12 Mid-Point and Its Converse

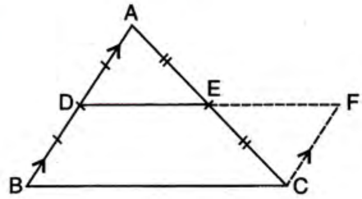
[Including Intercept Theorem]

12.1 MID-POINT THEOREM (Proof and simple applications and its converse)

Theorem 6

The line segment joining the mid-points of any two sides of a triangle is parallel to the third side, and is equal to half of it.

Given : D and E are the mid-points of sides AB and AC respectively of Δ ABC.



To Prove : DE // BC and DE = $\frac{1}{2}$ BC.

Construction : Draw CF parallel to BA which meets DE produced at F.

Proof :

Statement :	Reason :
1. In Δ ADE and Δ CEF :	
(i) AE = EC	Given, E is the mid-point of AC
(ii) ∠AED = ∠CEF	Vertically opposite angles
(iii) ∠EAD = ∠ECF	Alternate angles
∴ Δ ADE ≅ Δ CFE	A.S.A.
2. ∴ AD = CF	Corresponding parts of congruent Δs.
3. But, AD = BD	Given, D is mid-point of AB
4. ∴ CF = BD	From 2 and 3.
5. ∴ BCFD is a parallelogram	Opp. sides CF and BD are equal and parallel.
⇒ DF // BC and so, **DE // BC**.	Opp. sides of a // gm are parallel

(First part proved)

Now, DE = EF	Δ ADE ≅ Δ CFE
= $\frac{1}{2}$ DF	
= $\frac{1}{2}$ BC	In parallelogram BCFD; DF = BC.

∴ **DE // BC and DE = $\frac{1}{2}$ BC.**

Hence Proved

Theorem 7

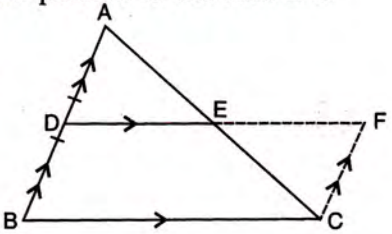
(Converse of Mid-point Theorem)

The straight line drawn through the mid-point of one side of a triangle parallel to another, bisects the third side.

Given : D is mid-point of side AB of a Δ ABC and DE is drawn parallel to the side BC.

To Prove : DE bisects AC, i.e. AE = EC.

Construction : Draw CF parallel to BA which meets DE produced at F.



Proof :

Statement :	Reason :
1. BCFD is a parallelogram	DF // BC and CF // BD
2. CF = BD	Opposite sides of a // gm are equal.
3. CF = DA	Since, BD = DA (given)
4. In Δ ADE and Δ CFE :	
(i) AD = CF	From (3)
(ii) ∠DAE = ∠ECF	Alternate angles
(iii) ∠ADE = ∠EFC	Alternate angles
∴ Δ ADE ≅ Δ CFE	A.S.A.
⇒ **AE = EC**	Corresponding parts of congruent Δs are equal.

Hence Proved

In the figure, given above, D is given to be the mid-point of AB and E is proved to be the mid-point of AC, therefore, DE will be half of the 3rd side *i.e.,* DE = $\frac{1}{2}$ BC.

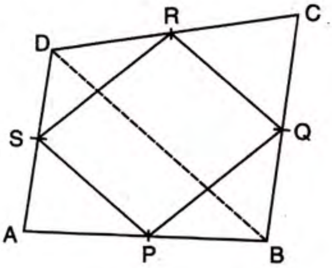
1 Prove that the figure obtained by joining the mid-points of the adjacent sides of a quadrilateral is a parallelogram.

Solution :

Given : P, Q, R and S are the mid-points of sides AB, BC, CD and DA respectively of quadrilateral ABCD.

To Prove : PQRS is a parallelogram.

Construction : Join B and D.



Proof :

1. In Δ ABD :

PS // BD and PS = $\frac{1}{2}$ BD [Line joining the mid-points of two sides of a Δ is parallel and half of third side]

2. In Δ BCD :

QR // BD and QR = $\frac{1}{2}$ BD [Line joining the mid-points of two sides of a Δ is parallel and half of third side]

$\therefore$ PS // QR and PS = QR [From (1) and (2)]

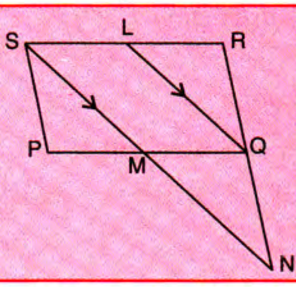
$\Rightarrow$ **PQRS is a parallelogram** [One pair of opp. sides are equal and parallel]

Hence Proved

2 In parallelogram PQRS, L is mid-point of side SR and SN is drawn parallel to LQ which meets RQ produced at N and cuts side PQ at M.

