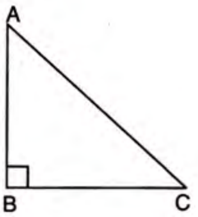
13 Pythagoras Theorem

[Proof and Simple Applications with Converse]

13.1 INTRODUCTION

Buddhayan, an Indian Mathematician (600 B.C.), developed a relationship between the squares described on the hypotenuse of a right-angled triangle and the sum of the squares described on the remaining two sides of the triangle. But the credit of the present form of this relationship goes to a Greek Mathematician **Pythagoras** and is named as *Pythagoras Theorem.*



13.2 PYTHAGORAS THEOREM

Theorem 9

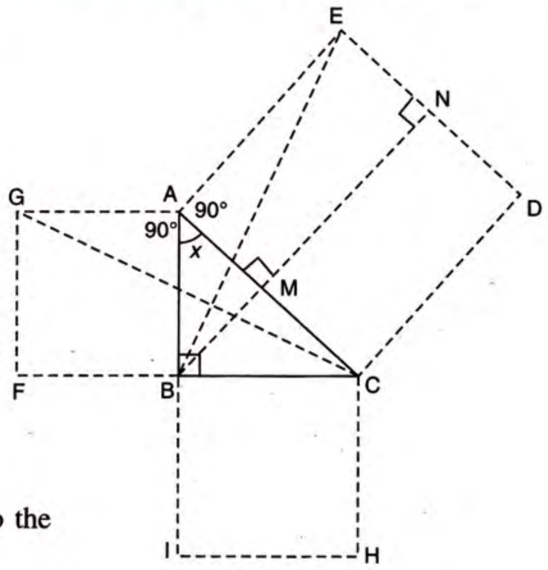
[Area Based Proof]

In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the remaining two sides.

Given : A triangle ABC in which $\angle ABC = 90°$.

To prove : $AC^2 = AB^2 + BC^2$

Construction : Draw squares ACDE, ABFG and BCHI on sides AC, AB and BC respectively. Draw BMN perpendicular to AC at point M and DE at point N. Join GC and BE.



Proof : $\angle FBC = \angle FBA + \angle ABC$

$= 90° + 90° = 180°$

$\Rightarrow$ FBC is a straight line.

Since, GA // FB [Opp. sides of the square]

$\Rightarrow$ GA is parallel to FC

Since, BMN and AE both are perpendicular to the same line AC

$\Rightarrow$ AE is parallel to BMN. As each angle of quadrilateral AMNE is 90°, AMNE is a rectangle

Now, let $\angle BAC = x$

$\therefore$	$\angle GAC = \angle BAE$	[Each $90° + x$]
	$AG = AB$	[Sides of the same square]
	$AC = AE$	[Sides of the same square]
$\therefore$	$\Delta GAC \cong \Delta BAE$	[By SAS]
	$\Rightarrow$ area of Δ GAC = area of Δ BAE .....(i)	[Congruent triangles are equal in area]

We know, the area of a triangle is half the area of a parallelogram (rectangle, square, etc.) if both are on the same base and between the same parallels.

Since, Δ GAC and square ABFG are on same the base (AG) and between the same parallels (AG // CF)

$\therefore$ area of Δ GAC = $\frac{1}{2}$ × area of square ABFG .....(ii)

Similarly, Δ BAE and rectangle AMNE are on same base (AE) and between the same parallels (AE//BN)

$\therefore$ area of Δ BAE = $\frac{1}{2}$ × area of rectangle AMNE .....(iii)

From equations (i), (ii) and (iii), we get :

area of square ABFG = area of rectangle AMNE ......(iv)

In the same way, it can be proved that :

area of square BCHI = area of rectangle CMND ......(v)

On adding equations (iv) and (v), we get :

area of square ABFG + area of square BCHI = area of rectangle AMNE + area of rectangle CMND

$\Rightarrow$ area of square ABFG + area of square BCHI = area of square ACDE

$\Rightarrow$ square on AB + square on BC = square on AC

$\Rightarrow$ $AB^2 + BC^2 = AC^2$ *i.e.*, $AC^2 = AB^2 + BC^2$ **Hence Proved.**

