 5. 68·2 6. 1457·14

Exercise 17.4

1. 891·2 2. 62·65 3. 266·25 4. 39·57 5. 23·25 6. 34·87 7. 68·2

Exercise 17.5

1. 27 2. 32 *and* 35 3. 30 4. 2 5. 35

Exercise 17.6

1. 45·45 2. 24·29 3. 45 4. 17·04

Exercise 17.7

1. (i) 5 (ii) 6 (iii) 2·5	2. (i) 5 (ii) 1·3	3. 6	4. 14	5.40
6. 23·46	7. 23·33	8. 43.89	9. 58·75	

Miscellaneous Exercise 17

1. (D)	2. (B)	3. (C)	4. (C)	5. (C)	6. (B)	7. (D)
8. (A)	9. (B)	10. (B)	11. 7·025	12. 21·25	13. 25	14. 2
15. 50·67	16. 0.4	17. 8	18. 49, 41 and 26	19. 26	20. 47·2	
21.	22.	23.	24.	25.61.5 and 62.4		

❑

18

Probability

18.01 Introduction

In our daily life many such incidents take place which have more than one result. It is natural that everyone has a curiosity to know the result. Persons try to take advantage by anticipating the result of these incidents. The science of knowing the result of an incident on the basis of prior information and circumstances is called probability.

The theory of probability was first propounded in the 17^{th} century in Europe. The gamblers and match fixers made attempts to know the results of their respective games before-hand in order to have maximum advantages. These people put this problem before their contemporary Mathematicians like Galilio, Pascal, Farma Kardeno etc. These mathematicians developed certain mathematical methods to solve these problems, and consequently this branch of Mathematics came to exit. Prominent mathematicians of the 18^{th} and 19^{th} centuries: Laplace, Gauss, Bernouli, etc. developed this principal further.

In the 20^{th} century, sampling theory, decision theory etc. based on probability theory were developed and credit of these got to R.S. Fisser and Karl Pearson.

In modern times the theory of probability is applied in various fields where decisions pertaining to future have to be taken. For example, in preparing the budget of any state or country, theory of probability is used. Insurance companies prepare death tables and make inference as to how long a person of particular age group in likely to survive and defence experts frame their strategies with the help of this theory. Many important policies in the fields of society, state administration, commerce and science are determined broadly on the basis of probability. First of all, we will attempt to define certain important terms used in the study of probability.

18.02 Some Definitions

1. **Random Experiment :** When all the possible result of an experiment are already known and no inference of any particular result is possible, it is called a random experiment.

   For example, the two result of tossing of a coin, either head or tail are already known. No definite result can be forecast, therefore toss of a coin is a random experiment.

2. **Trial and Event :** When out of many possible results (outcomes) of a random experiment is called a trial and the possible results are called events. For example:

   (i) Tossing of a coin is trial and getting head (H) or tail (T) are events.

   (ii) Throwing a dice is trial and getting any of 1, 2, 3, 4, 5 and 6 is event.

   (iii) Appearing in the examination of a candidate is trial and pass or fail is event.

3. **Simple Event :** When in a trial, only one event takes place at a time, it is a simple event.

   For example : Drawing a ball from a bag containing a few black and white balls is a simple event.

4. **Exhaustive Events or Total Number of Cases :** All possible results of a trails are called Exhaustive events or total number of cases of that trail.

   For Example :

(i) Tossing of a coin is a trial and head or tail can occur. So in this trail exhaustive events are 2.

(ii) On throwing of a dice 1, 2, 3, 4, 5 or 6 can occur so in this trail exhaustive events are 6.

5. Favourable events or cases : The number of cases favourable to a particular event in a trial is the number of outcomes which entail the happening of the particular event. For example :

(i) In throwing a die, the number of cases favourable to getting an even number is 2 ,4, 6 i.e., 3.

(ii) On drawing two cards from a pack of card, the number of cases favourable to getting king is 6.

(ii) In throwing of two dice, the number of cases favourable to getting a sum 5 is (1,4), (4, 1), (2,3), (3, 2), i.e. 4.

6. Independent and dependent events :

(i) Independent events : Two or more events are called independent events if the happening or not happening of any one does not depend on the happening or non happening of the other. For example, on tossing a coin and throwing a dice, the outcomes getting head on coin and 4 on dice are independent events.