Prove that :

(i) SP = $\frac{1}{2}$ RN (ii) SN = 2 LQ



Solution :

(i) In Δ SRN :

L is mid-point of SR [Given]

and LQ // SN [Given]

$\therefore$ LQ bisects RN [Line through mid-point of one side of a Δ and parallel to another side bisects the third side]

i.e. RQ = QN = $\frac{1}{2}$ RN

But, SP = RQ [Opposite sides of // gm PQRS]

$\therefore$ **SP = $\frac{1}{2}$ RN**

Hence Proved

(ii) In Δ SRN :

L is mid-point of SR [Given]

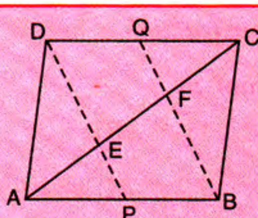
and Q is mid-point of RN [Proved in part (i)]

$\therefore$ LQ = $\frac{1}{2}$ SN [Line joining the mid-points of two sides of a Δ is half of the third side]

or, **SN = 2 LQ**

Hence Proved

3 The adjoining figure shows a parallelogram ABCD in which P is mid-point of AB and Q is mid-point of CD. Prove that : AE = EF = FC.



Solution :

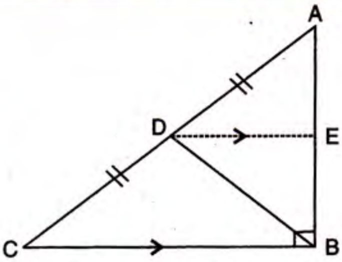
Since, PB = $\frac{1}{2}$ AB [Given, P is mid-point of AB]

$DQ = \frac{1}{2} DC$	[Given, Q is mid-point of DC]
$\therefore$ PB = DQ	[Since, AB = DC; the opp. sides of // gm ABCD]
Also, PB // DQ	[As AB // DC]
$\therefore$ DPBQ is a parallelogram	[Opp. sides PB and DQ are parallel and equal]
$\Rightarrow$ DP // QB	[Opp. sides of the // gm DPBQ]
Now in Δ ABF :	
P is mid-point of AB	[Given]
PE // BF	[As DP // QB]
$\therefore$ PE bisects AF	[Line passing through the mid-point of one side and parallel to another bisects the third side]
i.e. AE = EF I	
Similarly, in Δ CDE :	
QF bisects CE	[Q is mid-point of CD and QF // DE]
$\therefore$ EF = FC II	
$\therefore$ **AE = EF = FC**	[From I and II]

Hence Proved

4 In a right-angled triangle ABC, ∠ABC = 90° and D is mid-point of AC. Prove that : $BD = \frac{1}{2} AC$.

Solution :



According to the given statement, the figure will be as shown alongside :

Draw the line segment DE parallel to CB, which meets AB at point E.

Since, DE//CB and AB is transversal,

∠AED = ∠ABC [Corresponding angles]

= 90° = ∠DEB.

Also, as D is mid-point of AC and DE is parallel to CB; DE bisects side AB. *i.e.* AE = BE.

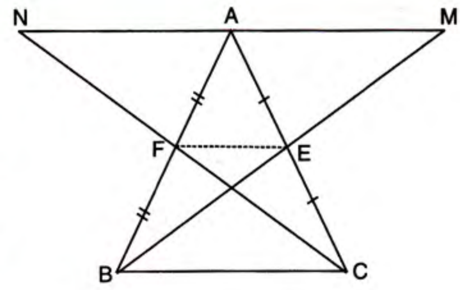
In Δ AED and Δ BED,

	∠AED = ∠BED	[Each 90°]
	AE = BE	[Proved above]
and,	DE = DE	[Common]
$\therefore$	Δ AED ≅ Δ BED	[By S.A.S.]
$\Rightarrow$	**BD** = AD	[C.P.C.T.C.]
	$= \frac{1}{2}$ **AC**	

Hence Proved.

5 In triangle ABC, BE and CF are medians. M is a point on BE produced such that BE = EM and N is a point on CF produced such that CF = FN. Prove that :
(i) NAM is a straight line (ii) A is the mid-point of MN

Solution :



According to the given statement, the figure will be as shown alongside.

It is clear from the given statement that E is mid-point of AC and BM whereas F is mid-point of AB and CN.

