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25 Complementary Angles

25.1 INTRODUCTION

Till now, we have studied trigonometic ratios of standard angles such as : 0°, 30°, 45°, 60° and 90°. Evaluation of trigonometric expressions, involing the trigonometric ratios of standard angles is also done.

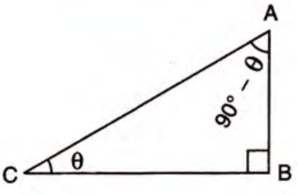
In this chapter, we shall be discussing and using the concept of trigonometric ratios of complementary angles and with their applications.

25.2 CONCEPT OF TRIGONOMETIC RATIOS OF COMPLEMENTARY ANGLES

1. Two acute angles are said to be complementary, if their sum is 90°.
   (i) 30° and 60° are complementary as 30° + 60° = 90°.
   (ii) 70° and 20° are complementary as 70° + 20° = 90°.
   (iii) $x°$ and $(90 - x)°$ are complementary as $x° + (90 - x)° = 90°$.
2. (i) For angle 48°, its complement = 90° – 48° = 42°.
   (ii) For angle θ°, its complement = (90 – θ)°.

25.3 COMPLEMENTARY ANGLES FOR SINE (sin) AND COSINE (cos)

Consider a triangle ABC, in which ∠B = 90°. If ∠ACB = θ; then by the knowledge of geometry, we know ∠CAB = 90° – θ.



$$\sin\theta = \sin\angle ACB = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{AB}{AC} \qquad \text{.... I}$$

$$\cos(90° - \theta) = \cos\angle CAB = \frac{\text{base}}{\text{hypotenuse}} = \frac{AB}{AC} \qquad \text{.... II}$$

From I and II, we get : $\sin\theta = \cos(90° - \theta)$ *i.e.* $\cos(90° - \theta) = \sin\theta$.

Now, $$\cos\theta = \cos\angle ACB = \frac{\text{base}}{\text{hypotenuse}} = \frac{BC}{AC} \qquad \text{.... III}$$

$$\sin(90° - \theta) = \sin\angle CAB = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{BC}{AC} \qquad \text{.... IV}$$

From III and IV, we get : $\sin(90° - \theta) = \cos\theta$

Thus,
1. $\cos(90° - \theta) = \sin\theta$
2. $\sin(90° - \theta) = \cos\theta$

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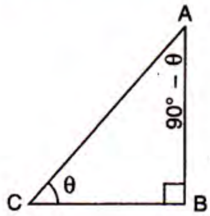
25.4 COMPLIMENTARY ANGLES FOR TANGENT (tan) AND COTANGENT (cot)

Consider a triangle ABC, in which $\angle B = 90°$. If $\angle ACB = \theta$; then $\angle CAB = 90° - \theta$.



$$\tan\theta = \tan\angle ACB = \frac{\text{perpendicular}}{\text{base}} = \frac{AB}{BC}$$

$$\cot(90° - \theta) = \cot\angle CAB = \frac{\text{base}}{\text{perpendicular}} = \frac{AB}{BC}$$

$\therefore$ $\tan\theta = \cot(90° - \theta)$ *i.e.* **$\cot(90° - \theta) = \tan\theta$**

Also, $\cot\theta = \cot\angle ACB = \dfrac{BC}{AB}$ $\left[\cot = \dfrac{\text{base}}{\text{perpendicular}}\right]$

and, $\tan(90° - \theta) = \tan\angle CAB = \dfrac{BC}{AB}$ $\left[\tan = \dfrac{\text{perpendicular}}{\text{base}}\right]$

$\therefore$ $\cot\theta = \tan(90° - \theta)$ *i.e.* **$\tan(90° - \theta) = \cot\theta$**

Thus : **1.** **$\cot(90° - \theta) = \tan\theta$**

$\Rightarrow$ (i) $\cot(90° - 20°) = \tan 20°$

(ii) $\cot(90° - 57°) = \tan 57°$

2. **$\tan(90° - \theta) = \cot\theta$**

$\Rightarrow$ (i) $\tan(90° - 70°) = \cot 70°$

(ii) $\tan(90° - 25°) = \cot 25°$

25.5 COMPLIMENTARY ANGLES FOR SECANT (sec) AND COSECANT (cosec)

Using the given figure, we find :

