

Standard X

MATHEMATICS

Part-1



Government of Kerala
Department of Education

State Council of Educational Research and Training (SCERT)
2016

THE NATIONAL ANTHEM

Jana-gana-mana adhinayaka, jaya he
Bharatha-bhagya-vidhata.
Punjab-Sindh-Gujarat-Maratha
Dravida-Utkala-Banga
Vindhya-Himachala-Yamuna-Ganga
Uchchala-Jaladhi-taranga
Tava subha name jage,
Tava subha asisa mage,
Gahe tava jaya gatha.
Jana-gana-mangala-dayaka jaya he
Bharatha-bhagya-vidhata.
Jaya he, jaya he, jaya he,
Jaya jaya jaya, jaya he!

PLEDGE

India is my country. All Indians are my brothers and sisters.

I love my country, and I am proud of its rich and varied heritage. I shall always strive to be worthy of it.

I shall give respect to my parents, teachers and all elders and treat everyone with courtesy.

I pledge my devotion to my country and my people. In their well-being and prosperity alone lies my happiness.

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Dear children,

Mathematics starts in counting and measuring. In the age of agriculture, it becomes the second degree equations of areas; rises to astronomy for weather prediction. Grows into the branch of mathematics called trigonometry. In Renaissance Europe, trigonometry forms the foundation of navigation. It becomes the basis of locating places using satellites in today's world. The mathematical principles which seventeenth century mathematicians developed as purely mathematical operations of numbers are now used to make security systems in e-transactions. I wish all of you would recognize the innumerable applications of mathematics and revel in its theoretical rhythms.

With love and regards

Dr. P. A. Fathima
Director, SCERT

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Certain icons are used in this textbook for convenience



Computer Work



Additional Problems



Project



Self Assessment



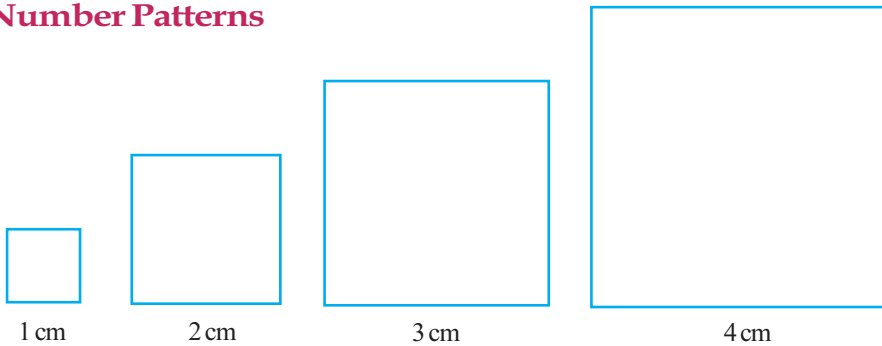
For Discussion



Arithmetic Sequences



Number Patterns



See the squares. What are their perimeters?

And areas?

As the lengths of the sides go

1 cm, 2 cm, 3 cm, 4 cm, ...

the perimeters are

4 cm, 8 cm, 12 cm, 16 cm, ...

And the areas

1 sq.cm, 4 sq.cm, 9 sq.cm, 16 sq.cm, ...

Let's look at the numbers alone.

The lengths of the sides are just the natural numbers, written in order;

1, 2, 3, 4, ...

The perimeters form the multiples of 4, in order

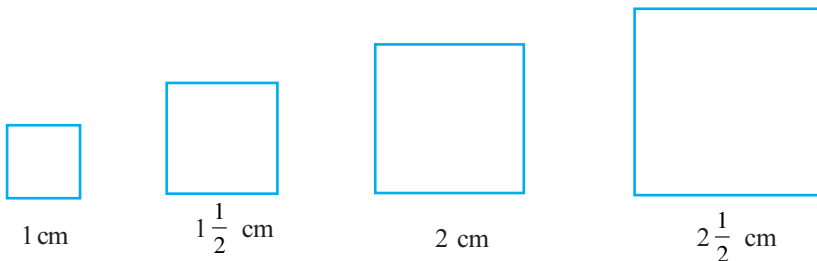
4, 8, 12, 16, ...

And the areas form the perfect squares in order

1, 4, 9, 16, ...

What about their diagonals? Write those numbers also.

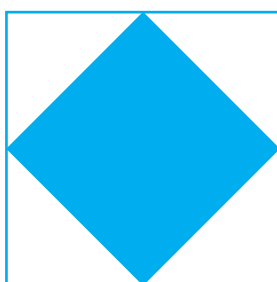
How about increasing the lengths of sides in steps of half a centimetre instead of one centimetre?



Sides	1,	$1\frac{1}{2}$,	2,	$2\frac{1}{2}$,	...
Perimeter	4,	6,	8,	10,	...
Area	1,	$2\frac{1}{4}$,	4,	$6\frac{1}{4}$,	...
Diagonal	$\sqrt{2}$,	$\frac{3}{2}\sqrt{2}$,	$2\sqrt{2}$,	$\frac{5}{2}\sqrt{2}$,	...

A set of numbers written like this, as the first, second, third and so on, is called a *sequence*.

We can make another sequence with squares. Imagine a square of side 1 metre. Joining the midpoints of the sides, we get another square:



1 metre

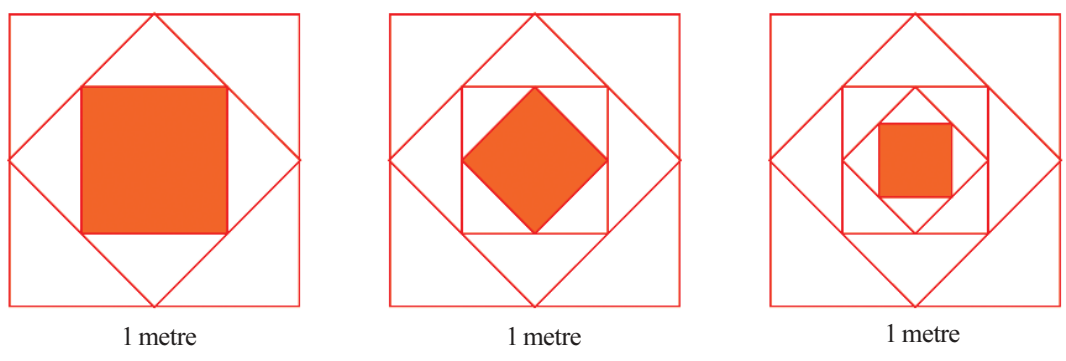
What is the area of this smaller square?

Its diagonal is 1 metre; and we know that the area of a square is half the square of its diagonal (The section, **Rhombus** of the lesson **Areas of Quadrilaterals**, in the Class 8 textbook).

So, the area of the small square is half a square metre.



Continuing, the area is halved each time:



What number sequence do we get from this?

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$$

We get sequences from physics also. The speed of an object falling from a height increases every instant. If the speed at t seconds is taken as v metres per second, the time–speed equation is

$$v = 9.8t$$

If the distance travelled in t seconds is taken as s metres, then the time–distance equation is

$$s = 4.9t^2$$

So, we get two sequences from this:

Time	1,	2,	3,	4,	...
Speed	9.8,	19.6,	29.4,	39.2,	...
Distance	4.9,	19.6,	44.1,	78.4,	...

We can form sequences from peculiarities of pure numbers, instead of numbers as measures. For example, the prime numbers written in order gives the sequence

$$2, 3, 5, 7, 11, 13, \dots$$

The digits in the decimal form of $\frac{21}{37}$, written in order is the sequence

$$5, 6, 7, 5, 6, 7, 5, 6, 7, \dots$$

If we take π instead, we get this sequence:

$$3, 1, 4, 1, 5, 9, 2, 6, \dots$$



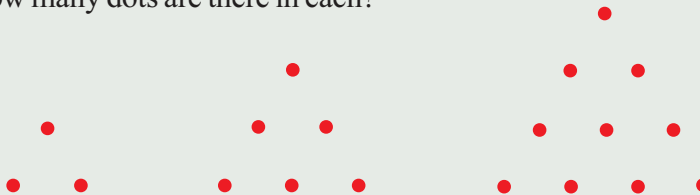
The same sequence can be described in different ways. For example, this is the sequence of natural numbers ending in 1:

$$1, 11, 21, 31, \dots$$

We can also say that this is the sequence of natural numbers which leave remainder 1 on division by 10.



- (1) Look at these triangles made with dots.
How many dots are there in each?



Compute the number of dots needed to make the next two triangles.

- (2) Make the following number sequences, from the sequence of equilateral triangles, squares, regular pentagons and so on, of regular polygons:
- | | |
|------------------------|--------------|
| Number of sides | 3, 4, 5, ... |
| Sum of interior angles | |
| Sum of exterior angles | |
| One interior angle | |
| One exterior angle | |
- (3) Write down the sequence of natural numbers leaving remainder 1 on division by 3 and the sequence of natural numbers leaving remainder 2 on division by 3.
- (4) Write down the sequence of natural numbers ending in 1 or 6 and describe it in two other ways.
- (5) One cubic centimetre of iron weighs 7.8 grams. Write as sequences, the volumes of weights of iron cubes of sides 1 centimetre, 2 centimetres and so on.

Algebra of sequences

We have seen that the perimeters of squares of sides 1 centimetre, 2 centimetres, 3 centimetres and so on, form the sequence

$$4, 8, 12, \dots$$

The numbers forming a sequence are called its *terms*.

Thus 4, 8, 12, ... are the terms of the above sequence. More precisely, 4 is the first term, 8 is the second term, 12 is the third term and so on.

We can write this as given below:

Position	1,	2,	3,	...
Term	4,	8,	12,	...

What is the 5th term? The 20th term?

What is the relation between the positions and the terms?

Each term is four times its position.

Using a bit of algebra, we can put it like this:

The n^{th} term of the sequence is $4n$

Usually the terms in a sequence are written in algebra as

x_1, x_2, x_3, \dots or y_1, y_2, y_3, \dots

So, we can shorten the above sequence rule further

$$x_n = 4n$$

When we take the natural numbers 1, 2, 3, ... as n , we get all the terms of the sequence as

$$x_1 = 4$$

$$x_2 = 8$$

$$x_3 = 12$$

...

We can compute the 100th term directly as

$$x_{100} = 400$$

The sequence of areas is

1, 4, 9, 16, ...

What is the relation between a term and its position here?

Each term is the square of its position.



We can draw sequences of polygons with a common side using GeoGebra.

Mark the points A, B and type

Sequence [Polygon [A, B, n], n, 3, 10]

in the Input Bar. This command means, like the numbers from 3 to 10 as n and draw the regular polygon of n sides with AB as a side.

To draw the polygon one by one make an Integer Slider m and change the command to

Sequence [Polygon [A, B, n + 2], n, 1, m]

This means, draw polygons of 3, 4, 5, ... sides as we move the slider to change m through, 1, 2, 3, ...

If instead of $n + 2$, we type $2n$ in the above command, what sort of polygons do we get? What if we type $2n + 1$ instead?

How about putting this in algebra?

$$x_n = n^2$$

We can also write the lengths of diagonals of these squares as a sequence. How do we put it in algebra?

Let's look at the sequences got on increasing the sides by half centimetre steps.

Side	1,	$1\frac{1}{2}$,	2,	$2\frac{1}{2}$,	...
Perimeter	4,	6,	8,	10,	...
Area	1,	$2\frac{1}{4}$,	4,	$6\frac{1}{4}$,	...
Diagonal	$\sqrt{2}$,	$\frac{3}{2}\sqrt{2}$,	$2\sqrt{2}$,	$\frac{5}{2}\sqrt{2}$,	...

How can we find the algebraic expression for the sequence of the lengths of sides?

First, let's write it like this:

$$1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, \dots$$

There are both natural numbers and fractions in it; and all the fractions have the denominator 2. How about writing the whole numbers also as fractions with denominator 2?

$$\frac{2}{2}, \frac{3}{2}, \frac{4}{2}, \frac{5}{2}, \frac{6}{2}, \frac{7}{2}, \dots$$

The sequence of numerators is

$$2, 3, 4, 5, 6, 7, \dots$$

What is the algebraic expression for this sequence? Try it!

So, how do we write the sequence of sides in algebra?

Writing the length of a side of the n^{th} square as s_n , we find

$$s_n = \frac{n+1}{2}$$

We can also write it like this

$$s_n = \frac{1}{2} (n+1)$$



What about the sequence of perimeters?

Perimeter is four times the length of a side. So, the algebraic expression for the sequence of perimeters is

$$p_n = 4 \times \frac{1}{2} (n + 1) = 2 (n + 1)$$

For example, the length of a side of the 25th square is

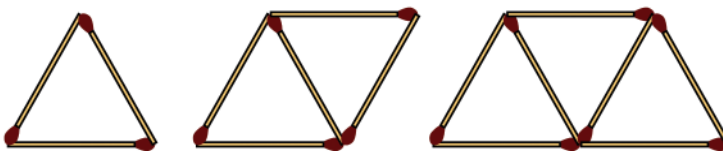
$$s_{25} = \frac{1}{2} \times (25 + 1) = 13$$

and the perimeter of the 50th square is

$$p_{50} = 2 \times (50 + 1) = 102$$

Like this, can't you write the algebraic expressions for the sequence of areas and the sequence of diagonals?

Let's look at another problem.



Three matchsticks to make one triangle, five to make two triangles, seven to make three triangles.

How many matchsticks are needed to make four triangles?

Five triangles?

Three matchsticks to make the first triangle. Thereafter, we need only two more matchsticks to make each triangle. Thus the number of matchsticks form the sequence

$$3, 5, 7, 9, 11, \dots$$

How many matchsticks do we need to make 10 triangles like this?

3 matchsticks for the first, 2 each for the remaining 9, which makes $9 \times 2 = 18$. Altogether $3 + 18 = 21$.

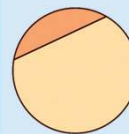
How many matchsticks do we require to make for 100 triangles?

$$3 + (99 \times 2) = 201$$

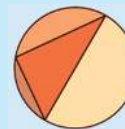
Let's now write this in algebra. How can we write the number of matchsticks we need to make n triangles?

Circle division

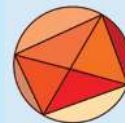
A line joining two points on a circle, divides the circle into two parts.



Taking three points on a circle and joining them in pairs, we get four parts



What if we take four points and join all pairs?



And five points? How many parts do you expect by joining six points like this? Check it by actually drawing such lines.



3 sticks for the first triangle and $2(n-1) = 2n - 2$ sticks for the remaining $n - 1$ triangles; altogether $3 + 2n - 2 = 2n + 1$

That is, the number of sticks we need to make n triangles is

$$x_n = 2n + 1$$

This is the algebraic expression for the sequence 3, 5, 7, ... got by adding 2 again and again to 3. Thus, we can easily compute the number of matchsticks needed to make 500 triangles:

$$x_{500} = (2 \times 500) + 1 = 1001$$



Once we have found out the algebraic expression for a sequence, we can use a computer to write down its terms. For example, the algebraic expression for the weights of iron cubes of sides 1 centimetre, 2 centimetres, 3 centimetres and so on is

$$x_n = 7.8n^3$$

To get a list of the weights of the first hundred cubes, we can use the python language (python 3) and write

```
for n in range (1,101):
    print (7.8*n**3)
```

If we save this piece of code in a file `weights.py` and run the command

```
python 3.2 weights.py > weights.txt
```

We get these numbers written in order in a file `weights.txt`



- (1) Write the algebraic expression for each of the sequences below:
 - i) Sequence of odd numbers
 - ii) Sequence of natural numbers which leave remainder 1 on division by 3.
 - iii) The sequence of natural numbers ending in 1.
 - iv) The sequence of natural numbers ending in 1 or 6.
- (2) For the sequence of regular polygons starting with an equilateral triangle, write the algebraic expressions for the sequence of the sums of interior angles, the sums of the exterior angles, the measures of an interior angle, and the measures of an exterior angle.

$$\frac{\pi + \sqrt{5}}{2}$$



(3) Look at these pictures:



The second picture is obtained by removing the small triangle formed by joining the midpoints of the first triangle. The third picture is got by removing such a middle triangle from each of the red triangles of the second picture.

- How many red triangles are there in each picture?
- Taking the area of the first triangle as 1, compute the area of a small triangle in each picture.
- What is the total area of all the red triangles in each picture?
- Write the algebraic expressions for these three sequences obtained by continuing this process.

Arithmetic sequences

When we computed the perimeters of squares with the length of a side 1, 2, 3, 4, ... we got the sequence

$$4, 8, 12, 16, \dots$$

Here the lengths of sides are increased in steps of 1 and so the perimeter increases in steps of 4. What if we take the side as $1, 1\frac{1}{2}, 2, 2\frac{1}{2}, \dots$?

Since the lengths of sides increase in steps of $\frac{1}{2}$, the perimeter increases in steps of $4 \times \frac{1}{2} = 2$. The sequence we get is

$$4, 6, 8, 10, \dots$$

Now look at the problem of matchstick triangles. We need 3 sticks for the first triangle and added two for each triangle thereafter. Thus we start with 3 and add 2 repeatedly to get the sequence

$$3, 5, 7, 9, \dots$$

$$(0, 1)$$

$$an + b$$

A sequence got by starting with any number and adding a fixed number repeatedly is called an *arithmetic sequence*.

In the second problem of square, the lengths of sides are

$$1, 1\frac{1}{2}, 2, 2\frac{1}{2}, \dots$$

This is also an arithmetic sequence: We start with 1 and add $\frac{1}{2}$ repeatedly.

The sums of exterior angles of regular polygons give the sequence

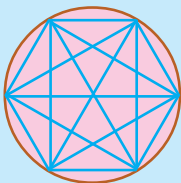
$$360, 360, 360, \dots$$

This again is an arithmetic sequence where we start with 360 and add 0 again and again.

Think before you leap!

We have seen how lines joining points on a circle dissect it (The sidebar, circle division). For 2, 3, 4, 5 points, we get 2, 4, 8, 16 parts. For 6 points? 32 may be our guess.

What do we actually get on drawing?



If the points are equally spaced, we get 30 parts; otherwise 31 parts.

Anyhow, the maximum number of parts is 31. We can show that the maximum number of parts, by joining n points is



$$\frac{1}{24}n(n-1)(n-2)(n-3) + \frac{1}{2}n(n-1) + 1$$

What is interesting is that for $n = 1, 2, 3, 4, 5$, this expression and 2^{n-1} give the same numbers 1, 2, 4, 8, 16. From $n = 6$ onwards, they give different numbers.

Let's look at another problem:

An object moves along a straight line at 10 metres/second. Applying a constant force in the opposite direction, the speed is reduced by 2 metres/second every second.

The sequence of speeds is

$$10, 8, 6, \dots$$

Here, the terms are got by subtracting 2 repeatedly from 10. This is also considered an arithmetic sequence. There are two ways to include such sequence among arithmetic sequences: either change "adding a fixed number repeatedly" to "adding or subtracting a fixed number repeatedly" in the definition of an arithmetic sequence or interpret "subtract 2" as "add -2" to justify the inclusion.

We can describe arithmetic sequences in another manner. In such a sequence, we add the same number to move from a term in any position to the term immediately after it. So, if we subtract from any term, the term immediately before it, we get this number.

An arithmetic sequence is a sequence in which we get the same number on subtracting from any term, the term immediately preceding it.



This constant difference got by subtracting from any term the just previous term, is called the *common difference* of an arithmetic sequence.

Very often, we find out whether a given sequence is an arithmetic sequence by checking whether the difference between the terms is constant. For example, consider the multiples of 3:

$$3, 6, 9, \dots$$

The difference of two consecutive multiples of 3 is 3 itself. So, it is an arithmetic sequence with common difference 3.

Now suppose we add 1 to each of these multiples. We get the sequence

$$4, 7, 10, \dots$$

This again is an arithmetic sequence with common difference 3.

Now look at the sequence of the powers of 3:

$$3, 9, 27, \dots$$

$9 - 3 = 6$ and $27 - 9 = 18$. Here the difference between consecutive terms is not the same.

So, it is not an arithmetic sequence.

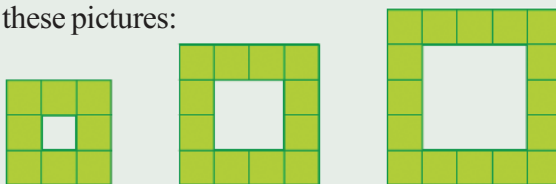
Now go through all the sequences we have discussed so far and pick out the arithmetic sequences from them.

?



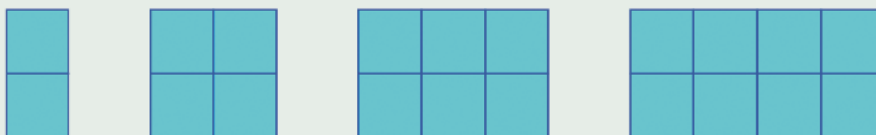
- (1) Check whether each of the sequences given below is an arithmetic sequence. Give reasons. For the arithmetic sequences, write the common difference also.
- Sequence of odd numbers
 - Sequence of even numbers
 - Sequence of fractions got as half the odd numbers
 - Sequence of powers of 2
 - Sequence of reciprocals of natural numbers

(2) Look at these pictures:



If the pattern is continued, do the numbers of coloured squares form an arithmetic sequence? Give reasons.

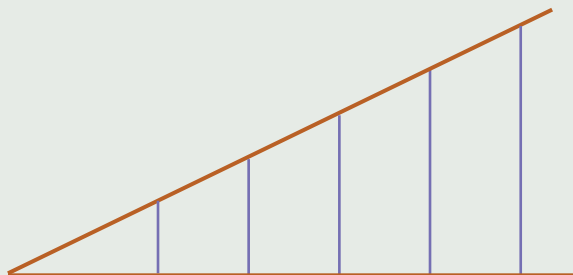
(3) See the pictures below:



- i) How many small squares are there in each rectangle?
- ii) How many large squares?
- iii) How many squares in all?

Continuing this pattern, is each such sequence of numbers, an arithmetic sequence?

(4) In this picture, the perpendiculars to the bottom line are equally spaced. Prove that, continuing like this, the lengths of perpendiculars form an arithmetic sequence.



(5) The algebraic expression of a sequence is

$$x_n = n^3 - 6n^2 + 13n - 7$$

Is it an arithmetic sequence?

Position and term

Can you make an arithmetic sequence with 1 and 11 as the first and second terms?

Simple, isn't it? To get 11 from 1, we must add 10. Adding 10 again and again, we get the arithmetic sequence

$$1, 11, 21, 31, \dots$$

Another question then: Can you make an arithmetic sequence with 1 and 11 as the first and third terms?

How do we calculate the common difference of such a sequence?

The common difference added to 1 gives the second term; and we don't know what it is. Adding it once more should give the third term, 11.

That is, 11 is got by adding twice the common difference to 1.

So, twice the common difference is 10, which means the common difference is 5.

Now we can write the sequence as

$$1, 6, 11, 16, 21, \dots$$

How about an arithmetic sequence with the 3rd term 37 and 7th term 72?

To get the 7th term from the 3rd term, we must add the common difference $7 - 3 = 4$ times.

And the actual number to be added is $72 - 37 = 36$

Thus 4 times the common difference is 36 and so the common difference is $36 \div 4 = 9$

How do we find the first term?

Subtract the common difference twice from the 3rd term; that is $37 - (2 \times 9) = 19$

Now we can write the sequence from the beginning

$$19, 28, 37, \dots$$

How do we compute the 25th term of this sequence?

There are several ways:

To the 3rd term, add $(25 - 3) = 22$ times the common difference: $37 + (22 \times 9) = 235$

To the 2nd term, add $(25 - 2) = 23$ times the common difference: $28 + (23 \times 9) = 235$

To the 1st term, add $(25 - 1) = 24$ times the common difference: $19 + (24 \times 9) = 235$

Sequence and remainder

The even numbers 2, 4, 6, ... form an arithmetic sequence. So do the odd numbers 1, 3, 5, The common difference is 2 in both. Even numbers are numbers divisible by 2, or numbers leaving the remainder 0 on division by 2. And odd numbers are those leaving remainder 1 on division by 2.

Similarly, we can make three arithmetic sequences, leaving remainders 0, 1, 2 on division by 3. What is the common difference of all these?

How about division by 4? Now let's think in reverse: in an arithmetic sequence of natural numbers, the difference of any two terms is a multiple of the common difference.

So they all leave the same remainder on division by the common difference (why?)

In general, if we know any two terms of an arithmetic sequence and their positions, we can compute the entire sequence.

What is the general principle used?

The difference between any two terms of an arithmetic sequence is the product of the difference of positions and the common difference.

We can put it like this also:

In an arithmetic sequence, term difference is proportional to position difference; and the constant of proportionality is the common difference.

Sequence rule

What is the next term of the sequence 3, 5, 7, ...?
 It's not said to be an arithmetic sequence. so the next term need not be 9. For example, if it is supposed to be the sequence of odd primes, the next term is 11. What is the moral here? Just by writing down the first few terms of a sequence, we cannot predict exactly the terms to come next. To do this either the rule of formation of the sequence or the context in which the sequence arises must be specified.

Consider these sequences:

$$x_n = 2n - 1$$

$$x_n = n^2 - n + 1$$

$$x_n = n^3 - 3n^2 + 4n - 1$$

All of them have 1 and 3 as the first two terms.

What about the third terms?

We can use this to check whether a given number is a term of a given arithmetic sequence.

For example, let's take a sequence seen before:

$$19, 28, 37, \dots$$

The difference between any two terms of this is a multiple of 9. On the other hand, what if the difference of a number from a term of this sequence is multiple of 9?

For example, the difference of the number 1000 with the first term 19 of this sequence is $1000 - 19 = 981 = 109 \times 9$, which is multiple of 9. Thus 1000 is got by adding 109 times the common difference 9 to the first term 19. So, it is the 110th term of the sequence.



Is every power of 10, a term of the arithmetic sequence 19, 28, 37, ...?



- (1) In each of the arithmetic sequences below, some terms are missing and their positions are marked with \bigcirc . Find them.
- i) 24, 42, \bigcirc , \bigcirc , ...
 - ii) \bigcirc , 24, 42, \bigcirc , ...
 - iii) \bigcirc , \bigcirc , 24, 42, ...
 - iv) 24, \bigcirc , 42, \bigcirc , ...
 - v) \bigcirc , 24, \bigcirc , 42, ...
 - vi) 24, \bigcirc , \bigcirc , 42, ...
- (2) The terms in two positions of some arithmetic sequences are given below. Write the first five terms of each:
- i) 3rd term 34 ii) 3rd term 43 iii) 3rd term 2
6th term 67 6th term 76 5th term 3
 - iv) 4th term 2 v) 2nd term 5
7th term 3 5th term 2
- (3) The 5th term of an arithmetic sequence is 38 and the 9th term is 66. What is its 25th term?
- (4) Is 101 a term of the arithmetic sequence 13, 24, 35, ...? What about 1001?
- (5) How many three-digit numbers are there, which leave a remainder 3 on division by 7?
- (6) Fill up the empty cells of the square below such that the numbers in each row and column form arithmetic sequences:

1			4
7			28

What if we use other numbers instead of 1, 4, 28 and 7?

- (7) In the table below, some arithmetic sequences are given with two numbers against each. Check whether each belongs to the sequence or not.

Sequence	Numbers	Yes/No
11, 22, 33, ...	123	
	132	
12, 23, 34, ...	100	
	1000	
21, 32, 43, ...	100	
	1000	
$\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \dots$	3	
	4	
$\frac{3}{4}, 1\frac{1}{2}, 2\frac{1}{4}, \dots$	3	
	4	

Algebra of arithmetic sequences

The simplest arithmetic sequence is that of natural numbers and we have seen many of its peculiarities in classes seven and eight. For example, look at these sums:

$$1 + 2 + 3 = 6 = 3 \times 2$$

$$2 + 3 + 4 = 9 = 3 \times 3$$

$$3 + 4 + 5 = 12 = 3 \times 4$$

The sum of any three consecutive natural numbers is three times the middle number. Why?

To understand this, let's take the middle number as x . Then the first number is $x - 1$ and the third number is $x + 1$. Their sum is

$$(x - 1) + x + (x + 1) = 3x$$

which is thrice the middle number.

Is this true for three consecutive odd numbers?

And even numbers?

Now what if we take three consecutive terms of some arithmetic sequence?

For example, consider the arithmetic sequence

$$2, 7, 12, 17, \dots$$



Pick three consecutive terms of this, say, 37, 42, 47, ...

Their sum is

$$37 + 42 + 47 = 126$$

Isn't this three times 42?

Is it true for all arithmetic sequences?

Let's use algebra to check.

Let x be the middle one of three consecutive terms of an arithmetic sequence.

To write the terms on either side, we must have the common difference.

Let's take it as y . Then the first term is $x - y$ and the third term is $x + y$. The sum of all three is

$$(x - y) + x + (x + y) = 3x$$

So, this is true for all arithmetic sequences:

For any arithmetic sequence, the sum of three consecutive terms is thrice the middle one.

We can observe another fact from this. The first and the last terms added to the middle term makes three times the middle one; so the sum of the first and the last should be twice the middle (Three times a number is double the number added to it, isn't it?)

We can put this in a slightly different way:

In any three consecutive terms of an arithmetic sequence, the middle one is half the sum of the first and the last.

We can write these in algebra.

If x, y, z are three consecutive terms of an arithmetic sequence, then

$$\bullet \quad x + y + z = 3y \quad \bullet \quad y = \frac{1}{2} (x + z)$$

What can you say about the sum of five consecutive terms of an arithmetic sequence and the sum of the first and last terms? What if we consider seven consecutive terms? What is the general conclusion?



Now let's look at the algebraic expressions for some arithmetic sequences.

First let's take the arithmetic sequence

$$19, 28, 37, \dots$$

To get the term at any position of this sequence, we must multiply the position difference from the first by the common difference 9, and add to the first term 19. For example, if we take the 15th term, the position difference from the first is $15 - 1 = 14$; so to get the 15th term, we must add 14 times the common difference 9 to the first term 19.

$$15^{\text{th}} \text{ term is } 19 + (14 \times 9) = 145$$

What about the 20th term?

In general, for any natural number n , the n^{th} term is

$$19 + (n - 1) \times 9 = 9n + 10$$

Thus the algebraic expression for this sequence is

$$x_n = 9n + 10$$

Now look at the arithmetic sequence,

$$\frac{1}{2}, \frac{3}{4}, 1, \dots$$

Thinking along the same lines as in the first problem, the n^{th} term is

$$\frac{1}{2} + (n - 1) \times \frac{1}{4} = \frac{1}{4}n + \frac{1}{4}$$

That is, the algebraic expression for the sequence is

$$x_n = \frac{1}{4} (n + 1)$$

In the first sequence, we multiply the position n by the common difference 9 and add 10; in the second, we multiply by $\frac{1}{4}$ and add $\frac{1}{4}$.

Is every arithmetic sequence in this form?

Taking the first term of an arithmetic sequence as f and the common difference as d , the n^{th} term is

$$f + (n - 1) d = dn + (f - d)$$

That is, we multiply n by number d and add the number $f - d$.



Thus each term of an arithmetic sequence is got by multiplying the position number by the common difference and adding a fixed number.

That is, the algebraic expression for any arithmetic sequence is of the form.

$$x_n = an + b$$

On the other hand, is any sequence $x_n = an + b$, an arithmetic sequence?

Any two consecutive terms of this sequence are of the form $an + b$ and $a(n + 1) + b$; and their difference is

$$a(n + 1) + b - (an + b) = a$$

That is the difference of any two consecutive terms is the same number a ; and so it is an arithmetic sequence.

Any arithmetic sequence is of the form

$$x_n = an + b$$

where a and b are fixed numbers; conversely, any sequence of this form is an arithmetic sequence.

We have also seen that in the arithmetic sequence $x_n = an + b$, the common difference is a .

Using this, we can easily find the algebraic expression for a given arithmetic sequence. For example, let's look at the sequence got by starting at $\frac{1}{2}$ and adding $\frac{1}{3}$ repeatedly:

$$\frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \dots$$

Since the common difference is $\frac{1}{3}$, its algebraic expression is of the form

$$\frac{1}{3}n + b. \text{ Take } n = 1, \text{ first term is } \frac{1}{3} + b.$$

So we get

$$\frac{1}{3} + b = \frac{1}{2}$$

from which we find

$$b = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

So the algebraic expression for the sequence is

$$\frac{1}{3}n + \frac{1}{6}$$

Mark a point A in GeoGebra and type the command

Sequence [Circle [A, n], n, 1, 10]

We get circles centered at A with radii 1 to 10. To change the number of circles, make an Integer slider m and change the command to

Sequence [Circle [A, n], n, 1, m]

Now by changing n to 2n or 2n + 1, we can draw circles with radii even or odd numbers.

Making two more sliders a, b with Min = 0 and changing the command to

Sequence [Circle [A, a + b], n, 1, m]

we can draw circles of radii in different arithmetic sequences by changing a and b.

Draw a regular hexagon and draw a sequence of circles like this, centred at each vertex.



We can write the algebraic expression for this sequence as a fraction like this;

$$x_n = \frac{2n+1}{6}$$

We can note many things from this. All fractions in this sequence have odd numbers as numerations and 6 as the denominator. The odd numbers don't have 2 as a factor and so don't have 6 as a factor either. Thus no term of this sequence is a natural number.

In other words, this sequence does not contain any natural number.

?



- (1) Write three arithmetic sequences with 30 as the sum of the first five terms.
- (2) The first term of an arithmetic sequence is 1 and the sum of the first four terms is 100. Find the first four terms.
- (3) Prove that for any four consecutive terms of an arithmetic sequence, the sum of the two terms on the two ends and the sum of the two terms in the middle are the same.
- (4) Write four arithmetic sequences with 100 as the sum of the first four terms.
- (5) The 8th term of an arithmetic sequence is 12 and its 12th term is 8. What is the algebraic expression for this sequence?
- (6) The Bird problem in Class 8 (The lesson, **Equations**) can be slightly changed as follows.

One bird said:

“We and we again, together with half of us and half of that, and one more is a natural number”

Write the possible number of birds in order. For each of these, write the sum told by the bird also.

Find the algebraic expression for these two sequences.

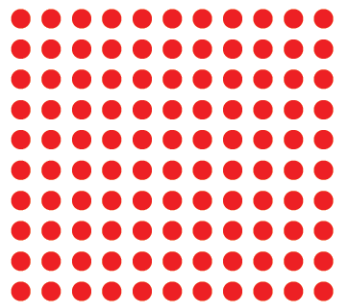
- (7) Prove that the arithmetic sequence with first term $\frac{1}{3}$ and common difference $\frac{1}{6}$ contains all natural numbers.
- (8) Prove that the arithmetic sequence with first term $\frac{1}{3}$ and common difference $\frac{2}{3}$ contains all odd numbers, but no even number.



- (9) Prove that the squares of all the terms of the arithmetic sequence 4, 7, 10, ... belong to the sequence.
- (10) Prove that the arithmetic sequence 5, 8, 11, ... contains no perfect squares.
- (11) The angles of a pentagon are in arithmetic sequence. Prove that its smallest angle is greater than 36° .
- (12) Write the whole numbers in the arithmetic sequence $\frac{11}{8}, \frac{14}{8}, \frac{17}{8}, \dots$
Do they form an arithmetic sequence?

Sums

See this picture:



How many dots are there in it?
We need not count one by one. There are 11 dots in each row and 10 such rows; total $10 \times 11 = 110$.

How many dots are there in this triangle?



We can count row by row and add:
 $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$

Is there an easier way?
How about changing this to a rectangle?

Language of law

We have noted that to compute the terms of a sequence, the rule or law of formation must be specified. And we have seen how such rules can be algebraically expressed.

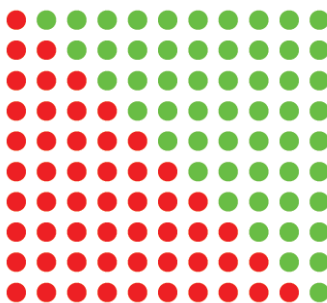
But the rule of forming some sequences cannot be written in algebra. For example, consider the sequence 2,3, 5, 7, 11, of primes. No algebraic expression has been found to compute the number in a specified position in this sequence. Again, consider the sequence 3, 1, 4, 1, 5, 9, ... of digits in the decimal form of π . There is no algebraic expression for this sequence also.

In such cases, we can only describe the rule of formation in ordinary language.

For that, make another triangle like this;



Turn it upside down and join with the first:



A math tale

You have heard about the great mathematician Gauss, haven't you? He showed extraordinary talent in mathematics from a very early age. There are several tales about this.

Gauss was 10. The teacher asked his class to add all numbers from 1 to 100 just to keep them quiet. Little Gauss gave the answer very quickly, 5050.

And he explained it like this: 1 and 100 make 101, so do 2 and 99. Thus 50 pairs each of sum 101 give $50 \times 101 = 5050$.

As seen before, this rectangle, $10 \times 11 = 110$ dots.

How many dots are there in each triangle?

Half of 110, which is 55.

We can write in numbers, what we did with pictures. Let's write

$$s = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

Writing in reverse,

$$s = 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$$

How about adding the numbers in the same position?

$$2s = 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11$$

So,

$$s = \frac{1}{2} \times 10 \times 11 = 55$$

We can add numbers from 1 to 100 like this:

$$s = 1 + 2 + 3 + \dots + 98 + 99 + 100$$

$$s = 100 + 99 + 98 + \dots + 3 + 2 + 1$$



Adding numbers in the same place,

$$\begin{aligned} 2s &= \overbrace{101 + 101 + 101 + \dots + 101 + 101 + 101 + 101}^{100 \text{ times}} \\ &= 100 \times 101 \end{aligned}$$

From this, we get

$$s = \frac{1}{2} \times 100 \times 101 = 5050$$

Can't we find the sum upto any number, instead of 100, like this?