(ii) Dependent event : Two or more events are called dependent events if the happening of any one does depend on the happening of the other. For example : A card drawn from an ordinary pack of card should be a heart card , after that without replacing it in pack, again a drawn card should be a spade card, both are dependent events.

7. Mutually exclusive or disjoint events : Two or more events are said to be mutually exclusive or disjoint events if no two or more occur simultaneously in the same trial i.e., if the occurrence of any one of them prevents the occurrence of all others. For example :

(i) On tossing of a coin occurring of head or tail are mutually exclusive events.

(ii) A card is drawn from a pack of card, it being a king or a queen are mutually exclusive events.

8. Equally likely events : If in an experiment, possibility of happening of all events is same then such events are called equally likely events. For example likely events.

(i) In tossing a coin, getting of head or tail are equally

(ii) In drawing a card from a pack of card it will be red or black card, is equally likely events.

9. Compound events : If two or more events happen at a time then they are, called compound events or joint event.

For example : In two bags, there are some blue and some red balls. Selection of a beg and then drawing a ball from it is a compound event because selection of one bag from two bags and then a ball is drawn from selected bag is happening at a time.

10. Sample point and sample space : Each outcome of a trial is called sample point and set of all sample point of a trial is called its sample space. It is generally denoted by S.

(i) The sample point in tossing of two coins are

(H,H), (H,T), (T,H), (T,T)

and S = {(H,H), (H,T), (T,H), (T,T)} is sample space.

(ii) Two children are selected from 3 boys and 2 girls. The sample space of this trial is (boys B_1, B_2, B_3, girls G_1, G_2) :

$S = \{B_1B_2, B_2B_3, B_3B_1, B_1G_1, B_1G_2, B_2G_1, B_2G_2, B_3G_1, B_3G_2, G_1G_2\}$

18.03 Mathematical definition of Probability :

If the results of a trial *n* are equally likely, mutually exclusive and exhaustive cases and m of them are favourable to the happening of an event A, then the probability of A is defined as the ratio *m/n* and is denoted by P(A).

Thus $P(A) = \dfrac{\text{favourable cases of A}}{\text{exhaustive cases of A}} = \dfrac{m}{n}$, (numerical measure)

If in a trial, happening of event A is sure then $m = n$ and

$$P(A) = \frac{n}{n} = 1,$$

If happening of event A is impossible then $m = 0$ and

$$P(A) = \frac{0}{n} = 0,$$

Therefore, for any event A, $0 \leq P(A) \leq 1$

i.e., Probability of any event can not be less than 0 and greater than 1 and limit of probability is from 0 to 1. Probability of non happening of event A is denoted as $P(\bar{A})$

So, $P(\bar{A}) = \dfrac{\text{unfavourable cases of event A}}{\text{exhaustive cases of event A}} = \dfrac{n-m}{n} = 1 - \dfrac{m}{n}$

$\Rightarrow \quad P(\bar{A}) = 1 - P(A)$

18.04 Notation :

(1) P(A) = Probability of happening an event A

(2) P(A) = Probability of non happening an event A.

Illustrative Examples

Example 1 : Find the probability of getting in throwing an a dice.

Solution : In a throw of a dice, 6 types of numbers can occur. Hence the number of exhaustive events = 6, even number 2, 4, 6 will occur for the required event, which is in number 3. So number of favourable events = 3.

$\therefore$ Required probability $= \dfrac{3}{6} = \dfrac{1}{2}$

Example 2 : In a single throw of two dice, determine probability of getting a total of 7.

Solution : On throwing two dice 6x 6 = 36 output can be obtained. So exhaustive cases for required event = 36

The following pairs are possible for a total of 7.

(1,6), (2,5), (3,4), (4,3), (5,2), (6,1) which number is 6.

So favourable number of events = 6

$\therefore$ Required probability $= \dfrac{6}{36} = \dfrac{1}{6}$

Example 3 : Find the probability that a leap year, selected at random, will contain 53 Mondays.

Solution : We know that in a leap year there are 366 days. So 52complete week and 2 days are remaining. The seven possibilities of these 2 days are as follows :

(1) Sunday and Monday (2) Monday and Tuesday (3) Tuesday and Wednesday

(4) Wednesday and Thursday

(5) Thursday and Friday

(6) Friday and Saturday

(7) Saturday and Sunday.

So exhaustive cases for required event = 7

Out of seven possible cases in two cases, Monday occurs.