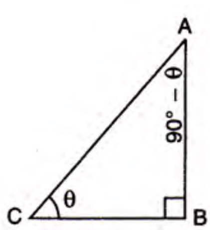
$$\sec\theta = \sec\angle ACB = \frac{\text{hypotenuse}}{\text{base}} = \frac{AC}{BC}$$

and, $\text{cosec}(90° - \theta) = \text{cosec}\angle CAB = \dfrac{AC}{BC}$ $\left[\text{cosec} = \dfrac{\text{hypotenuse}}{\text{perpendicular}}\right]$

$\therefore$ $\sec\theta = \text{cosec}(90° - \theta)$

i.e. **$\text{cosec}(90° - \theta) = \sec\theta$**

Also, $\text{cosec}\,\theta = \text{cosec}\angle ACB = \dfrac{AC}{AB}$



and, $\sec(90° - \theta) = \sec\angle CAB = \dfrac{AC}{AB}$

$\therefore$ $\text{cosec}\,\theta = \sec(90° - \theta)$

i.e. **$\sec(90° - \theta) = \text{cosec}\,\theta$**

Thus : **1.** **$\text{cosec}(90° - \theta) = \sec\theta$**

$\Rightarrow$ (i) $\text{cosec}(90° - 40°) = \sec 40°$

(ii) $\text{cosec}(90° - 67°) = \sec 67°$

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2. **sec (90° – θ) = cosec θ**

⇒ (i) sec (90° – 35°) = cosec 35°

(ii) sec (90° – 82°) = cosec 82°

1. sin (90° – θ) = cos θ and cos (90° – θ) = sin θ
2. tan (90° – θ) = cot θ and cot (90° – θ) = tan θ
3. cosec (90° – θ) = sec θ and sec (90° – θ) = cosec θ

1 Evaluate :

$$\left(\frac{\cos 47^\circ}{\sin 43^\circ}\right)^2 + \left(\frac{\sin 72^\circ}{\cos 18^\circ}\right)^2 - 2\cos^2 45^\circ$$

Solution :

$\because$ $\cos 47^\circ = \cos(90^\circ - 43^\circ) = \sin 43^\circ$

and, $\sin 72^\circ = \sin(90^\circ - 18^\circ) = \cos 18^\circ$

$$\therefore \left(\frac{\cos 47^\circ}{\sin 43^\circ}\right)^2 + \left(\frac{\sin 72^\circ}{\cos 18^\circ}\right)^2 - 2\cos^2 45$$

$$= \left(\frac{\sin 43^\circ}{\sin 43^\circ}\right)^2 + \left(\frac{\cos 18^\circ}{\cos 18^\circ}\right)^2 - 2\times\left(\frac{1}{\sqrt{2}}\right)^2$$

$$= (1)^2 + (1)^2 - 2 \times \frac{1}{2} = 2 - 1 = 1$$ **Ans.**

If A and B are complimentary angles *i.e.* if A + B = 90

(i) sin A = cos B and cos A = sin B

(ii) tan A = cot B and cot A = tan B

(iii) sec A = cosec B and cosec A = sec B

2 Evaluate : (i) cosec 82° – sec 8° (ii) sec 70° sin 20° + cos 20° cosec 70°

Solution :

(i) cosec 82° – sec 8° = cosec (90° – 8°) – sec 8°

= sec 8° – sec 8° = **0** **Ans.**

(ii) sec 70° sin 20° + cos 20° cosec 70°

= sec (90° – 20°) sin 20° + cos 20° cosec (90° – 20°)

= cosec 20° sin 20° + cos 20° sec 20°

$$= \frac{1}{\sin 20^\circ} \times \sin 20^\circ + \cos 20^\circ \times \frac{1}{\cos 20^\circ}$$

= 1 + 1 = **2** **Ans.**

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3 Prove that : (i) $\cos 55^\circ \sin 35^\circ + \sin 55^\circ \cos 35^\circ = 1$

(ii) $\dfrac{\tan 72^\circ}{\cot 18^\circ} - \dfrac{\cot 72^\circ}{\tan 18^\circ} = 0$

(iii) $\sec 70^\circ \sin 20^\circ + \text{cosec } 70^\circ \cos 20^\circ = 2.$

Solution :