The sum of any number of consecutive natural numbers, starting with one, is half the product of the last number and the next natural number.

In the language of algebra,

$$1 + 2 + 3 + \dots + n = \frac{1}{2} n(n + 1)$$

Using this, we can find sums of consecutive terms of other arithmetic sequences also.

For example, let's take the even numbers 2, 4, 6, ..., 100. Even numbers are got by multiplying natural numbers by 2. So, we can write

$$2 + 4 + 6 + \dots + 100 = 2(1 + 2 + 3 + \dots + 50)$$

We have seen that

$$1 + 2 + 3 + \dots + 50 = \frac{1}{2} \times 50 \times 51$$

From this, we get

$$2 + 4 + 6 + \dots + 100 = 2 \times \frac{1}{2} \times 50 \times 51 = 2550$$

In general, the first n even natural numbers are

$$2, 4, 6, \dots, 2n$$

And their sum is

$$2 + 4 + 6 + \dots + 2n = 2(1 + 2 + 3 + \dots + n) = n(n + 1)$$

Can't you find the sum of the first n multiples of 3 in this manner?

How do we find the sum of the first n odd numbers?



First let's write the algebraic expression for the sequence of odd numbers:

$$x_n = 2n - 1$$

Taking $n = 1, 2, 3, \dots$ in this, we get the sequence of odd numbers. So, we can write the sequence like this:

$$x_1 = (2 \times 1) - 1$$

$$x_2 = (2 \times 2) - 1$$

.....

$$x_n = (2 \times n) - 1$$

How about adding all this from top to bottom?

$$\begin{aligned} x_1 + x_2 + \dots + x_n &= ((2 \times 1) + (2 \times 2) + \dots + (2 \times n)) - \overbrace{(1+1+\dots+1)}^{n \text{ times}} \\ &= 2(1 + 2 + \dots + n) - n \end{aligned}$$

Now, if we use

$$1 + 2 + \dots + n = \frac{1}{2} n(n + 1)$$

we get,

$$x_1 + x_2 + \dots + x_n = 2 \times \frac{1}{2} n(n + 1) - n = n^2$$

That is, the sum of consecutive odd numbers starting from 1 is the square of the number of odd numbers added.

We have already noted this in Class 7, in the section **Perfect squares** of the lesson **Square and Square root**.

Like this, we can calculate the sum of any arithmetic sequence.

An arithmetic sequence is of the form

$$x_n = an + b$$

To calculate the sum of its first n terms, we put $n = 1, 2, 3, \dots$, in this and add:

$$x_1 = a + b$$

$$x_2 = 2a + b$$

.....

$$x_n = na + b$$

$$\begin{aligned}
 x_1 + x_2 + \dots + x_n &= (a + 2a + \dots + na) + \overbrace{(b + b + \dots + b)}^{n \text{ times}} \\
 &= a(1 + 2 + \dots + n) + nb \\
 &= a \frac{n(n+1)}{2} + nb \\
 &= \frac{1}{2}an(n+1) + nb
 \end{aligned}$$

For the arithmetic sequence

$$x_n = an + b$$

the sum of the first n terms is

$$x_1 + x_2 + \dots + x_n = \frac{1}{2}an(n+1) + nb$$

For example, let's see how we find the sum of the first n terms of the arithmetic sequence

$$1, 4, 7, \dots$$

The algebraic expression of this sequence is

$$x_n = 3n - 2$$

So, the sum of the first 100 terms is

$$\left(\frac{1}{2} \times 3 \times 100 \times 101\right) + (100 \times (-2)) = 14950$$

In general, the sum of the first n terms of this sequence

$$\frac{1}{2} \times 3 \times n(n+1) - 2n = \frac{1}{2}(3n^2 - n)$$

We can calculate the sum of an arithmetic sequence in another way. For this, we write the algebraic expression for the sum like this:

$$\begin{aligned}
 \frac{1}{2}an(n+1) + nb &= \frac{1}{2}n(a(n+1) + 2b) \\
 &= \frac{1}{2}n((an+b) + (a+b))
 \end{aligned}$$

Here $an + b$ is the n^{th} term x_n of the sequence and $a + b$ is the first term x_1 .

The sum of the first n terms x_1, x_2, \dots, x_n of an arithmetic

Sum of squares

We have seen the identity

$$(x+1)^2 = x^2 + 2x + 1$$

Like this, we have the identity

$$(x+1)^3 = x^3 + 3x^2 + 3x + 1$$

From this, we see that for any number x ,

$$(x+1)^3 - x^3 = 3x^2 + 3x + 1$$

Taking $x = 1, 2, 3, \dots, n$ in this, and adding all these, we get

$$\begin{aligned}
 (n+1)^3 - 1 &= 3(1^2 + 2^2 + 3^2 + \dots + n^2) \\
 &\quad + 3(1 + 2 + 3 + \dots + n) + n
 \end{aligned}$$

That is,

$$\begin{aligned}
 n^3 + 3n^2 + 3n \\
 &= 3(1^2 + 2^2 + 3^2 + \dots + n^2) + \frac{3}{2}n(n+1) + n
 \end{aligned}$$

So,

$$\begin{aligned}
 1^2 + 2^2 + 3^2 + \dots + n^2 \\
 &= \frac{1}{3}(n^3 + 3n^2 + 3n - \frac{3}{2}n(n+1) - n)
 \end{aligned}$$

The expression on the right can be simplified to give

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

sequence is

$$x_1 + x_2 + \dots + x_n = \frac{1}{2}n(x_n + x_1)$$

In ordinary language we can put it like this:

The sum of any number of consecutive terms of an arithmetic sequence is half the product of the number of terms and the sum of the first and last terms.

To compute the sum of the first 100 terms of the arithmetic sequence 1, 3, 5, ... we first find the 100th term as

$$1 + (99 \times 3) = 298$$

Now the sum of the first 100 terms is

$$\frac{1}{2} \times 100 \times (298 + 1) = 14950$$

The sum of the first n terms of any arithmetic sequence has a definite algebraic form. To see it, we write the expression for the sum like this:

$$\frac{1}{2}an(n+1) + nb = \frac{1}{2}an^2 + \left(\frac{1}{2}a + b\right)n$$

In this, $\frac{1}{2}a$ and $\frac{1}{2}a + b$ are constants associated with the sequence. Thus the sum is the sum of products of n^2 and n with definite numbers.

In other words, the sum of the first n terms of an arithmetic sequence is of the form $pn^2 + qn$.



To find the sum of consecutive terms of a sequence with known algebraic expression, we can use the sum function in python. For example, the command.

```
sum(x**2 for x in range(1,101))
```

gives the sum of the squares of the first hundred natural numbers.



(1) Find the sum of the first 25 terms of each of the arithmetic sequences below.

- i) 11, 22, 33, ... ii) 12, 23, 34, ... iii) 21, 32, 43, ...
- iv) 19, 28, 37, ... v) 1, 6, 11, ...





- (2) What is the difference between the sum of the first 20 terms and the next 20 terms of the arithmetic sequence 6, 10, 14, ...?
- (3) Calculate the difference between the sum of the first 20 terms of the arithmetic sequences 6, 10, 14, ... and 15, 19, 23, ...
- (4) Find the sum of all three digit numbers, which are multiples of 9.
- (5) Find n in the equation $5^2 \times 5^4 \times 5^6 \times \dots \times 5^{2n} = (0.008)^{-30}$.
- (6) The expressions for the sum to n terms of some arithmetic sequences are given below. Find the expression for the n^{th} term of each:
 - i) $n^2 + 2n$ ii) $2n^2 + n$ iii) $n^2 - 2n$
 - iv) $2n^2 - n$ v) $n^2 - n$
- (7) Calculate in head, the sums of the following arithmetic sequences.
 - i) $51 + 52 + 53 + \dots + 70$
 - ii) $1\frac{1}{2} + 2\frac{1}{2} + \dots + 12\frac{1}{2}$
 - iii) $\frac{1}{2} + 1 + 1\frac{1}{2} + 2 + 2\frac{1}{2} + \dots + 12\frac{1}{2}$
- (8) The sum of the first 10 terms of an arithmetic sequence is 350 and the sum of the first 5 terms is 100. Write the algebraic expression for the sequence.
- (9) Prove that the sum of any number of terms of the arithmetic sequence 16, 24, 32, ... starting from the first, added to 9 gives a perfect square.
- (10)

7	10		
13	16	19	
22	25	28	31

.....

.....

.....

Write the next two lines of the pattern above. Calculate the first and last terms of the 20th line.



Project

- Find those arithmetic sequences, which contain all powers of their terms.
- Find those arithmetic sequences in which the sum of any number of consecutive terms, starting from the first is a perfect square.

Looking back

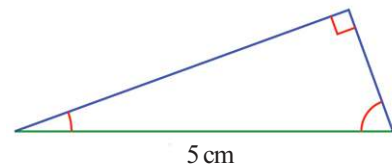


Learning outcomes	On my own	With teacher's help	Must improve
<ul style="list-style-type: none"> ● Writing the rule of formulation of a sequence and writing the algebraic expression for it. ● Recognising arithmetic sequence from other sequences. ● Formulating the relation between terms, position of terms and common difference in arithmetic sequences ● Using various methods to compute the sum of arithmetic sequences. 			

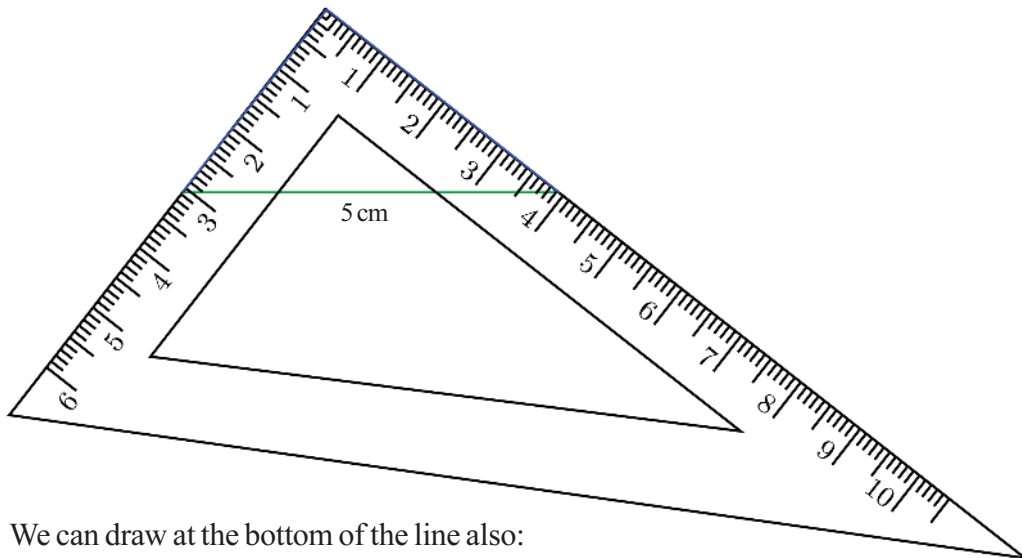


Can you draw a right triangle of hypotenuse 5 centimetres? The perpendicular sides can be of any length.

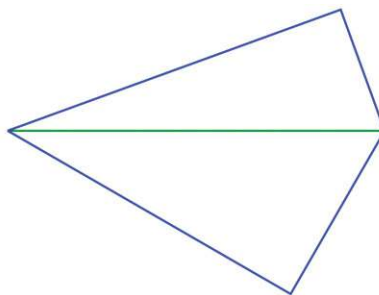
Draw a line 5 centimetres long. Draw any angle at one end and draw 90° minus this angle at the other end to make a triangle.



We can also use a set square to do this. Place it with the right angle on top and edges passing through the ends of the line. Try it!

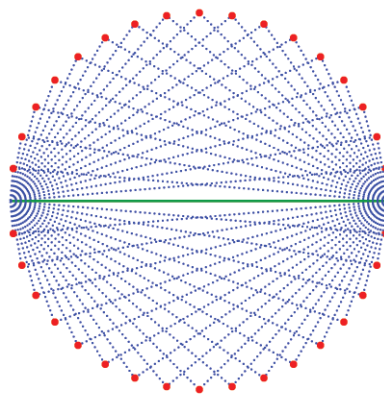


We can draw at the bottom of the line also:



Draw several such triangles and look at the third vertices.

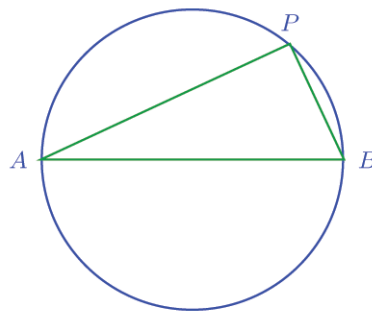
Why do all of them lie on a circle?
Let's think about this.



Using GeoGebra, we can actually see how right triangles make a circle. First draw a line 5 centimetres long, using **Line with Given Length**. Make slider a from 0 to 180 with increment 5. Make a 90° angle at one end and $(90 - a)^\circ$ clockwise at the other end, using **Angle with Given Size**. Join the points so got with the ends of the line, using **Line**. Join the intersection of these lines and the ends of the first line to make a triangle. Apply **Trace On** to the top vertex and sides of the triangle and **Animation** on to the slider.

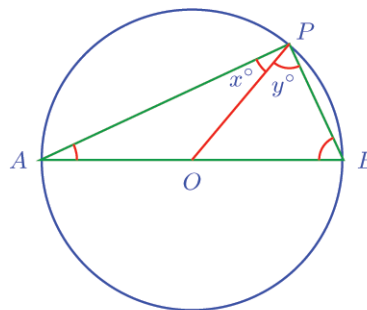
Right angle and circles

Remember the discussion on the angle got by joining a point on a circle with the ends of a diameter, at the end of the lesson, **Equal Triangles** in Class 8.

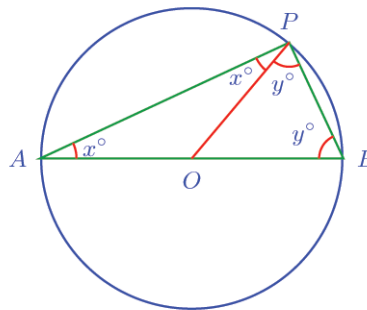


How did we show that the angle at P is a right angle?

Join P to the centre of the circle. Now the angle at P is split into two:



The triangles AOP and BOP on the left and right are isosceles (Why?). So the angle at A is x° and the angle at B is y° .



In the large triangle ABP , the sum of the angles is 180° . So,

$$x + y + (x + y) = 180$$

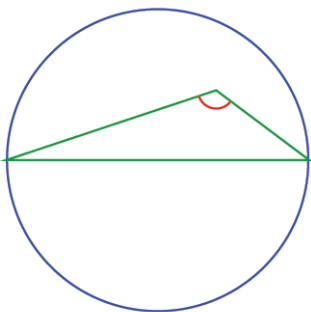
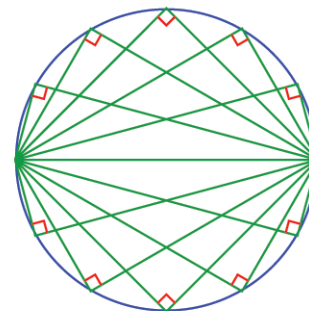
From this, we get

$$x + y = 90$$

If we join the ends of a diameter of a circle to a point on the circle, we get a right angle.

We can shorten this:

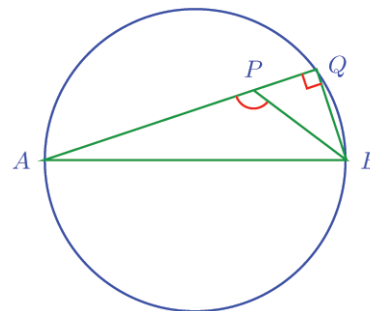
Angle in a semicircle is right.



We can think a bit more about this. Joining ends of a diameter to a point on the circle gives a right angle. What if we join the ends to a point inside the circle?

Extend one of the lines to meet the circle. Join this point to the other end of the diameter.

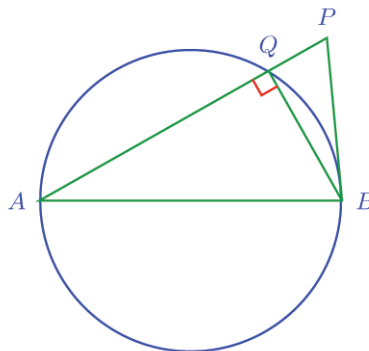
Now APB is the exterior angle at P of ΔAQB . So, it is the sum of the interior angles at Q and B .



Of these, the angle at Q is right. So $\angle APB$ is larger than a right angle.

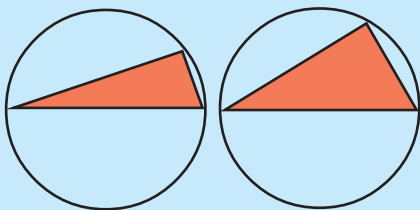
What about a point outside the circle? In this case, APB is an interior angle of $\triangle PQB$; and the right angle AQB is an exterior angle. So we can see that $\angle APB$ is less than right angle.

Now suppose that the angle obtained when ends of a diameter joined to some point is a right angle. This point can't be inside the circle. (For any point inside the circle, such an angle is larger than a right angle). It can't be outside the circle either (for any point outside, such an angle is less than a right angle). So the point must be on the circle.



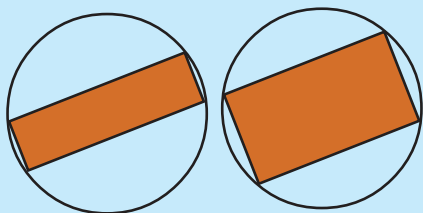
Special square

We can draw several right triangles in a circle, by joining any point on it to the ends of a diameter.



Which point on the circle gives a triangle of maximum area?

Another question: We can draw several rectangles with all four corners on the circle.



What's the speciality of the rectangle of maximum area?

What do we see here?

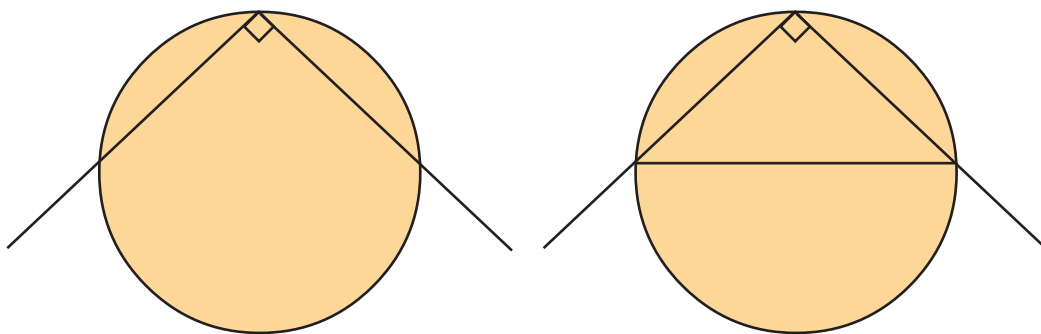
If a pair of lines drawn from the ends of a diameter of a circle are perpendicular to each other, then they meet on the circle.

We can change this slightly so that the circle appears at the end:

All pairs of mutually perpendicular lines, drawn from the ends of a fixed line, meet on the circle with that line as diameter.

Now don't you see why the third corners of right triangles in the first picture form a circle?

Now let's think in reverse. If we draw mutually perpendicular lines from a point on a circle and join the points where they cut the circle, do we get a diameter?



Then we get a right triangle and its circumcircle, and we have seen in Class 9 that the circumcentre of a right triangle is the midpoint of the hypotenuse.

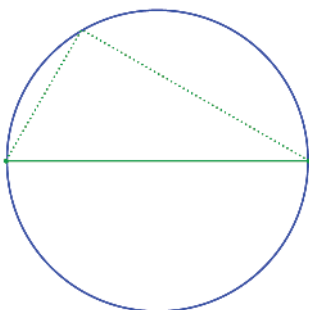
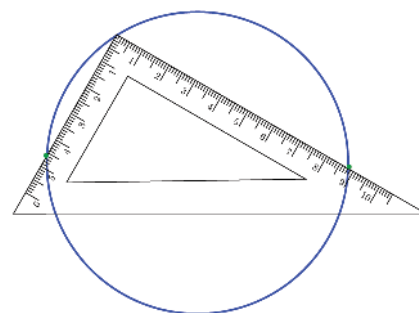
So, the bottom line is a diameter.

Now let's see some applications of these ideas.

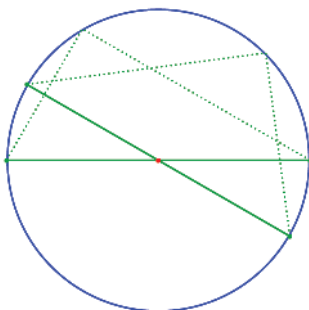
Remember the method to locate the centre of a circle drawn using a bangle or a lid?

There's another way. Place a set square with its right angle on the circle and mark the points where the perpendicular edges cross the circle.

The line joining them is a diameter.



Now change the position of the set square and draw another diameter:

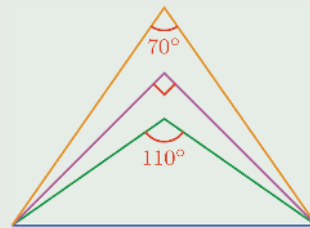


The point where these diameters cross is the centre.

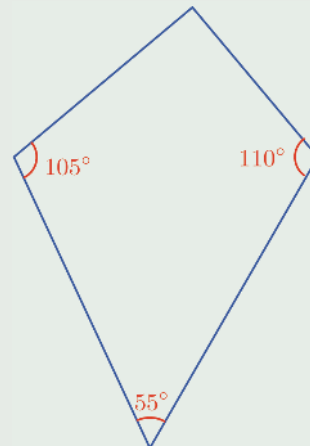
Draw a circle centred at A and mark two points B and C on it. Draw a line from B through C, using **Ray**. Draw a perpendicular to this through B and mark the point D where it meets the circle. Join CD. Is it a diameter? Change the positions of B, C and see.



- (1) Suppose we draw circle with the bottom side of the triangles in the picture as diameter. Find out whether the top corner of each triangle is inside the circle, on the circle or outside the circle.

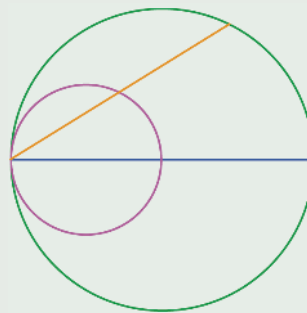


- (2) For each diagonal of the quadrilateral shown, check whether the other two corners are inside, on or outside the circle with that diagonal as diameter.



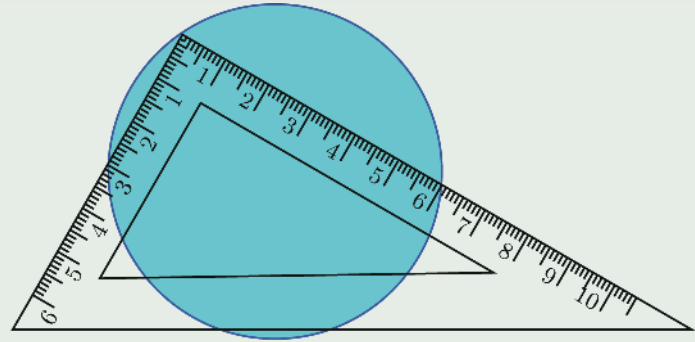
- (3) If circles are drawn with each side of a triangle of sides 5 centimetres, 12 centimetres and 13 centimetres, as diameters, then with respect to each circle, where would be the third vertex?

- (4) In the picture, a circle is drawn with a line as diameter and a smaller circle with half the line as diameter. Prove that any chord of the larger circle through the point where the circles meet is bisected by the small circle.

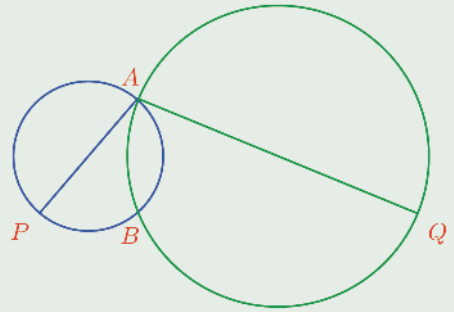


- (5) Prove that the two circles drawn on the two equal sides of an isosceles triangle as diameters pass through the mid point of the third side.

- (6) Use a calculator to determine upto two decimal places, the perimeter and the area of the circle in the picture.

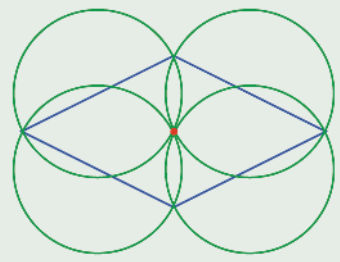


- (7) The two circles in the picture cross each other at A and B . The points P and Q are the other ends of the diameters through A .

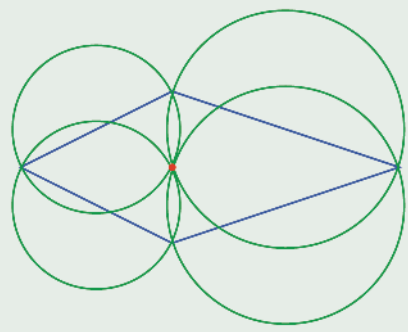


- i) Prove that P, B, Q lie on a line.
- ii) Prove that PQ is parallel to the line joining the centres of the circles and is twice as long as this line.

- (8) Prove that all four circles drawn with the sides of a rhombus as diameters pass through a common point.

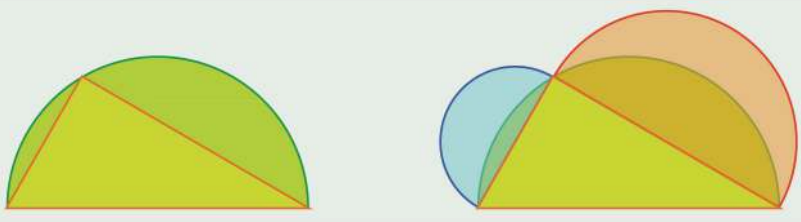


Prove that this is true for any quadrilateral with adjacent sides equal, as in the picture.





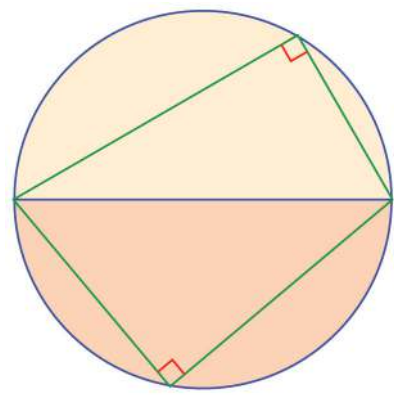
(9) A triangle is drawn by joining a point on a semicircle to the end of the diameter. Then semicircles are drawn with the other two sides as diameter.



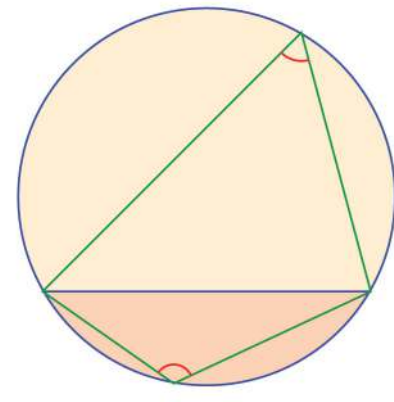
Prove that the sum of the areas of the blue and red crescents in the second picture is equal to the area of the triangle.

Chord, angle and arc

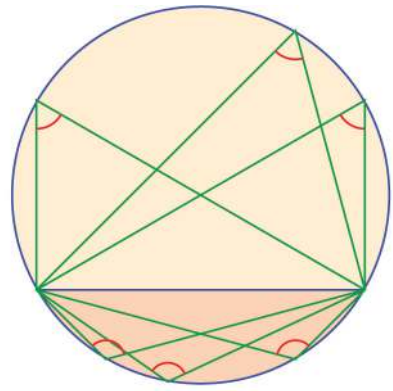
A diameter of a circle divides it into two equal parts, and joining the ends of the diameter to a point in any part gives a right angle.




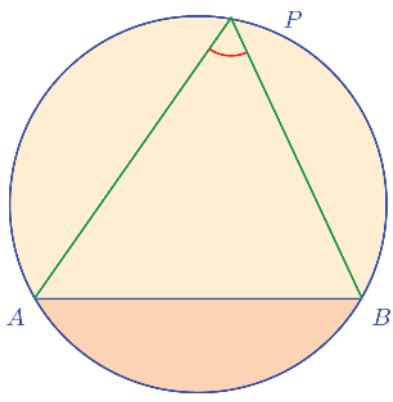
What about a chord, which is not a diameter?



The parts are not equal; nor are the angles right.
But here also, are all the angles on the same side equal?

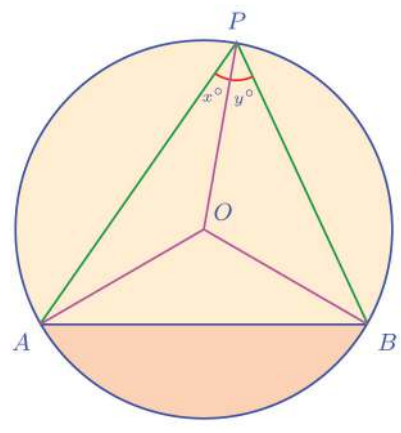


Let's see. First look at an angle at the top.



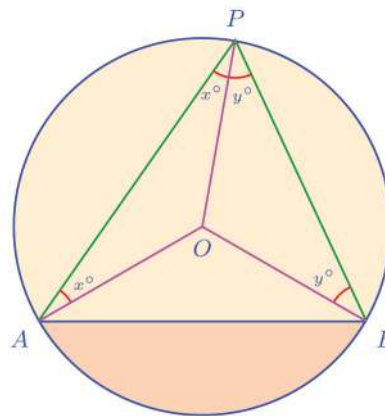
Draw a circle centred on a point A and mark three points B, C, D on it. Mark $\angle D$. Move D around the circle. What happens to the measure of this angle? Change the positions of B and C. When is $\angle D$ right? When is it acute or obtuse?

As in the case of the diameter, let's join a point P to the centre O of the circle. This line splits the angle at P into two. Let's take the measures of these parts as x° and y° . Here the centre is not on the chord. So, let's join OA and OB also.



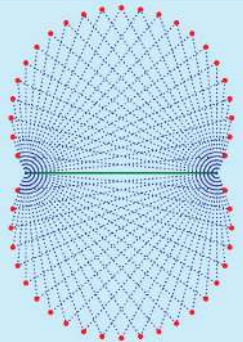
As in the case of a diameter, here also $\triangle OAP$ and $\triangle OBP$ are isosceles. So, we can write a part of the angles at A and B .

Here unlike the case with the diameter, the two isosceles triangles together do not form a single large triangle. So, the earlier trick of summing up the angles of a triangle won't work.



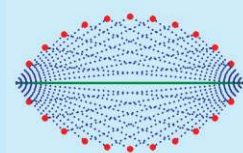
Circle trick

See this picture obtained by drawing angles of the same side above and below a line.



The angle is 60° in this picture.

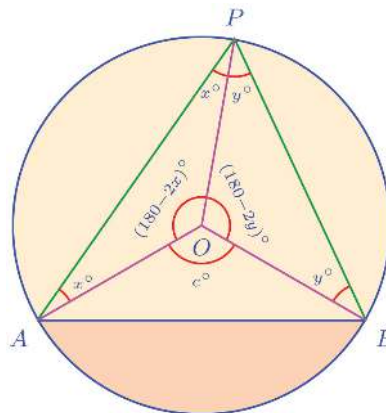
For 120° , we get this picture:



If we take 60° above the line and 120° below the line, we get a full circle. Why is this so?

If we take 30° above, what angle should we take below to get a full circle?

Instead, let's write all the angles around O :



If we take $\angle AOB = c^\circ$ as in the picture, then

$$(180 - 2x) + (180 - 2y) + c = 360$$

From this, we get

$$2(x + y) = c$$

and this gives

$$\angle APB = (x + y)^\circ = \frac{1}{2}c^\circ$$

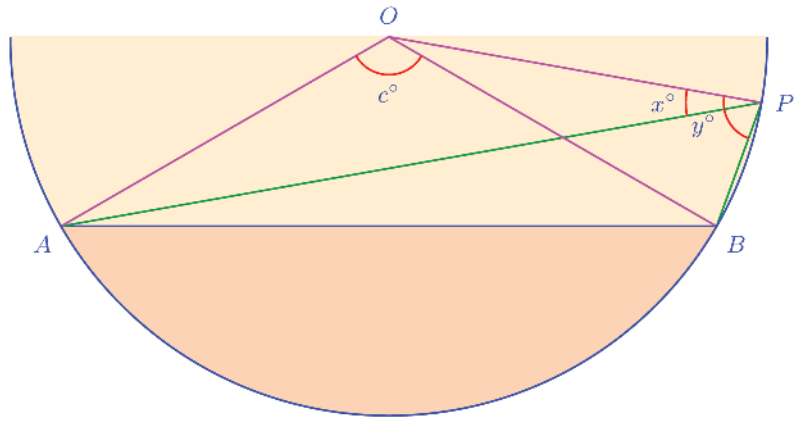
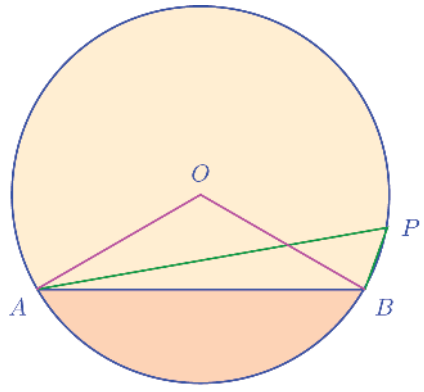
The point to note here is that once we fix A and B , changing the position of P on the circle changes x and y , but not c .

Will we get $\angle APB = \frac{1}{2} c^\circ$, if the position of P is anywhere above AB on the circle?

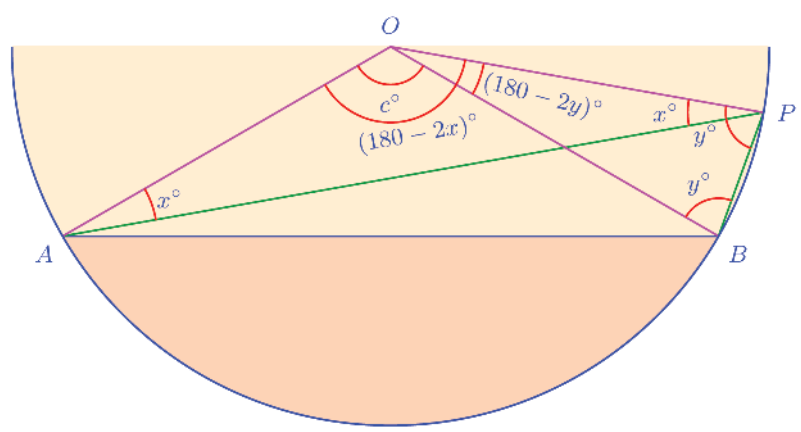
What if its like this?

Let's take $\angle APO = x^\circ$ and $\angle BPO = y^\circ$ as before.

To see the angles clearly, let's zoom in onto the part of the picture we're interested in:



Using the fact that OAP and OBP are isosceles triangles, we can compute some other angles as before:



From the angles at P , we can see that

$$\angle APB = (y - x)^\circ$$

And from the angles at O we see that

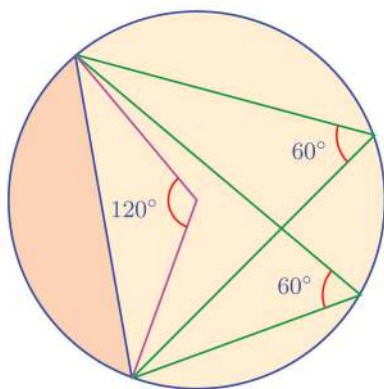
$$c = (180 - 2x) - (180 - 2y) = 2(y - x)$$

So, again we get,

$$\angle APB = \frac{1}{2}c^\circ$$

Thus joining the ends of a non-diametrical chord to any point on the larger part of the circle, we get an angle half the size of the angle we get by joining them to the a centre of the circle.

For example, see this picture:

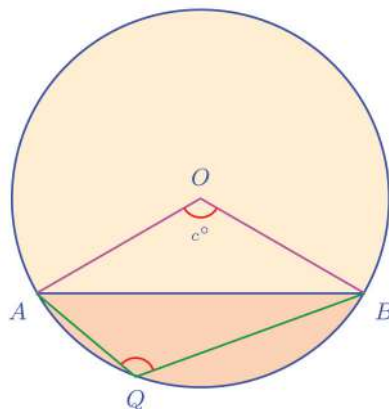


Now let's look at the angles in the smaller part.



Draw a circle centred at a point A and mark points B, C, D on the circle. Join B and C to A and D . Mark angles BDC and BAC . Is there any relation between these angles?

Change the positions of B, C, D and see.



Here also, we get two isosceles triangles by joining OQ . So, we can compute some angles as before.