So favourable cases for required event = 2

$\therefore$ Required probability $= \dfrac{2}{7}$

Example 4 : From 12 tickets marked 1 to 12, if one ticket is selected at random. Find the probability that the number on it is a multiple of 2 or 3.

Solution : The multiple of 2 or 3, in number 1 to 12 is 2, 3, 4, 6,8 9 10, 12

So out of 12 equally likely cases 8 are favourable.

$\therefore$ Required probability $= \dfrac{8}{12} = \dfrac{2}{3}$

Example 5 : A coin is tossed once. Find the probability of getting tail.

Solution : In tossing a coin possible outcome are two, head (H) and tail (T). Let E be event of getting tail, then favourable event for E is 3,

$$\text{So,} \quad P(E) = P(T) = \frac{\textit{Favourable cases} \text{ of events E}}{\textit{Total} \text{ no. of cases}} = \frac{1}{2}$$

Example 6 : A bag contains one white ball, one black ball and one red ball of same size. Savita taken out one ball without looking in the bag. What will be the probability that drawn ball is red?

Solution : Drawing a ball by Savita is equally likely event and total number of balls (events) is 3. Let drawing red ball is event R, then favourable outcome of the event R is 1.

$$\text{So,} \quad P(R) = \frac{1}{3}$$

Example 7: A dice is thrown once. What will be the probability that the number on the dice will be 5 or less than 5.

Solution : In throwing a dice possible outcomes are 1, 2, 3, 4, 5, 6. Let to obtain the number less than or equal to 5 is event E than favourable out comes will be 1, 2, 3, 4, and 5.

$$\therefore \quad P(E) = \frac{5}{6}$$

Example 8 : A card is drawn at random form a well-shuffled deck of 52 cards. Find the probability of its being a king.

Solution : In a deck of 52 cards there are 4 kings. Let a king card is an event E, so number of favourable events is 4. Total number of events are 52.

$$\therefore \quad P(E) = \frac{4}{52} = \frac{1}{13}$$

Example 9 : Two players Ram and Shyam plays a chess match. It is given that probability of winning the match by Ram is $\frac{4}{5}$. What will be the probability of winning the match by Shyam?

Solution : Let R and S are the event of winning the match by Ram and Shyam respectively.

$\therefore$ Probability of winning Ram = P (R) = $\frac{4}{5}$

$\therefore$ Probability of winning Shyam = P (S) = 1 - P (R)

$$= 1 - \frac{4}{5}$$

$$= \frac{1}{5}$$

Example 10 : A coin is tossed twice. Find the probability of getting at least one head.

Solution : Possible outcomes in tossing a coin twice are (H,H), (H,T), (T,H), (T,T). Let E be the event of getting at least one head. Therefore the favourable outcomes are (H,T), (T,H), (T,T).

Total number of favourable cases = 3.

$\therefore$ P(E) $= \frac{3}{4}$

Example 11 : Two dices are thrown together. What is the probability that sum of two numbers on the faces is 7 ?

Solution : On throwing two dice together, the possible 36 outcomes are -

{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5) (6,6)}

Let E be the event when sum on the faces is 7.

Favourable events are = {(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)}

Total number of favourable events = 6

$\therefore$ P (E) $= \frac{6}{36} = \frac{1}{6}$

Exercise 18.1

1. In a throw of a die, determine the probability of getting a number more than four.
2. A coin is tossed twice. Find the probability of getting tails both times.
3. One number is selected at random from natural number from 1 to 17. Find the probability that the number is prime.
4. Find the probability of throwing head or tail alternatively in 3 successive tossing of a coin.
5. Find the probability that a non-leap year should have only 52 Sunday.
6. If P (A) = 0.65 than what will be the probability of "not A".
7. On tossing two coins find the probability of getting at most one tail.
8. A dice is thrown twice. What will be the probability that the sum on the faces is-

   (i) 9 (ii) 13
9. A bag contains 5 red and 3 white balls. From this bag one ball is drawn randomly. What will be the probability that the drawn ball is

   (i) White (ii) Not White

10. Due to some reason 12 faulty pens are mixed in 132 good pens. By inspection are can not judge that which pen is faulty. If one pen is drawn at random then what is the probability that this pen is good one?