Since, the line joining the mid-points of any two sides of a triangle is parallel and half of the third side

∴ In Δ ACN, EF // AN and EF = $\frac{1}{2}$ AN I

In Δ ABM, EF // AM and EF = $\frac{1}{2}$ AM II

(i) From I and II, we get AN // AM (both are parallel to EF)
As, AN and AM are parallel and have a common point (point A), this is possible only if **NAM is a straight line**.

Hence Proved.

(ii) From equations I and II, we have : EF = $\frac{1}{2}$ AN and

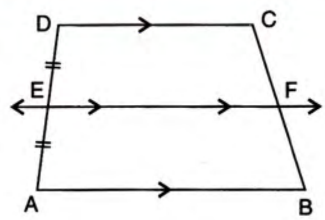
EF = $\frac{1}{2}$AM ⇒ AN = AM ⇒ **A is mid-point of MN**

Hence Proved.

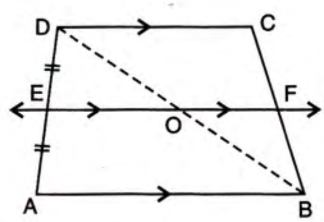
6 In a trapezium ABCD, AB//DC, E is mid-point of AD. A line through E and parallel to AB intersects BC at point F. Show that :
(i) F is mid-point of BC. (ii) 2EF = AB + DC.

Solution :

(i) According to the given statement, the figure, will be as shown alongside :



Draw diagonal BD which intersects EF at point O.



In triangle ABD, E is mid-point of AD and EO//AB (as EF//AB).

∴ O is mid-point of BD
[By the converse of mid-point theorem]

In Δ BCD, O is mid-point of BD [Proved above]

and OF//DC (as EF//AB//DC)

$\therefore$ **F is mid-point of BC** [By the converse of mid-point theorem]

Hence Proved.

(ii) In Δ ABD,

E is mid-point of AD and O is mid-point of BD

$\therefore$ $EO = \frac{1}{2}AB$ ....I [By mid-point theorem]

Also, O is mid-point of BD and F is mid-point of BC

$\therefore$ $OF = \frac{1}{2}DC$ .....II [By mid-point theorem]

$\Rightarrow$ $EO + OF = \frac{1}{2}AB + \frac{1}{2}DC$ [Adding equations I and II]

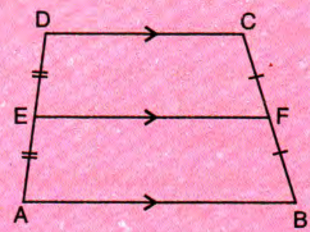
$\Rightarrow$ $EF = \frac{1}{2}(AB + DC)$

i.e. **2EF = AB + DC** **Hence Proved.**

In every trapezium, the length of the line segment joining the mid-points of the non-parallel sides is equal to half of the sum of lengths of the parallel sides.

According to the result of example 7 (Proved above).

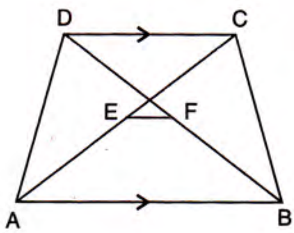
$2EF = AB + DC \Rightarrow EF = \frac{1}{2}(AB + DC)$.



7 The line segment joining the mid-points of the diagonals of a trapezium is parallel to the parallel sides of the trapezium and is equal to half the difference between the parallel sides. Prove it.

Solution :

According to the given statement, the figure will be as shown alongside :



In the figure, AB//DC, E is the mid-point of diagonal AC and F is mid-point of diagonal BD.

Required to prove : (i) EF//AB//DC.

(ii) $EF = \frac{1}{2}(AB - DC)$

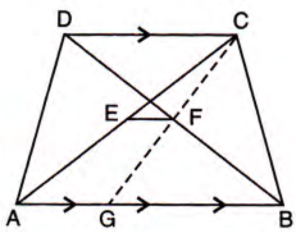
Join CF and produce to meet AB at point G.

Consider the triangles DFC and BFG

DF = FB [F is mid-poind of BD]

$\angle$DFC = $\angle$BFG [Vertically opposite angles]

and, $\angle$CDF = $\angle$GBF [Alternate angles]



$\therefore \quad \Delta DFC \cong \Delta BFG$ [By A.S.A.]

$\Rightarrow \quad DC = GB$ and $CF = GF$ [By C.P.C.T.C.]