(i) Since, $\cos 55^\circ = \cos(90^\circ - 35^\circ) = \sin 35^\circ$

and, $\sin 55^\circ = \sin(90^\circ - 35^\circ) = \cos 35^\circ$

$\therefore$ $\cos 55^\circ \sin 35^\circ + \sin 55^\circ \cos 35^\circ$

$= \sin 35^\circ \sin 35^\circ + \cos 35^\circ \cos 35^\circ$

$= \sin^2 35^\circ + \cos^2 35^\circ = 1$ **Hence Proved.**

(ii) Since, $\tan 72^\circ = \tan(90^\circ - 18^\circ) = \cot 18^\circ$

and, $\cot 72^\circ = \cot(90^\circ - 18^\circ) = \tan 18^\circ$

$\therefore \dfrac{\tan 72^\circ}{\cot 18^\circ} - \dfrac{\cot 72^\circ}{\tan 18^\circ} = \dfrac{\cot 18^\circ}{\cot 18^\circ} - \dfrac{\tan 18^\circ}{\tan 18^\circ}$

$= 1 - 1 = 0$ **Hence Proved.**

(iii) Since, $\sec 70^\circ = \sec(90^\circ - 20^\circ) = \text{cosec } 20^\circ$

and, $\text{cosec } 70^\circ = \text{cosec}(90^\circ - 20^\circ) = \sec 20^\circ$

$\Rightarrow \sec 70^\circ \sin 20^\circ + \text{cosec } 70^\circ \cos 20^\circ$

$= \text{cosec } 20^\circ \sin 20^\circ + \sec 20^\circ \cos 20^\circ$

$= \dfrac{1}{\sin 20^\circ} \cdot \sin 20^\circ + \dfrac{1}{\cos 20^\circ} \cdot \cos 20^\circ = 1 + 1 = 2$

Hence Proved.

4 Evaluate : $\dfrac{2\tan 53^\circ}{\cot 37^\circ} - \dfrac{\cot 80^\circ}{\tan 10^\circ}$

Solution :

We know that $\tan(90^\circ - \theta) = \cot\theta$ and $\cot(90^\circ - \theta) = \tan\theta$

$\therefore \dfrac{2\tan 53^\circ}{\cot 37^\circ} - \dfrac{\cot 80^\circ}{\tan 10^\circ} = \dfrac{2\tan(90^\circ - 37^\circ)}{\cot 37^\circ} - \dfrac{\cot(90^\circ - 10^\circ)}{\tan 10^\circ}$

$= \dfrac{2\cot 37^\circ}{\cot 37^\circ} - \dfrac{\tan 10^\circ}{\tan 10^\circ}$

$= 2 - 1 = 1$ **Ans.**

Alternative method :

$\therefore \dfrac{2\tan 53^\circ}{\cot 37^\circ} - \dfrac{\cot 80^\circ}{\tan 10^\circ} = \dfrac{2\tan 53^\circ}{\cot(90^\circ - 53^\circ)} - \dfrac{\cot 80^\circ}{\tan(90^\circ - 80^\circ)}$

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$$= \frac{2\tan 53^\circ}{.\tan 53^\circ} - \frac{\cot 80^\circ}{\cot 80^\circ}$$

$= 2 - 1 = 1$ **Ans.**

5 Prove that : (i) $\sin(90^\circ - A)\cos(90^\circ - A) = \dfrac{\tan A}{1+\tan^2 A}$

(ii) $\dfrac{\cos(90^\circ - A).\cos A}{\cot A} - \sin^2 A = 0$

Solution :

(i) **L.H.S.** $= \cos A \sin A$

R.H.S. $= \dfrac{\tan A}{\sec^2 A}$ $\quad$ $1 + \tan^2 A = \sec^2 A$

$$= \frac{\frac{\sin A}{\cos A}}{\frac{1}{\cos^2 A}} = \frac{\sin A}{\cos A} \times \cos^2 A = \sin A \cos A$$

∴ **L.H.S. = R.H.S.** **Hence Proved.**

(ii) **L.H.S.** $= \dfrac{\sin A \cos A}{\frac{\cos A}{\sin A}} - \sin^2 A$

$= \sin^2 A - \sin^2 A = 0 =$ **R.H.S.** **Hence Proved.**

6 Given : $\cos 38^\circ \sec(90^\circ - 2A) = 1$; find the value of angle A.

Solution :

$\cos 38^\circ \sec(90^\circ - 2A) = 1 \Rightarrow \cos 38^\circ \text{cosec } 2A = 1$ $\quad [\because \sec(90^\circ - \theta) = \text{cosec } \theta]$

$\Rightarrow \cos 38^\circ \times \dfrac{1}{\sin 2A} = 1$ $\quad$ $\text{cosec } \theta = \dfrac{1}{\sin\theta}$