From the angles at O , we get,

$$c = (180 - 2x) + (180 - 2y)$$

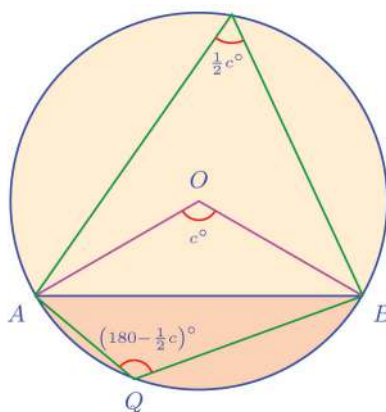
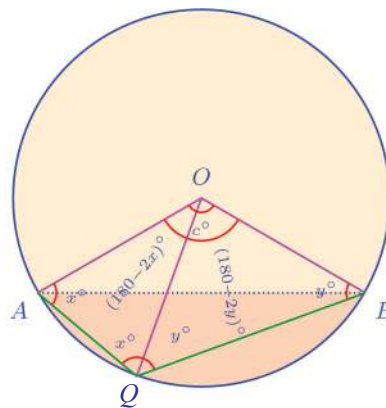
From this we get

$$2(x + y) = 360 - c$$

And this gives

$$\angle AQB = (x + y)^\circ = \left(180 - \frac{1}{2}c\right)^\circ$$

Now let's look at the angles in the two parts of the circle and the angle at the centre together:

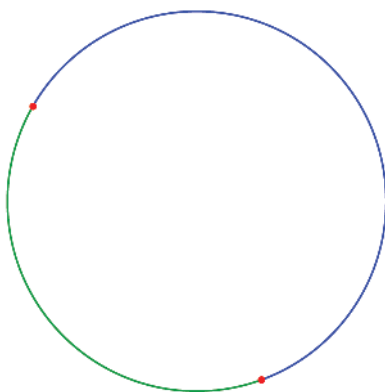
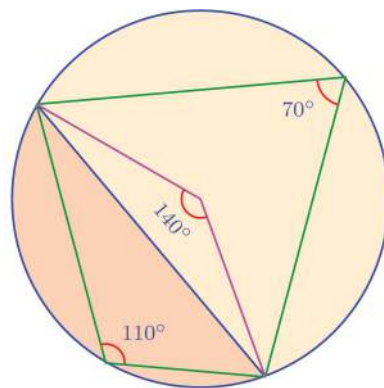


Any chord which is not a diameter splits the circle into unequal parts. The angle got by joining any point on the larger part to the ends of the chord is half the angle got by joining the centre of the circle to these ends. The angle got by joining any point on the smaller part to the ends of the chord is half the angle at the centre subtracted from 180° .

In short, if we know the angle which a chord makes at the centre, then the angle it makes on either side can be computed.



For example, a chord making 140° at the centre makes $\frac{1}{2} \times 140^\circ = 70^\circ$ on the larger part and $180^\circ - \left(\frac{1}{2} \times 140^\circ\right) = 110^\circ$ on the smaller part.

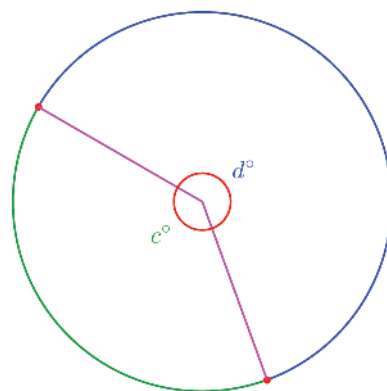


We've talked about central angles of an arc in Class 9. We can put the above results in terms of this idea.

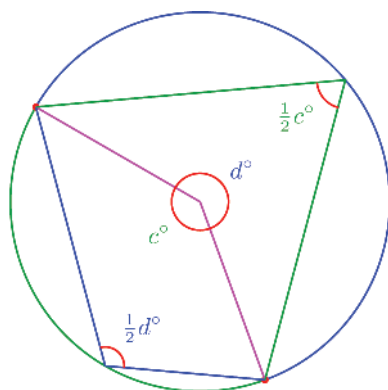
Any two points on a circle divides it into two arcs.

Each of these two arcs can be called the *alternate arc* or *complementary arc* of the other. If we take their central angles as c° and d° , then

$$c + d = 360$$



Each of these two arcs can be called the *alternate arc* or *complementary arc* of the other. If we take their central angles as c° and d° , then



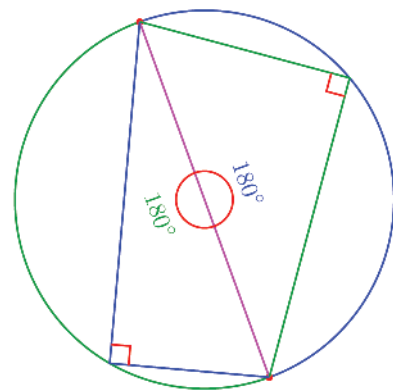
Now let's look at the angles which the first two points make at a point on each arc. As seen above, the angle on the larger arc is $\frac{1}{2} c^\circ$.

And the angle on the smaller arc is

$$\left(180 - \frac{1}{2}c\right)^\circ = \frac{1}{2}(360 - c)^\circ = \frac{1}{2}d^\circ$$

Suppose we start with the end points of a diameter?
 The picture is like this.
 So we can say this about any arc of a circle.

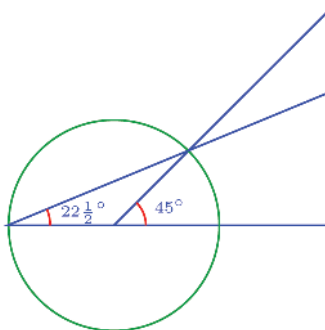
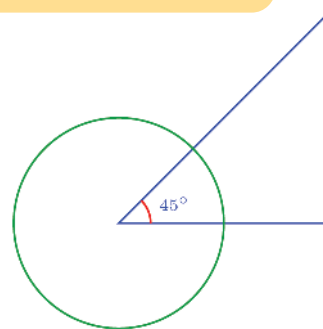
The angle made by any arc of a circle on the alternate arc is half the angle made at the centre.



From this, we also see that all angles made by an arc on the alternate arc are equal; moreover, if we recall the fact that $\frac{1}{2}d^\circ = \left(180 - \frac{1}{2}c\right)^\circ$ as shown in the first picture, we can see that sum of the angles on alternate arcs is 180° . Pairs of angles of sum 180° are usually called *supplementary angles*. Thus we have the following:

All angles made by an arc on the alternate arc are equal, and a pair of angles on alternate arcs are supplementary.

We can use this principle to halve angles. See this picture.

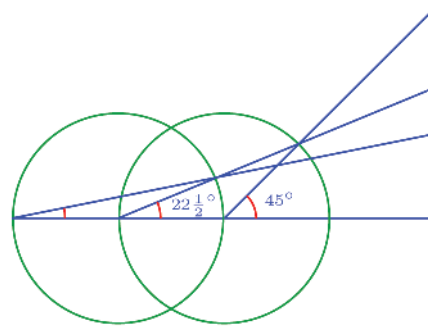


The corner of the angle is the centre of the circle. Now extend one side of the angle to meet the circle and join this point to the point where the other side cuts the circle. This gives us half the angle.



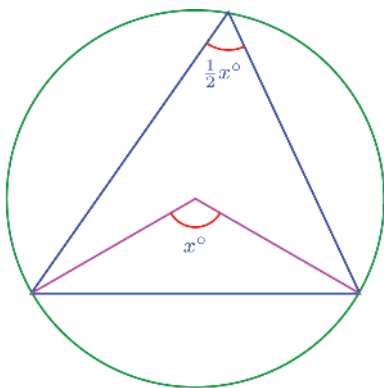
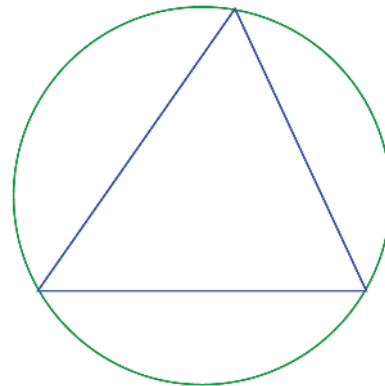
What if we do this again?

How much is the third angle?



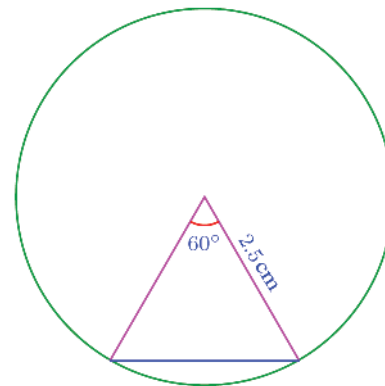
We can also use this to draw the triangle with specified angles and circumradius. For example, let's see how we can draw a triangle of angles 30°, 70°, 80° and of circumradius 2.5 centimetres.

The three sides of the triangle are chords of the circum circle.



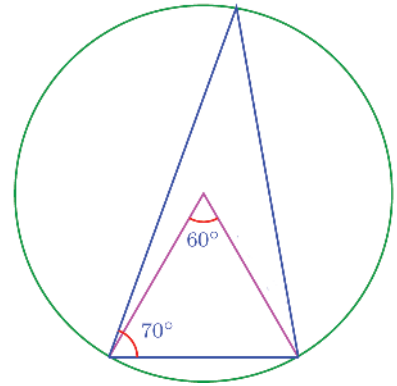
So, the angle opposite each side is half the angle it makes at the centre.

So to draw the triangle we require, we first draw a circle of radius 2.5 centimetres and draw a 60° angle at the centre. Joining its ends, we get the side opposite to 30° angle in our triangle.

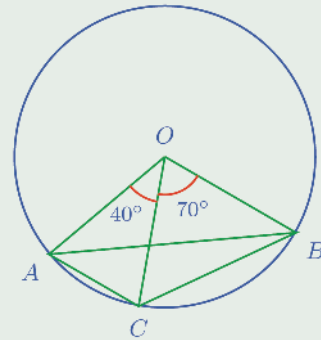
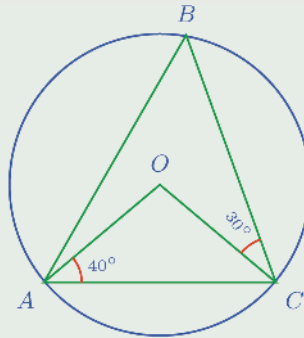
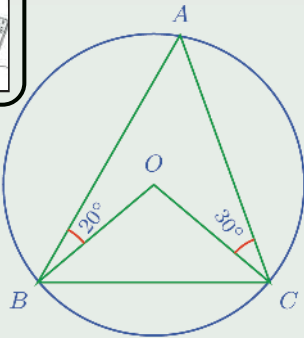


Now draw an angle of 70° at one of its ends and extend the other side to meet the circle. Joining this point to the other end of the line completes the triangle. Aren't the other two angles of this triangle 30° and 80° ? (Why?)

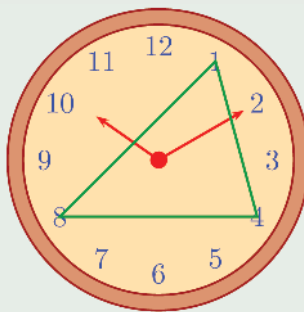
We note another thing here. We can draw several triangles with the same three angles. Fixing the circumradius also, we determine the triangle completely.



- (1) In all the pictures given below, O is the centre of the circle and A, B, C are points on it. Calculate all angles of $\triangle ABC$ and $\triangle OBC$ in each.



- (2) The numbers 1, 4, 8 on a clock's face are joined to make a triangle.



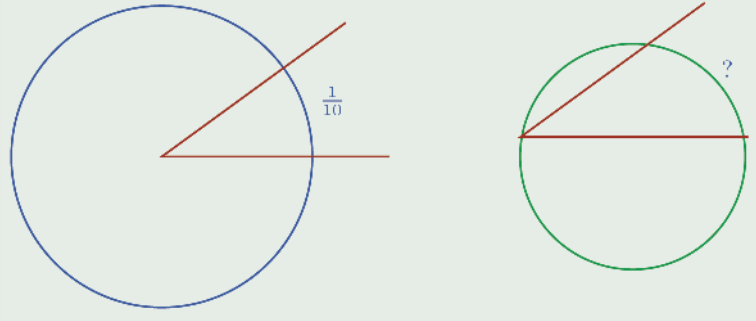
Calculate the angles of this triangle.

How many equilateral triangles can we make by joining numbers on the clock's face?

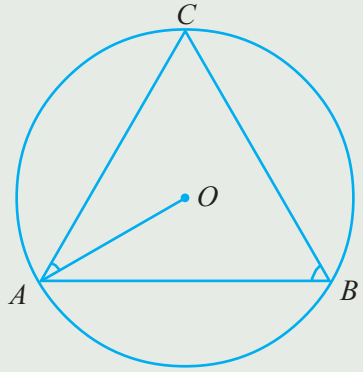
- (3) In each problem below, draw a circle and a chord to divide it into two parts such that the parts are as specified;

- i) All angles on one part 80° .
- ii) All angles on one part 110° .
- iii) All angles on one part half of all angles on the other.
- iv) All angles on one part, one and a half times the angles on the other.

(4) A rod bent into an angle is placed with its corner at the centre of a circle and it is found that $\frac{1}{10}$ of the circle lies within it. If it is placed with its corner on another circle, what part of the circle would be within it?

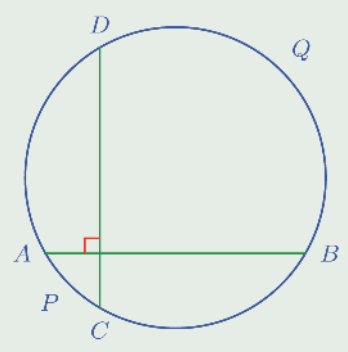


(5) In the picture, O is the centre of the circle and A, B, C , are points on it. Prove that $\angle OAC + \angle ABC = 90^\circ$.

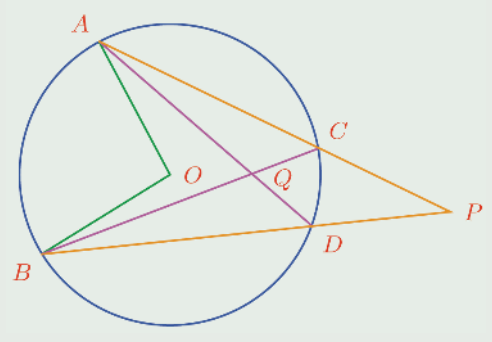


(6) Draw a triangle of circumradius 3 centimetres and two of the angles $32\frac{1}{2}^\circ$ and $37\frac{1}{2}^\circ$.

- (7) In the picture, AB and CD are mutually perpendicular chords of the circle. Prove that the arcs APC and BQD joined together would make half the circle.

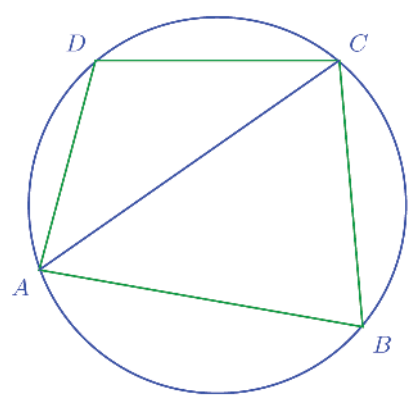
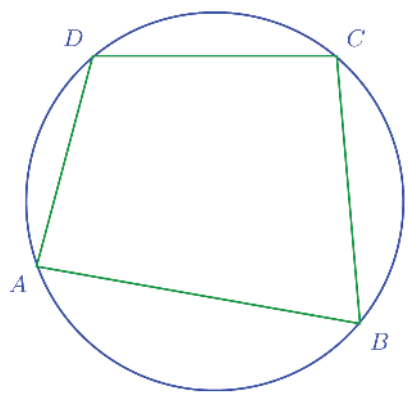


- (8) In the picture, A, B, C, D are points on a circle centred at O . The lines AC and BD are extended to meet at P . The line AD and BC intersect at Q . Prove that the angle which the small arc AB makes at O is the sum of the angles it makes at P and Q .



Circle and quadrilateral

Look at this picture:



Is there a relation between the angles at A, B, C, D ?
Let's join AC .



Using **Polygon**, draw a quadrilateral joining four points on a circle. Click on **Angle** to measure all angles. Do you see any relation between the angles? Change the positions of the points and see.

Now the angles at B and D are the angles on two parts into which the chord AC divides the circle, so they are supplementary.

Similarly, drawing BD , we see the angles at A and C are supplementary.

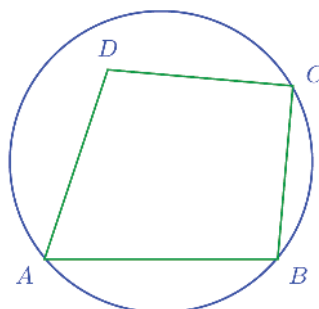
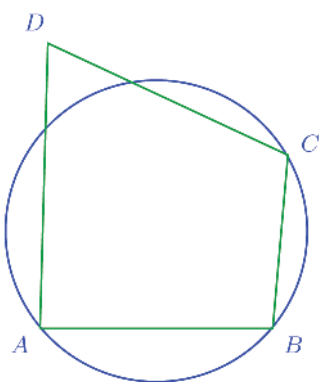
So what can we say in general?

If all four vertices of a quadrilateral are on a circle, then its opposite angles are supplementary.

Is the converse true? That is, if the opposite angles of a quadrilateral are supplementary, can we draw a circle through all four vertices?

To answer this question, let's see how we can check practically whether the four vertices of a given quadrilateral are on a circle.

We can draw a circle through three of the vertices. (Remember how in Class 9, we drew a circle through any three points not on a line). Now, if the fourth vertex is also on the circle, we are done. But this vertex may be outside the circle or inside it.



Look at the first picture. Joining A and the point where CD cuts the circle, we get a quadrilateral with all four vertices on the circle:

So,

$$(1) \quad \angle B + \angle AEC = 180^\circ$$

Now as in the discussion in the section **Circle and right angle**, we see that

$$\angle AEC = \angle EAD + \angle D$$

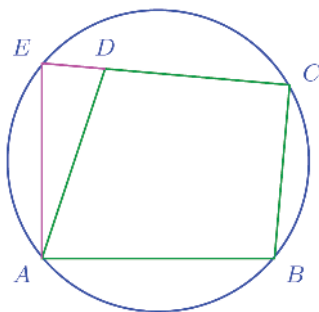
and so

$$(2) \quad \angle D < \angle AEC$$

Here, thinking a bit about the meanings of the relations marked (1) and (2), we see that

$$\angle B + \angle D < 180^\circ$$

Next in the second picture, let's extend CD to meet the circle at E and join AE :



Here we see that

$$(3) \quad \angle B + \angle E = 180^\circ$$

and from $\triangle EAD$

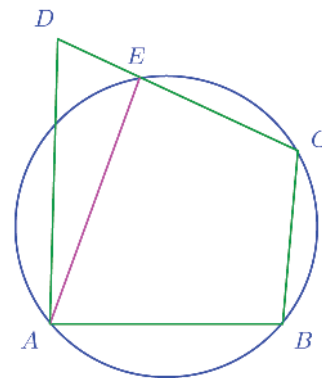
$$\angle ADC = \angle E + \angle EAD$$

so that

$$(4) \quad \angle ADC > \angle E$$

From the relations (3) and (4) we get

$$\angle B + \angle ADC > 180^\circ$$



Mark three points A, B, C on a circle and a point D outside the circle. Use **Polygon** to draw the quadrilateral A, B, C, D and use **Angle** to measure all angles. When D is on the circle, the sum of $\angle B$ and $\angle D$ is 180° . What happens when D is outside? What happens as D is moved away from the centre or towards the centre? What if D is moved inside the circle?

What do we see here?

If one vertex of a quadrilateral is outside the circle drawn through the other three vertices, then the sum of the angles at this vertex and the opposite vertex is less than 180° , if it is inside the circle, the sum is more than 180° .

(We have already seen that if the vertex is on the circle, then the sum of the angles is 180°)

Now suppose in quadrilateral A, B, C, D , we have $\angle B + \angle D = 180^\circ$.

Draw the circle through A, B, C .

Can D be outside the circle? If it is outside, we must have the sum of $\angle B$ and $\angle D$ less than 180° so it is not outside the circle.

Is it inside that circle? If it is inside, the sum of $\angle B$ and $\angle D$ must be greater than 180° , so it is not inside the circle.

Since it is neither outside nor inside the circle, D must be on the circle.

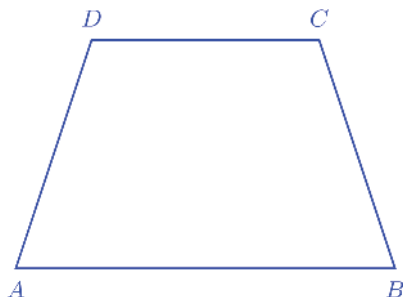
That is,

If the opposite angles of a quadrilateral are supplementary, we can draw a circle passing through all four of its vertices.

Instead of saying, a quadrilateral for which a circle can be drawn through all four vertices, we call it a *cyclic quadrilateral*. As seen now, cyclic quadrilaterals are those quadrilaterals with opposite angles supplementary.

All rectangles are cyclic quadrilaterals, right? So are isosceles trapeziums.

Look at this figure:



$ABCD$ is an isosceles trapezium. So

$$\angle A = \angle B$$

Also, AB and CD are parallel, so that

$$\angle A + \angle D = 180^\circ$$

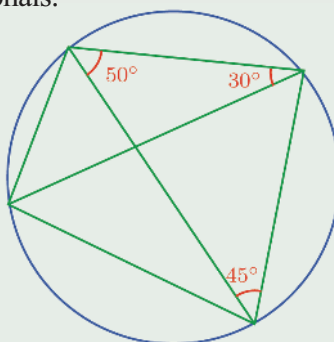
From these two equations, we get

$$\angle B + \angle D = 180^\circ$$

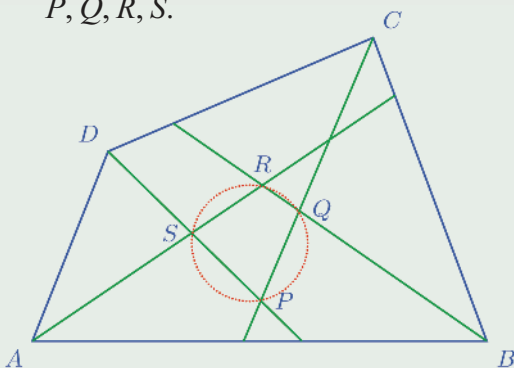
So $ABCD$ is a cyclic quadrilateral.



- (1) Calculate the angles of the quadrilateral in the picture and also the angles between their diagonals:



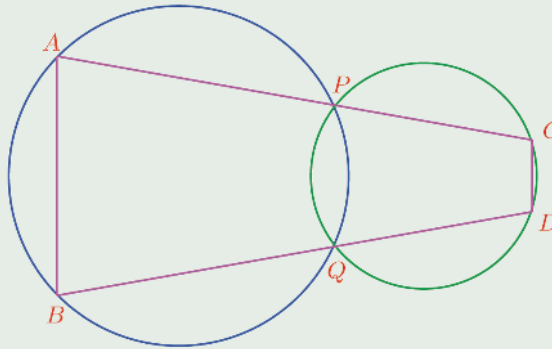
- (2) Prove that any exterior angle of a cyclic quadrilateral is equal to the interior angle at the opposite vertex.
- (3) Prove that a parallelogram which is not a rectangle is not cyclic.
- (4) Prove that a non-isosceles trapezium is not cyclic.
- (5) In the picture, bisectors of adjacent angles of the quadrilateral $ABCD$ intersect at P, Q, R, S .



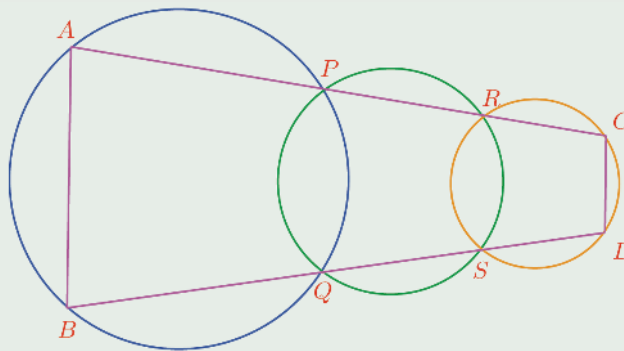
Prove that $PQRS$ is a cyclic quadrilateral.

Draw a quadrilateral and the bisectors of its angles in GeoGebra. Mark the points of intersection of bisectors of adjacent angles and draw the quadrilateral joining these points. Check whether it is cyclic. For this use **Circle Through 3 Points** to draw the circle through three vertices and see whether it passes through the fourth. Change the positions of the vertices of the first quadrilateral to make it a parallelogram, rectangle, square, isosceles trapezium and see what happens to the inner quadrilateral (use **Grid** for this).

- (6) i) The two circles below intersect at P, Q and lines through these points meet the circles at A, B, C, D . Prove that if AC and BD are of equal length, then $ABCD$ is a cyclic quadrilateral.

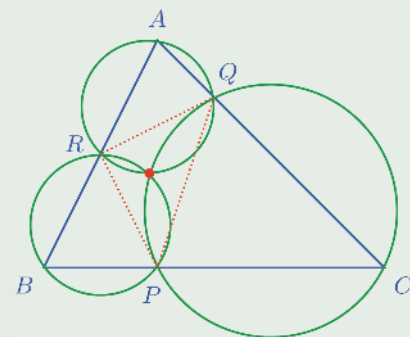


- ii) In the picture, the circles on the left and right intersect the middle circle at P, Q, R, S ; the lines joining them meet the left and right circles at A, B, C, D . Prove that $ABCD$ is a cyclic quadrilateral.



- (7) In the picture, points P, Q, R are marked on the sides BC, CA, AB of $\triangle ABC$ and the circumcircles, of $\triangle AQR, \triangle BRP, \triangle CPQ$ are drawn.

Prove that they pass through a common point.

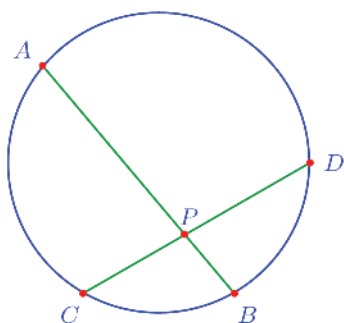
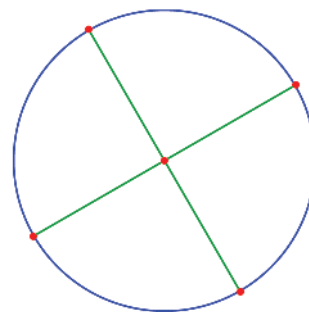


Two chords

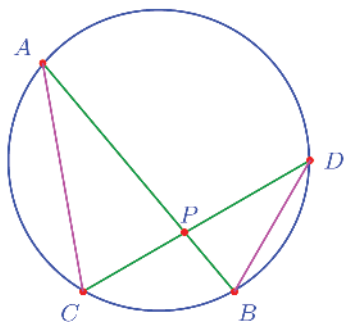
Any two diameters of a circle intersect at the centre, and the length of the four pieces are equal.

What happens when two chords which are not diameters intersect?

See this picture:



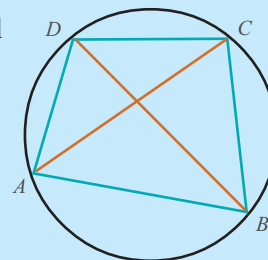
The pieces are not equal; yet there are some relations between them. To see this, we first join AC and BD.



The angles which the small arc BC makes at the points A and D on the alternate arc are equal. So are the angles which the small arc AD makes at B and C.

Ptolemy's Theorem

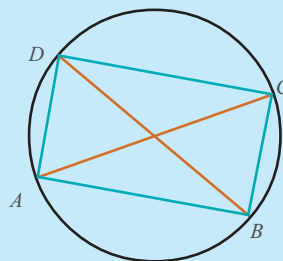
We can prove that the sum of the products of the opposite sides of a cyclic quadrilateral is equal to the product of the diagonals. That is if ABCD is a cyclic quadrilateral, then



$$(AB \times CD) + (AD \times BC) = AC \times BD$$

Conversely, if this is true in a quadrilateral, then it is cyclic. This is known as Ptolemy's Theorem.

Now a rectangle is a cyclic quadrilateral and its opposite sides are equal and so are the diagonals. Thus in a rectangle ABCD Ptolemy's Theorem becomes



$$AB^2 + BC^2 = AC^2$$

This is Pythagoras Theorem!



Thus ΔPAC and ΔPDB have the same angles. So their sides are in the same ratio. So,

$$\frac{PA}{PC} = \frac{PD}{PB}$$

We write it in multiplicative form like this:

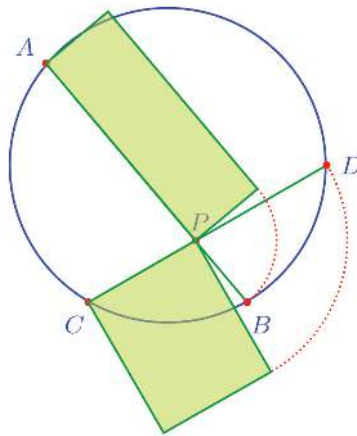
$$PA \times PB = PC \times PD$$

In this, PA, PB are parts of the chord AB and PC, PD are parts of the chord CD . So we can put it like this:

If two chords of a circle intersect within the circle, then the products of the parts of the two chords are equal.

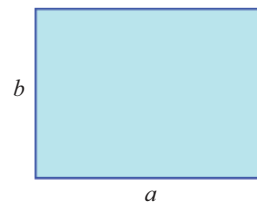
We can interpret the product of two lengths as an area. So this relation can be put in geometric language as below:

If the chords of a circle intersect within a circle, then the rectangles formed by the parts of the same chord have equal area.



We can solve some problems on areas using this.

For example, see this rectangle:



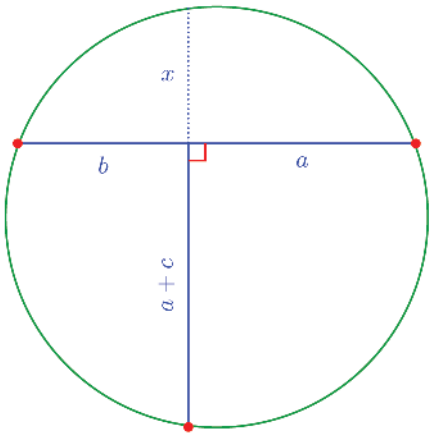
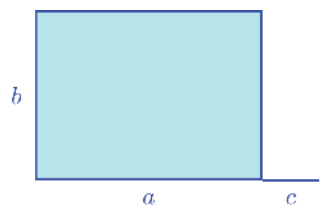
We want to increase one side a little and draw a rectangle, without changing the area.

In algebraic language, the problem is this:

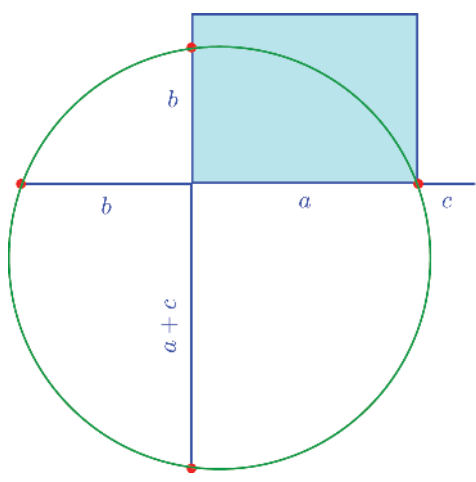
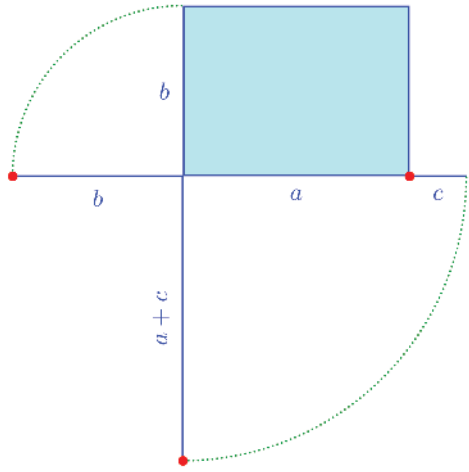
$$\text{Find } x \text{ so that } (a + c)x = ab$$

How about using our result on chords?

We have to draw a picture like this:



So, first we extend the bottom side of the rectangle to the left by b and the left side downwards by $a + c$.



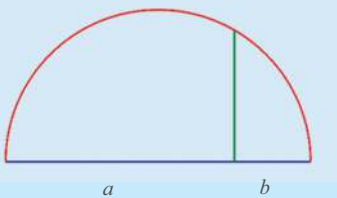
Now draw the circle through the three red dots in the picture. (You know how to draw the circumcircle of a triangle, don't you?)



The part of the left side of the rectangle cut by this circle is the other side of the required rectangle.

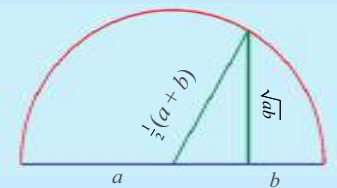
Geometry, algebra and numbers

See this picture:



What is the height of the perpendicular? If we write it as x , then we get $ab = x^2$ or $x = \sqrt{ab}$.

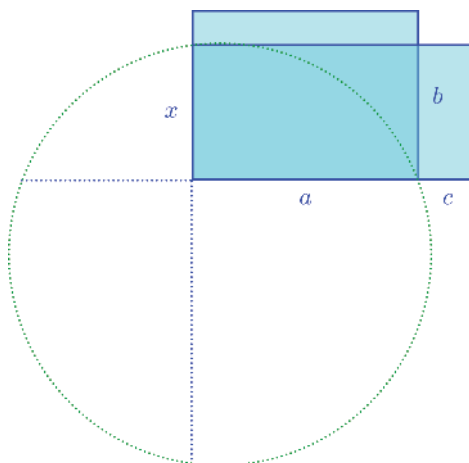
What is the radius of this semicircle? Since diameter is $a + b$, radius is $\frac{1}{2}(a + b)$.



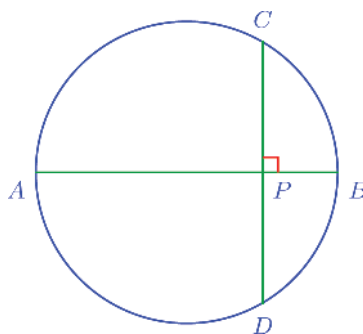
In the picture, radius is longer than the perpendicular. Are they equal in some case?

So, what do we get? For any pair a, b of different numbers

$$\frac{1}{2}(a + b) > \sqrt{ab}$$



A special case of intersecting chords is also of interest. In the picture below, AB is a diameter of the circle and CD is a chord perpendicular to it.

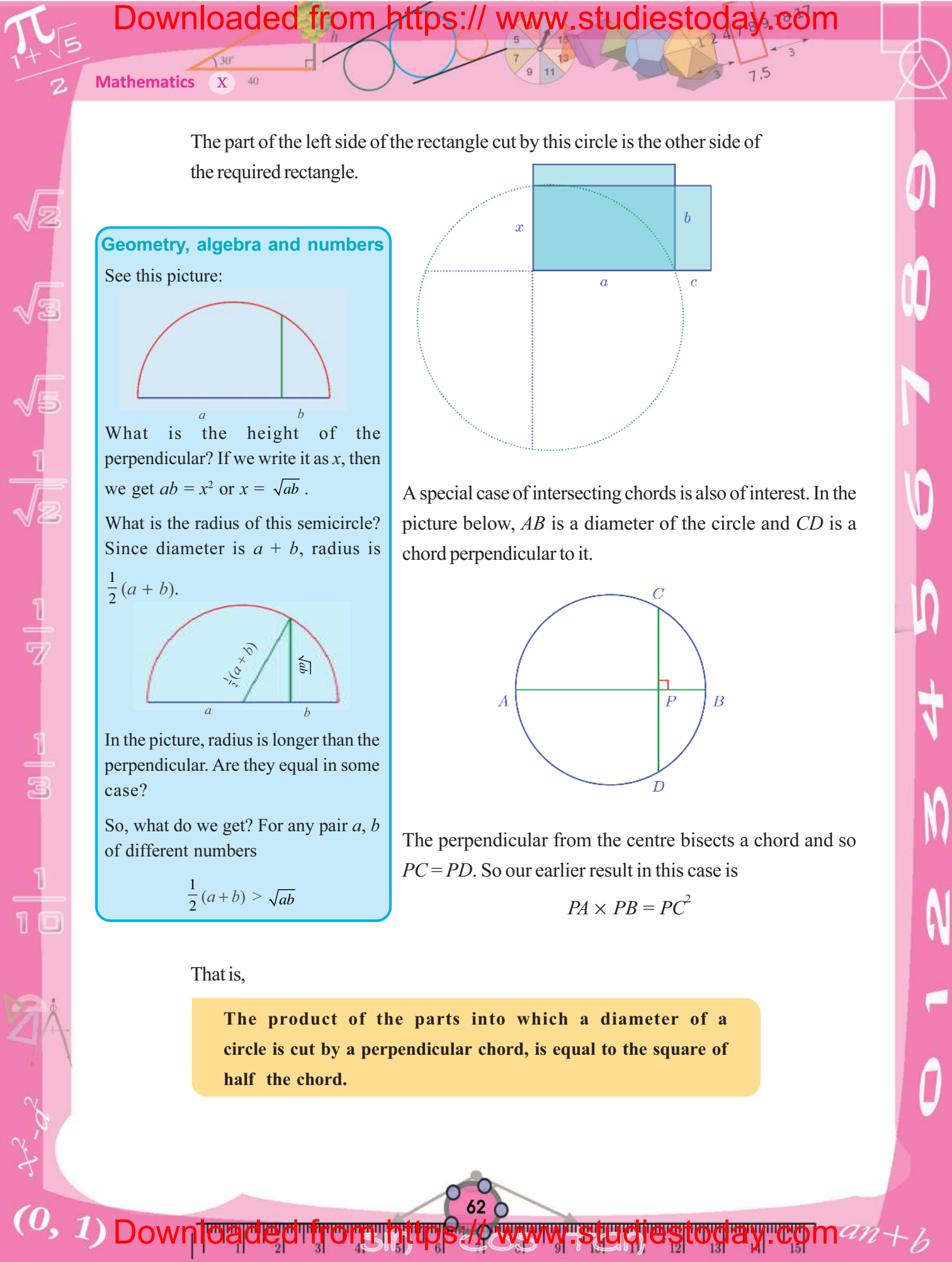


The perpendicular from the centre bisects a chord and so $PC = PD$. So our earlier result in this case is

$$PA \times PB = PC^2$$

That is,

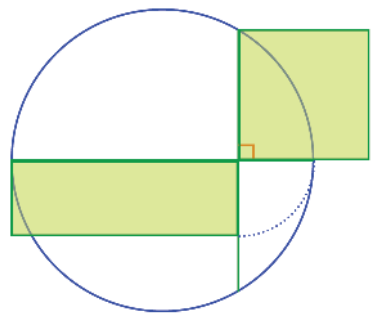
The product of the parts into which a diameter of a circle is cut by a perpendicular chord, is equal to the square of half the chord.