In ΔACG, E is mid-point of AC and F is mid-point of CG

$\Rightarrow EF // AG$ and $EF = \frac{1}{2}AG$ [By mid-point theorem]

$EF // AG \Rightarrow$ **EF//AB//DC** **Hence Proved.**

Now, $EF = \frac{1}{2}AG$

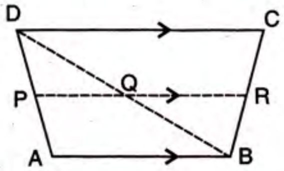
$= \frac{1}{2}(AB - GB)$

$= \frac{1}{2}(AB - DC)$ **Hence Proved.**

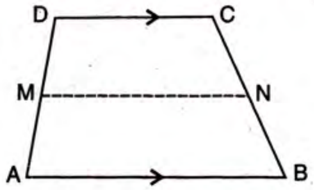
EXERCISE 12(A)

1. In triangle ABC, M is mid-point of AB and a straight line through M and parallel to BC cuts AC at N. Find the lengths of AN and MN, if BC = 7 cm and AC = 5 cm.
2. Prove that the figure obtained by joining the mid-points of the adjacent sides of a rectangle is a rhombus.
3. D, E and F are the mid-points of the sides AB, BC and CA of an isosceles Δ ABC in which AB = BC. Prove that Δ DEF is also isosceles.
4. The following figure shows a trapezium ABCD in which AB // DC. P is the mid-point of AD and PR // AB. Prove that :

$$PR = \frac{1}{2}(AB + CD)$$



5. The figure, given below, shows a trapezium ABCD. M and N are the mid-points of the non-parallel sides AD and BC respectively. Find :

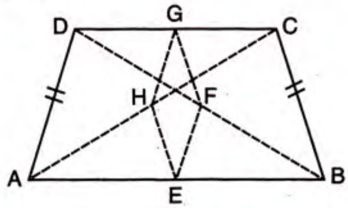


(i) MN, if AB = 11 cm and DC = 8 cm.

(ii) AB, if DC = 20 cm and MN = 27 cm.

(iii) DC, if MN = 15 cm and AB = 23 cm.

6. The diagonals of a quadrilateral intersect at right angles. Prove that the figure obtained by joining the mid-points of the adjacent sides of the quadrilateral is a rectangle.
7. L and M are the mid-points of sides AB and DC respectively of parallelogram ABCD. Prove that segments DL and BM trisect diagonal AC.
8. ABCD is a quadrilateral in which AD = BC. E, F, G and H are the mid-points of AB, BD, CD and AC respectively. Prove that EFGH is a rhombus.

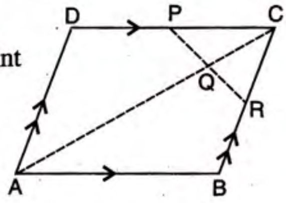


9. A parallelogram ABCD has P the mid-point of DC and Q a point of AC such that $CQ = \frac{1}{4}AC$. PQ produced meets BC at R.

Prove that :

(i) R is the mid-point of BC,

(ii) $PR = \frac{1}{2}DB$.



10. D, E and F are the mid-points of the sides AB, BC and CA respectively of Δ ABC. AE meets DF at O. P and Q are the mid-points of OB and OC respectively. Prove that DPQF is a parallelogram.

11. In triangle ABC, P is the mid-point of side BC. A line through P and parallel to CA meets AB at point Q; and a line through Q and parallel to BC meets median AP at point R. Prove that : (i) AP = 2AR (ii) BC = 4QR

12. In trapezium ABCD, AB is parallel to DC; P and Q are the mid-points of AD and BC respectively. BP produced meets CD produced at point E. Prove that :
    (i) point P bisects BE,
    (ii) PQ is parallel to AB.

13. In a triangle ABC, AD is a median and E is mid-point of median AD. A line through B and E meets AC at point F.
    Prove that : AC = 3AF

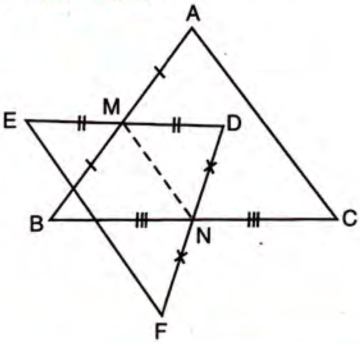
Draw DG parallel to BF, which meets AC at point G.

14. D and F are mid-points of sides AB and AC of a triangle ABC. A line through F and parallel to AB meets BC at point E.
    (i) Prove that BDFE is a parallelogram.
    (ii) Find AB, if EF = 4·8 cm.

15. In triangle ABC, AD is the median and DE, drawn parallel to side BA, meets AC at point E. Show that BE is also a median.