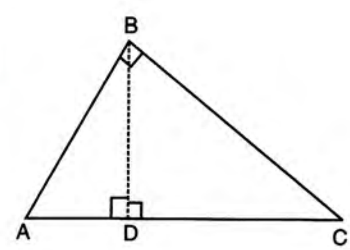
Conversely, if in any triangle, the square on the largest side of the triangle is equal to the sum of the squares on remaining two sides, then the triangle is a right-angled triangle and the angle opposite to the largest side is a right-angle.

i.e. if in triangle ABC; BC is the largest side and $BC^2 = AB^2 + AC^2$, then $\angle A = 90°$.

Theorem 10

[Alternative Proof for Theorem 9]

In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the remaining two sides.



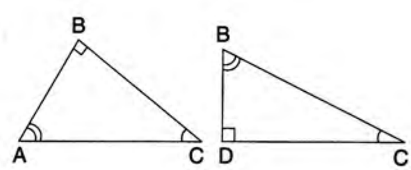
Given : A triangle ABC in which $\angle ABC = 90°$.

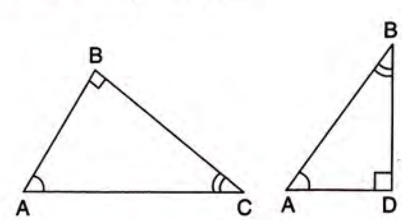
To Prove : $AC^2 = AB^2 + BC^2$

Construction : Draw BD ⊥ AC.

Proof :

Statement	Reason
In Δ ABC and Δ BDC,	
(i) $\angle ABC = \angle BDC$	Each is 90°
(ii) $\angle BCA = \angle BCD$	Common
$\therefore$ $\Delta ABC \sim \Delta BDC$	A.A. postulate.
$\Rightarrow \frac{BC}{DC} = \frac{AC}{BC}$	Corresponding sides of similar Δs are proportional
$\Rightarrow BC^2 = AC \times DC$ ...I	
Now, in Δ ABC and Δ ADB,	
(i) $\angle ABC = \angle ADB$	Each is 90°
(ii) $\angle BAC = \angle BAD$	Common
$\therefore$ $\Delta ABC \sim \Delta ADB$	A.A. postulate





$\Rightarrow \dfrac{AB}{AD} = \dfrac{AC}{AB}$	Corresponding sides of similar Δs are proportional
$\Rightarrow AB^2 = AC \times AD$ ...II	
$\therefore AB^2 + BC^2 = AC \times AD + AC \times DC$	Adding I and II
$\Rightarrow$ **$AB^2 + BC^2$** $= AC\ (AD + DC)$	
$= AC \times AC$	$AD + DC = AC$
$=$ **AC^2**	

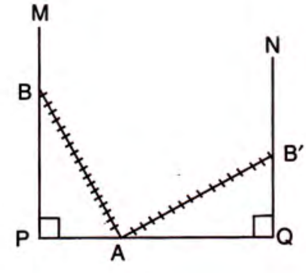
Hence Proved.

1. In a right-angled triangle, the hypotenuse is the largest side.
2. If AB is the largest side of a triangle ABC, then :
   (i) $AB^2 = AC^2 + BC^2$
   $\Rightarrow$ ABC is a right-angled triangle with AB as hypotenuse and $\angle ACB = 90°$.
   (ii) $AB^2 > AC^2 + BC^2$
   $\Rightarrow$ ABC is an obtuse-angled triangle with angle ACB greater than 90°.
   (iii) $AB^2 < AC^2 + BC^2$
   $\Rightarrow$ ABC is an acute-angled triangle.

1 A ladder reaches a window which is 15 metres above the ground on one side of the street. Keeping its foot at the same point, the ladder is turned to the other side of the street to reach a window 8 metre high. Find the width of the street, if the length of the ladder is 17 metres.