In geometric language, this can be put like this:

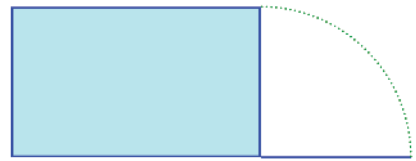
The area of the rectangle formed of parts into which a diameter of a circle is cut by a perpendicular chord is equal to the area of the square formed by half the chord.



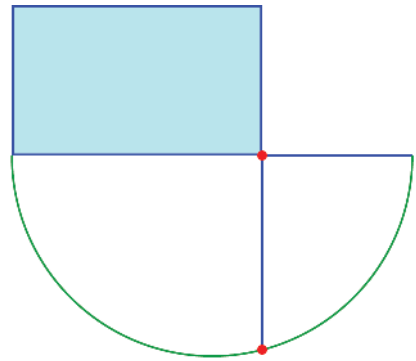
We can use this to change a rectangle into a square of the same area. For example see this rectangle.



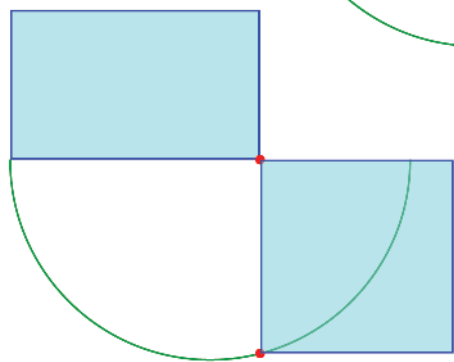
To draw a square of the same area, first extend the width by the height.



Now draw a circle below with this line as diameter. Extend the right side of the rectangle to meet this semi circle.



This line is the side of the required square (why?)



We can also use this to draw a square of specified area.

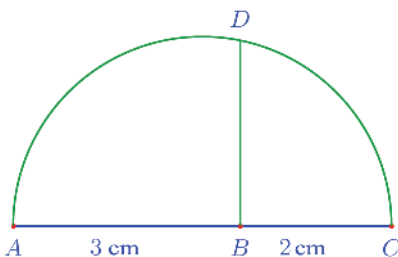




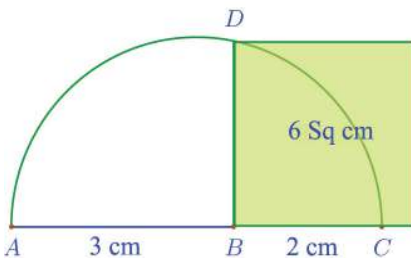
For example, let's see how we can draw a square of area 6 square centimetres. Since $6 = 3 \times 2$ we need only draw a square of area equal to that of a rectangle of side 3 centimetres and 2 centimetres. We need not draw the rectangle, only a line like this:



Now draw a semicircle with AC as diameter and draw the perpendicular to AC at B. Let it meet the semicircle at D.



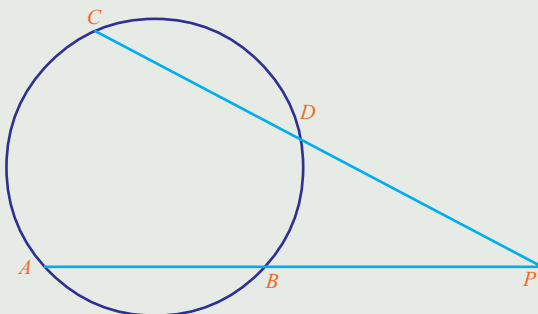
As seen in the general result, $BD^2 = AB \times BC = 6$, so that the square of side BD has an area 6 square centimetres.



Here the length of BD is $\sqrt{6}$ centimetres. Thus we can use this method to draw some lines of irrational lengths.

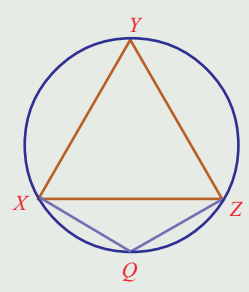
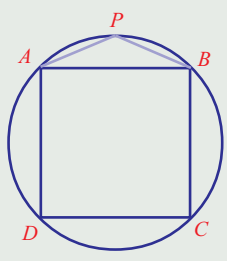


- (1) In the picture, chords AB and CD of the circle are extended to meet at P.





- i) Prove that the angles of $\triangle APC$ and $\triangle PBD$, formed by joining AC and BD , are the same.
 - ii) Prove that $PA \times PB = PC \times PD$.
 - iii) Prove that if $PB = PD$, then $ABDC$ is an isosceles trapezium.
- (2) Draw a rectangle of width 5 centimetres and height 3 centimetres.
 - i) Draw a rectangle of the same area with width 6 centimetres.
 - ii) Draw a square of the same area.
 - (3) Draw a square of area 15 square centimetres.
 - (4) Draw a square of area 5 square centimetres in three different ways.
 - (5) Draw a triangle of sides 4, 5, 6 centimetres and draw a square of the same area
 - (6) Draw an equilateral triangle of height 3 centimetres.
 - (7) Draw an isosceles right triangle of hypotenuse 4 centimetres.
 - (8) In the picture below, $ABCD$ is a square with vertices on a circle and XYZ is such an equilateral triangle. P and Q are points on the circles:



- i) How much is $\angle APB$?
- ii) How much is $\angle XQZ$?



Project

- What is the relation between the angles of a polygon with vertices on a circle and the number of sides even?

Looking back



Learning outcomes	On my own	With teacher's help	Must improve
<ul style="list-style-type: none"> Understanding the relation between the angles which an arc of a circle makes at the centre and at various points on the circle. Realising that of the two parts into which a chord divides a circle, angles on the same part are equal and angles on different parts are supplementary. Recognising quadrilaterals which have circum circles. Understanding the relation between the parts into which chords of a circle cut each other. Converting a rectangle into another rectangle or square without changing the area. 			



Mathematics of Chance

Possibilities as numbers

There are ten beads in a box, nine black and only one white. If we pick a bead (without looking)...

It is most likely to be black; can be white though.

There are eight black beads and two white in another box. How about picking one from this?

Again it is more likely to be black.

Five black and five white in the third. What if we pick a bead from this? Could be black or white; can't say anything more?

We can say all these in a different way. From the first and the second box, the probability of getting a black bead is more. From the third box, the probabilities are the same.

Let's have a game with beads. Five black and five white beads in one box. Six black and four white in another. One has to choose a box and pick a bead. If it is black, he wins. Which box is the better choice?

The second box contains more blacks. So don't we have a greater probability of getting a black from it?

Suppose we take a black bead from the second box and put it in the first.

So this is how the boxes are now:

First box : 6 black 5 white

Second : 5 black 4 white

Now to win the game, which box would you choose?

The first box contains more black beads. Is the probability of getting a black from it also more?

Let's think in terms of totals. The first box has 11 beads in all, of which 6 are black. That is, $\frac{6}{11}$ of the total are black.

What about the second? $\frac{5}{9}$ of the total are black.

Which is larger, $\frac{6}{11}$ or $\frac{5}{9}$?

$\frac{5}{9}$, isn't it?

Thus, the second box has a larger black part. So isn't it still the better choice?

In other words, the probability of getting a black bead from the second box is larger. We can go even further and say that the probability of getting a black bead from the first box is $\frac{6}{11}$ and the probability of getting a black bead from the second box is $\frac{5}{9}$.

What about the probabilities of getting a white bead? $\frac{5}{11}$ from the first and $\frac{4}{9}$ from the second. Which is larger? So if it is white for a win, which box is the better choice?

Let's write down the various probabilities in a table:

		First box		Second box	
		Black	White	Black	White
First	Number	5	5	6	4
	Probability	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{2}{5}$
Later	Number	6	5	5	4
	Probability	$\frac{6}{11}$	$\frac{5}{11}$	$\frac{5}{9}$	$\frac{4}{9}$

And another question. We have seen that at the beginning and after transferring a bead, the probability of getting a black bead from the second box is larger.

Is this probability less or more after the transfer?

Let's look at another problem.

Numbers from 1 to 25 are written on paper slips and put in a box. One slip is taken from it. What is the probability that it is an even number?

Of all the 25 numbers, 13 are odd and 12 even, right? So the probability of getting an even number is $\frac{12}{25}$.

What about the probability of getting an odd number?

What is the probability of getting a multiple of three? A multiple of six?

?



- (1) A box contains 6 black and 4 white balls. If a ball is taken from it, what is the probability of it being black? And the probability of it being white?
- (2) There are 3 red balls and 7 green balls in a bag, 8 red and 7 green balls in another.
 - i) What is the probability of getting a red ball from the first bag?
 - ii) From the second bag?
 - iii) If all the balls are put in a single bag, what is the probability of getting a red ball from it?
- (3) One is asked to say a two-digit number. What is the probability of it being a perfect square?
- (4) Numbers from 1 to 50 are written on slips of paper and put in a box. A slip is to be drawn from it; but before doing so, one must make a guess about the number: either prime number or a multiple of five. Which is the better guess? Why?
- (5) A bag contains 3 red beads and 7 green beads. Another contains one red and one green more. The probability of getting a red from which bag is more?

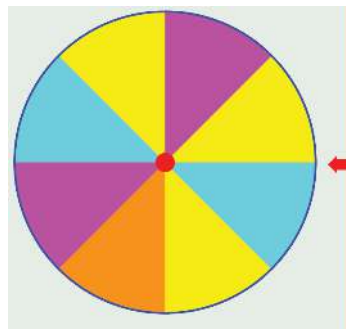
Geometrical probability

A multicoloured disc spins on a board.

What is the probability of getting yellow against the arrow when it stops?

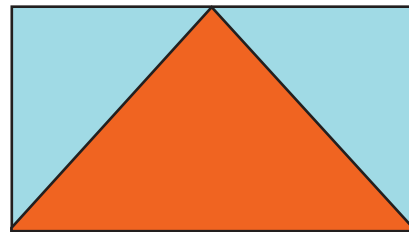
Any of the eight sectors can come against the arrow. Of these, three are yellow.

So, the probability of yellow is $\frac{3}{8}$.



Can you calculate the probabilities of other colours?

Let's try another problem. A cardboard rectangle is cut out and the midpoint of one side is joined to the ends of the opposite sides to make a triangle. If you shut your eyes and put a dot in this rectangle, what is the probability that it would be within the red triangle?



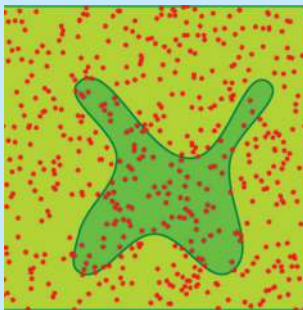
The triangle and rectangle have the same base and height, haven't they?

So the triangle is half the rectangle.

That is, the area of the triangle is $\frac{1}{2}$ of the area of the rectangle. So the probability of the dot falling within the triangle is also $\frac{1}{2}$.

Area and probability

Probability can be used to find areas of complex regions approximately. Draw the region within a specific square and mark points within the square, without any order or pattern.

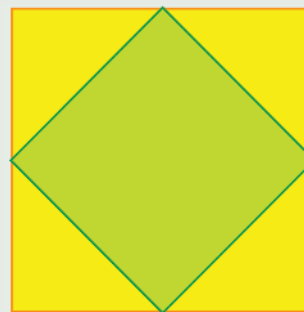


The number of dots within our region divided by the total number of dots would be approximately equal to the area of the region divided by the area of the square. And it becomes better as we increase the number of dots. The generation of dots and the calculations can be done quickly using a computer. It is called the Monte carlo method.

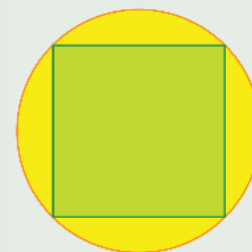


In each picture below, the explanation of the green part is given. Calculate in each, the probability of a dot put without looking to be within the green part.

- (1) A square got by joining the mid points of a bigger square.

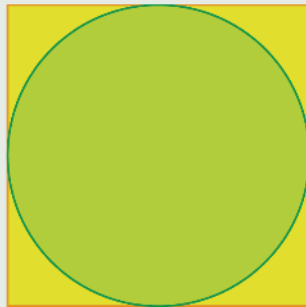


- (2) A square with all vertices on a circle.

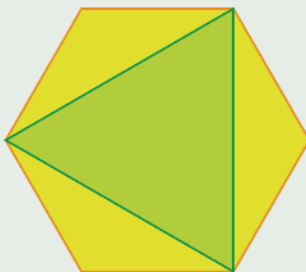




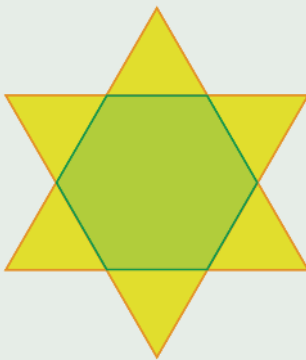
(3) Circle exactly fitting inside a square.



(4) A triangle got by joining alternate vertices of a regular hexagon.

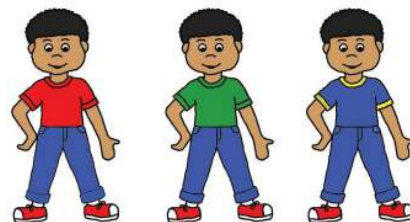


(5) A regular hexagon formed by two overlapping equilateral triangles.

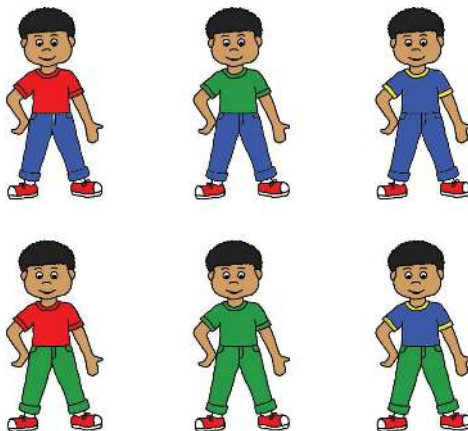


Pairs

Looking for a clean dress, Johnny found a pair of blue pants and three shirts, red, green and blue. "In how many ways can I dress?", thought Johnny.



Searching again he found a pair of green pants also. Now this can be worn with each of the three shirts, giving three more ways to dress, Johnny calculated.



A problem

Galileo has written about a problem posed by a gambler friend. This friend calculated six different ways of getting the sum 9 and sum 10 from three dice:

	9	10
1.	1 + 2 + 6	1 + 3 + 6
2.	1 + 3 + 5	1 + 4 + 5
3.	1 + 4 + 4	2 + 2 + 6
4.	2 + 2 + 5	2 + 3 + 5
5.	2 + 3 + 4	2 + 4 + 4
6.	3 + 3 + 3	3 + 3 + 4

But his experience showed that 10 occurs more. Galileo's solution was like this. The gambler had taken 1, 2, 6 to mean 1 from any one die, 2 from another and 3 from yet another. Instead of this, write (1, 2, 6) to mean 1 from the first die, 6 from the second and 2 from the third. Thus (1, 6, 2), (1, 6, 2), (2, 1, 6), (2, 6, 1), (6, 1, 2), (6, 2, 1) gives six different ways for the sum 9. Expanding all possible triples like this, Galileo computes the total number of ways of getting 9 as 25 and getting 10 as 27. (Try it!)

Thus Johnny can dress in six different ways.

In how many of these are the colours of shirt and pants the same?

So, what is the probability of Johnny wearing shirt and pants of the same colour?

$$\frac{2}{6} = \frac{1}{3}, \text{ right?}$$

Let's look at another problem. A box contains four paper slips numbered 1, 2, 3, 4 and another contains two slips numbered 1, 2. One slip is picked from each. What are the possible pairs?

With 1 from the first box, we can combine 1 or 2 from the second box to form two pairs. Let's write them as (1, 1) and (1, 2).

How about writing down all such pairs, combine each number from the first box with the two possibilities from the second?

- (1, 1), (1, 2)
- (2, 1), (2, 2)
- (3, 1), (3, 2)
- (4, 1), (4, 2)

8 pairs in all.

In how many of these are both numbers odd?

Only in (1, 1) and (3, 1), right? So if we take one slip from each, the probability of both being odd is $\frac{2}{8} = \frac{1}{4}$.

Can you compute the probability of both being even? The probability of one odd and the other even? And both being the same number?

?



- (1) Rajani has three necklaces and three pairs of earrings, of green, blue and red stones. In what all different ways can she wear them? What is the probability of her wearing the necklace and earrings of the same colour? Of different colours?
- (2) A box contains four slips numbered 1, 2, 3, 4 and another box contains two slips numbered 1, 2. If one slip is taken from each, what is the probability of the sum of numbers being odd? What is the probability of the sum being even?
- (3) A box contains four slips numbered 1, 2, 3, 4 and another contains three slips numbered 1, 2, 3. If one slip is taken from each, what is the probability of the product being odd? The probability of the product being even?
- (4) From all two-digit numbers with either digit 1, 2, or 3 one number is chosen.
 - i) What is the probability of both digits being the same?
 - ii) What is the probability of the sum of the digits being 4?
- (5) A game for two players. First, each has to decide whether he wants odd number or even number. Then both raises some fingers of one hand. If the sum is odd, the one who chose odd at the beginning wins; if it is even, the one who chose even wins. In this game, which is the better choice at the beginning, odd or even?

More pairs

Two boxes again, one containing ten slips numbered 1 to 10 and the other, five slips from 1 to 5. One slip is taken from each box, as usual. What is the probability of both being odd?

The method of solution is simple. Calculate the number of all possible pairs of numbers and then find how many of them have both odd, as required. The second number divided by the first gives the probability.

Easier said than done. It is not very interesting to list all pairs and count.

Probability and frequency

Usually if a coin is tossed a number of times, head and tails would come up more or less equally. But sometimes because of some defect in manufacture or something else, heads may come up more often. How do we find it out?

When the coin is repeatedly tossed, if the number of times each face comes up differs considerably from half, such a doubt is reasonable. So the method is to toss again and again and record the number times each face comes up. For example, look at this table.

Toss	Head	Tail
10	6	4
100	58	42
1000	576	424
10000	5865	4135

Here we can see that it is better to take the probability of a head as 0.6 and probability of a tail as 0.4 instead of 0.5 for both. Methods of making such computations precise are part of the branch of mathematics called Probability Theory.

Let's think. The number from the first box can be any one of the ten numbers; and the number from the second, any one of the five. So how many pairs are there with first number 1? How many with first number 2?

In short, after fixing the first number, we can make 5 different pairs, by changing the second. The first number can be fixed in 10 ways. Thus we can imagine all the pairs arranged like this:

	◀ 5 pairs. ▶			
↑ 10 pairs. ↓	(1, 1)	(1, 2)	...	(1, 5)
	(2, 1)	(2, 2)	...	(2, 5)

	(10, 1)	(10, 2)	...	(10, 5)

Of these 50 pairs, how many have both odd numbers? For that, the first number must be one of the 5 numbers 1, 3, 5, 7, 9; and the second number one of the 3 numbers 1, 3, 5. Taking each of the first 5 numbers with each of the second 3 numbers, how many different pairs can we form?

5×3 , isn't it? (Imagine them as rows and columns if you want). So the probability of getting odd numbers from both these boxes is $\frac{15}{50} = \frac{3}{10}$. Similarly, can you calculate the

probability of getting even numbers from both and odd from one and even from the other?

One more problem: there are 50 mangoes in a basket, 20 of which are unripe. Another basket contains 40 mangoes, with 15 unripe. If we take one mango from each basket, what is the probability of both being ripe?

In how many different ways can we choose a pair of mangoes, one from each basket? (If you want, imagine the mangoes from each basket laid on a line; you can also imagine them to be numbered).

So there are $50 \times 40 = 2000$ ways of taking a pair of mangoes. How many of these have both ripe?

The first basket has $50 - 20 = 30$ ripe ones and the second, $40 - 15 = 25$.

Each ripe mango from the first basket paired with a ripe mango from the second gives $30 \times 25 = 750$ pairs. So the probability of both being ripe is $\frac{750}{2000} = \frac{3}{8}$.

Similarly, can't you compute the probability of both being unripe?

What is the probability of getting at least one ripe mango?

At least one ripe means one ripe and the other unripe, or both ripe. In these, one ripe and the other unripe can occur in two ways: the first ripe and the second unripe or the other way round. Thus the total number of pairs with only one ripe is

$$\begin{aligned}(30 \times 15) + (20 \times 25) &= 450 + 500 \\ &= 950\end{aligned}$$

We have already seen that the number of pairs with both ripe is 750. Taking all these together, the number of pairs with at least one ripe is

$$950 + 750 = 1700$$

Measure of uncertainty

Have you noted the time of sunrise and sunset for each day given in a calendar? These can be computed since the Earth and Sun move according to definite mathematical laws. Similarly summer months and rainy months can be computed. But it may not be possible to predict a sudden shower in summer season. The large number of factors influencing rainfall and the complexities of their inter relations make such predictions difficult.

But analysing the context mathematically, probabilities can be calculated. That is why daily weather forecasts are often given as probabilities. Unexpected changes in the circumstances sometimes make them wrong.

If we look at it rationally, we can see that such probabilistic predictions are more reliable than supposedly certain predictions with no scientific basis.

So, the probability of getting at least one ripe mango is $\frac{1700}{2000} = \frac{17}{20}$.

This can be found in a different way. To have at least one ripe, both cannot be unripe. Of all the 2000 possible pairs, both are unripe in $20 \times 15 = 300$ pairs.

So, in the remaining $2000 - 300 = 1700$ pairs, at least one must be ripe.

Thus the probability of getting at least one ripe mango is $\frac{1700}{2000} = \frac{17}{20}$.

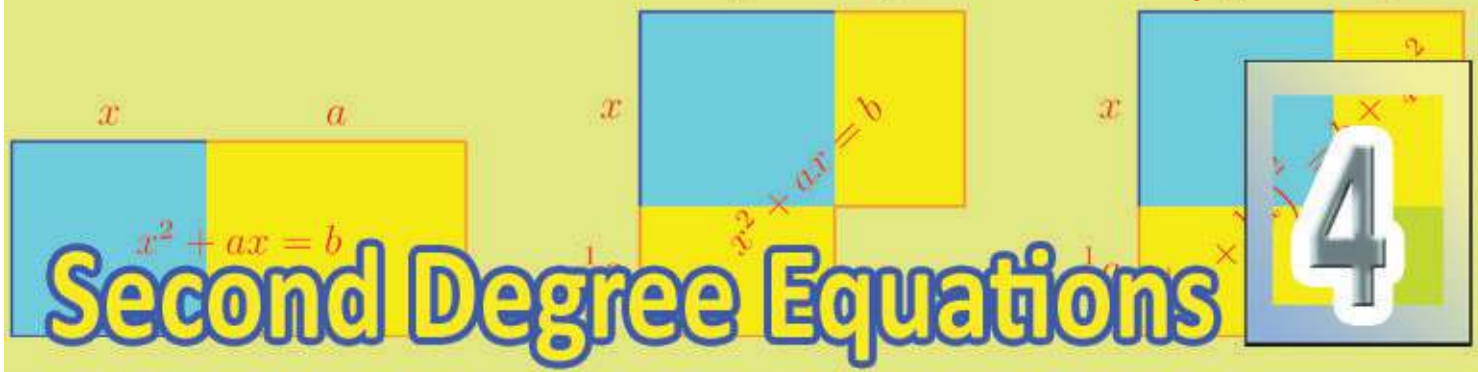


- (1) In class 10A, there are 20 boys and 20 girls. In 10B, there are 15 boys and 25 girls. One student is to be selected from each class.
 - i) What is the probability of both being girls?
 - ii) What is the probability of both being boys?
 - iii) What is the probability of one boy and one girl?
 - iv) What is the probability of at least one boy?
- (2) One is asked to say a two-digit number.
 - i) What is the probability of both digits being the same?
 - ii) What is the probability of the first digit being larger?
 - iii) What is the probability of the first digit being smaller?
- (3) Two dice with faces numbered from 1 to 6 are rolled together. What are the possible sums? Which of these sums has the maximum probability?

Looking back



Learning outcomes	On my own	With teacher's help	Must improve
<ul style="list-style-type: none"> • Explaining probability as a number. • Explaining the method to calculate probability in different contexts. • Justifying the need to analyze probability using numbers in practical situations. 			



Second Degree Equations

Square problems

Let's start with a problem:

When each side of a square is increased by 1 metre, the perimeter becomes 36 metres. What is the length of a side of the original square?

We can easily do this by calculating a side of the new square as $36 \div 4 = 9$ metres and then, a side of the original square as $9 - 1 = 8$ metres.

Suppose we change the question like this:

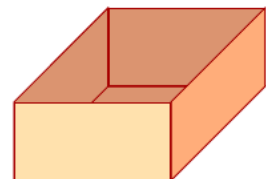
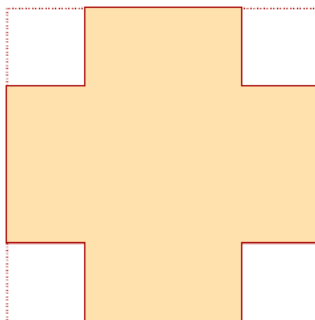
When each side of a square is increased by 1 metre, the area becomes 36 square metres. What is the length of a side of the original square?

What is the length of a side of the new square?

$\sqrt{36} = 6$ metres, isn't it? So the length of a side of the original square is $6 - 1 = 5$ metres.

Now look at this problem:

A box is to be made by cutting off small squares from each corner of a square of thick paper, and bending upwards. The height of the box is to be 5 centimetres and volume $\frac{1}{2}$ litre.



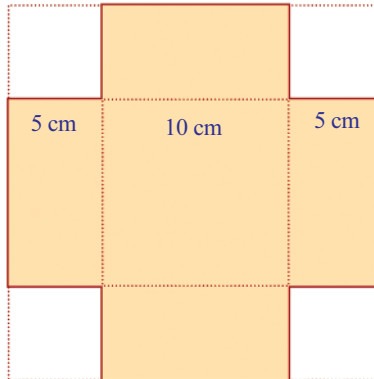
What should be the length of a side of the square sheet we start with?

The volume of the box is the product of the base area and height, right?

We want the volume to be $\frac{1}{2}$ litre, which is 500 cubic centimetres, and the height 5 centimetres.



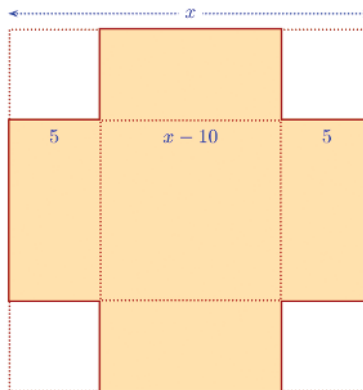
So, the base area of the box must be $500 \div 5 = 100 = 500$ square centimetres.
Since the base is a square (why?) each of its sides must be 10 centimetres.



This square is got by subtracting $2 \times 5 = 10$ centimetres from each side of the square we started with.

So, a side of the original square must be $10 + 10 = 20$ centimetres.

Instead of such reverse thinking, we can first think directly and put the problem in algebraic form. If the length of a side of the original square is taken as x centimetres, the base of the box can be seen to be a square of side $(x - 10)$ centimetres:



Since the height of the box is 5 centimetres, its volume is $5(x - 10)^2$ cubic centimetres.

Thus the problem is translated to algebra like this:

What should be x to get $5(x - 10)^2 = 500$?

Then we can think in reverse like this:

- If $5(x - 10)^2 = 500$, then $(x - 10)^2 = 500 \div 5 = 100$
- If $(x - 10)^2 = 100$, then $x - 10 = \sqrt{100} = 10$
- If $x - 10 = 10$, then $x = 10 + 10 = 20$



Do these problems using algebra or otherwise:

- (1) When each side of a square was reduced by 2 metres, the area became 49 square metres. What was the length of a side of the original square?
- (2) A square ground has 2 metre wide path all around it. The total area of the ground and path is 1225 square metres. What is the area of the ground alone?
- (3) The square of a term in the arithmetic sequence 2, 5, 8, ..., is 2500. What is its position?
- (4) 2000 rupees was deposited in a scheme in which interest is compounded annually. After two years the amount in the account was 2200 rupees. What is the rate of interest?

Square completion

See the picture:



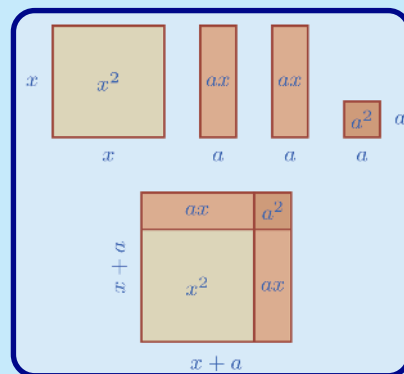
A green square, two red rectangles of the same height and a small yellow square are kept together. The width of the red rectangles and the side of the yellow square are all 1 metre. And the total area of the entire figure is 100 square metres.

Geometric square

We have seen that

$$x^2 + 2ax + a^2 = (x + a)^2$$

In the case where x and a are positive numbers, this can be given a geometric form:





We have to calculate the length of a side of the green square. Not very easy to calculate directly, is it?

Let's try algebra. Take the length of a side of the green square to be x centimetres.

Then the total area is

$$x^2 + x + x + 1 = x^2 + 2x + 1$$

We are told that the total area is 100 square metres. So the algebraic form of the problem is this:

$$\text{If } x^2 + 2x + 1 = 100, \text{ then what is } x?$$

Does the expression $x^2 + 2x + 1$ look familiar?

We have seen in the lesson, **Identities** of Class 8, that

$$(x + 1)^2 = x^2 + 2x + 1$$

We can also see this by rearranging the pieces of our figure.

So we can rephrase our problem:

$$\text{If } (x + 1)^2 = 100 \text{ then what is } x?$$

Now we can see that, $x + 1 = 10$ and so $x = 9$.

Thus the length of a side of the green square is 9 metres.

Let's look at another problem:

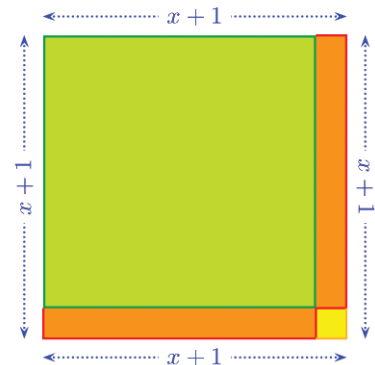
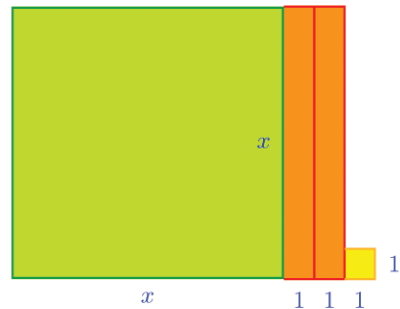
One side of a rectangle is 2 metres longer than the other side and its area is 224 square metres. What are the lengths of the sides?

We first put the problem in algebra. Taking the length of the shorter side as x metres, the length of the longer side is $x + 2$ metres, the area is $x(x + 2) = x^2 + 2x$ square metres.

Now we translate the geometric problem to an algebraic problem:

$$\text{If } x^2 + 2x = 224, \text{ then what is } x?$$

What should we do next?



Look at the first problem again. We changed $x^2 + 2x + 1$ to $(x + 1)^2$ and went ahead. In this problem we have only $x^2 + 2x$.

We need only to add 1, right?

So we can proceed like this:

- If $x^2 + 2x = 224$, then $x^2 + 2x + 1 = 224 + 1 = 225$
- That is, $(x + 1)^2 = 225$
- If $(x + 1)^2 = 225$, then $x + 1 = \sqrt{225} = 15$
- If $x + 1 = 15$, then $x = 14$

Thus we get the shorter side as 14 metres. So, the longer side is 16 metres.

Suppose we change the question slightly like this:

One side of a rectangle is 20 metres longer than the other and its area is 224 square metres. What are the lengths of the sides?

The algebraic form gets changed like this:

$$\text{If } x^2 + 20x = 224, \text{ then what is } x?$$

Here also, by adding 1, the number on the right side of the equation becomes $225 = 15^2$; but then the left side becomes $x^2 + 20x + 1$. Can we write it as $(x + a)^2$ for some a ?

How do we change $x^2 + 20x$ into the form of a square?

We know that for any number a ,

$$(x + a)^2 = x^2 + 2ax + a^2.$$

In our problem, we have $20x$ in the place of $2ax$ in the general identity.

So, how about taking a as 10?

$$(x + 10)^2 = x^2 + 20x + 100$$

In our problem, $x^2 + 20x = 224$. As seen now, let's add 100 to proceed:

$$x^2 + 20x = 224$$

$$x^2 + 20x + 100 = 324$$

Different way

There is another method to solve $x(x + 20) = 224$. Write $x + 20$ as $(x + 10) + 10$ and x as $(x + 10) - 10$.

Then we have,

$$x(x + 20) = ((x + 10) - 10)((x + 10) + 10) = (x + 10)^2 - 10^2$$

So, our original equation becomes

$$(x + 10)^2 - 100 = 224$$

From this we get

$$(x + 10)^2 = 324$$

We can now find x as before. See if you can solve $x^2 + 10x = 3000$ by this method

$$(x + 10)^2 = 324$$

$$x + 10 = \sqrt{324} = 18$$

$$x = 8$$

Thus we can find the lengths of the sides of the rectangle as 8 metres and 28 metres.

Here's another rectangle problem:

Square again!

You know that among all rectangle of perimeter 20 centimetres, the square of side 5 centimetres has the maximum area.

We can see it like this also. Taking a side of such a rectangle as x centimetres, the area is,

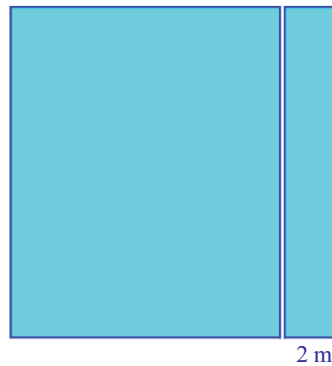
$$p(x) = x(10 - x) = 10x - x^2 = -(x^2 - 10x)$$

We can find the areas of all such rectangles from this polynomial. We can write it as

$$p(x) = -((x - 5)^2 - 25) = 25 - (x - 5)^2$$

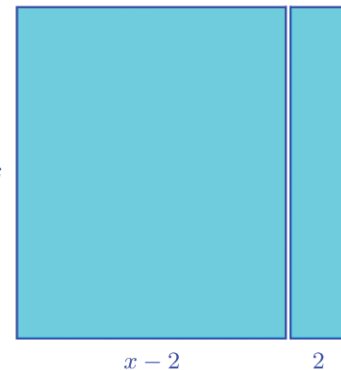
Now whatever number we take as x in this, $(x - 5)^2$ cannot be a negative number. So the number $p(x)$ is never greater than 25. And taking $x = 5$, we do get $p(x) = 25$.

A 2 metre wide strip is cut off from a square as shown below:



The area of the remaining rectangle is 99 square metres. What is the length of a side of the square?

Take the length of a side of the square as x metres, then the length of the sides of the remaining rectangle are x metres and $(x - 2)$ metres.



So, the area of the remaining rectangle is $x(x - 2) = x^2 - 2x$ square metres.

Thus the problem becomes this:

$$\text{If } x^2 - 2x = 99, \text{ then what is } x?$$

Can we change $x^2 - 2x$ to a square form, as we did with $x^2 + 2x$?

Recall another identity from Class 8:

$$x^2 - 2x + 1 = (x - 1)^2$$

Now we can find x as follows:

$$x^2 - 2x = 99$$

$$x^2 - 2x + 1 = 100$$

$$(x - 1)^2 = 100$$

$$x - 1 = 10$$

$$x = 11$$

Thus the length of a side of the original square is 11 metres.

Look at this problem:

A rectangle is to be made with perimeter 100 metres and area 525 square metres. What should be the length of its sides?

The sum of the lengths of the sides is 50 metres, right? So, taking the length of a side as x metres, the length of the other side is $(50 - x)$ metres; and the area is $x(50 - x) = 50x - x^2$ square metres. So we can write the problem like this.

If $50x - x^2 = 525$, then what is x ?

Had the left side of the equation been $x^2 - 50x$, we could have proceeded as before. So let's rewrite the equation in a slightly different form. The number x^2 subtracted from the number $50x$ should give 525.