11. A card is drawn from a well shuffled deck of 52 card. Find the probability of getting the following

(i) Jack of red colour (ii) Card of red colour (iii) Ace of heart

(iv) Queen of diamond (v) Card of spade

IMPORTANT POINTS

1. Trail and event: In any reference out of all the possible result (out comes) of a random experiment is called a trial and the possible results are called events.

2. Exhaustive events or total number of cases: All possible results of a trials are called exhaustive events or total number of cases of that trail.

3. Favourable events or cases: The numbers of cases favourable to a particular event in a trial is the number of outcomes which entail the happening of the particular event.

4. Probability :

Probability of happening event A is

$$P(A) = \frac{\text{favourable cases of A}}{\text{exhaustive cases of A}} = \frac{m}{n}$$

Probability of not happening the event A is

$$P(\bar{A}) = \frac{\text{unfavourable cases of event A}}{\text{exhaustive cases of event A}} = \frac{n-m}{n} = 1 - \frac{m}{n}$$

or $P(\bar{A}) = 1 - P(A)$

or $P(A) + P(A) \leq 1$

Limit of Probability is $0 \leq P(A) \leq 1$

Answers

Excercise 18.1

(1) $\frac{1}{3}$ (2) $\frac{1}{4}$ (3) $\frac{7}{17}$ (4) $\frac{1}{4}$ (5) $\frac{6}{7}$ (6) 0.35 (7) $\frac{1}{2}$

(8) (i) $\frac{1}{9}$ (ii) 0 (9) (i) $\frac{3}{8}$ (ii) $\frac{5}{8}$ (10) $\frac{11}{12}$

(11) (i) $\frac{1}{52}$ (ii) $\frac{1}{2}$ (iii) $\frac{1}{52}$ (iv) $\frac{1}{52}$ (v) $\frac{1}{4}$

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Road Safety Education

Arithmetic Progression

Objective :

Forming Arithmetic Progression with the time taken and distance covered while crossing traffic signals.

Content :

In an arithmetic progression we deal with the sequence and series of numbers which can include distances and time as well. As the distance from one place to another is covered by road also may be by car, or any other light vehicles or heavy vehicle. So an A.P. can be formed by time taken and the distance covered.

Exercise :

Distance from A to B is 150 Km and has 10 traffic signals. If a car travels at a uniform speed of 60 Km/hr. With all green signals it reaches pt. B in 2 hrs. 30 mins; but on another day on heavy traffic day it gets a stoppage as follows :

First signal - 1 min.

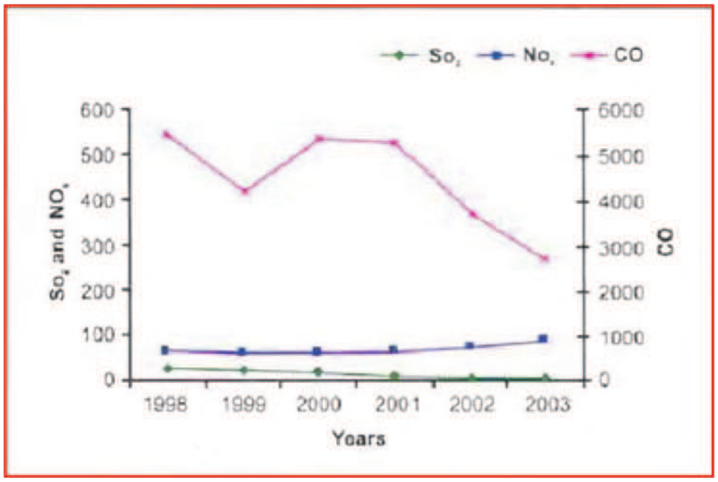
Second signal - 2 min. ....Till Tenth signal .... 10 min.

Calculate the total time taken by the same car if it is obeying all traffic signals (ignore other hurdles) and moves with the same speed of 60 Km/hr.

Data Collection

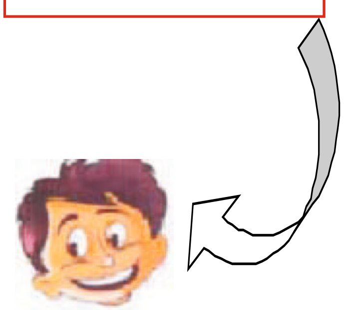
Objective :

Control of vehicular pollution is necessary. Measures adopted to reduce pollution are emphasized.



The above graph shows concentration of major atmospheric pollutants. In which year was decrease of major pollutants noted? What do you attributed this to?