Solution :

According to the given statement, the figure will be as shown alongside. In the figure, PQ is the width of the street, PM and QN are two buildings, A is the foot of the ladder, AB is the first position of the ladder and AB′ is its second position.



Clearly, $AB = AB' = 17$ m, $PB = 15$ m

and, $QB' = 8$ m

In Δ PAB, $PA^2 + PB^2 = AB^2$

$\Rightarrow \quad PA^2 + 15^2 = 17^2$

i.e. $\quad PA^2 = 17^2 - 15^2 = 289 - 225 = 64 \qquad \Rightarrow PA = \sqrt{64}$ m $= 8$ m

In Δ QAB′, $QA^2 + (QB')^2 = (AB')^2$

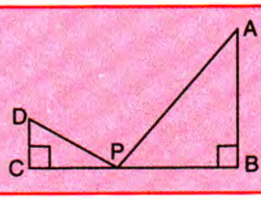
$\Rightarrow \quad QA^2 + 8^2 = 17^2$

i.e. $\quad QA^2 = 17^2 - 8^2 = 289 - 64 = 225 \qquad \Rightarrow QA = \sqrt{225}$ m $= 15$ m

$\therefore$ **The width of the street** = PQ

= PA + QA = 8 m + 15 m = **23 m** **Ans.**

2 In the given diagram, AB = 3CD = 18 cm and 3BP = 4CP = 36 cm. Show that the measure of angle APD is 90°.

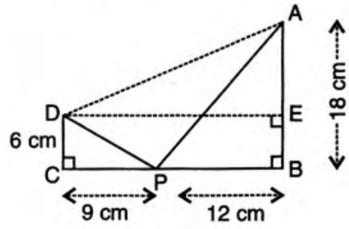


Solution :

> The measure of angle APD will be 90°,
> if, $AP^2 + DP^2 = AD^2$

$AB = 3CD = 18$ cm $\Rightarrow AB = 18$ cm and $CD = \frac{18}{3}$ cm $= 6$ cm

$3BP = 4CP = 36$ cm $\Rightarrow BP = \frac{36}{3}$ cm $= 12$ cm and $CP = \frac{36}{4}$ cm $= 9$ cm



In Δ DPC,

$DP^2 = DC^2 + CP^2 = 6^2 + 9^2 = 36 + 81 = 117$

In Δ APB,

$AP^2 = AB^2 + BP^2$

$= 18^2 + 12^2 = 324 + 144 = 468$

$\therefore DP^2 + AP^2 = 117 + 468 = 585$ I

Draw DE ⊥ to AB and join DA

DE = CB = 9 cm + 12 cm = 21 cm

and AE = AB – BE = AB – CD = 18 cm – 6 cm = 12 cm

$\therefore$ In Δ AED, $AD^2 = AE^2 + DE^2$

$= 12^2 + 21^2 = 144 + 441 = 585$ II

From I and II, we get : $DP^2 + AP^2 = AD^2$ $\therefore$ **Angle APD = 90°.** **Ans.**

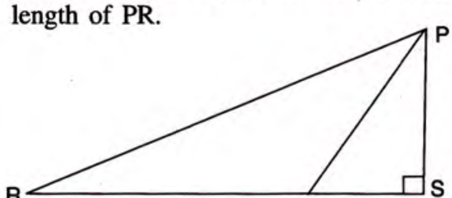
> **Pythagorean triplets**
>
> Consider three positive numbers *a*, *b* and *c* with *c* as the largest of *a*, *b* and *c*.
>
> If $a^2 + b^2 = c^2$; *a*, *b* and *c* are called *Pythagorean triplets.*
>
> *For example :*
>
> 3, 4 and 5 are Pythagorean triplets as $3^2 + 4^2 = 5^2$.

EXERCISE 13(A)

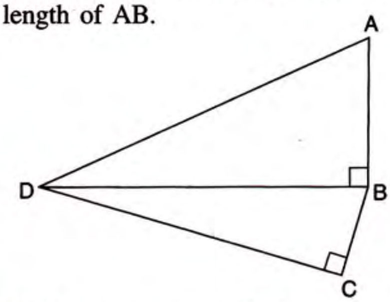
1. A ladder 13 m long rests against a vertical wall. If the foot of the ladder is 5 m from the foot of the wall, find the distance of the other end of the ladder from the ground.