So, subtracting the other way round should give its negative, -525 .

We can therefore rewrite the problem like this:

If $x^2 - 50x = -525$, then what is x ?

Now we add a number to $x^2 - 50x$ to put it in square form. What is the number?

$$(x - 25)^2 = x^2 - 50x + 625$$

Adding and subtracting

There is another method to find the rectangle of perimeter 100 metres.

Since the sum of the lengths of the side is 50 metres, we can take the length of the side as $25 + x$ metres and $25 - x$ metres.

Then the area is $(20 - x)(25 + x) = 625 - x^2$ square metres.

Now we can compute x as follows

$$625 - x^2 = 525$$

$$x^2 = 100$$

$$x = 10$$

The lengths of the sides
 $25 - 10 = 15$ metres,
 $25 + 10 = 35$ metres

Now we can solve our problem as follows:

$$x^2 - 50x = -525$$

$$x^2 - 50x + 625 = -525 + 625 = 100$$

$$(x - 25)^2 = 100$$

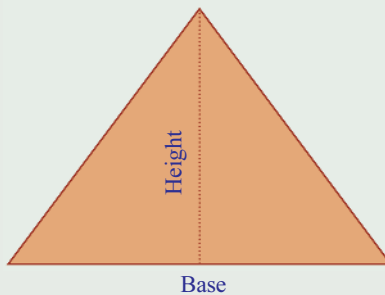
$$x - 25 = 10$$

$$x = 35$$

Thus the lengths of the sides of the rectangle are 35 metres and 15 metres.

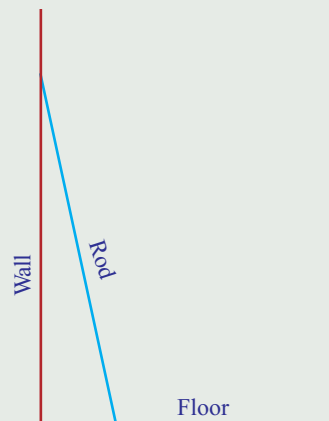


- (1) 1 added to the product of two consecutive even numbers gives 289. What are the numbers?
- (2) 9 added to the product of two consecutive multiples of 6 gives 729. What are the numbers?
- (3) An isosceles triangle has to be made like this



The height should be 2 metres less than the base. What should be the length of its sides?

- (4) A 2.6 metres long rod leans against a wall, its foot 1 metre from the wall. When the foot is moved a little away from the wall, its upper end slides the same length down. How much farther is the foot moved?



- (5) 16 added to the sum of the first few terms of the arithmetic sequence 9, 11, 13, ... gave 256. How many terms were added?

- (6) How many terms of the arithmetic sequence 5, 7, 9, ..., must be added to get 140?
- (7) A mathematician travelled three hundred kilometres to attend a conference. During his talk he said:

“Had my average speed been increased by 10 kilometres per hour, I could have reached here one hour earlier.”

What was the average speed?

- (8) Thirty sweets were distributed equally among some kids. Sucking in the sweetness, a budding mathematician said,

“Had we been one less, each would have got one more sweet.”

How many kids were there?

Two answers

You have learnt something about the relation between time and speed, haven't you? To calculate the distance travelled by an object moving along a straight line with unchanging speed, we need only multiply the speed and distance. This can be written as an algebraic equation like this; for an object moving along a straight line at the speed of u metre / second, the distance s metres travelled in time t seconds is given by

$$s = ut$$

Now suppose the speed changes with time. If the speed increases continuously, the distance travelled in each second also increases. If the speed decreases continuously, the distance travelled in each second also decreases. There is a mathematical rule in such changes of distance. Starting with a speed of u metre / second, if the speed increases at the rate of a metre / second every second, then the distance s metres from the start, at t seconds is given as,

$$s = ut + \frac{1}{2} at^2$$

If the speed decreases at the rate of a metres/second every second, then

$$s = ut - \frac{1}{2} at^2$$



Now look at this problem:

An object starts to move along a straight line at a speed of 40 metres/second and the speed decreases at the rate of 8 metres/second every second. What is the relation between the time of travel and distance from the starting point?

Taking the distance at t second as s metres, we have, as mentioned above,

$$s = 40t - \frac{1}{2} \times 8 \times t^2 = 40t - 4t^2$$

Using this, we can calculate how far the object is from the starting point at any time:

Time:	1	2	3	4	5	6
Distance:	36	64	84	96	100	96

Why does the distance increase for sometime and then decrease?

The starting speed is 40 metres/second and speed decreases each second by 5 metres/second. So, at 5 seconds, the speed becomes 0. Thereafter, it travels in the reverse direction. (The very cause of the decrease in speed is a force in the opposite direction).

Extending our table a little more we can clearly see the return journey:

Time:	1	2	3	4	5	6	7	8	9	10	11
Distance:	36	64	84	96	100	96	84	64	36	0	-44

Thus at 10 seconds it comes back to the starting position; at 11 seconds, it is 44 metres away on the other side. This is what the negative sign indicates.

Using this table, we can find the distance at any time; on the other hand, how do we find the time to reach a specified distance?

For example, we can ask a question like this:

At what time is it 99 metres away from the start?

For that, $40t - 4t^2$ must be 99. As before, we rewrite the equation $40t - 4t^2 = 99$ as

$$4t^2 - 40t = -99$$

Here the coefficient of t^2 is 4. First we make it 1. (It was 1 in all our earlier problems.) For that we divide by 4.

$$t^2 - 10t = \frac{-99}{4}$$

Now we add a number to $t^2 - 10t$ to change it to square form. What is the number?

$$(t - 5)^2 = t^2 - 10t + 25$$

So we add 25 in our equation and proceed.

(Here 25 is the square of half the coefficient of t)

$$t^2 - 10t + 25 = 25 - \frac{99}{4} = \frac{1}{4}$$

$$(t - 5)^2 = \frac{1}{4}$$

$$t - 5 = \frac{1}{2}$$

$$t = 5 \frac{1}{2}$$

So, at $5 \frac{1}{2}$ seconds, the object would be 99 metres from the start.

But then the pattern of the table shows that at the same time before and after 5 seconds, the object is at the same position. For example, at 4 seconds and 6 seconds, it is 96 metres from the start. Likewise at 3 seconds and 7 seconds, it is 84 metres from the start. So it seems as if it should be at the same position $\frac{1}{2}$ second before and after 5 seconds. We have seen that after $\frac{1}{2}$ second, that is at $5 \frac{1}{2}$ seconds, the object is 99 metres away. As seen now, shouldn't it be 99 metres away at $4 \frac{1}{2}$ seconds also?

Taking $t = 4\frac{1}{2}$ in our equation, we get

$$40t - 4t^2 = \left(40 \times 4\frac{1}{2}\right) - 4 \times \left(4\frac{1}{2}\right)^2 = \left(40 \times \frac{9}{2}\right) - \left(4 \times \frac{81}{4}\right) = 180 - 81 = 99$$

Why didn't we get this second solution $t = 4\frac{1}{2}$ when we calculated t such that $40t - 4t^2 = 99$?

Let's check the steps in the computation which gave us $t = 5\frac{1}{2}$. At one stage we had to find t such that $(t - 5)^2 = \frac{1}{4}$ and we went ahead, taking $t - 5 = \frac{1}{2}$. Is $\frac{1}{2}$ the only number whose square is $\frac{1}{4}$?

What is the square of $-\frac{1}{2}$?

$$\left(-\frac{1}{2}\right)^2 = \left(-\frac{1}{2}\right) \times \left(-\frac{1}{2}\right) = \frac{1}{4}$$

Thus if we get the square of a number as $\frac{1}{4}$, we can only say that the number is $\frac{1}{2}$ or $-\frac{1}{2}$.

So in our problem, from the equation,

$$(t - 5)^2 = \frac{1}{4}$$

We can only say that

$$t - 5 = \frac{1}{2} \text{ or } t - 5 = -\frac{1}{2}$$

If we take $t - 5 = \frac{1}{2}$, in this, we get $t = 5\frac{1}{2}$ as found out first. If we take $t - 5 = -\frac{1}{2}$, then we get $t = 4\frac{1}{2}$ as found out later.

Then another question arises: would we have got a second solution in all our earlier problems, had we taken the negative square root also?

For example let's look at the rectangle problem discussed earlier: one side is 2 metres longer than the other and area is 224 square metres.

To find the length of its sides, we started with x metres as the length of the shorter side and got the equation $(x + 1)^2 = 225$.

Then we took $x + 1 = 15$ and found the length of the shorter side as 14 metres.

Considering algebra alone, $x + 1 = -15$ also, so that x can be -16 .

But in this problem, x is the length of a rectangle and so must be positive. Thus the solution $x = -16$ is not suitable to this problem.

Let's look at another rectangle problem done earlier. Perimeter 100 metres and area 525 square metres.

Taking the length of one side as x , we get equation $(x - 25)^2 = 100$. We took $x - 25 = 10$ and calculated the length of one side as 35 metres and the other side as $50 - 35 = 15$ metres.

What if we take the negative root? We get $x - 25 = -10$ which gives $x = 15$, that is we get the length of one side as 15 metres and the length of the other side as $50 - 15 = 35$ metres.

Thus for this problem, we get the same solution whether we take the positive or negative root.

In general, when a practical problem is turned into an algebraic problem and we think only mathematically, we may get more than one answer. It may happen that some of these solutions or sometimes even all of them, may not be suitable for the original practical problem.

So, usually all solutions are found algebraically and then those which suit the context are chosen.

?



- (1) The product of a number and 2 more than that is 168, What are the numbers?
- (2) Find two numbers with sum 4 and product 2.
- (3) How many terms of the arithmetic sequence 99, 97, 98, ... must be added to get 900?
- (4) The sum of a number and its reciprocal is $2\frac{1}{6}$. What is the number?
- (5) Two taps open into a tank. If both are opened, the tank would be filled in 12 minutes. The time taken to fill the tank by the smaller tap alone is 10 minutes more than the time taken to fill it by the larger tap alone. If the smaller tap alone is opened, what would be the time taken to fill the tank?

Equations and Polynomials

In the polynomial $p(x) = 4x^2 + 24x + 11$, if we take different numbers as x , we get different numbers as $p(x)$. For example

$$p(1) = 4 + 24 + 11 = 39$$

$$p\left(\frac{1}{2}\right) = \left(4 \times \frac{1}{4}\right) + \left(24 \times \frac{1}{2}\right) + 11 = 1 + 12 + 11 = 24$$

$$p(-1) = 4 - 24 + 11 = -9$$

We can also ask a reverse question as what number we should take as x to get a specified number as $p(x)$. For example

In the polynomial $p(x) = 4x^2 + 24x + 11$, what number should we take as x to get $p(x) = 0$?

We can simplify the question like this:

$$\text{What should be } x \text{ to get } 4x^2 + 24x = -11?$$

We have done several such problems. The stages of finding x are as follows:

$$\text{Change the coefficient of } x^2 \text{ to 1: } x^2 + 6x = -\frac{11}{4}$$

Add the square of half the

coefficient of x : $x^2 + 6x + 9 = -\frac{11}{4} + 9$

Write as a square : $(x + 3)^2 = \left(\frac{5}{2}\right)^2$

Take square roots : $x + 3 = \frac{5}{2}$

or

$$x + 3 = -\frac{5}{2}$$

Calculate x : $x = \frac{5}{2} - 3 = -\frac{1}{2}$

or

$$x = -\frac{5}{2} - 3 = -5\frac{1}{2}$$

That is, to get $p(x) = 0$ we must take $x = -\frac{1}{2}$ or

$$x = -5\frac{1}{2}.$$

Now suppose we want to find that x for which $p(x) = 1$.

We can write $p(x) = 1$ as $p(x) - 1 = 0$. That is,

$$4x^2 + 24x + 10 = 0$$

If we write the polynomial $4x^2 + 24x + 10$ as $q(x)$, the problem can be put like this:

In the polynomial $q(x) = 4x^2 + 24x + 10$, what number should be taken as x to get $q(x) = 0$?

We can proceed as in the first problem:

$$4x^2 + 24x + 10 = 0$$

$$4x^2 + 24x = -10$$

$$x^2 + 6x = -\frac{5}{2}$$

$$x^2 + 6x + 9 = 9 - \frac{5}{2} = \frac{13}{2}$$

$$(x + 3)^2 = \frac{13}{2}$$

$$x + 3 = \sqrt{\frac{13}{2}} \text{ or } -\sqrt{\frac{13}{2}}$$

$$x = -3 + \sqrt{\frac{13}{2}} \text{ or } -3 - \sqrt{\frac{13}{2}}$$

A bit of history

The method of solving second degree equations by completing squares is quite old. We can see this method was used to solve problems related to area by Babylonians around 1500 BC.

But then the method of changing a problem to an algebraic equation was not used at that time (Such methods are at best only five hundred years old) Problems and their solutions were described in ordinary language. Solutions of geometric problems were found using geometric methods.

In other words, most of what we now present as geometric interpretations of algebraic methods, were originally the prototypes of the algebraic methods.

We shorten $-3 + \sqrt{\frac{13}{2}}$ or $-3 - \sqrt{\frac{13}{2}}$ as $-3 \pm \sqrt{\frac{13}{2}}$. That is to get

$p(x) = 1$, we must take x as one of the numbers $-3 \pm \sqrt{\frac{13}{2}}$

Now let's look at the general method of finding that number which gives 0 from a second degree polynomial. Any second degree polynomial can be put in the form

$$p(x) = ax^2 + bx + c$$

The stages of finding the number x for which $p(x) = 0$, can be written like this, as done before.

Diagonal problem

The method completing square was used in ancient times not only to solve second degree equations but to compute approximate values of square root also.

For example, an ancient Babylonian tablet gives the method of computing the diagonal of a tall, slender rectangle as follows,

Divide the square of the width by height and add half of it to the height.

In current algebraic notation, this means

$$\sqrt{a^2 + b^2} \approx a + \frac{b^2}{2a}$$



Can you justify this using our algebra?

- Rewrite $ax^2 + bx + c = 0$:

$$ax^2 + bx = -c$$

- Change the coefficient of x^2 to 1:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

- Add the square of half the coefficient $\frac{b}{a}$ of x :

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$

- Write as a square:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

- Take square roots:

$$\left(x + \frac{b}{2a}\right) = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

- Compute x :

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In the polynomial $p(x) = ax^2 + bx + c$, the number to take as x , to get $p(x) = 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We can shorten this a bit:

To get $ax^2 + bx + c = 0$, we must take

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This can be used to combine the various stages of our solutions of earlier problems into a single line.

For example, consider the first problem in the section, **Two answers**. In this problem, to find the times at which the object is 99 metres from the start, we computed the t 's which make $40t - 4t^2 = 99$.

This problem can be put like this:

What number should be taken as t to get $4t^2 - 40t + 99 = 0$?

To find it, we need only take a, b, c in the general result above as 4, -40, 99.

From this,

$$t = \frac{-(-40) \pm \sqrt{(-40)^2 - 4 \times 4 \times 99}}{2 \times 4}$$

$$t = \frac{40 \pm \sqrt{1600 - 16 \times 99}}{8} = \frac{40 \pm \sqrt{16}}{8}$$

That is

$$t = \frac{40 \pm 4}{8} = \frac{44}{8} \text{ or } \frac{36}{8}$$

From this, we get as a before, $t = 5\frac{1}{2}$ or $4\frac{1}{2}$

Now look at this problem:

A stone is thrown straight up with a speed of 30 metres / second.

If we take the height above the ground at t second as s metres, then the relation between s and t is

$$s = 30t - 4.9t^2$$

At what time would the stone be 20 metres above the ground?



Here we have to find the t for which $30t - 4.9t^2 = 20$. In other words the problem is this:

What number should be t to get $4.9t^2 - 30t + 20 = 0$?

As before, we can answer in one line:

$$t = \frac{30 \pm \sqrt{900 - 4 \times 4.9 \times 20}}{9.8}$$

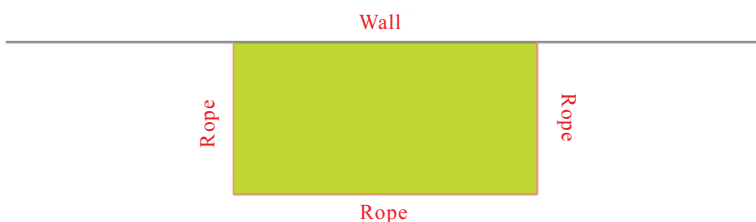
It's better to use a calculator or computer to calculate this. We can then find up to two decimal places,

$$t \approx 5.36 \text{ or } 0.76$$

Here, 0.76 is the time at which the stone reaches 20 metres in the upward journey and 0.76 is the time at which it comes down to 20 metres in the downward journey.

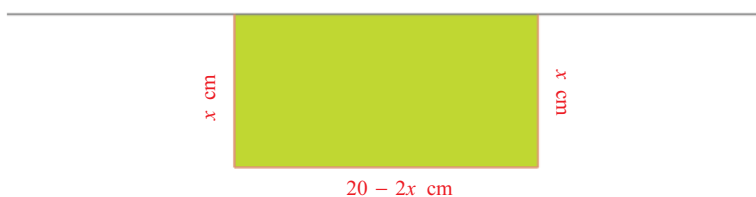
Next look at this problem:

A rectangle is to be made on the ground using a 20 metre long rope, with a wall as one side:



The area enclosed must be 50 square metres. What should be the length of the sides?

If the length of the left and right sides is taken as x metres, the length of the bottom side is $20 - 2x$ metres and the area is $x(20 - 2x) = 2x(10 - x)$ square metres.



So the problem in algebra is this:

What should be x to get $2x(10 - x) = 50$?

First we write the equation like this:

$$x^2 - 10x + 25 = 0$$

That is,

$$(x - 5)^2 = 0$$

If the square of a number is zero, so is the number. Thus $x - 5 = 0$, or $x = 5$.

So the length of the side of the rectangle are 5 metres and $20 - 10 = 10$ metres.

Can we change the length to get a bit more area? At least one square metre?

$$2x(10 - x) = 51$$

$$2x^2 - 20x + 51 = 0$$

It is not easy to complete squares in this. Let's try the formula.

$$x = \frac{20 \pm \sqrt{400 - 408}}{4} = \frac{20 \pm \sqrt{-8}}{2}$$

What does this means? Negative numbers don't have square roots.

(Whether a number is positive or negative, its square is positive, right?)

This means our equation does not have a solution. In other words, whatever number we take as x , the number $x^2 - 20x + 51$ will not be 0.

Had we used completion of squares instead of the formula, we would have got

$$x^2 - 10x + 25 \frac{1}{2} = 0$$

$$x^2 - 10x + 25 = -\frac{1}{2}$$

$$(x - 5)^2 = -\frac{1}{2}$$

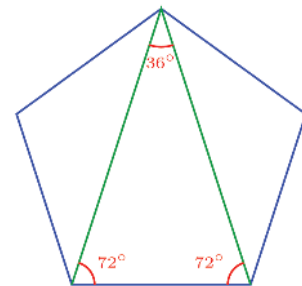
And at this stage we would have recognized that there is no solution, since the square of a number cannot be negative. Returning to our rectangle problem, the essence of our discussion above is that the area cannot be made 51 square centimetres.

Thinking along similar lines, we can see the area cannot be increased even a little bit from 50 square metres.

Another problem:

How many times the side of a regular pentagon is its diagonal?

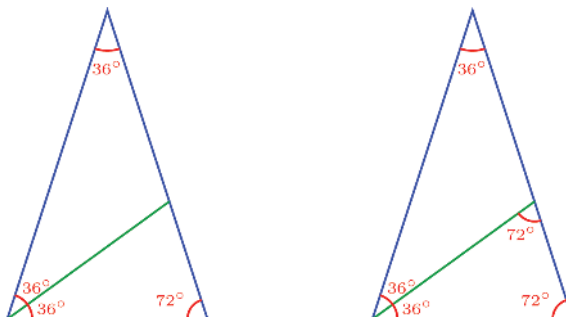
We have seen that a triangle is formed by a side and two diagonals of a regular pentagon has angles 36° , 72° and 72° (The lesson, **Polygons** in Class 8).



So the question is this:

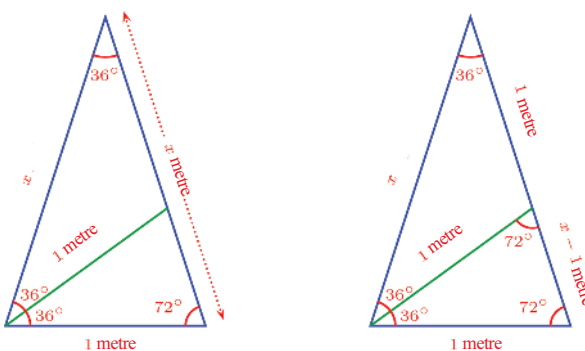
In a triangle with angles 36° , 72° , 72° how many times the base are the equal sides?

Such triangles have a peculiarity. To see it, draw the bisector of a base angle to meet the opposite side. Then we can calculate the angles of the small triangle thus got:



So the original big triangle and the small lower triangle within it have the same angles. In other words, they are similar (Now suppose we draw the bisector of a base angle to meet the opposite side and then draw the bisector of a 72° angle of the smaller triangle..., let's stop here.)

Returning to our problem, let's take the base to be 1 metre and the equal sides to be x metres. Since the two triangles formed by the angle bisector are also isosceles, we can calculate some more lengths.



Since the large triangle and the lower triangle inside have the same angles, the change in the length of the sides are in the same scale. (The lesson, **Similar Triangles** of Class 9).

The equal sides of the small triangle are 1 metre each and those of the large triangle are x metres each.

The third side of the small triangle is $x - 1$ metres and that of the large triangle is 1 metre.

Since the scale of changes is the same, we have

$$\frac{x}{1} = \frac{1}{x-1}$$

Using cross multiplication (The lesson, **Fraction** of Class 9), we can rewrite this equation as

$$x(x - 1) = 1$$

And then as a polynomial equation like this:

$$x^2 - x - 1 = 0$$

Now using the solution formula:

$$x = \frac{1 \pm \sqrt{1 - (-4)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

In our problem, x is positive, so that the solution $\frac{1}{2}(1 - \sqrt{5})$ is not right.

So the lengths of the equal sides of our triangle are $\frac{1}{2}(\sqrt{5} + 1)$ metres each.

Thus the diagonal of a regular pentagon is $\frac{1}{2}(\sqrt{5} + 1)$ times its side.

?



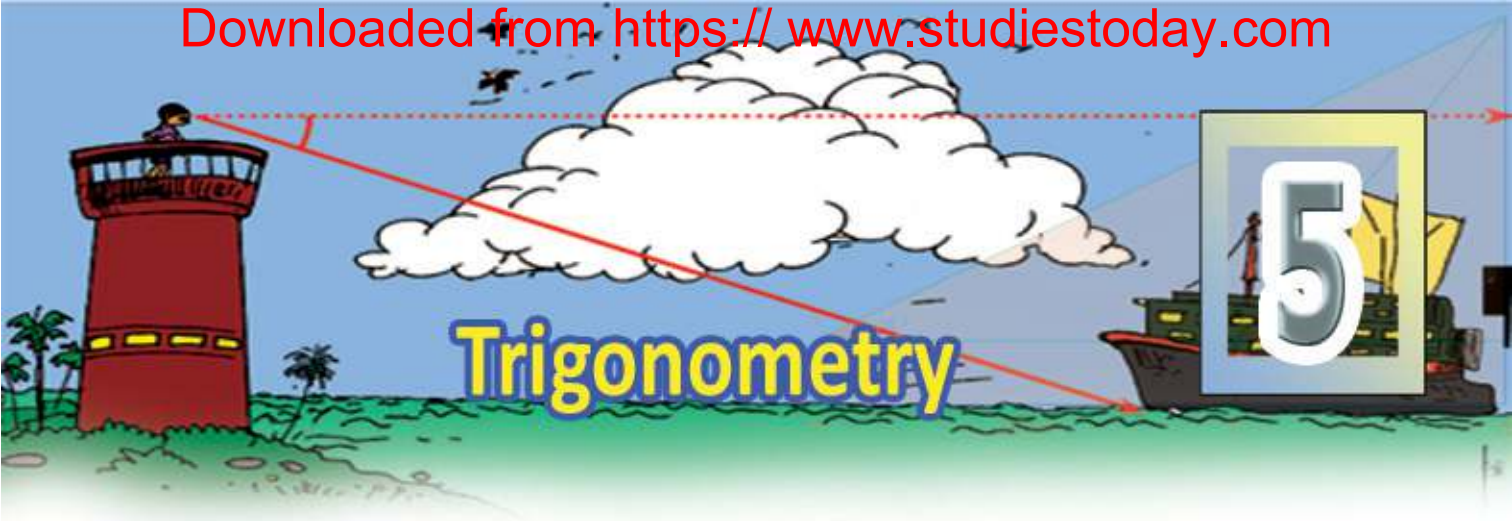
- (1) The perimeter of a rectangle is 42 metres and its diagonal is 15 metres. What are the lengths of its sides?
- (2) How many consecutive natural numbers starting from 1 should be added to get 300?
- (3) The reciprocal of a positive number, subtracted from the number itself gives $1\frac{1}{2}$. What is the number?
- (4) Can the sum of a number and its reciprocal be $1\frac{1}{2}$? Why?

- (5) In writing the equation to construct a rectangle of specified perimeter and area, the perimeter was wrongly written as 24 instead of 42. The length of a side was then computed as 10 metres. What is the area in the problem? What are the lengths of the rectangle in the correct problem?
- (6) In copying a second degree equation, the number without x was written as 24 instead of -24 . The answers found were 4 and 6. What are the answers of the correct problem?

Looking back

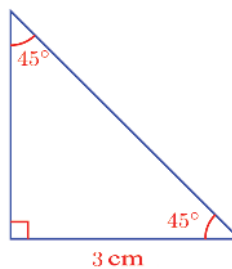


Learning outcomes	On my own	With teacher's help	Must improve
<ul style="list-style-type: none"> Translating practical problems into algebraic equations. Explaining the method of solving a second degree equation by completion of square. Explaining the method of finding both solutions of some practical problems. Recognising situations where solutions of equations may or may not be solutions of physical problems. 			



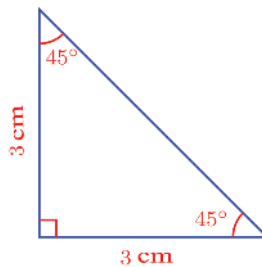
Angles and sides

See the triangle.



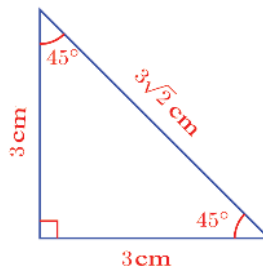
What are the lengths of its other sides?

We know that sides opposite to equal angles are also equal. So the perpendicular side opposite the 45° angle on the right is also 3 centimetres long.



What about the hypotenuse?

According to Pythagoras Theorem, the square of the hypotenuse $3^2 + 3^2 = 18$. So hypotenuse is $\sqrt{18} = 3\sqrt{2}$ centimetres.



(The unit, New Numbers Class 9)

Now suppose we draw another triangle like this with the bottom side a little longer, say 5 centimetres. What would be the lengths of the other sides?

In general, whatever be the length of one of the perpendicular sides of such a triangle, the other perpendicular side would be of the same length; and the length of the hypotenuse would be $\sqrt{2}$ times this length.

Earth and sky

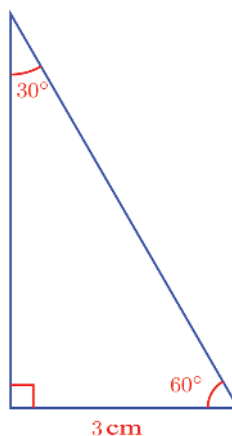
Trigonometry is the study of the relation between angles and sides of triangles. We have seen that angles are used as measures of slant, spread or turn. Slants first make their appearance in history in constructions on the earth and measures of turn in the study of planets in the sky.

The first astronomical studies were also for earthly needs. Food production, that is agriculture, depended largely on weather. And a factor determining weather is earth's revolution around the sun. To determine this correctly, the positions of other planets and stars have to be determined. This is why astronomy was an important area of study in all ancient agricultural societies. And for that, mathematics, especially geometry, is essential.

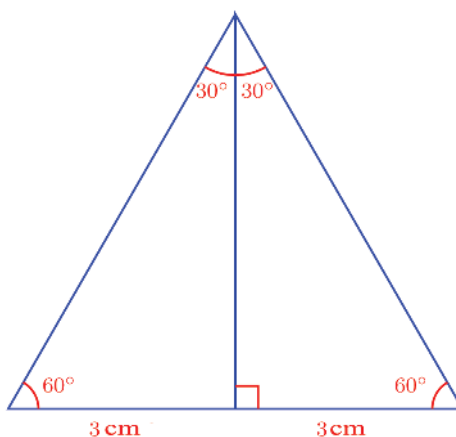
We can state this concisely using ratios:

The sides of any triangle of angles $45^\circ, 45^\circ, 90^\circ$ are in the ratio $1:1:\sqrt{2}$.

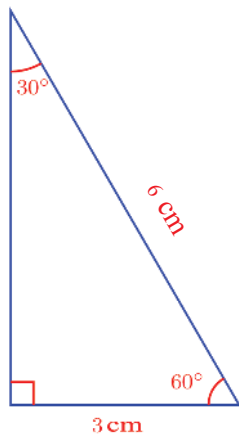
Now see this triangle:



Can you visualize it as half an equilateral triangle?



Since the base of this equilateral triangle is 6 centimetres, so are the other two sides. This gives the hypotenuse of our right triangle.



What about the third side?

$$\sqrt{6^2 - 3^2} = \sqrt{(6+3)(6-3)} = \sqrt{9 \times 3} = 3\sqrt{3}$$

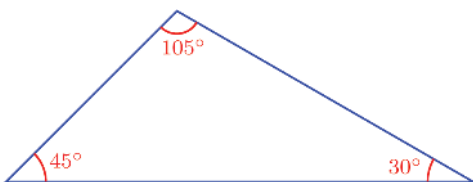
What if we draw a triangle like this with the base reduced to 2 centimetres?

The hypotenuse would be 4 centimetres and the third side would be $2\sqrt{3}$ centimetres.

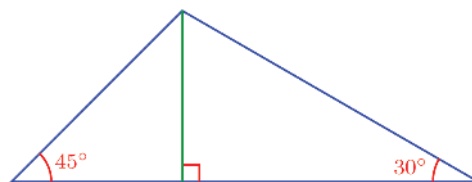
So in all such triangles, the longest side would be twice as long as the shortest side; and the side in between, $\sqrt{3}$ times the shortest.

In any triangle of angles $30^\circ, 60^\circ, 90^\circ$ the sides are in the rates $1 : \sqrt{3} : 2$.

Using these two kinds of triangles, we can compute the ratios of the sides of some non-right triangles also. For example, see this triangle:



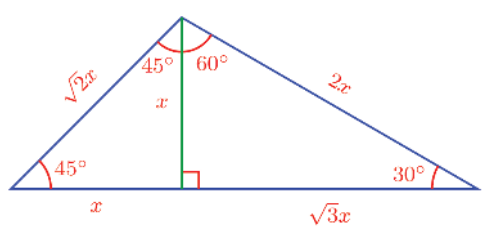
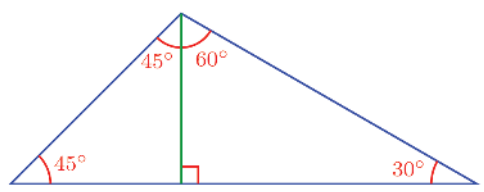
By drawing the perpendicular from the top vertex to the base, we can split it into two right triangles.



If the sides of a triangle are in the ratio $1 : \sqrt{3} : 2$, would its angles be $30^\circ, 60^\circ, 90^\circ$? Let's check using GeoGebra. First we draw triangles with sides in the ratio $1 : \sqrt{3} : 2$. Make a slider a with $\text{Min} = 0$, **Choose Segment with Given Length** and click on a point. In the ensuing window, type the length as $a * \text{sqrt}(3)$ (It means $\sqrt{3} a$). With centre at the ends of these lines, draw circles of radii a and $2a$. Draw a triangle with vertices as the ends of the line and the point of intersection of the circles. Mark its angles. Change a using the slider and watch the angles.

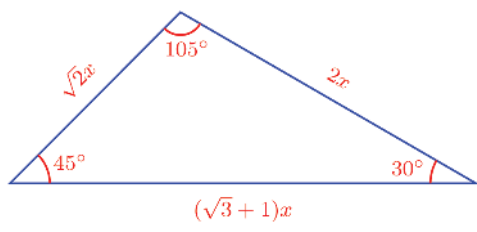
Likewise draw triangles of sides in the ratio $2 : \sqrt{5} + 1 : \sqrt{5} + 2$ and find the angles.

What are the top angles of these right triangles?



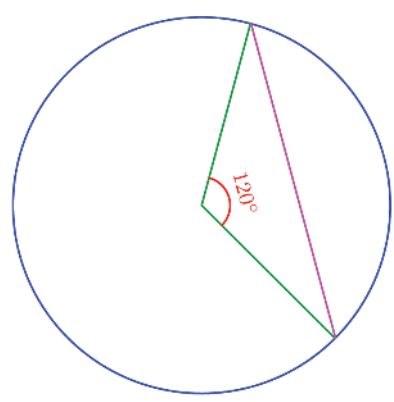
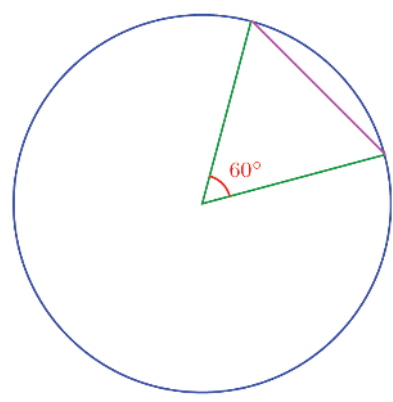
To calculate the ratio of the sides, take x as the length of their common side. Then using the ratios seen earlier, we can write the lengths of the other sides.

So the sides of the original triangle are like this.



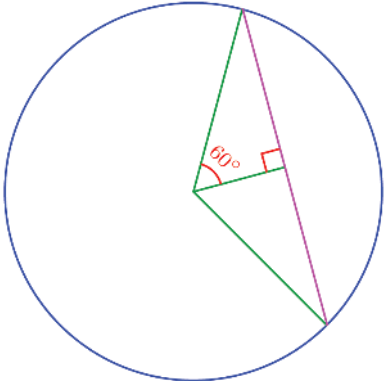
In other words, the ratio of the sides is $\sqrt{2} : 2 : \sqrt{3} + 1$

Ratios of sides of a triangle are useful in calculating other lengths also. For example, we can easily see that in a circle, the length of a chord of central angle 60° is equal to the radius.

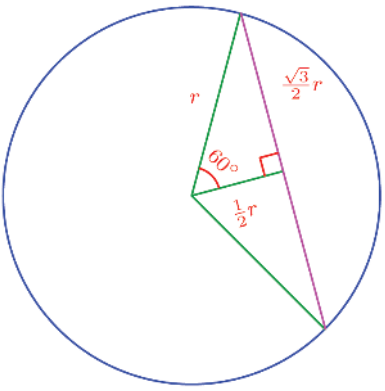


What about the length of a chord of central angle 120° ?

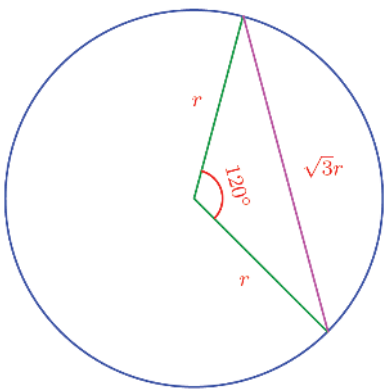
To compute it, we draw the perpendicular from the centre to the chord. It bisects the chord and the central angle (Reason?)



The angles of the triangle made by the radius, the perpendicular and the half-chord are 30° , 60° , 90° . So, if the radius is taken as r , the length of the perpendicular is $\frac{1}{2}r$ and that of the half-chord is $\frac{\sqrt{3}}{2}r$.



Thus the length of a chord of central angle 120° is $\sqrt{3}$ times the radius.



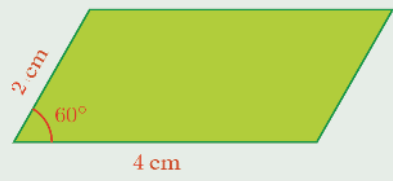
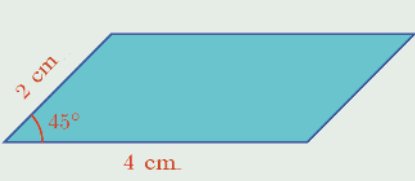
We have seen that as the central angle is doubled, so is the length of the arc; but note that the length of the chord is not doubled.

Straight way?

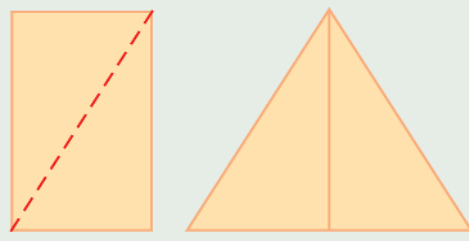
The size of an angle is usually measured by the length of an arc. The astronomer Hippocrates, who lived in Greece during the second century BC, used chords instead in computations. Many later mathematicians say that he computed a large table of chord lengths of different angles; but it has not been found. But such a table by Claudius Ptolemy, an astronomer who lived in Egypt during second century AD, has been found. In it he has accurately computed the lengths of chords of central angles from 0° to 180° differing by $\frac{1}{2}^\circ$, in a circle of radius 60.