$\Rightarrow \sin 2A = \cos 38^\circ$

$= \cos(90^\circ - 52^\circ)$

$\Rightarrow \sin 2A = \sin 52^\circ$ $\quad$ $\cos(90^\circ - \theta) = \sin\theta$

∴ $2A = 52^\circ$ and $A = 26^\circ$ **Ans.**

7 For triangle ABC, prove that : $\sec\left(\dfrac{A+B}{2}\right) = \text{cosec } C.$

Solution :

In Δ ABC, $\quad A + B + C = 180^\circ$

$\Rightarrow \quad A + B = 180^\circ - C$

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$$\Rightarrow \qquad \frac{A+B}{2} = 90° - \frac{C}{2}$$

$$\therefore \qquad \sec\frac{A+B}{2} = \sec\left(90° - \frac{C}{2}\right)$$

$$= \operatorname{cosec}\frac{C}{2} \qquad [\sec(90° - \theta) \operatorname{cosec}\theta]$$

Hence Proved.

EXERCISE 25

1. Evaluate :

 (i) $\frac{\cos 22°}{\sin 68°}$ (ii) $\frac{\tan 47°}{\cot 43°}$

 (iii) $\frac{\sec 75°}{\operatorname{cosec} 15°}$ (iv) $\frac{\cos 55°}{\sin 35°} + \frac{\cot 35°}{\tan 55°}$

 (v) $\sin^2 40° - \cos^2 50°$

 (vi) $\sec^2 18° - \operatorname{cosec}^2 72°$

 (vii) $\sin 15° \cos 15° - \cos 75° \sin 75°$

 (viii) $\sin 42° \sin 48° - \cos 42° \cos 48°$

2. Evaluate :

 (i) $\sin(90° - A) \sin A - \cos(90° - A) \cos A$

 (ii) $\sin^2 35° - \cos^2 55°$

 (iii) $\frac{\cot 54°}{\tan 36°} + \frac{\tan 20°}{\cot 70°} - 2$

 (iv) $\frac{2 \tan 53°}{\cot 37°} - \frac{\cot 80°}{\tan 10°}$

 (v) $\cos^2 25° - \sin^2 65° - \tan^2 45°$

 (vi) $\left(\frac{\sin 77°}{\cos 13°}\right)^2 + \left(\frac{\cos 77°}{\sin 13°}\right)^2 - 2\cos^2 45°$

3. Show that :

 (i) $\tan 10° \tan 15° \tan 75° \tan 80° = 1$

 (ii) $\sin 42° \sec 48° + \cos 42° \operatorname{cosec} 48° = 2$

4. Express each of the following in terms of angles between 0° and 45° :

 (i) $\sin 59° + \tan 63°$

 (ii) $\operatorname{cosec} 68° + \cot 72°$

 (iii) $\cos 74° + \sec 67°$

5. For triangle ABC, show that :

 (i) $\sin\frac{A+B}{2} = \cos\frac{C}{2}$

 (ii) $\tan\frac{B+C}{2} = \cot\frac{A}{2}$

6. Evaluate :

 (i) $3\frac{\sin 72°}{\cos 18°} - \frac{\sec 32°}{\operatorname{cosec} 58°}$

 (ii) $3 \cos 80° \operatorname{cosec} 10° + 2 \sin 59° \sec 31°$.

 (iii) $\frac{\sin 80°}{\cos 10°} + \sin 59° \sec 31°$

 (iv) $\tan(55° - A) - \cot(35° + A)$

 (v) $\operatorname{cosec}(65° + A) - \sec(25° - A)$

 (vi) $2\frac{\tan 57°}{\cot 33°} - \frac{\cot 70°}{\tan 20°} - \sqrt{2}\cos 45°$

 (vii) $\frac{\cot^2 41°}{\tan^2 49°} - 2\frac{\sin^2 75°}{\cos^2 15°}$

 (viii) $\frac{\cos 70°}{\sin 20°} + \frac{\cos 59°}{\sin 31°} - 8\sin^2 30°$

 (ix) $14 \sin 30° + 6 \cos 60° - 5 \tan 45°$.

7. A triangle ABC is right angled at B; find the value of $\frac{\sec A \cdot \sin C - \tan A \cdot \tan C}{\sin B}$

8. In each case, given below, find the value of angle A, where $0° \le A \le 90°$.

 (i) $\sin(90° - 3A) \cdot \operatorname{cosec} 42° = 1$

 (ii) $\cos(90° - A) \cdot \sec 77° = 1$

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