2. A man goes 40 m due north and then 50 m due west. Find his distance from the starting point.

3. In the figure : ∠PSQ = 90°, PQ = 10 cm, QS = 6 cm and RQ = 9 cm. Calculate the length of PR.

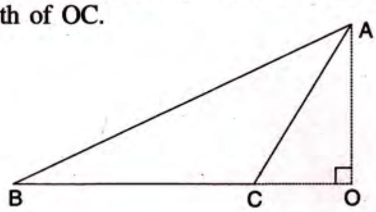


4. The given figure shows a quadrilateral ABCD in which AD = 13 cm, DC = 12 cm, BC = 3 cm and ∠ABD = ∠BCD = 90°. Calculate the length of AB.



5. AD is drawn perpendicular to base BC of an equilateral triangle ABC. Given BC = 10 cm, find the length of AD, correct to 1 place of decimal.

6. In triangle ABC, given below, AB = 8 cm, BC = 6 cm and AC = 3 cm. Calculate the length of OC.

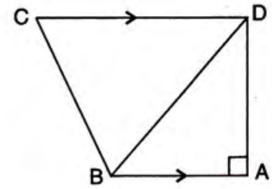


7. In triangle ABC,
AB = AC = x; BC = 10 cm and the area of the triangle is 60 cm^2. Find x.

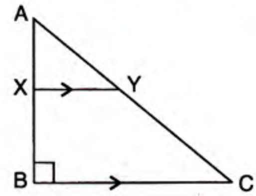
8. If the sides of a triangle are in the ratio $1 : \sqrt{2} : 1$, show that it is a right-angled triangle.

9. Two poles of heights 6 m and 11 m stand vertically on a plane ground. If the distance between their feet is 12 m; find the distance between their tips.

10. In the given figure, AB//CD, AB = 7 cm, BD = 25 cm and CD = 17 cm; find the length of side BC.

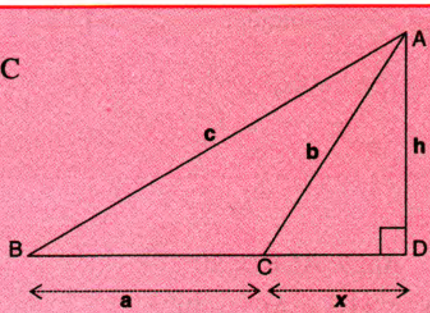


11. In the given figure, $\angle B = 90°$, XY//BC, AB = 12 cm, AY = 8 cm and AX : XB = 1 : 2. Find the lengths of AC and BC.



12. In ΔABC, $\angle B = 90°$. Find the sides of the triangle, if :
   (i) AB = $(x - 3)$ cm, BC = $(x + 4)$ cm and AC = $(x + 6)$ cm
   (ii) AB = x cm, BC = $(4x + 4)$ cm and AC = $(4x + 5)$ cm

3 In the given figure; AD is perpendicular to BC produced. Prove that :
$c^2 = a^2 + b^2 + 2ax$.



Solution :

Applying Pythagoras Theorem in right-angled Δ ABD;

We get : $AB^2 = BD^2 + AD^2$

$\Rightarrow \quad c^2 = (a + x)^2 + h^2$

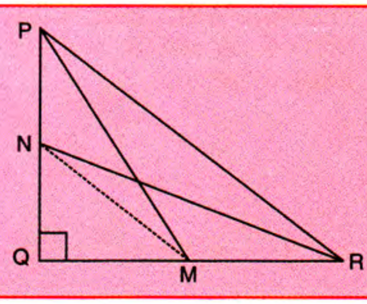
$= a^2 + 2ax + x^2 + h^2$

$= a^2 + (x^2 + h^2) + 2ax$

$= a^2 + b^2 + 2ax$ [In right Δ ADC, $x^2 + h^2 = b^2$]

Hence Proved.