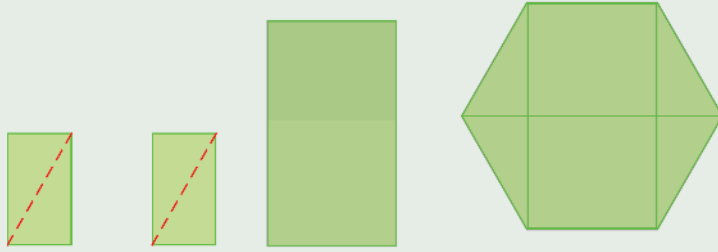
(1) Calculate the areas of the parallelograms shown below:



(2) A rectangular board is to be cut along the diagonal and the pieces rearranged to form an equilateral triangle; and the sides of the triangle must be 50 centimetres. What should be the length and breadth of the rectangle?

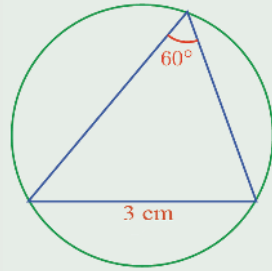


(3) Two rectangles are cut along the diagonal and the triangles got are to be joined to another rectangle to make a regular hexagon as shown below:

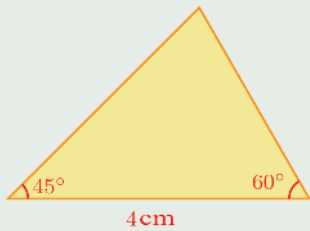


If the sides of the hexagon are 30 centimetres, what would be the length and breadth of the rectangles?

(4) The picture shows a triangle and its circumcircle. What is the radius of the circle?



(5)



Calculate the area of the triangle shown.

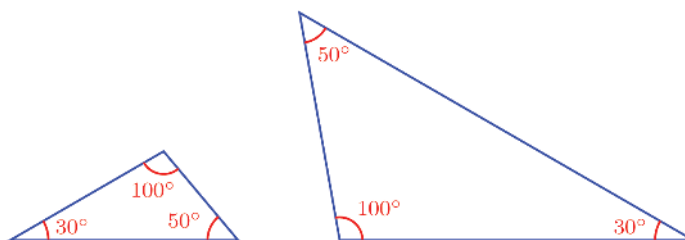
(6) What is the circumradius of an equilateral triangle of sides 8 centimetres?



New measure of angles

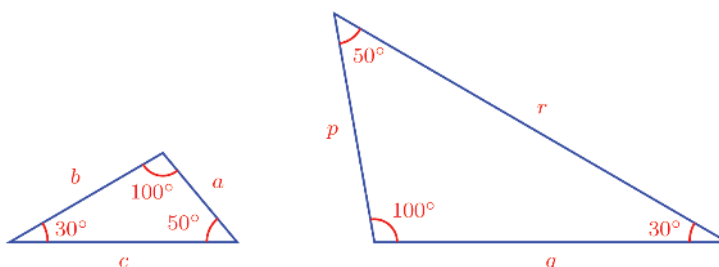
We have calculated the ratios of the sides of some triangles of specific angles. Do the angles of any triangle determine the ratio of its sides?

Let's make the question clear by means of an example. See these triangles:



They have the same angles. Is the ratio of the sides of the small triangle the same as the ratio of the sides of the large triangle?

Let's write the sides of the small triangle as a, b, c in increasing size and those of the large triangle as p, q, r :



We have seen that the lengths of the pairs of sides opposite equal angles are changed in the same scale.

That is, the numbers p, q, r are got by multiplying the numbers a, b, c by the same number. Taking this number as k , we get

$$p = ak \quad q = bk \quad r = ck$$

Using ratios, this means.

$$a : b : c = p : q : r$$

In triangles of the same angles drawn in different sizes, the lengths of the sides are different, but their ratios are the same.



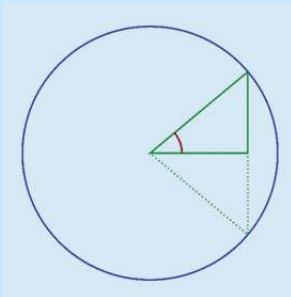
In other words.

The angles of a triangle determine the ratio of its sides.

For example, as we have seen earlier, the sides of a triangle of angles 30° , 60° , 90° are in the ratio $1 : \sqrt{3} : 2$. We have also found the ratios of sides of some other triangles of specific angles.

Half-chord

To compute the side opposite an angle in a right triangle using Ptolemy's table, the angle must be doubled and the chord halved. To avoid this, a table relating each angle with half the chord of double the angle may be constructed.

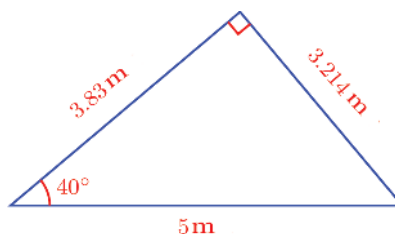


Such a table can be seen in *Suryasiddhanta*, book on astronomy written in India, during the fifth century AD. Computations using such tables can also be seen in *Aryabhateeya*, a book written in the same period by the famous Indian astronomer Aryabhata. He calls this measure of an angle as *ardhajya* (The word *jya* in Sanskrit is used for chord)

It is not easy in general to compute the ratios of sides from the angles. However, mathematicians from very early times have found techniques to compute such ratios for right triangles and arranged them in special tables .

For example, these tables show that for a right triangle of one angle 40° , the side opposite this angle is approximately 0.6428 times the hypotenuse and the other perpendicular side is approximate 0.7660 times the hypotenuse.

So in a right triangle of hypotenuse 5 metres and one angle 40° , the side opposite this angle is $5 \times 0.6428 = 3.214$ metres and the third side is $5 \times 0.766 = 3.83$ metres, correct to a millimetre:



The numbers in these tables have special names. In the example above the number 0.6428 is got by dividing the side opposite 40° by the hypotenuse. It is called the *sine* of 40° and is written $\sin 40^\circ$.

The other number 0.7660 is got by dividing the smaller side of 40° (is called its adjacent side) by the hypotenuse, it is called the *cosine* of 40° and is written $\cos 40^\circ$.

Thus,

$$\sin 40^\circ \approx 0.6428$$

$$\cos 40^\circ \approx 0.7660$$

Tables of sine and cosine of angles differing by 1° are available. A part of such a table is shown below:

Angle	sin	cos
35°	0.5736	0.8192
36°	0.5878	0.8090
37°	0.6018	0.7986
38°	0.6157	0.7880
39°	0.6293	0.7771
40°	0.6428	0.7660

(A full table is given at the end of the lesson)

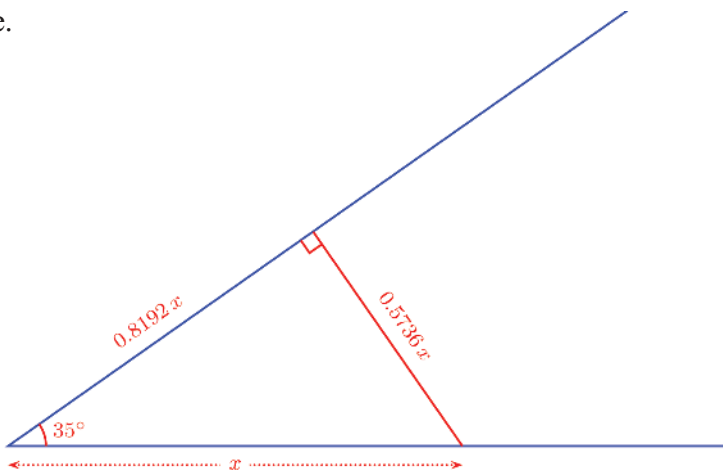
From this table, we find for example,

$$\sin 35^\circ \approx 0.5736$$

$$\cos 35^\circ \approx 0.8192$$

These numbers can be explained in a different manner.

Suppose that a 35° angle is drawn and from some point on one of its sides, a perpendicular is drawn to the other side. The length of this perpendicular is 0.5736 times the distance of the point from the corner of the angle; and the foot of the perpendicular is 0.8192 times this distance away from the corner of the angle.



What is in a name?

What we now call sin is what Aryabhata called *ardhajya*. The name evolved in different stages.

Aryabhata himself discarded the adjective *ardha* (half) later and used only the word *jya*. In the years following seventh century AD, the rulers of the Arab world encouraged translations of science texts from Greece and India. The translations of Indian texts transliterated the word *jya* into *jiba* in Arabic.

In old Arabic texts, vowels are not written and so this was written as *jb*.

Later in the thirteenth century AD, these Arab books reached Europe and were translated to Latin. The word written in arabic as *jb* was mistaken for the Arab word *jaib* meaning bend or fold and translated it to the Latin word *sinus* with this meaning. In the course of time, it got reduced to just sine.

The word *kotijya* used by Aryabhata became cosine.



If we take a 40° angle instead, these lengths would be 0.6428 and 0.7660 of the first distance.

Looking at sine and cosine in this manner, we see that they are actually measures of an angle.

Now we can write the ratios we have found earlier for two types of right triangles, in terms of sine and cosine:

$$\sin 45^\circ = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}}$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{1}{2}$$

$$\sin 30^\circ = \frac{1}{2}$$

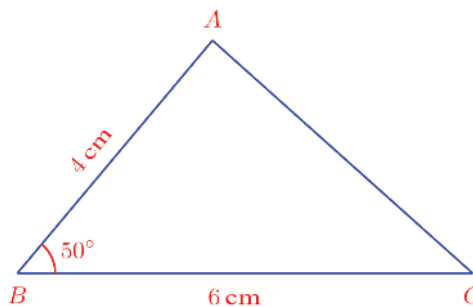
$$\cos 30^\circ = \frac{\sqrt{3}}{2}$$



Let's look at some problems worked out using sine and cosine.

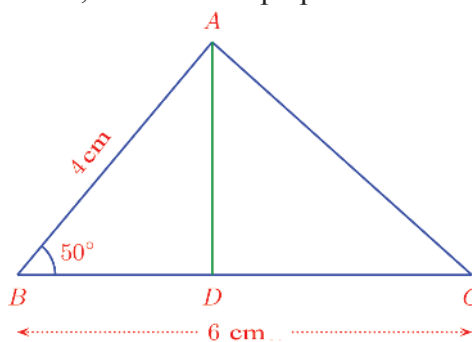
Sine and Cosine

Draw a line AB of length 1 in GeoGebra. Make an angle slider α . Choose **Angle with Given Size** and click on B,A in that order. In the window opening up, give the measure of angle as α . A new point B' is got. From B, draw a perpendicular to AB' and mark its foot as C. Draw $\triangle ABC$ and hide the line AB'. Mark the lengths of the sides of $\triangle ABC$. We can see how these change as the angle is changed. In this, the length of BC is the sine of α and AC is the cosine of α . Make a table of sine and cosine using this. What are the maximum of sine and cosine?



What is the area of this triangle?

To calculate it, let's draw the perpendicular from A to BC.



The area is

$$\frac{1}{2} \times BC \times AD = \frac{1}{2} \times 6 \times AD = 3 \times AD$$

How do we find AD?

From the right triangle ABD

$$AD = AB \times \sin 50^\circ = 4 \sin 50^\circ$$

From the table, we get

$$\sin 50^\circ \approx 0.7660$$

So,

$$AD \approx 4 \times 0.7660 = 3.064 \text{ centimetres}$$

Now we can calculate the area as

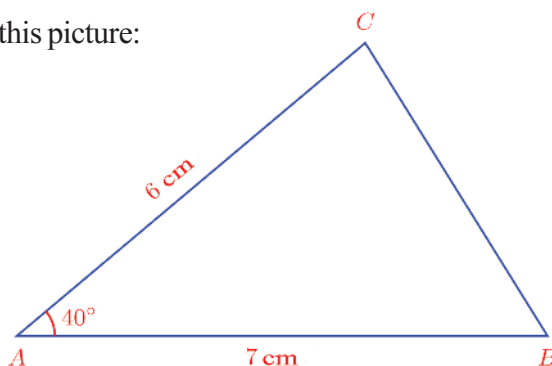
$$3 \times AD = 3 \times 3.064 \approx 9.19 \text{ centimetres}$$

Thus the area is approximately 9.19 square centimetres.

If the angle at B is 130° instead of 50° in this problem, how do we compute the area?

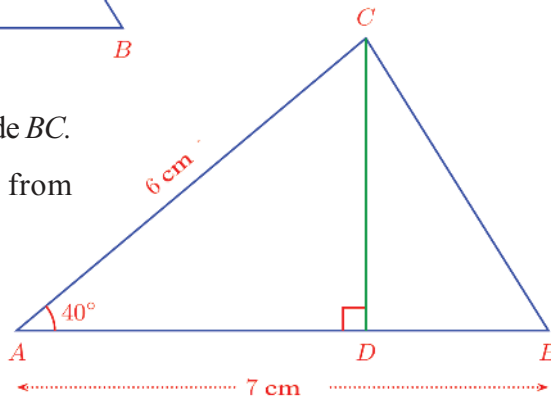


Now see this picture:



We have to calculate the length of the third side BC .

The trick is to draw the perpendicular from C to AB .



Now from the right triangle BCD , we get,

$$BC^2 = BD^2 + DC^2$$

Next we compute BD and DC .

From the right triangle ACD ,

$$DC = AC \sin 40^\circ \approx 6 \times 0.6428 \approx 3.86 \text{ centimetres}$$

and again from this triangle,

$$AD = AC \cos 40^\circ \approx 6 \times 0.766 \approx 4.6 \text{ centimetres}$$

So,

$$BD = AB - AD \approx 7 - 4.6 = 2.4 \text{ centimetres}$$

Now we can calculate BC :

$$BC = \sqrt{BD^2 + DC^2} \approx \sqrt{3.86^2 + 2.4^2} = 4.54 \text{ centimetres}$$

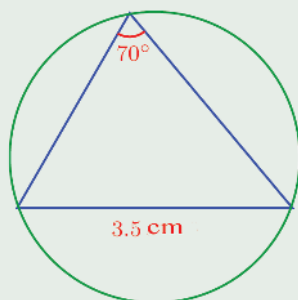
Thus the length of BC is approximately 4.5 centimetres.



How do we compute the length of BC if the angle at A is 100° ?



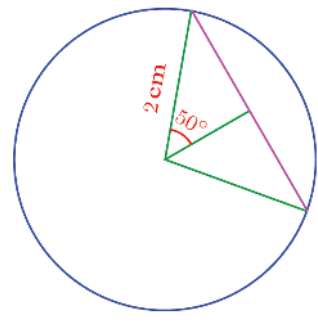
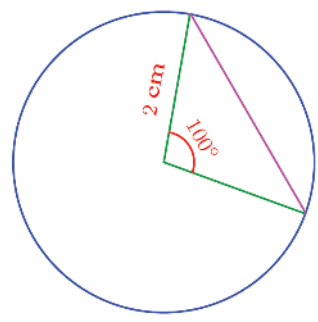
- (1) Without drawing pictures or looking up the tables, arrange the numbers $\sin 1^\circ$, $\cos 1^\circ$, $\sin 2^\circ$, $\cos 2^\circ$ in ascending order.
- (2) The sides of a rhombus are 5 centimetres long and one of its angles is 100° . Compute its area.
- (3) The sides of a parallelogram are 8 centimetres and 12 centimetres and the angle between them is 50° . Calculate its area.
- (4) The sides of a parallelogram are 6 centimeters and 14 centimetres and the angle between them is 30° . What are the lengths of its diagonals?
- (5) A triangle is to be drawn with one side 8 centimetres and an angle on it 40° . What is the minimum length of the side opposite this angle?
- (6) A regular pentagon is drawn with vertices on a circle of radius 15 centimetres. Calculate the length of its sides.
- (7) The lengths of two sides of a triangle are 8 centimetres and 10 centimetres and the angle between them is 40° . Calculate its area. What is the area of the triangle with sides of the same length, but angle between them 140° ?
- (8) The picture shows a triangle and its circumcircle. What is the radius of the circle?





Triangle and circle

We have to calculate the length of the chord in this picture.



As we did in a similar problem earlier, we draw the perpendicular from the centre to the chord, splitting it and the central angle into halves.

In this picture, the side opposite 50° in each right triangle is $\sin 50^\circ$ times the hypotenuse; and from the table of sines,

$$\sin 50^\circ \approx 0.7660$$

Thus half the chord is

$$2 \times 0.766 \approx 1.53 \text{ cm.}$$

and so the full chord is

$$2 \times 1.53 = 3.06 \approx 3.1 \text{ cm.}$$

We can compute the chord of any central angle like this. What all things did we do to get it?

We found the sine of half the central angle and multiplied it by the radius to find half the chord. Doubling it gives the full chord.

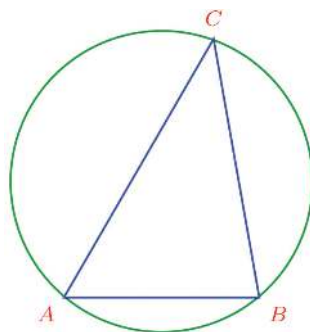
In a circle, the length of a chord is double the product of the sine of half the central angle by the radius.

We can write this using algebra as follows:

In a circle of radius r , the length of a chord of central angle c° is

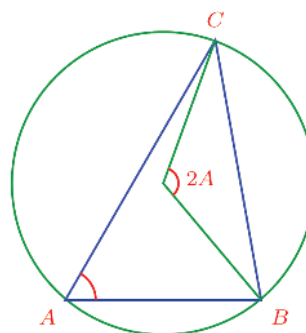
$$2r \sin \left(\frac{c}{2} \right)^\circ$$


Now look at this picture:



ΔABC and its circumcircle. The sides AB, BC, CA of the triangle are chords of the circle, right?

The central angle of the chord BC is twice the angle at A in the triangle, as seen in the lesson, **Circles**. So writing the measure of this angle in degrees also as A , the central angle of the chord is $2A$.



If the radius of the circle is taken as r , the length of BC is $2r \sin A$, by the general principle we saw just now.

Similarly, the length of CA is $2r \sin B$ and the length of AB is $2r \sin C$.

Is it true for all triangles? What if one angle is larger than 90° ?

New meanings

If the diameter of ΔABC is taken as the unit of length, what would be the lengths of the sides?

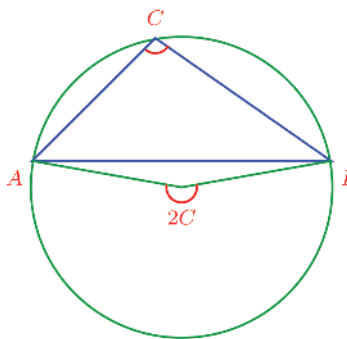
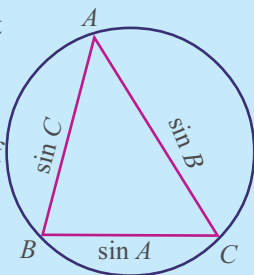
$BC = d \sin A = \sin A$.

Similarly we get

$AB = \sin C$;

$AC = \sin B$.

Thus the sine of an angle is the length of a chord making this angle at some point on a circle of diameter 1.



In the picture, the central angle of AB is $360 - 2C$ right? So, the length of AB is $2r \sin (180 - C)$.



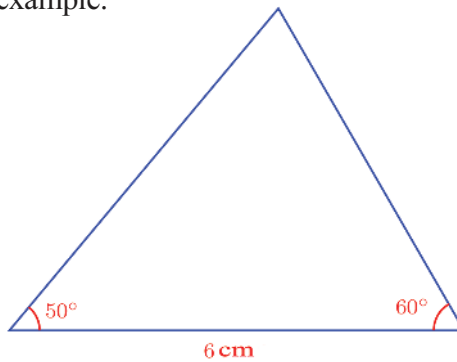
In general, we can say this:

The length of the sides of a triangle are the sines of its angles multiplied by the diameter of the circumcircle. If any angle is greater than a right angle, the sine of its supplementary angle should be taken; if any angle is right its opposite side is equal to this diameter.

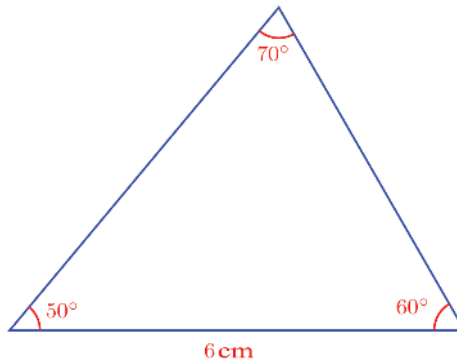
Now we know how the angles of a triangle determine the ratio of its sides.

The ratio of the sides of a triangle is the ratio of the sines of the opposite angles. For any angle larger than a right angle, the sine of the supplementary angle must be taken and for a right angle, the opposite side must be taken as 1.

Let's look at an example.



The other two sides of the triangle are to be calculated. The angle opposite the bottom side is 70° .



If we take the diameter of the circumcircle as d , we have $d \sin 70^\circ = 6$.

From this, we can find

$$d = \frac{6}{\sin 70^\circ} \approx \frac{6}{0.94} = 6.38$$

Now we can calculate the other two sides:

$$6.38 \times \sin 50^\circ \approx 6.38 \times 0.77 \approx 4.9$$

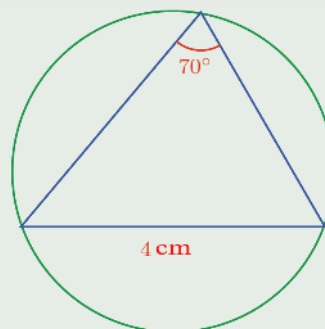
$$6.38 \times \sin 60^\circ \approx 6.38 \times 0.87 \approx 5.5$$

Thus the other two sides are approximately 4.9 centimetres and 5.5 centimetres.

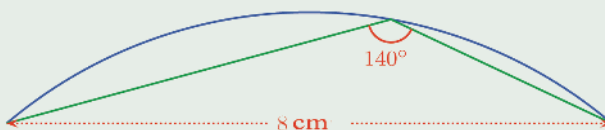


Using the sine and cosine tables, and if needed a calculator, do these problems.

- (1) A triangle and its circumcircle are shown in the picture. Calculate the diameter of the circle.

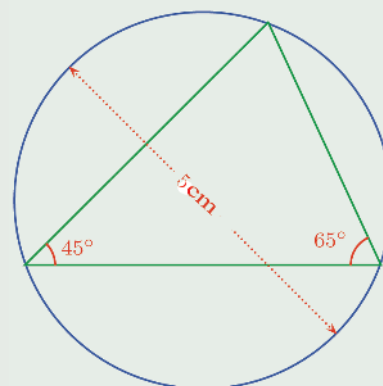


- (2) A circle is to be drawn, passing through the ends of a line, 5 centimetres long; and the angle on the circle on one side of the line should be 80° . What should be the radius of the circle?
- (3) The picture below shows part of a circle:



What is the radius of the circle?

- (4) Draw the picture shown in your notebook and explain how it was drawn. Calculate the lengths of all three sides.



(5) A triangle is made by drawing angles of 50° and 65° at the ends of a line 5 centimetres long. Calculate its area.

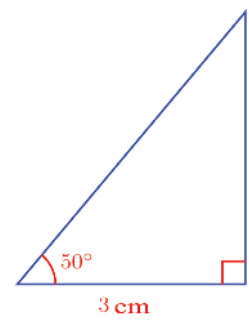
Another measure

A right triangle is to be drawn with one of the shorter sides 3 centimetres and a 50° angle at one of its ends.

Not very difficult to draw, is it?

What is the length of the other short side?

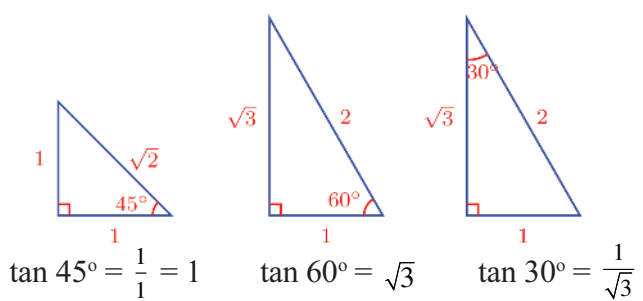
We can look up $\cos 50^\circ$ in the table and then use it to calculate the hypotenuse. Then use Pythagoras Theorem to find the third side.



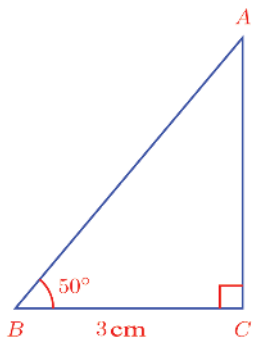
There is another table which we can use to do this directly. The table of numbers got by dividing the opposite side of an angle by the adjacent side in right triangles.

This number is called the *tangent* of the angle and is shortened as tan.

For example, let's look at some triangles seen earlier:



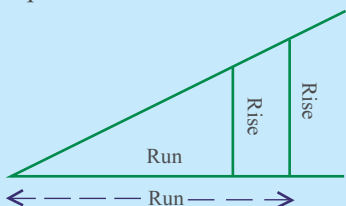
Now back to our original problem:



Measure of slant

The method of measuring angles by dividing a circle into 360 equal parts is due to the ancient Babylonians. And it is related to astronomy. This was used in Babylonia from around the third century BC. This is the degree measure we follow.

But for constructions on earth, another method was used to measure slant. See this picture.



As shown in it, at different points on a side of the angle, the “run” and “rise” are different, but rise divided by run gives the same number. (Why?). And this number depends on the angle. This was taken as a measure of slant. Such computations can be seen in the Ahmos Papyrus from ancient Egypt. The slant a face of square pyramids to the base is calculated in this manner. We may also observe the numbers got by dividing the hypotenuse of certain right triangles by another side given in a clay tablet from ancient Babylon.

As mentioned now,

$$\frac{AC}{BC} = \tan 50^\circ$$

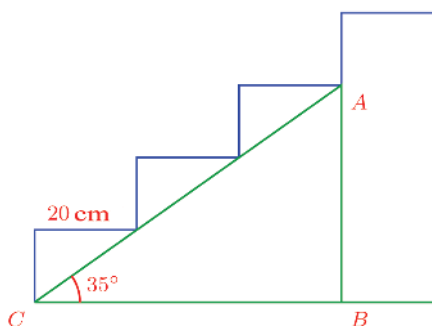
Now we can use our picture and the table to get.

$$AC = BC \times \tan 50^\circ \approx 3 \times 1.1918 = 3.5754 \approx 3.6$$

Here is an instance of using the tan measure of an angle. We have to find out how high the man in the picture is standing:



The various measures of the stairs are like this:



AB is the height we want to calculate.

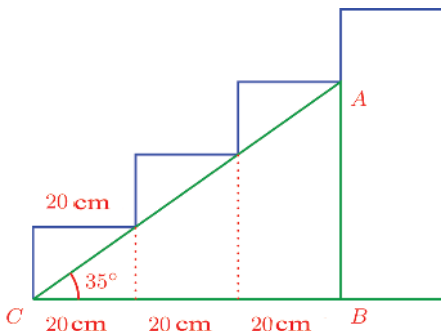
From the picture we get

$$AB = BC \times \tan 35^\circ$$

We get from the tan table,

$$\tan 35^\circ \approx 0.7002$$

What about the length of BC ?



From this picture, we see that BC is 60 centimetres. So

$$AB = BC \times \tan 35^\circ \approx 60 \times 0.7002 \approx 42.01$$

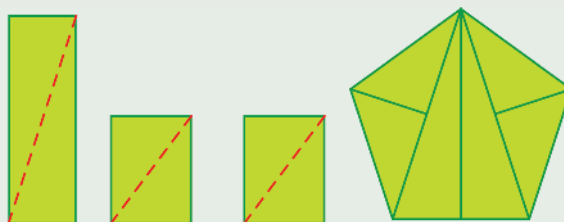
Thus the height is about 42 centimetres.

Other measures

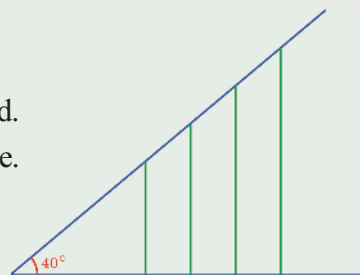
We have seen how various measures like \sin , \cos and \tan of an angle are got by taking the quotient of various combinations of sides of a right triangle containing the angle. There are other possible quotients. They also have special names in trigonometry. The reciprocals of sine and cosine are called *cosecant* and *secant* and the reciprocals of tangent are called *cotangent* . They are shortened as *cosec* , *sec* and *cot* .



- (1) One angle of a rhombus is 50° and one diagonal is 5 centimetres. What is its area?
- (2) A ladder leans against a wall, with its foot 2 metres away from the wall and the angle with the floor 40° . How high is the top end of the ladder from the ground?
- (3) Three rectangles are to be cut along the diagonals and the triangles so got rearranged to form a regular pentagon, as shown in the picture. If the sides of the pentagon are to be 30 centimetres, what should be the length and breadth of the rectangles?



- (4) In the picture, the vertical lines are equally spaced. Prove that their heights are in arithmetic sequence. What is the common difference?

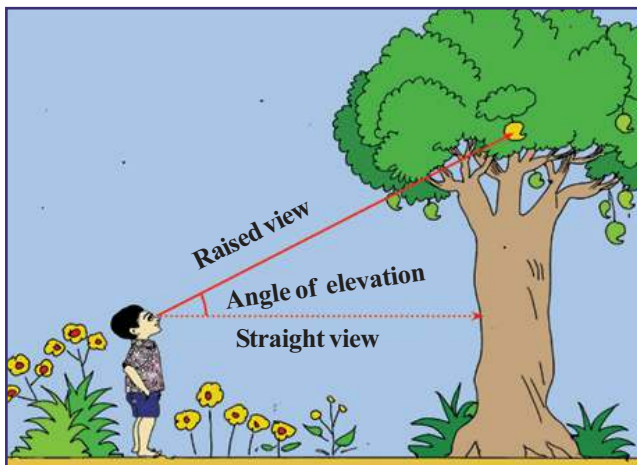
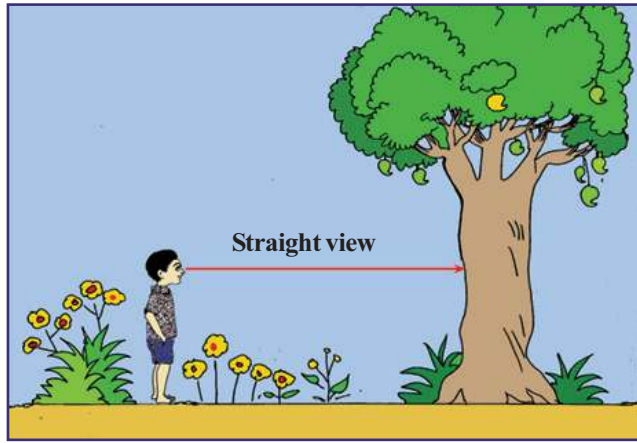


- (5) One side of a triangle is 6 centimetres and the angles at its ends are 40° and 65° . Calculate its area.



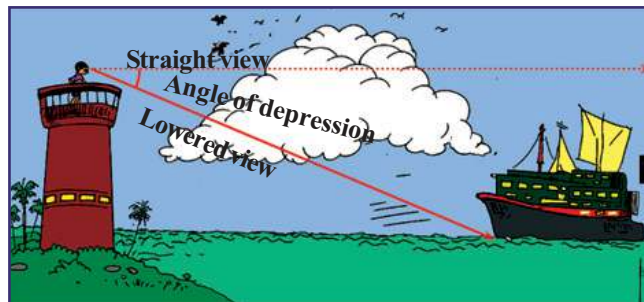
Distances and heights

To see things taller than us, we have to lift our heads. See these pictures:



Usually our line of vision is parallel to the ground. To see things a bit high, we have to raise (elevate) this. The angle between these two lines is called the *angle of elevation*.

Similarly, when we stand at a high place, we have to lower (depress) our sight to see things below.



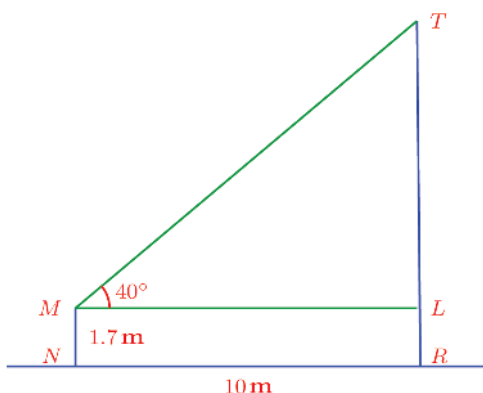
The angle so formed is called the *angle of depression*.

Such angles can be measured using an instrument called clinometer. Distances and heights which cannot be measured directly are computed by measuring angles using a clinometer and calculating lengths using sine, cosine and tangent.

Let's look at some examples.

A man standing 10 metres away from the foot of a tree sees its top at an elevation of 40° . His height is 1.7 metres. What is the height of the tree?

In the picture, MN is the man and TR is the tree.



From the picture (and using tables) we get,

$$TL = ML \tan 40^\circ \approx 10 \times 0.8391 = 8.391$$

So,

$$TR = TL + LR = TL + MN \approx 8.391 + 1.7 = 10.091$$

Thus the height of the tree is about 10.09 metres.

Another problem:

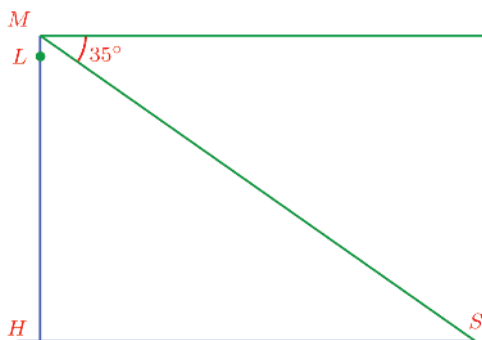
A man, 1.8 metres tall, stands on top of a light house 25 metres high and sees a ship at sea at a depression of 35° . How far is it from the foot of the light house?

Tan measure



Let's see how we can find \tan of angles using GeoGebra. Draw a line AB of length 1 and the perpendicular to AB at B . Make an angle slider α . Choose **Angle with Given Size** and click on B and A and in the window give α as the angle. We get a new point B' . Join AB' and mark the point where it intersects the perpendicular at B as C . Draw $\triangle ABC$ and hide all other points and lines. Mark the length BC . It is $\tan A$ (Why?). How large can be the \tan measure?

Let's draw a picture:



In this, LH is the light house and ML is the man standing on it. S denotes the ship.

The length to compute is HS .

From the given details,

$$MH = ML + LH = 25 + 1.8 = 26.8$$

Also, $\angle HMS = 55^\circ$

So from the right triangle MHS , we have.

$$HS = MH \tan 55^\circ \approx 26.8 \times 1.4281 \approx 38.27$$

Thus the ship is about 38.27 metres away from the foot of the light house.

Another problem:

A boy standing at the edge of a canal sees the top of a tree at an elevation of 50° . Stepping 10 metres back, he sees it at an elevation of 25° . The boy is 1.5 metres tall. How wide is the canal and how tall is the tree?

Slant and spread

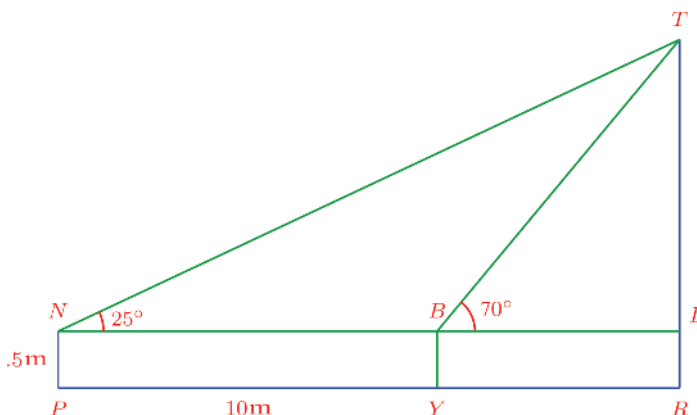
Sine and cosine arise from the need to see angle as spread. Tangent of an angle arises when we connect these to angle as slant. (It is just the old way of measuring slant as rise by run).

Such a relation and a tan table was first given by the Arab mathematician Ahmed Ibn Abdullah Al Morvasi. The name tan was introduced in the 14th century.





In the picture below, TR is the tree, BY is the first position of the boy, NP is his new position:



We have to calculate the lengths TR and YR . From the picture,

$$YR = BL \quad TR = TL + LR = TL + 1.5$$

So, we need only find BL and TL . Let

$$BL = x \quad TL = y$$

Then from the right triangle BTL ,

$$y = x \tan 70^\circ \approx 2.7475x$$

and from the right triangle NTL ,

$$y = (x + 10) \tan 25^\circ \approx 0.4663(x + 10) = 0.4663x + 4.663$$

So, we have

$$2.7475x \approx 0.4663x + 4.663$$

Which gives

$$x \approx \frac{4.663}{2.2818} \approx 2.044$$

using a calculator. From this we get

$$y \approx 2.7475 \times 2.044 \approx 5.616$$

Thus the width of the canal is about 2.04 metres and the height of the tree is about $5.62 + 1.75 = 7.12$ metres.

?