Do you know that a Pollution Under Control (PUC) certificate is mandatory for every vehicle.



Pollution Control Certificate (P.U.C.)

Applications of Trigonometry

Objective :

Application of trigonometry on road in light of increasing traffic and accidents.

Content :

As the heights and distances deals with the measurement of heights or distances of the objects like towers, buildings etc., this can be related to the traffic and road as the traffic on roads is increasing and the number of accidents is increasing.

Exercise :

A CCTV Camera is to be placed on top of an erect pole of height 13m such that it captures footages of all the vehicles passing beyond the line of sight if viewed from top such that the line of sight is of measure 12m. In such a case

(1) What is the distance from the feet of pole beyond which the traffic on road is visible?

(2) What is the area around the pole that has to turned into a green patch?

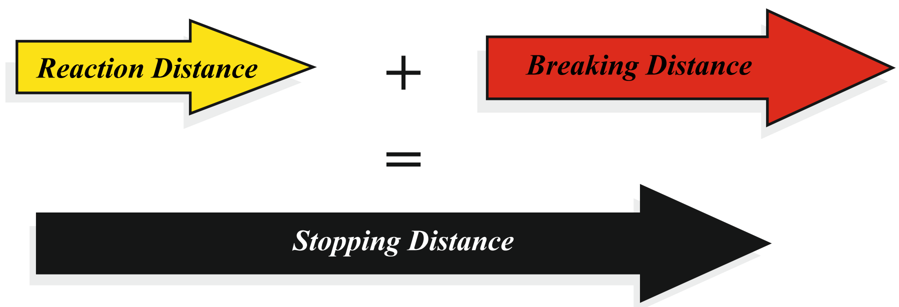
(3) Do you think CCTV cameras are really useful to manage the traffic sense? If yes, then How?

Problems on Two Varibles

Objective :

Problem related to road scenarios are used to solve equations.

Stopping Distance = Reaction Distance + Breaking Distance



A car moves with a speed of 50km/hr.

If the stopping distance = 40 m and retardation is 4.4 m/sec^2 calculate reach time.

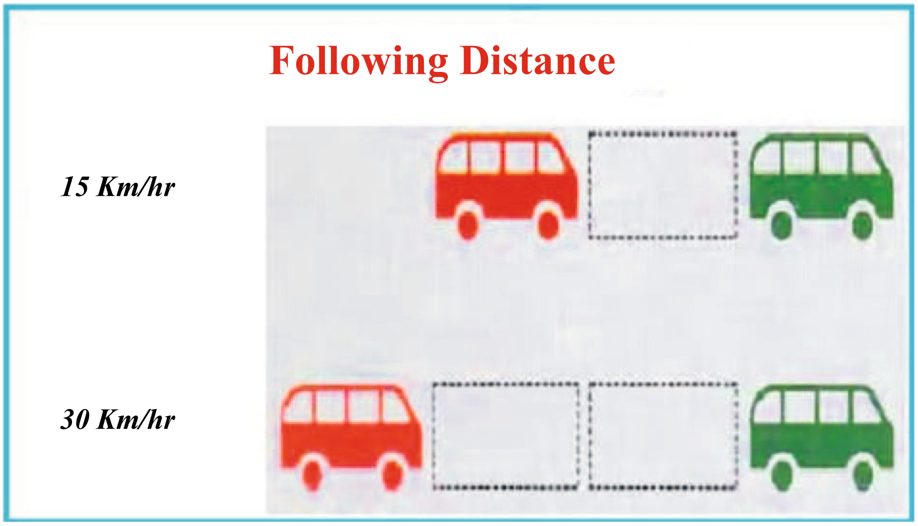
1. Will the stopping distance change with the speed of vehicles?
2. Howe will it change on wet slippery road?

Following Distance :

How much distance in seconds you should keep in following the vehicle in front of you, can be calculated with reference to speed, stopping distance and reaction time.

A simple way of calculating seconds is suggested as :

Firstly you could count in a rhythmic manner 19-20-21 i.e. nineteen, twenty, twenty one and normally each rhythmic count is equal to one second.



How much distance in seconds must be kept between you and the vehicle you are following?

This is calculated as :

Speed (Km/hr)	Total Stopping Distance (m)	Reaction Distance (m)	Following Distance (sec.)
(i)	(ii)	(iii)	(iv)
30	18	9	2
60	54	18	-
90	108	-	4

Find the missing values.