4 M and N are points on sides QR and PQ respectively of a Δ PQR, right-angled at Q. Prove that : $PM^2 + RN^2 = PR^2 + MN^2$



Solution :

1. In right-angled Δ PQM;
   $PM^2 = PQ^2 + QM^2$ [Pythagoras Theorem]
2. In right-angled Δ NQR;
   $RN^2 = NQ^2 + QR^2$ [Pythagoras Theorem]

$\therefore PM^2 + RN^2 = PQ^2 + QM^2 + NQ^2 + QR^2$ [Adding 1 and 2]

$= (PQ^2 + QR^2) + (QM^2 + NQ^2)$

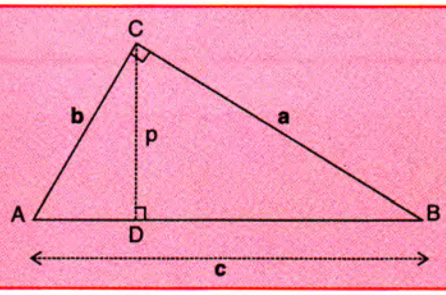
Since, in Δ PQR, $PR^2 = PQ^2 + QR^2$ and in Δ MQN, $MN^2 = QM^2 + NQ^2$

$\therefore$ **$PM^2 + RN^2 = PR^2 + MN^2$**

Hence Proved.

5 In triangle ABC, ∠ACB = 90°, AB = c unit, BC = a unit, AC = b unit, CD is perpendicular to AB and CD = p unit.

Prove that : $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$



Solution :

Angle ACB = 90°

$\Rightarrow AB^2 = AC^2 + BC^2$ [Pythagoras Theorem]

$\Rightarrow c^2 = b^2 + a^2$ ...I

Area of Δ ABC = $\frac{1}{2}$AB × CD = $\frac{1}{2}$BC × AC

$\Rightarrow \frac{1}{2}c \times p = \frac{1}{2}a \times b \Rightarrow c = \frac{ab}{p}$

Substituting this value of c in equation I, we get :

$$\frac{a^2b^2}{p^2} = b^2 + a^2$$

$$\Rightarrow \frac{1}{p^2} = \frac{b^2 + a^2}{a^2b^2}$$ [Dividing each term by a^2b^2]

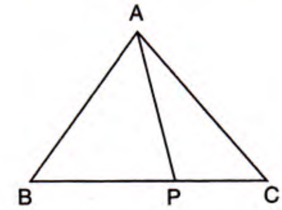
$$\Rightarrow \frac{1}{p^2} = \frac{b^2}{a^2b^2} + \frac{a^2}{a^2b^2} \Rightarrow \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

Hence Proved.

6 ABC is an equilateral triangle, P is a point in BC such that BP : PC = 2 : 1. Prove that : $9\ AP^2 = 7\ AB^2$

Solution :

According to the given statement, the figure will be as shown alongside.

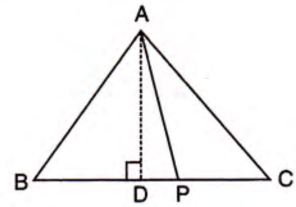


Draw AD perpendicular to BC. In order to find AP, which is a side of right ΔADP; find DP and AD.

To find DP :

Given : BP : PC = 2 : 1 $\Rightarrow BP = \frac{2}{3}BC = \frac{2}{3}AB$ $[\because AB = BC = AC]$

$AD \perp BC \Rightarrow BD = \frac{1}{2}BC = \frac{1}{2}AB$



$\therefore DP = BP - BD$

$$= \frac{2}{3}AB - \frac{1}{2}AB = \frac{4AB - 3AB}{6} = \frac{AB}{6}$$

To find AD :