- (1) When the sun is at an elevation of 40° , the length of the shadow of a tree is 18 metres. What is the height of the tree?
- (2) A 1.75 metre tall man, standing at the foot of a tower, sees the top of a hill 40 metres away at an elevation of 60° . Climbing to the top of the tower, he sees it at an elevation of 50° . Calculate the heights of the tower and the hill.
- (3) A 1.5 metre tall boy saw the top of a building under construction at an elevation of 30° . The completed building was 10 metres higher and the boy saw its top at an elevation of 60° from the same spot. What is the height of the building?
- (4) A man 1.8 metre tall standing at the top of a telephone tower, saw the top of a 10 metre high building at a depression of 40° and the base of the building at a depression of 60° . What is the height of the tower? How far is it from the building?
- (5) From the top of an electric post, two wires are stretched to either side and fixed to the ground, 25 metres apart. The wires make angles 55° and 40° with the ground. What is the height of the post?
- (6) When the sun is at an elevation of 35° , the shadow of a tree is 10 metres. What would be the length of the shadow, when the sun is at an elevation of 25° ?



Project

- Find the relation of the perimeter and area of a regular polygon with its circumradius.
- Find a sequence of numbers getting closer and closer to π using \sin .

Trigonometric values

Angle	sin	cos	tan	Angle	sin	cos	tan
0°	0.0000	1.0000	0.0000	46°	0.7193	0.6947	1.0355
1°	0.0175	0.9998	0.0175	47°	0.7314	0.6820	1.0724
2°	0.0349	0.9994	0.0349	48°	0.7431	0.6891	1.1106
3°	0.0523	0.9986	0.0524	49°	0.7547	0.6561	1.1504
4°	0.0698	0.9976	0.0699	50°	0.7660	0.6428	1.1918
5°	0.0872	0.9962	0.0875	51°	0.7771	0.6293	1.2349
6°	0.1045	0.9945	0.1051	52°	0.7880	0.6157	1.2799
7°	0.1219	0.9925	0.1228	53°	0.7986	0.6018	1.3270
8°	0.1392	0.9903	0.1405	54°	0.8090	0.5878	1.3764
9°	0.1564	0.9877	0.1584	55°	0.8192	0.5736	1.4281
10°	0.1736	0.9848	0.1763	56°	0.8290	0.5592	1.4826
11°	0.1908	0.9816	0.1944	57°	0.8387	0.5446	1.5399
12°	0.2079	0.9781	0.2126	58°	0.8480	0.5299	1.6003
13°	0.2250	0.9744	0.2309	59°	0.8572	0.5150	1.6643
14°	0.2419	0.9703	0.2493	60°	0.8660	0.5000	1.7321
15°	0.2588	0.9659	0.2679	61°	0.8746	0.4848	1.8040
16°	0.2756	0.9613	0.2867	62°	0.8829	0.4695	1.8807
17°	0.2924	0.9563	0.3057	63°	0.8910	0.4540	1.9626
18°	0.3090	0.9511	0.3249	64°	0.8988	0.4384	2.0503
19°	0.3256	0.9455	0.3443	65°	0.9063	0.4226	2.1445
20°	0.3420	0.9397	0.3640	66°	0.9135	0.4067	2.2460
21°	0.3584	0.9336	0.3839	67°	0.9205	0.3907	2.3559
22°	0.3746	0.9272	0.4040	68°	0.9272	0.3746	2.4751
23°	0.3907	0.9205	0.4245	69°	0.9336	0.3584	2.6051
24°	0.4067	0.9135	0.4452	70°	0.9397	0.3420	2.7475
25°	0.4226	0.9063	0.4663	71°	0.9455	0.3256	2.9042
26°	0.4384	0.8988	0.4877	72°	0.9511	0.3090	3.0777
27°	0.4540	0.8910	0.5095	73°	0.9563	0.2924	3.2709
28°	0.4695	0.8829	0.5317	74°	0.9613	0.2756	3.4874
29°	0.4848	0.8746	0.5543	75°	0.9659	0.2588	3.7321
30°	0.5000	0.8660	0.5774	76°	0.9703	0.2419	4.0108
31°	0.5150	0.8572	0.6009	77°	0.9744	0.2250	4.3315
32°	0.5299	0.8480	0.6249	78°	0.9781	0.2079	4.7046
33°	0.5446	0.8387	0.6494	79°	0.9816	0.1908	5.1446
34°	0.5592	0.8290	0.6745	80°	0.9848	0.1736	5.6713
35°	0.5736	0.8192	0.7002	81°	0.9877	0.1564	6.3138
36°	0.5878	0.8090	0.7265	82°	0.9903	0.1392	7.1154
37°	0.6018	0.7986	0.7536	83°	0.9925	0.1219	8.1443
38°	0.6157	0.7880	0.7813	84°	0.9945	0.1045	9.5144
39°	0.6293	0.7771	0.8098	85°	0.9962	0.0872	11.4301
40°	0.6428	0.7660	0.8391	86°	0.9976	0.0698	14.3007
41°	0.6561	0.7547	0.8693	87°	0.9986	0.0523	19.0811
42°	0.6691	0.7431	0.9004	88°	0.9994	0.0349	28.6363
43°	0.6820	0.7314	0.9325	89°	0.9998	0.0175	57.2900
44°	0.6947	0.7193	0.9657	90°	1.0000	0.0000
45°	0.7071	0.7071	1.0000				

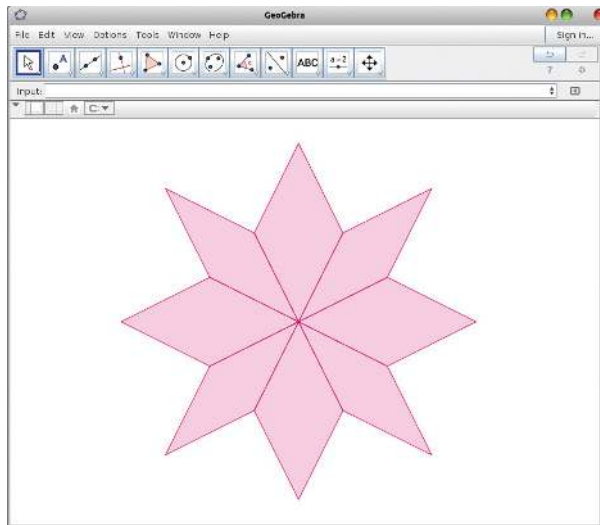
Looking back



Learning outcomes	On my own	With teacher's help	Must improve
<ul style="list-style-type: none"> • Interpreting sine, cosine and tangent as measures of an angle using straight lines. • Expressing the relation between the length of chord and its central angle using sine. • Recognising the fact that the side of a triangle are proportional to the sine of its angles. • Using trigonometric measures to calculate some measures of a triangle from other measures. • Explaining how distances and heights which cannot be directly measured can be computed using trigonometry. 			



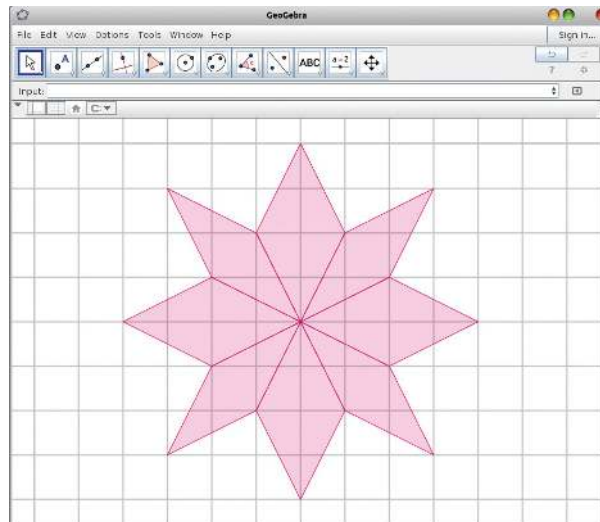
See this picture drawn in GeoGebra:



How is it drawn?

Many things used to draw this are hidden, after completing it.

See this picture:





First we draw square cells and then join some of the corners to draw the picture.



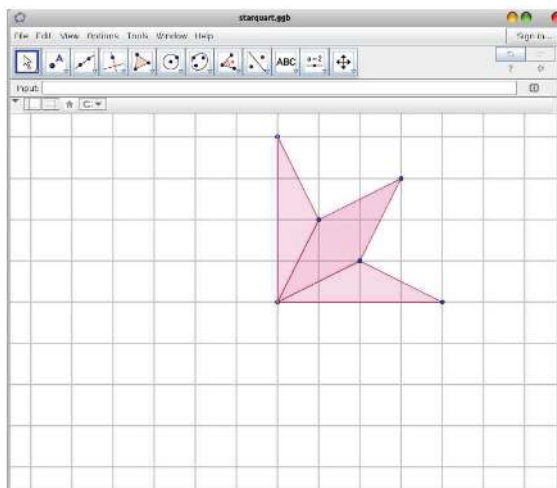
To get these cells, we use **Grid** in GeoGebra.

Now suppose we want to draw a larger copy of this picture on paper.

First we draw a grid like this and join the necessary corners.

There is a trick to draw the picture without marking all the needed corners.

See this picture:



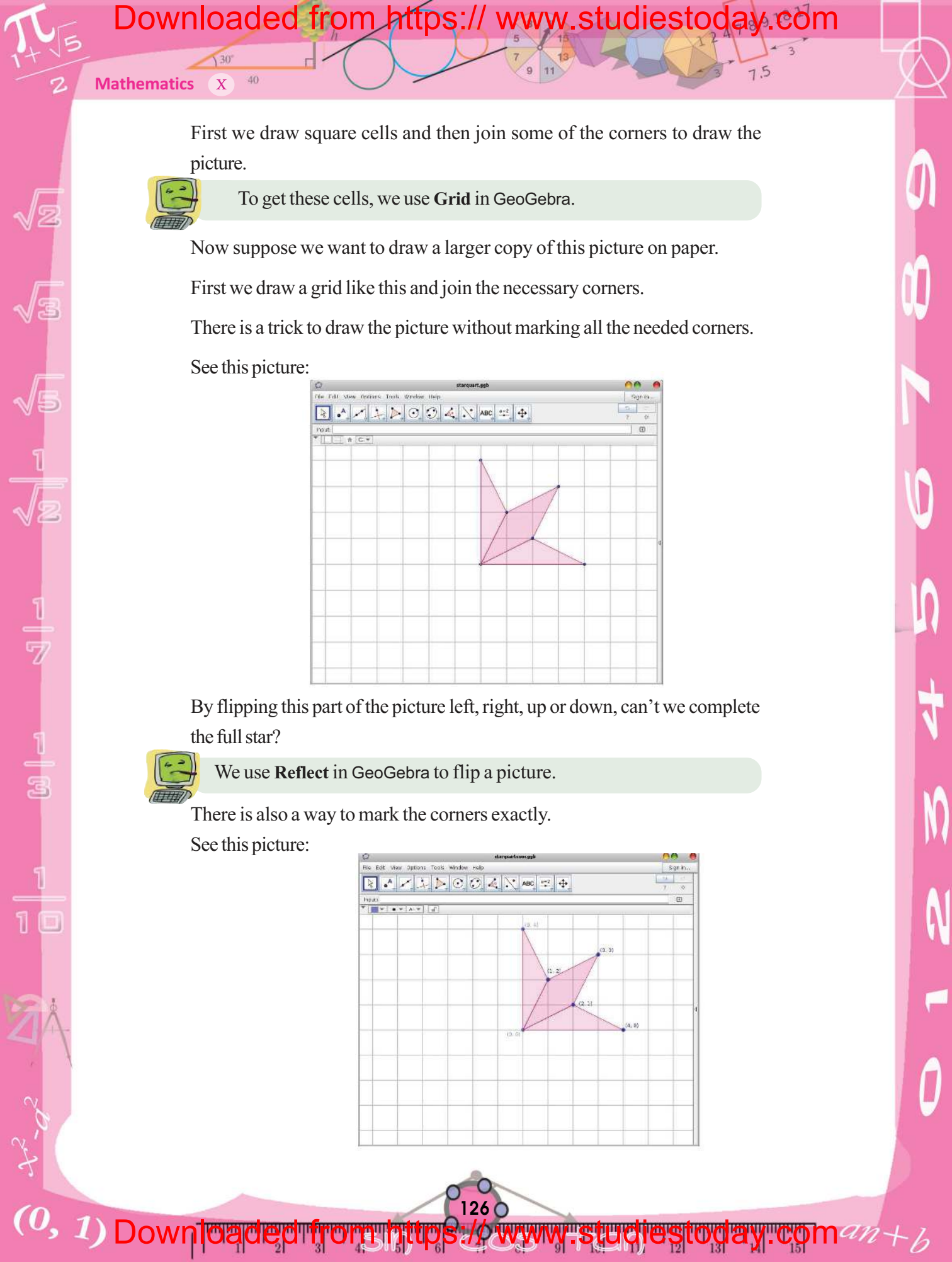
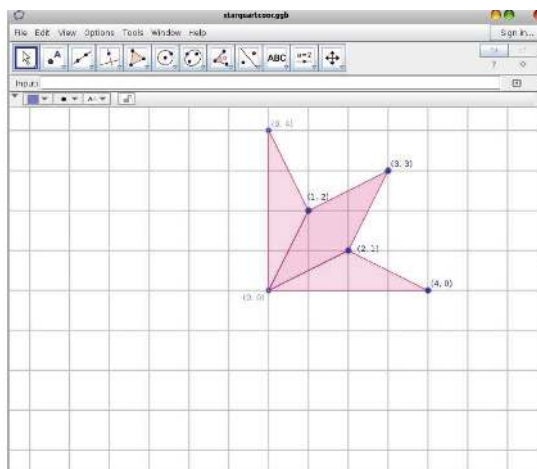
By flipping this part of the picture left, right, up or down, can't we complete the full star?



We use **Reflect** in GeoGebra to flip a picture.

There is also a way to mark the corners exactly.

See this picture:



$$\frac{1 + \sqrt{5}}{2}$$



Don't you see a pair of numbers at each corner of the picture?

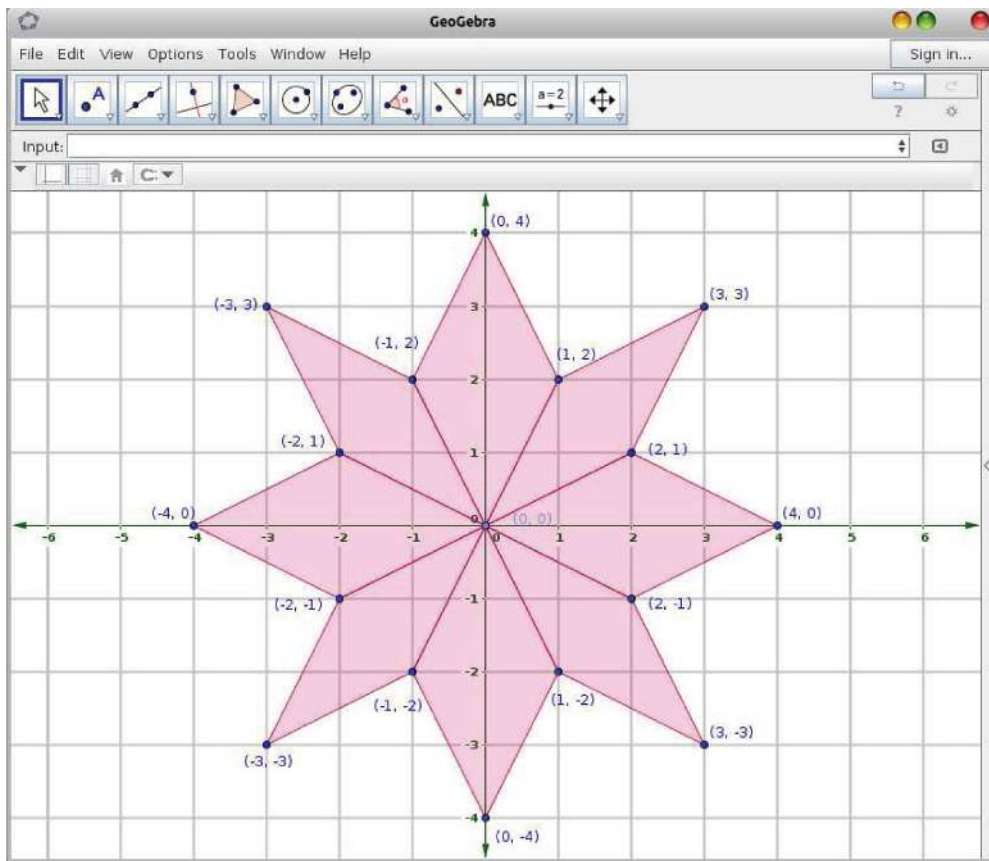
What do they mean?

For example, look at the corner marked (2, 1). This corner is reached from the centre of our star by moving 2 cells to the right and then 1 cell up.



We can mark points anywhere in GeoGebra by choosing a point and just clicking. To mark a point exactly at a specific place, it is better to type the pairs of numbers as above in the Input Bar.

We can find the number pairs for all corners of our star.



Look at the top left part of the pictures. All number pairs here have the first number negative, right?

Distances to the left of the centre are usually taken to be negative. It is a device to distinguish between left and right using numbers. (Recall the **Number line** discussed in Class 9).

$$x^2 - a^2$$

$$(0, 1)$$

$$an + b$$

Similarly, note that for points below the centre, the second numbers of the pairs are negative.

Thus when we denote points by number pairs like this, the first number shows distance to the right or left and the second number shows distance up or down. Distances to the left or downward are to be taken negative.

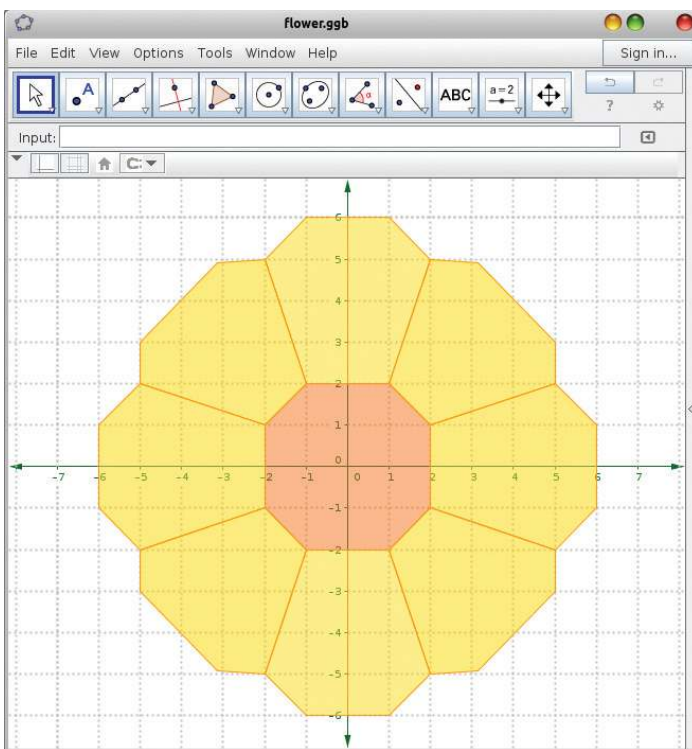
To see these numbers easily, distances are marked on a horizontal and a vertical line through the centre.



We use **Axes** in GeoGebra to see these lines.

Now try to copy the picture to your notebook.

Here's another picture drawn in GeoGebra.



Can you mark all its corners using number pairs? Then try to draw it on paper.



To mark points in GeoGebra using number pairs, type them one by one in the Input Bar. To draw the polygon connecting them, use Polygon. For example, type the following command in the Input Bar.

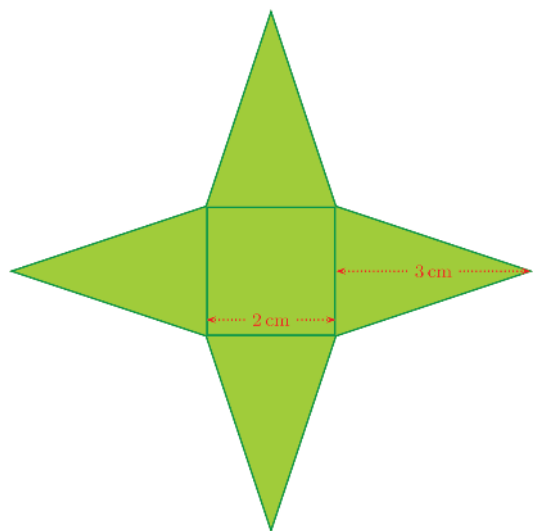
Polygon [(-1, -1), (1, -1), (1, 1), (-1, 1)]

$$\frac{\pi + \sqrt{5}}{2}$$

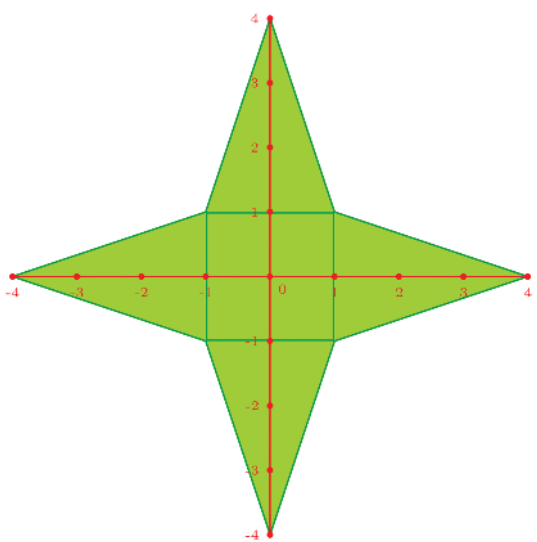


Positions and numbers

We want to draw a picture like this:



How about first marking all the corners with number pairs? We need not draw a grid. Suppose that we draw a horizontal line and a vertical line through the centre and mark distances on it, one centimetre apart:



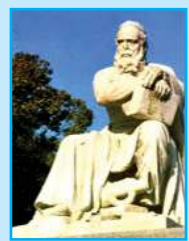
Can you write the number pairs of all corners?

For example, the top-right corner of the square is 1 centimetre to the right and then 1 centimetre up, from the centre. So its number pair is (1, 1).

A bit of history

Even as early as the second century BC, the Greek mathematician Apollonius used the method of denoting points by numbers to solve some geometric problems. These were distances from fixed lines.

Then in the eleventh century AD, we see the Persian poet and mathematician Omar Khayyam denoting number pairs as points, to convert certain algebraic problems to geometry.



Such connections between geometry and algebra started to develop as a systematic branch of mathematics after the French philosopher Rene Descartes published his *Geometry* in the seventeenth century.



$$\sqrt{2}$$

$$\sqrt{3}$$

$$\sqrt{5}$$

$$\frac{1}{\sqrt{2}}$$

$$\frac{1}{7}$$

$$\frac{3}{11}$$

$$\frac{1}{10}$$



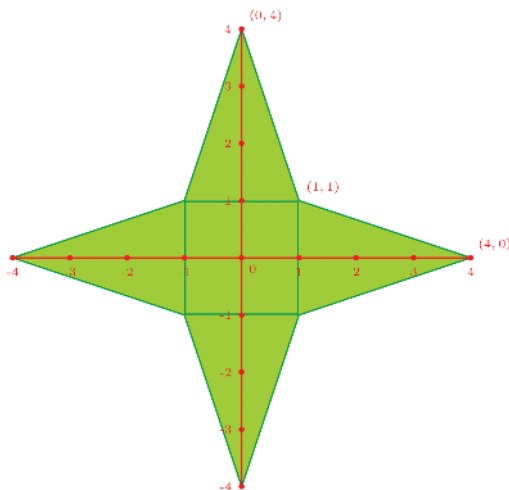
$$x^2 - a^2$$

$$(0, 1)$$

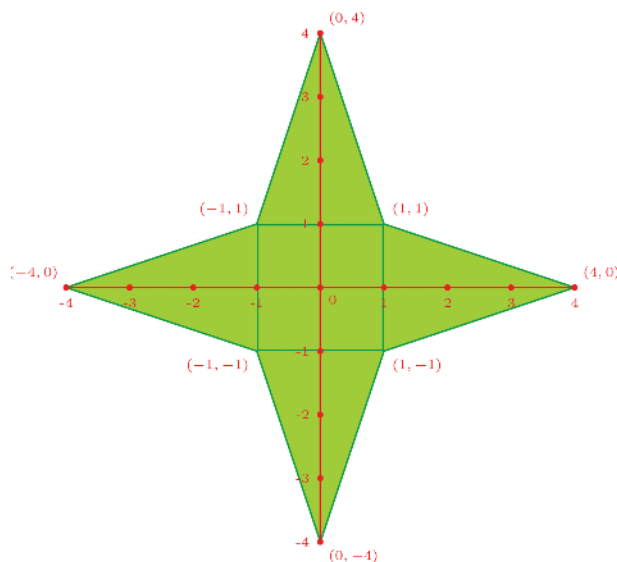
$$an + b$$



What about the rightmost point of the picture? 4 centimetres to the right of the centre, neither up nor down. So, we write its number pair as $(4,0)$. The situation is quite the opposite for the top most point: 4 centimetres straight up from the centre, neither left nor right. We write its number pair as $(0,4)$.



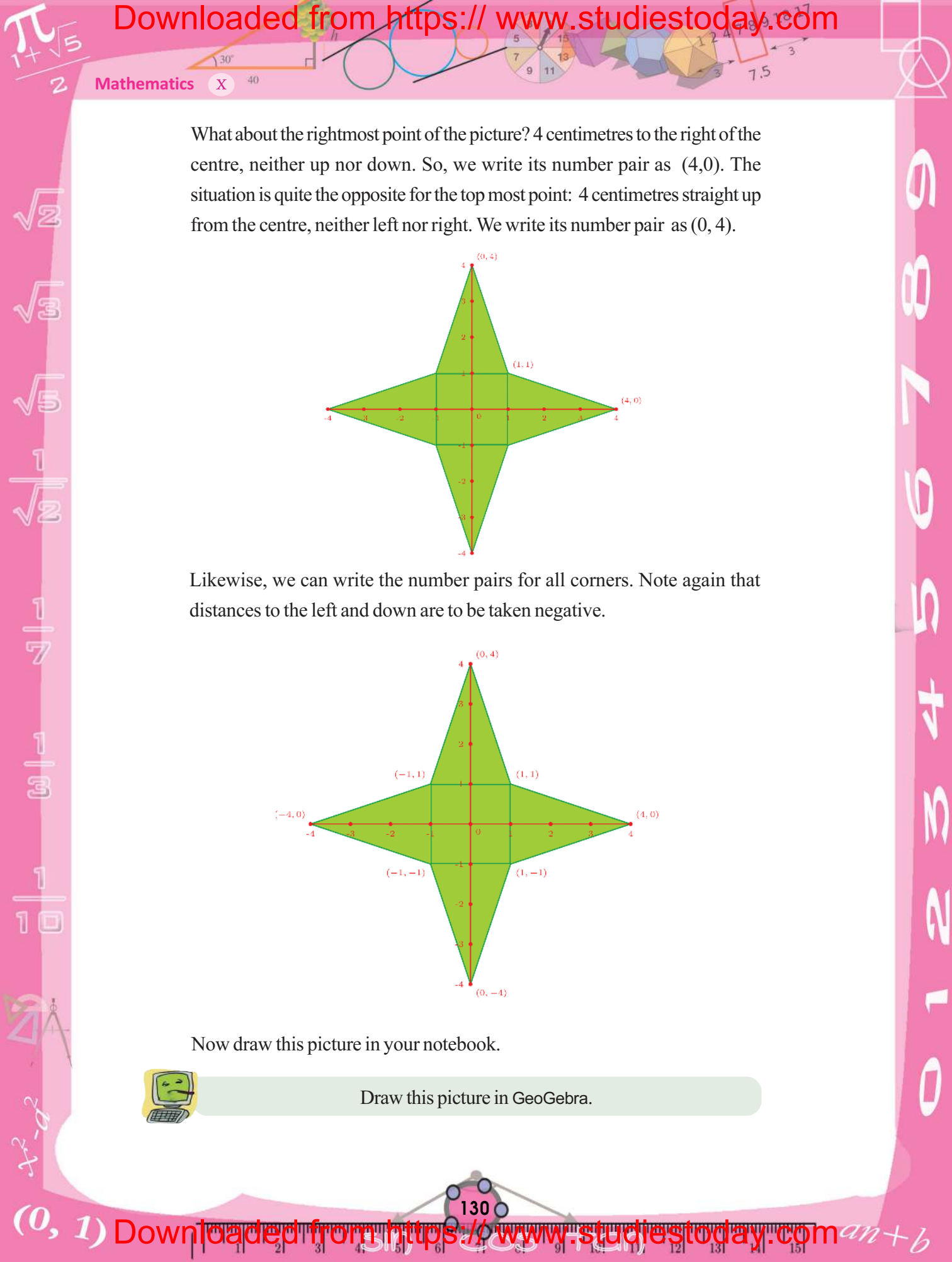
Likewise, we can write the number pairs for all corners. Note again that distances to the left and down are to be taken negative.



Now draw this picture in your notebook.



Draw this picture in GeoGebra.

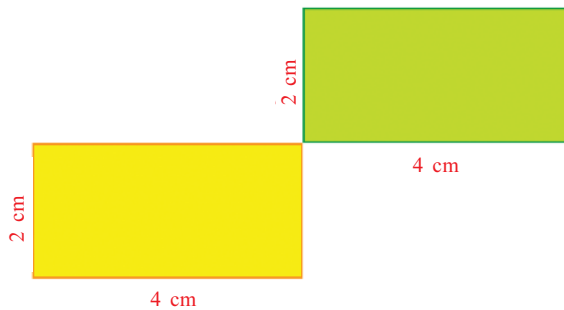


The perpendicular lines we draw to mark the positions of points are called the *axes of coordinates*. The horizontal line is the *x axis* and the vertical line is the *y axis*.

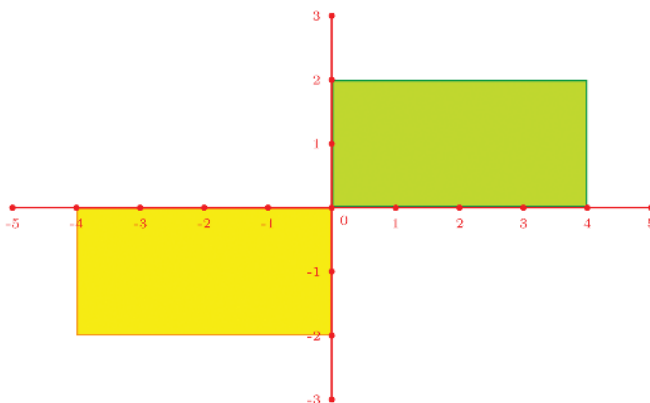
Once axes are drawn, we can denote the position of any point using a pair of numbers. These numbers are called the coordinates of the point.

To draw a picture, we can draw the axes of coordinates anywhere and in anyway we like. (They should be mutually perpendicular, that's all).

For example, see this picture:

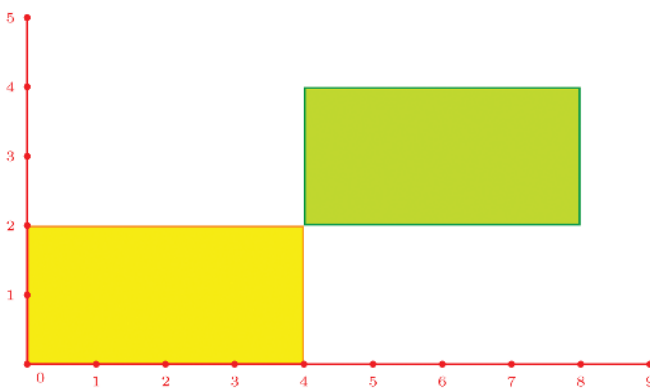


We can draw axes like this:



What are the coordinates of the corners of the two rectangles?

Now suppose we draw the axes like this:



Earth division

The Earth rotates by itself. On any spinning sphere, two points do not move. They are the poles. And the line joining the poles is the axis of rotation. We can draw circles on a sphere; and those circles whose centre is the centre of the sphere itself are called great circles. The great circle at the same distance from either pole is the equator. Circles parallel to the equator are the lines of longitude.

Great circles through the poles are called lines of longitude or meridians. Of these, the one through Greenwich in England is taken as the prime meridian.



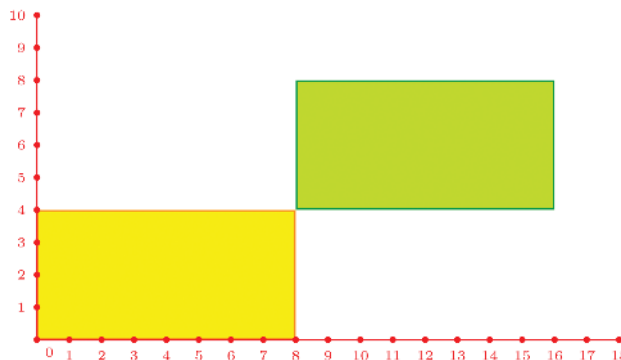
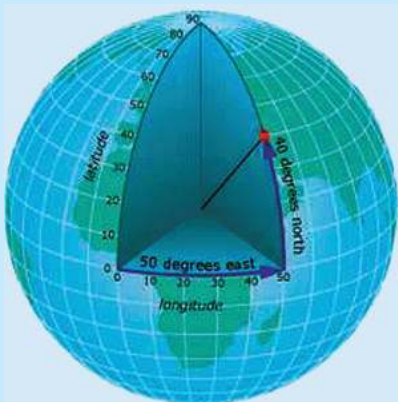
What are the coordinates of the corners of the rectangles in terms of these axes?

After drawing the axes, we must mark equally spaced points on them. The spacing need not be one centimetre, it can be any convenient distance.

For example, this is the last picture above, with points half a centimetre apart.

Positions on Earth

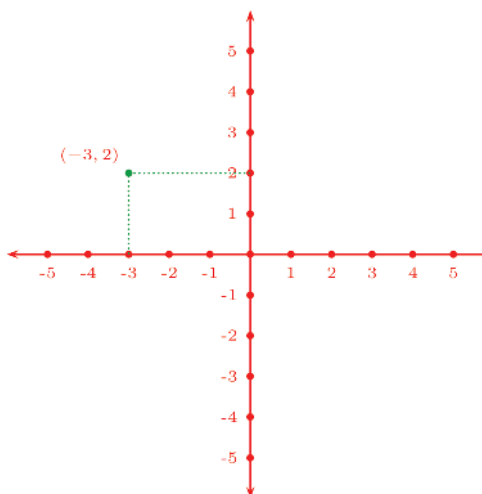
Imagine a the point of intersection of the equator and the Greenwich meridian and a line joining this point to the centre of the Earth. To get to another latitude from this point, we must move north or south and the line from the centre should rotate up or down by a certain angle. Such angles are used to denote latitudes (with the adjective north or south). Suppose we want to move to another longitude from the first point. We must move east or west and the line should rotate right or left through a certain angle. Such angles are used to denote longitudes.



Now what are the coordinates of the corners?

Once we have drawn the axes and marked distances along them, how do we mark points using coordinates?

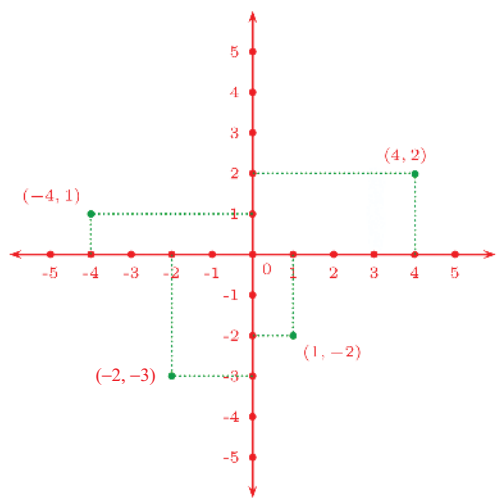
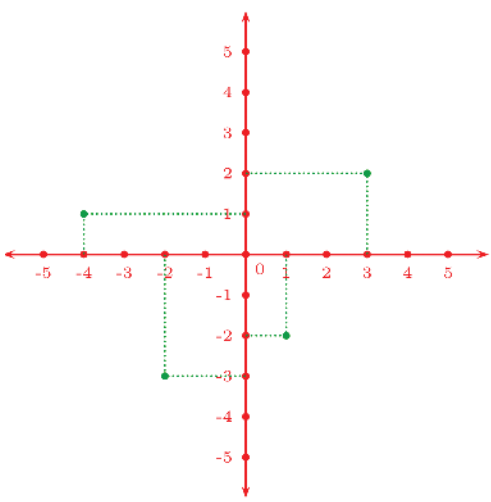
For example, see how a point with coordinates $(-3, 2)$ is marked:



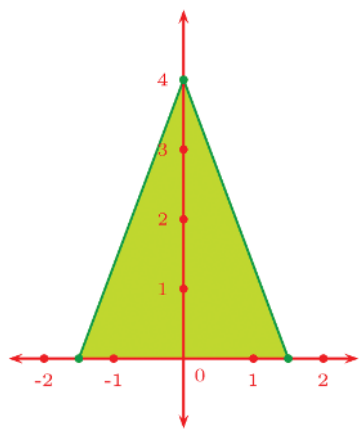
The point with coordinates $(-3, 2)$ is the intersection of perpendiculars from the point marked -3 on the x axis and the point marked 2 on the y axis.



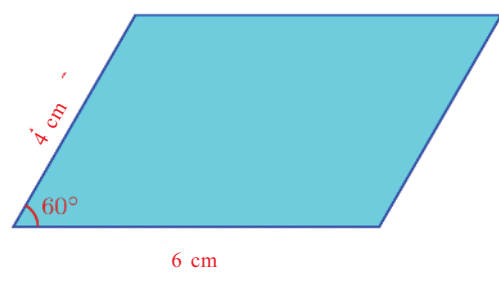
On the other hand, to find the coordinates of a point marked, we need only draw perpendiculars from the point to the x and y axes:



Coordinates need not be whole numbers. For example, to draw an isosceles triangle of base 3 centimetres and height 4 centimetres, we can draw axes like this:



Then, what are the coordinates of the vertices of the triangle?
How about this parallelogram?