In right Δ ABD,

$AD^2 + BD^2 = AB^2 \Rightarrow AD^2 + \left(\frac{AB}{2}\right)^2 = AB^2$ $[\because BD = \frac{1}{2}BC = \frac{1}{2}AB]$

$$\Rightarrow AD^2 = AB^2 - \frac{AB^2}{4} = \frac{3AB^2}{4}$$

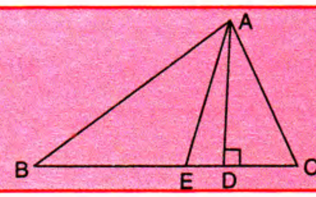
Now, in right Δ ADP,

$$AP^2 = AD^2 + DP^2 \Rightarrow AP^2 = \frac{3AB^2}{4} + \left(\frac{AB}{6}\right)^2$$

i.e. $AP^2 = \frac{3AB^2}{4} + \frac{AB^2}{36} = \frac{27AB^2 + AB^2}{36}$

i.e. $AP^2 = \frac{28AB^2}{36} = \frac{7AB^2}{9} \Rightarrow$ **$9\ AP^2 = 7\ AB^2$** **Hence Proved.**

7 The given figure shows a triangle ABC, in which AB > AC. E is the mid-point of BC and AD is perpendicular to BC.
Prove that : $AB^2 - AC^2 = 2BC \times ED$



Solution :

In rt. triangle ABD, $AB^2 = AD^2 + BD^2$ and in rt. triangle ACD, $AC^2 = AD^2 + CD^2$

$\therefore$ **$AB^2 - AC^2$** $= AD^2 + BD^2 - AD^2 - CD^2$

$= BD^2 - CD^2$

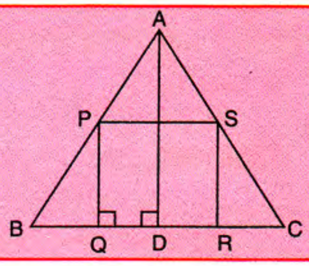
$= (BD + CD)(BD - CD)$
$= BC(BE + ED - CE + ED)$
$= BC(BE + ED - BE + ED)$
$= BC \times 2ED$
$= 2BC \times ED$

| $BD + CD = BC$,
| $BD - CD$
| $= (BE + ED) - (CE - ED)$
| $= BE + ED - CE + ED$
| $= 2ED$, as $BE = CE$

Hence proved.

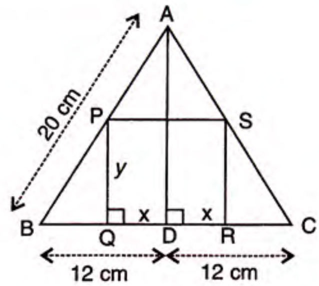
8 ABC is an isosceles triangle in which AB = AC = 20 cm and BC = 24 cm. PQRS is a rectangle drawn inside the isosceles triangle. Given PQ = SR = y cm and PS = QR = $2x$ cm.

Prove that : $y = 16 - \frac{4x}{3}$



Solution :

∵ In an isosceles triangle, the perpendicular from vertex bisects the base.



$\therefore BD = CD = \frac{24}{2}$ cm = 12 cm

By ASA, $\Delta PBQ \equiv \Delta SCR$

$\Rightarrow BQ = CR$

$\therefore BD - BQ = CD - CR \Rightarrow DQ = DR = \frac{2x}{2}$ cm = x cm

and, $BQ = CR = (12 - x)$ cm

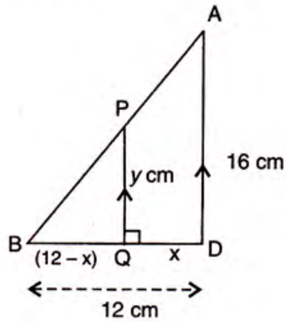
In right Δ ABD, $AD^2 = AB^2 - BD^2$ [Using Pythagoras Theorem]

$= 20^2 - 12^2 = 256$

$\Rightarrow$ AD = 16 cm

Since, PQ and AD both are perpendicular to the same line BD, therefore PQ is parallel to AD.

In Δ ABD, PQ // AD



$\Rightarrow \frac{PQ}{AD} = \frac{BQ}{BD} \Rightarrow \frac{y}{16} = \frac{12 - x}{12}$

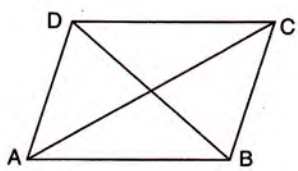
$\Rightarrow y = 16 - \frac{4x}{3}$

Hence Proved.

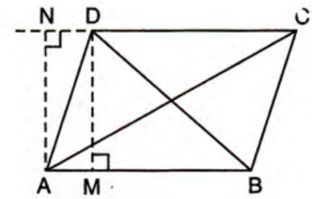
9 Prove that the sum of the squares on the diagonals of a parallelogram is equal to the sum of the squares on its sides.

Solution :

According to the given statement, the figure will be as shown alongside.



Required to prove :



$AC^2 + BD^2 = AB^2 + BC^2 + CD^2 + DA^2$

Draw AN ⊥ CD (produced) and DM ⊥ AB.

In right triangle ANC,

$AC^2 = AN^2 + NC^2$

$= AN^2 + (CD + DN)^2$

$= AN^2 + CD^2 + DN^2 + 2CD \times DN$

$= (AN^2 + DN^2) + CD^2 + 2CD \times DN$

$= AD^2 + CD^2 + 2CD \times DN$ I

In right Δ DMB,

$BD^2 = DM^2 + MB^2$

$= DM^2 + (AB - AM)^2$

$= DM^2 + AB^2 + AM^2 - 2AB \times AM$

$= (DM^2 + AM^2) + AB^2 - 2CD \times DN$ [In // gm ABCD, AB = CD and in rectangle AMDN, AM = DN]

$= AD^2 + AB^2 - 2CD \times DN$ ...II

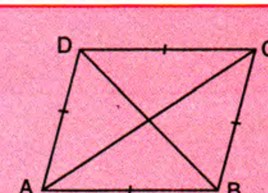
On adding equations I and II, we get :

$\mathbf{AC^2 + BD^2} = AD^2 + CD^2 + AD^2 + AB^2$

$= \mathbf{AB^2 + BC^2 + CD^2 + DA^2}$ [∵ AD = BC]

Hence Proved.

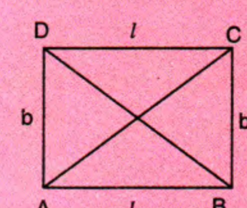
1. Since, rhombus is also a parallelogram.



$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$

$\Rightarrow$ $\mathbf{4 \times (side)^2 = AC^2 + BD^2}$

2. Since, rectangle is also a parallelogram.



$\therefore$ $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$

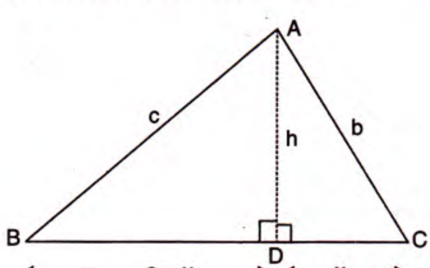
$\Rightarrow$ $l^2 + b^2 + l^2 + b^2 = d^2 + d^2$

[∵ AC = BD = d = diagonal]

$\Rightarrow$ $2l^2 + 2b^2 = 2d^2$ and $\mathbf{l^2 + b^2 = d^2}$

EXERCISE 13(B)

1. In the figure, given below, AD ⊥ BC.
   Prove that : $c^2 = a^2 + b^2 - 2ax$.



2. In equilateral Δ ABC, AD ⊥ BC and BC = x cm. Find, in terms of x, the length of AD.
3. ABC is a triangle, right-angled at B. M is a point on BC. Prove that :
   $AM^2 + BC^2 = AC^2 + BM^2$.
4. M and N are the mid-points of the sides QR and PQ respectively of a Δ PQR, right-angled at Q. Prove that :
   (i) $PM^2 + RN^2 = 5\ MN^2$