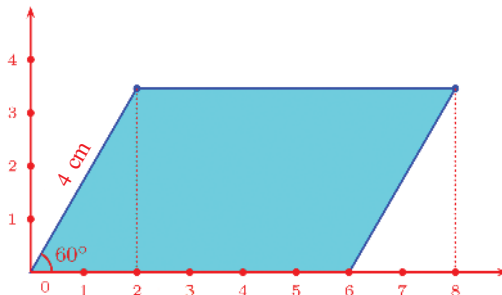
(0, 1)

$an + b$

Let's draw axes like this:



Make a slider a in GeoGebra. Type $(a, 0)$ in the Input Bar. Change a using the slider. What is the path of the point? Likewise, draw points such as $(a, 2)$, $(a, -1)$, $(0, a)$, $(3, a)$, $(-2, a)$ and so on and see the path of travel of each as a is changed. Apply Trace On to the point and see.



We know the ratio of sides of a triangle with angles 30° , 60° , 90° . So the coordinates of the top left vertex are $(2, 2\sqrt{3})$.

What about the top right vertex?



To get the point with coordinates $(2, 2\sqrt{3})$ in GeoGebra, type $(2, 2 \text{ sqrt } 3)$ in GeoGebra.

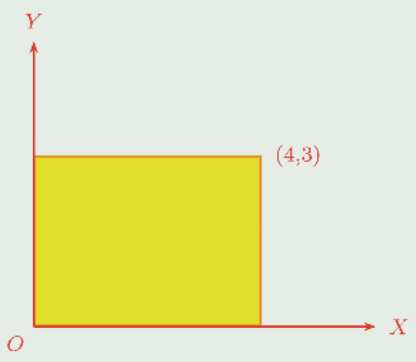
In drawing pictures using axes of coordinates, the x axis is labelled $X'X$ (from left to right) and the y axis, YY' (from top to bottom). Their point of intersection is labelled O and is called the origin.



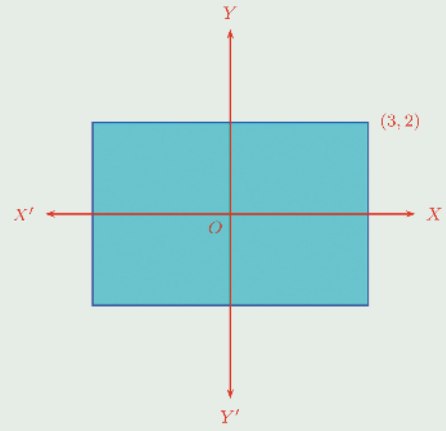
- (1) Find the following:
 - i) The y coordinate of any point on the x axis.
 - ii) The x coordinate of any point on the y axis.
 - iii) The coordinates of the origin.
 - iv) The y coordinate of any point on the line through $(0,1)$, parallel to the x axis.
 - v) The y coordinate of any point on the line through $(1, 0)$, parallel to the y axis.



(2) Find the coordinates of the other three vertices of the rectangle below:

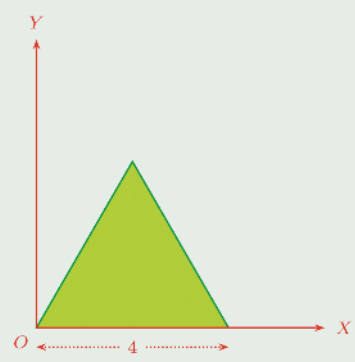


(3) In the rectangle shown below, the sides are parallel to the axes and origin is the mid point:



What are the coordinates of the other three vertices?

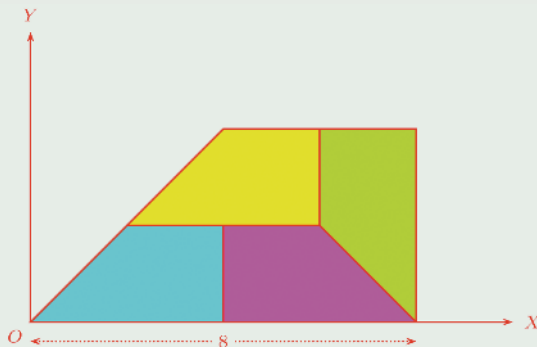
(4) The triangle shown below is equilateral:



Find the coordinates of its vertices.



(5) A large trapezium made up of four equal trapeziums:



Find the coordinates of the vertices of all these trapeziums.

Draw this picture in GeoGebra.



In the Input Bar of GeoGebra, type

Sequence [(a, a + 1), a, 0, 5]

This command asks to mark all points with coordinates (a, a + 1), taking a to be the whole numbers from 0 to 5, that is the points (0,1), (1, 2), (2,3), (3,4), (4,5), (5,6).

Change the command slightly as

Sequence [(a, a + 1), a, 0, 5, 0.5]

In this, the last 0.5 asks a to be taken as the sequence of numbers starting from 0 and increasing by 0.5 at every step, upto 5 (If the number to be added at each step is 1, we need not specify it). Thus with this command, we get the points (0, 1), (0.5, 1.5), (1,2), ... upto (5, 6).

Discuss the speciality of the points got from each of these commands:

Sequence [(a, 0), a, 0, 5, 0.5]

Sequence [(a, 2a), a, -3, 4, 0.25]

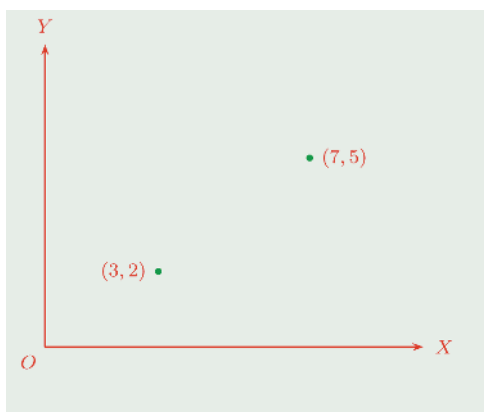
Sequence [(a, a²), a, -3, 3, 0.2]

Sequence [(a, -a²), a, -3, 3, 0.2]

Sequence [(a², a), a, -4, 4, 0.1]

Rectangles

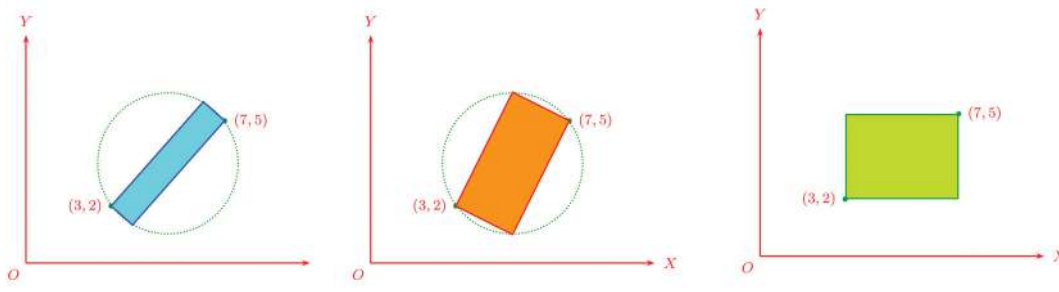
See this picture:



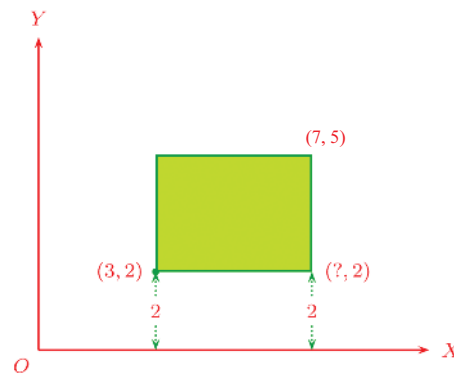
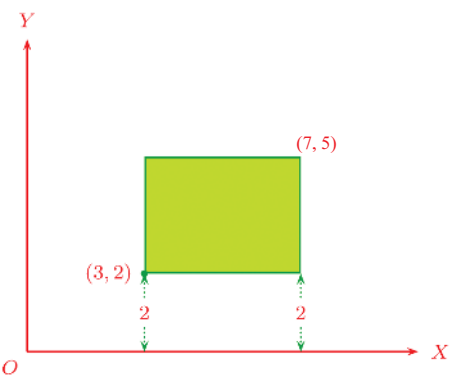
We want to draw a rectangle with these points as two opposite vertices.

Can't we draw any number of such rectangles?





Only one among these has sides parallel to the axes.
 What are the coordinates of its other vertices? For that, let's look at the figure in a little more detail.
 Since the bottom left corner has y coordinate 2, it is at height 2 above the x axis, and since the bottom side is parallel to the x axis, the other corner on this side is also at the same height.



That is, the y coordinate of this point is also 2.

To find its x coordinate, look at the top right corner. Since its x coordinate is 7, its distance from the y axis is 7.

Circle pictures

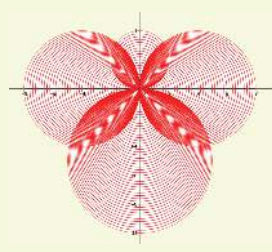
Typing circle [(1, 3), 2] in the Input Bar we get the circle centered at (1, 3) and of radius 2 in GeoGebra.

The command
 Sequence [circle [(a, 0), 1], a, 0, 5, 0.2]
 produces circles centered at the points (0, 0), (0.2, 0), (0.4, 0), ..., (5, 0) all with the radius 1.

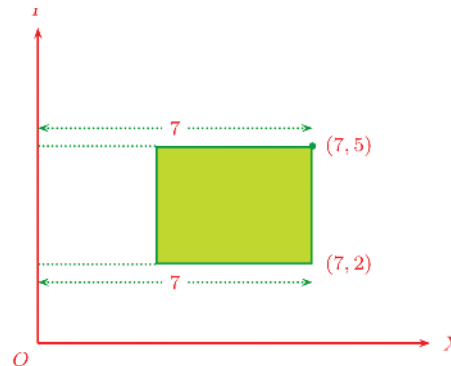
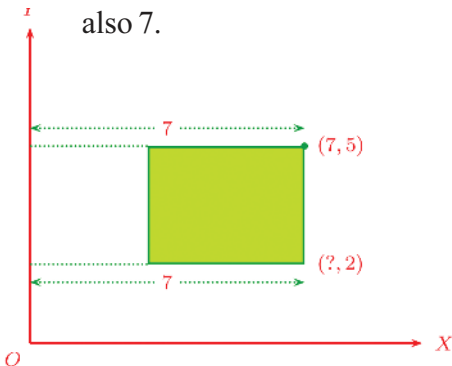
Now try to visualize the pictures got by each of these commands. Then check them using GeoGebra.

- Sequence [circle [(a, 0), a], a, 0, 10, 0.1]
- Sequence [circle [(a, 0), $\frac{a}{4}$], a, 0, 10, 0.1]

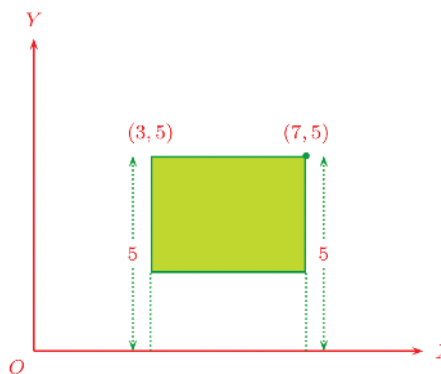
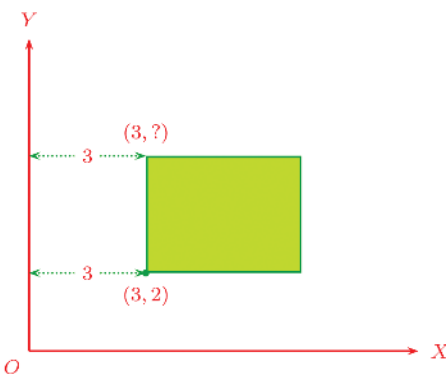
Can you think of the command to produce this picture?



Since the right side of the rectangle is parallel to the y axis, the other corner on the line is also at a distance 7 from the y axis. That is, its x coordinate is also 7.



Similarly, we can find the top left corner also.



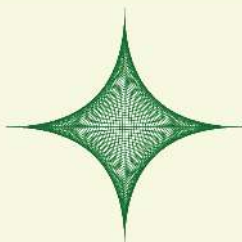
The command

Segment $[(2, -1), (3, 5)]$

produces the line segment joining the points $(2, -1)$ and $(3, 5)$. Discuss the speciality of the lines got from each of these commands.

- Sequence [segment $[(a, 0), (a, 3)], a, 0, 5, 0.2]$
- Sequence [segment $[(a, 0), (a, a)], a, 0, 5, 0.2]$
- Sequence [segment $[(0, 3), (a, 0)], a, -4, 4, 0.1]$
- Sequence [segment $[(a, 0), (0, a)], a, -3, 3, 0.2]$
- Sequence [segment $[(a, 0), (0, 5-a)], a, 0, 5, 0.1]$

Can you think of a command to get this picture?



Now look at the coordinates of all four vertices together:

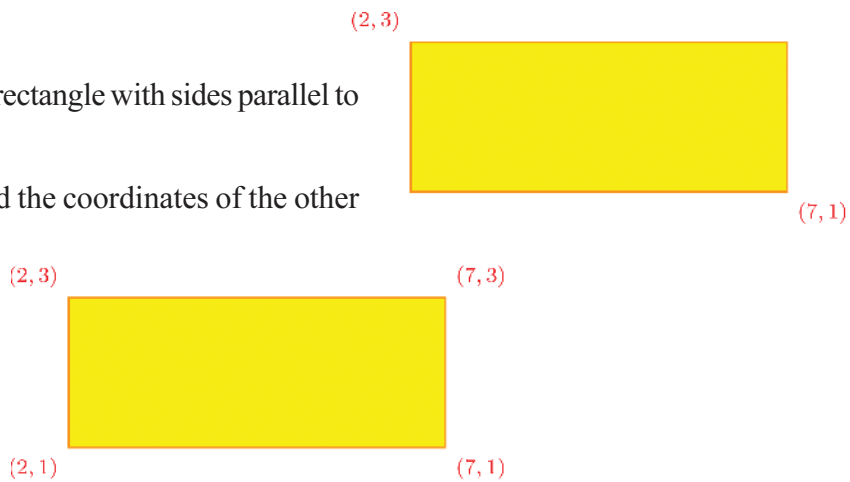


And also consider the method of finding these coordinates. What can we say in general?

Moving parallel to the x axis does not change the y coordinate, moving parallel to the y axis does not change the x coordinate.



Here's another rectangle with sides parallel to the axes. How do we find the coordinates of the other two vertices?



?

(1) All rectangles below have sides parallel to the axes. Find the coordinates of the remaining vertices of each.

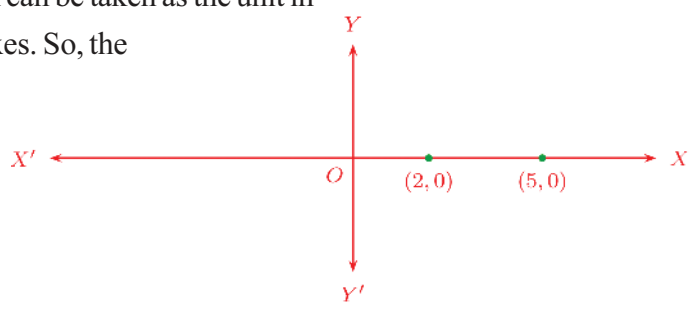
(2) Without drawing coordinate axes, mark each pair of points below with left-right, top-bottom position correct. Find the other coordinates of the rectangles drawn with these as opposite vertices and sides parallel to the axes.

i) (3, 5), (7, 8)	ii) (6, 2), (5, 4)
iii) (-3, 5), (-7, 1)	iv) (-1, -2), (-5, -4)

Distances

We have noted that any length can be taken as the unit in marking distance along the axes. So, the distance between two points on an axis can be said only as a multiple of this unit.

For example, look at these two points on the x axis.

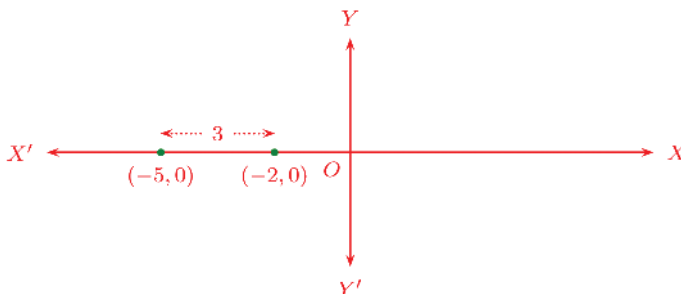




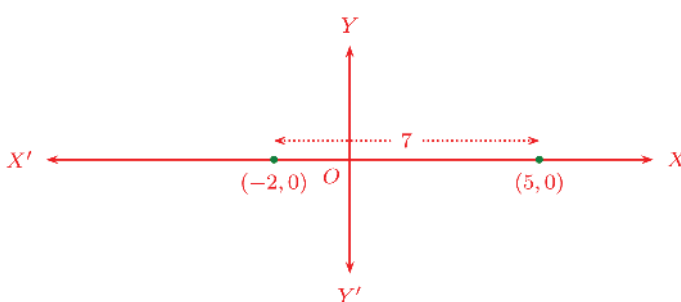
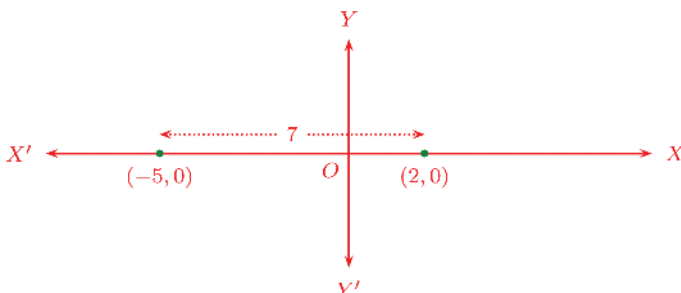
The distance from the origin to the first point is 2 times this unit; and the distance to the second point is 5 times this unit. We shorten this and say, the distance from the origin to the first point is 2 and the distance to the second point is 5.

So the distance between them is $5 - 2 = 3$.

What if the points are like this?



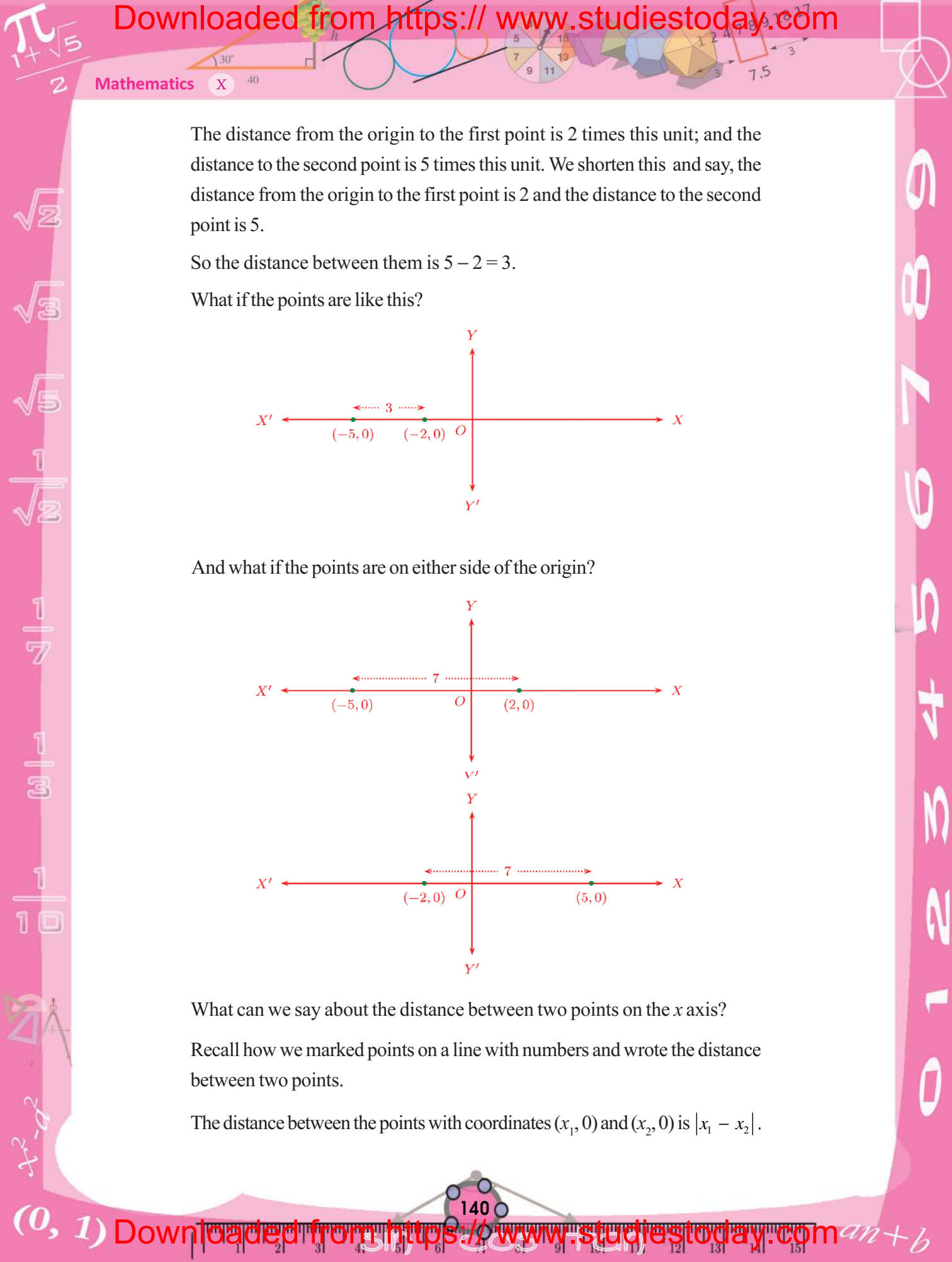
And what if the points are on either side of the origin?



What can we say about the distance between two points on the x axis?

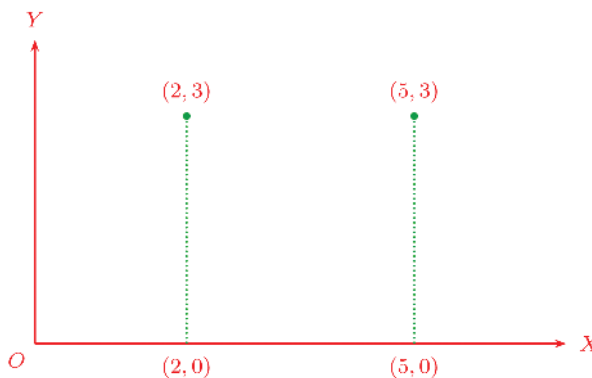
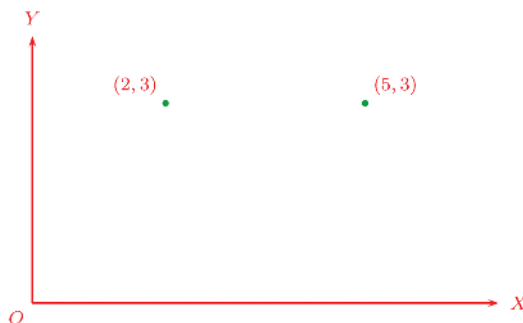
Recall how we marked points on a line with numbers and wrote the distance between two points.

The distance between the points with coordinates $(x_1, 0)$ and $(x_2, 0)$ is $|x_1 - x_2|$.



Now look at these points:

The distance between them is equal to the distance between the feet of the perpendiculars from these to the x axis, isn't it? (Why?)



In general, we can say this:

All points with the same y coordinate are on a line parallel to the x axis; and the distance between any pair of such points is the difference of their x coordinates.

We have something similar for points with same x coordinate also:

All points with the same x coordinate are on a line parallel to the y axis; and the distance between any pair of such points is the difference of their y coordinates.

We can put these in algebraic language:

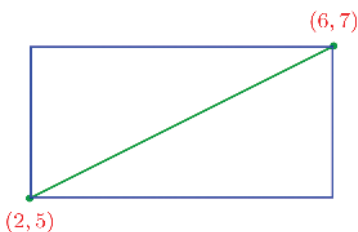
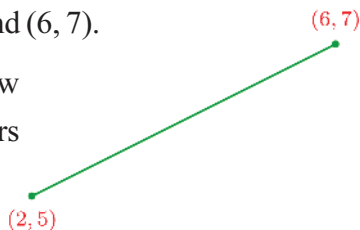
The distance between the points with coordinates (x_1, y) and (x_2, y) is $|x_1 - x_2|$.

The distance between the points with coordinates (x, y_1) and (x, y_2) is $|y_1 - y_2|$.

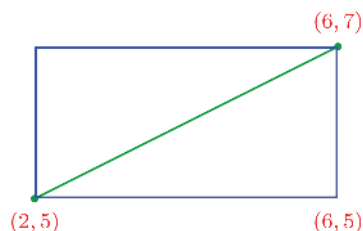
How do we find the distance between two points with x and y coordinates different?

For example, let's take the points (2, 5) and (6, 7).

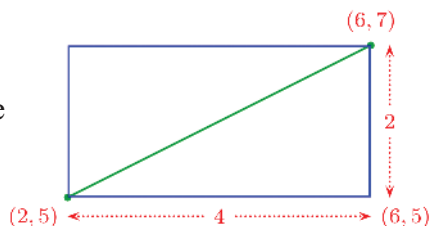
To find the distance between them, we draw the rectangle with these as opposite corners and sides parallel to the axes.



We want to find the length of its diagonal. To calculate it, we need the length of its sides. And for that, let's write the coordinates of the bottom right corner.



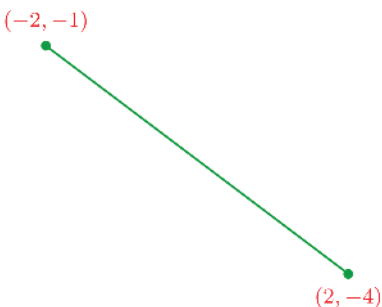
Using it can't we find the length of the sides of the rectangle?



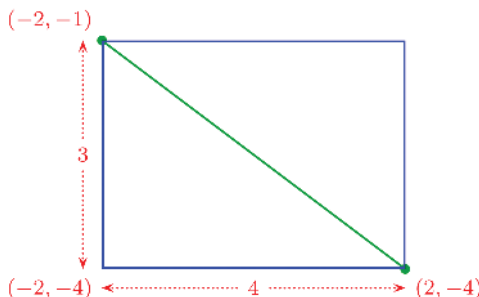
Now we can find the distance we want using Pythagoras theorem:

$$\sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$$

What if the points are like this?



Again we can draw a rectangle and calculate the distance:

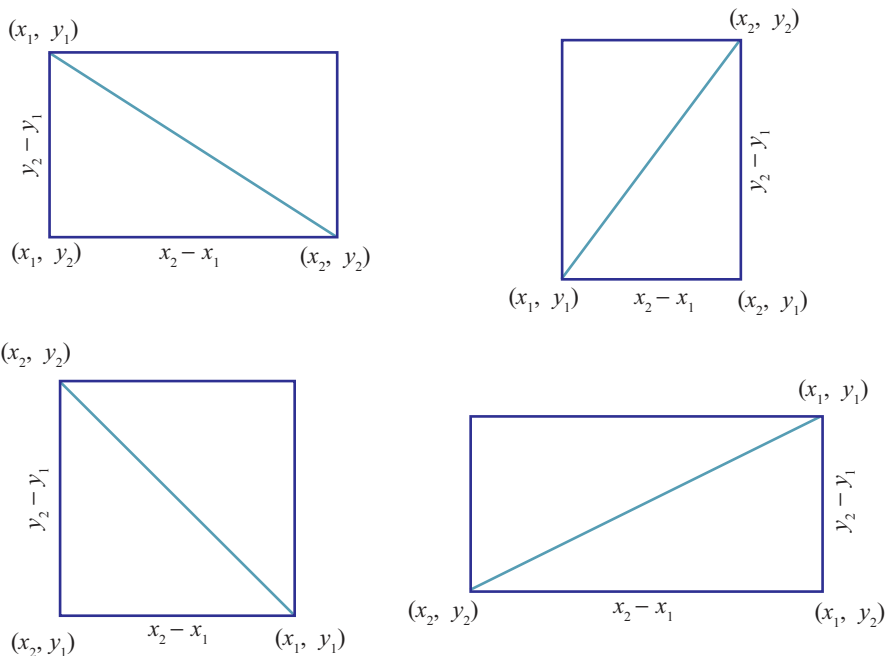


The distance we want is

$$\sqrt{4^2 + 3^2} = 5$$

Like this, we can find the distance between any two points with the x coordinates and y coordinates different. (If any coordinates are the same there is no such rectangle either).

In general, we can take two such points as (x_1, y_1) and (x_2, y_2) . We can draw a rectangle with these as opposite vertices and sides parallel to the axes. We then see that the other two vertices are (x_1, y_2) and (x_2, y_1) .



And find the lengths of the sides of the rectangle as $|x_1 - x_2|, |y_1 - y_2|$. So the distance between the first two points is

$$\sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}$$

We have seen in Class 9 that the square of a number and the square of its absolute value are equal.

So this distance is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

In this, if we take $y_1 = y_2$, then we get

$$\sqrt{(x_1 - x_2)^2} = |x_1 - x_2|$$

and if we take $x_1 = x_2$, then we get

$$\sqrt{(y_1 - y_2)^2} = |y_1 - y_2|$$

Thus, we can write the distance in this form, even if any coordinates are equal

The distance between two points with coordinates (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

For example, the distance between points with coordinates $(4, -2)$ and $(-3, -1)$ is

$$\sqrt{(4 - (-3))^2 + (-2 - (-1))^2} = \sqrt{7^2 + 1^2} = \sqrt{50} = 5\sqrt{2}$$

What about the distance between the point with coordinates $(-2, -1)$ and the origin?

$$\sqrt{(-2 - 0)^2 + (-1 - 0)^2} = \sqrt{5}$$

In general

The distance between the point with coordinates (x, y) and the origin is

$$\sqrt{x^2 + y^2}$$

Now look at the problem:

Are the points with coordinates $(-1, 2)$, $(3, 5)$, $(9, -3)$ on the same line?

If three points are on the same line, then among the distances between various pairs of them, the largest must be the sum of the other two.

Taking the three points as A, B, C we get

$$AB = \sqrt{(-1 - 3)^2 + (2 - 5)^2} = \sqrt{16 + 9} = 5$$

$$BC = \sqrt{(3 - 9)^2 + ((5 - (-3))^2} = \sqrt{36 + 64} = 10$$

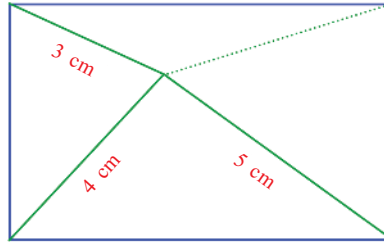
$$AC = \sqrt{(-1 - 9)^2 + ((2 - (-3))^2} = \sqrt{100 + 25} = 5\sqrt{5}$$

The largest among these is AC (How do we get this?) Now sum of the lengths AB and BC is 15, which is not the length AC . So, A, B and C are not on the same line.

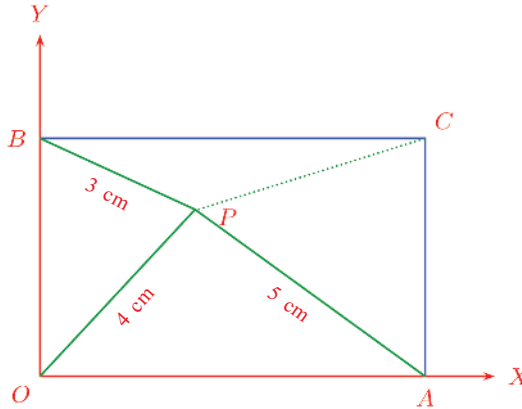
Let's look at another problem:

The distances from a point inside a rectangle to three consecutive vertices are 3 centimetres, 4 centimetres and 5 centimetres. What is the distance to the fourth vertex?

Let's draw a picture.



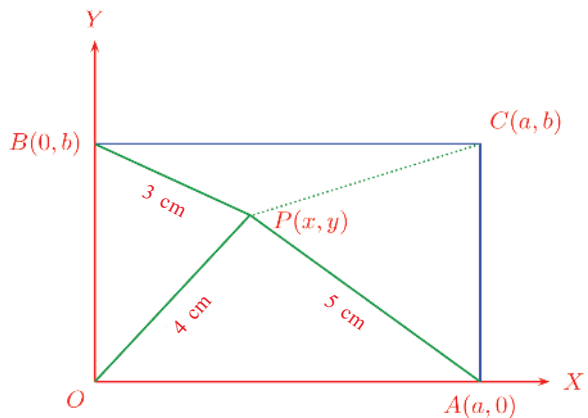
Let's take the bottom left vertex of the rectangle as the origin and the two sides through it as the axes.



Since the vertex A is on the x axis, its y coordinate is 0. Taking its x coordinate as a , we can write its coordinates as $(a, 0)$.

Similarly, taking the y coordinate of B as b its coordinates become $(0, b)$. So the coordinates of C are (a, b) .

Let's take the coordinates of P as (x, y) .





Now let's write the squares of the known distances in terms of the coordinates:

$$x^2 + (y - b)^2 = 9$$

$$x^2 + y^2 = 16$$

$$(x - a)^2 + y^2 = 25$$

What we want is PC . Its square is

$$(x - a)^2 + (y - b)^2$$

Can we calculate this from the three equations above?

From the first and second of these, we get

$$(x^2 + (y - b)^2) - (x^2 + y^2) = 9 - 16$$

That is,

$$(y - b)^2 - y^2 = -7$$

Using this and the third equation above, we get

$$(x - a)^2 + y^2 + ((y - b)^2 - y^2) = 25 - 7$$

That is

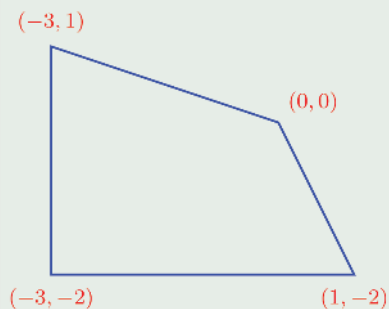
$$(x - a)^2 + (y - b)^2 = 18$$

So the length of PC is $\sqrt{18} = 3\sqrt{2}$ centimetres

?

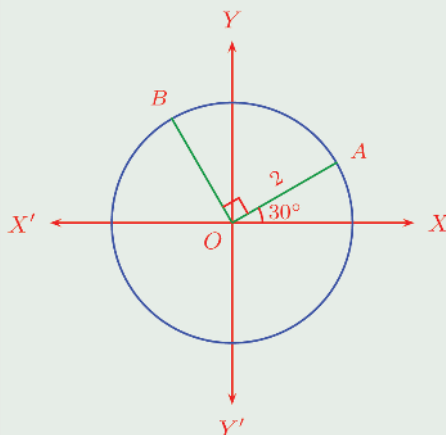


(1) Calculate the length of the sides and diagonals of the quadrilateral below:



(2) Prove that by joining the points $(2, 1)$, $(3, 4)$, $(-3, 6)$ we get a right triangle.

- (3) A circle of radius 10 is drawn with the origin as centre.
- Check whether each of the points with coordinates $(6, 9)$, $(5, 9)$, $(6, 8)$ is inside, outside or on the circle.
 - Write the coordinates of 8 points on this circle.
- (4) Find the coordinates of the points where a circle of radius $\sqrt{2}$, centred on the point with coordinates $(1, 1)$ cut the axes.
- (5) The coordinates of the vertices of a triangle are $(1, 2)$, $(2, 3)$, $(3, 1)$. Find the coordinates of the centre of its circumcircle and the circumradius.
- (6) The centre of the circle below is the origin and A, B are points on it.



Calculate the length of the chord AB .

Looking back



Learning outcomes	On my own	With teacher's help	Must improve
<ul style="list-style-type: none"> • Explaining the method of denoting the position of points using pairs of numbers. • Drawing pictures using number pairs to denote points. • Finding the coordinates of various points of a figure by drawing axes of coordinates. • Explaining the method to create geometric patterns in a computer, using GeoGebra. • Finding the method to compute the distance between two points, in terms of coordinates. • Using the coordinates of vertices of geometric figures to determine various lengths. 			



Notes

A large rectangular area with a red border, containing 15 horizontal red lines for writing notes.



$$\frac{1 + \sqrt{5}}{2}$$



Notes

Lined area for writing notes.

$$\sqrt{2}$$
$$\sqrt{3}$$
$$\sqrt{5}$$
$$\frac{1}{\sqrt{2}}$$

$$\frac{1}{7}$$
$$\frac{1}{3}$$
$$\frac{1}{10}$$

$$x^2 - a^2$$

$$(0, 1)$$

9
8
7
6
5
4
3
2
1
0



$$a + b$$

$$\frac{1 + \sqrt{5}}{2}$$



Notes

A large rectangular area with horizontal red lines, intended for writing notes.

- 9
- 8
- 7
- 6
- 5
- 4
- 3
- 2
- 1
- 0

Mathematical symbols and formulas along the left margin: $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\frac{1}{\sqrt{2}}$, $\frac{1}{7}$, $\frac{3}{1}$, $\frac{1}{10}$, $x^2 - a^2$, and $(0, 1)$.



$an + b$



Notes

Lined writing area for notes.