(ii) $4\ PM^2 = 4\ PQ^2 + QR^2$

(iii) $4\ RN^2 = PQ^2 + 4\ QR^2$

(iv) $4\ (PM^2 + RN^2) = 5\ PR^2$

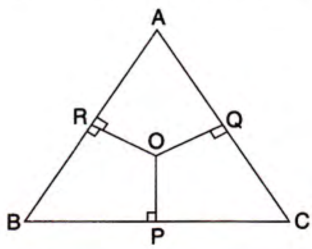
5. In triangle ABC, $\angle B = 90°$ and D is the mid-point of BC. Prove that : $AC^2 = AD^2 + 3CD^2$.

6. In a rectangle ABCD, prove that : $AC^2 + BD^2 = AB^2 + BC^2 + CD^2 + DA^2$.

7. In a quadrilateral ABCD, $\angle B = 90°$ and $\angle D = 90°$. Prove that : $2AC^2 - AB^2 = BC^2 + CD^2 + DA^2$.

8. O is any point inside a rectangle ABCD. Prove that : $OB^2 + OD^2 = OC^2 + OA^2$.

9. In the following figure, OP, OQ and OR are drawn perpendiculars to the sides BC, CA and AB respectively of triangle ABC. Prove that : $AR^2 + BP^2 + CQ^2 = AQ^2 + CP^2 + BR^2$

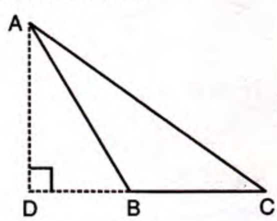


Join OA, OB and OC. Now find the values of $AR^2 + BP^2 + CQ^2$ and $AQ^2 + CP^2 + BR^2$

10. Diagonals of rhombus ABCD intersect each other at point O. Prove that :

$$OA^2 + OC^2 = 2AD^2 - \frac{BD^2}{2}$$

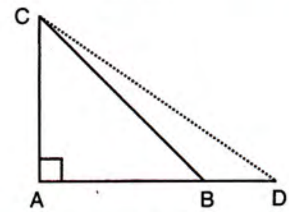
11. In the figure AB = BC and AD is perpendicular to CD. Prove that : $AC^2 = 2.BC.DC$.



$AC^2 = AD^2 + DC^2$
$= (AB^2 - DB^2) + (DB + BC)^2$
$= BC^2 - DB^2 + DB^2 + BC^2 + 2DB.BC$ (As, AB = BC)
$= 2BC^2 + 2DB.BC$
$= 2BC\ (BC + DB) = 2BC.DC$

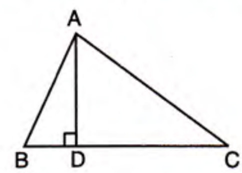
12. In an isosceles triangle ABC; AB = AC and D is a point on BC produced. Prove that : $AD^2 = AC^2 + BD.CD$.

13. In triangle ABC, angle A = 90°, CA = AB and D is a point on AB produced. Prove that : $DC^2 - BD^2 = 2AB.AD$.



14. In triangle ABC, AB = AC and BD is perpendicular to AC. Prove that : $BD^2 - CD^2 = 2CD \times AD$

15. In the following figure, AD is perpendicular to BC and D divides BC in the ratio 1 : 3.



Prove that : $2AC^2 = 2AB^2 + BC^2$

$BD : DC = 1 : 3 \Rightarrow BD = \frac{1}{4}\ BC$ and

$CD = \frac{3}{4}\ BC$

$AC^2 = AD^2 + CD^2$ and $AB^2 = AD^2 + BD^2$

$\Rightarrow AC^2 - AB^2 = CD^2 - BD^2$

$= (\frac{3}{4}BC)^2 - (\frac{1}{4}BC)^2$

$= \frac{9}{16}BC^2 - \frac{1}{16}BC^2$

$= \frac{8}{16}BC^2 = \frac{1}{2}BC^2$

$\therefore\ 2AC^2 - 2AB^2 = BC^2$

i.e. $2AC^2 = 2AB^2 + BC^2$