16. In Δ ABC, E is mid-point of the median AD and BE produced meets side AC at point Q. Show that BE : EQ = 3 : 1.

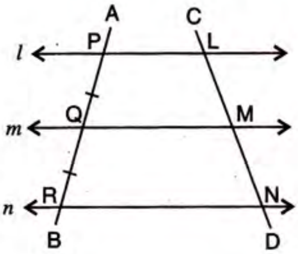
17. In the given figure, M is mid-point of AB and DE, whereas N is mid-point of BC and DF. Show that : **EF = AC**.



12.2 EQUAL INTERCEPT THEOREM (Proof and simple application)

Theorem 8

If a transversal makes equal intercepts on three or more parallel lines, then any other line cutting them will also make equal intercepts.



Given : Transversal AB makes equal intercepts on three parallel lines *l*, *m* and *n*.

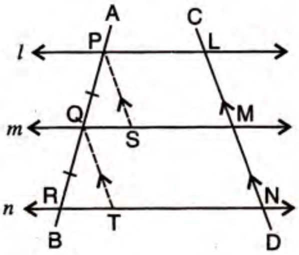
i.e. *l* // *m* // *n* and PQ = QR

CD is another transversal which makes intercepts LM and MN.

To Prove : LM = MN

Construction : Draw PS and QT parallel to CD.

Proof :



Statement :				Reason :
1. In Δ PQS and Δ QRT :				
(i)	PQ	=	QR	Given
(ii)	∠PQS	=	∠QRT	Corresponding angles
(iii)	∠QPS	=	∠RQT	Corresponding angles as PS // CD // QT.
∴	Δ PQS	≅	Δ QRT	A.S.A.

2.	∴ PS = QT	Corresponding parts of congruent Δs are equal.
3.	PSML is a parallelogram	Both the pairs of opp. sides are parallel.
	∴ PS = LM	Opp. sides of a // gm are equal.
4.	QTNM is a parallelogram	Both the pairs of opp. sides are // .
	∴ QT = MN	Opposite sides of a // gm are equal.
5.	∴ **LM = MN**	From (2), (3) and (4).

Hence Proved

8 Use the Intercept Theorem to prove the converse of the Mid-point Theorem

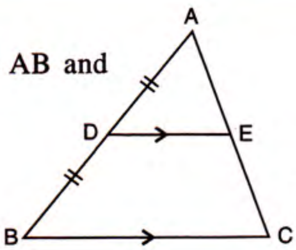
Solution :

Converse of Mid-point Theorem is :

The straight line drawn through the mid-point of one side of a triangle and parallel to another side bisects the third side.

Given :

In triangle ABC, D is mid-point of side AB and DE is parallel to BC

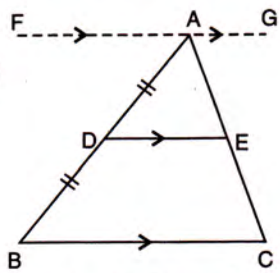


To Prove :

DE bisects AC *i.e.* AE = CE

Construction :

Through vertex A, draw FG parallel to BC so that FG // BC // DE.



Proof :

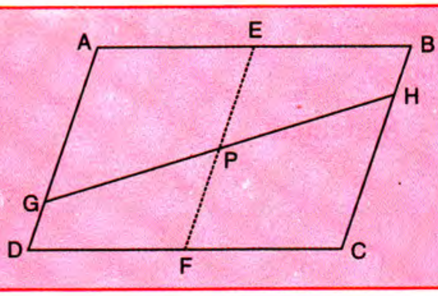
Since, FG // DE // BC and the transversal AB makes equal intercepts on these three parallel lines *i.e.* AD = DB.

Also, AC is an another transversal. According to Intercept Theorem, if a transversal makes equal intercepts on three or more parallel lines, then any other transversal, for the same parallel lines, will also make equal intercepts.

∴ **AE = CE**

Hence Proved

9 ABCD is a parallelogram. E is the mid-point of AB and F is the mid-point of CD. GH is any line that intersects AD, EF and BC at G, P and H respectively.
Prove that : GP = PH.



Solution :

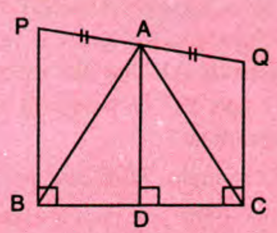
$AE = \frac{1}{2} AB$ [E is mid-point of AB]

$DF = \frac{1}{2} DC$	[F is mid-point of DC]
$\therefore$ AE = DF	[Since, AB = DC; opp. sides of a // gm]
Also, AE // DF	[As, AB // DC; opp. sides of // gm]
$\therefore$ AEFD is a parallelogram	[Opposite sides AE and DF are equal and parallel]
$\Rightarrow$ AD // EF // BC	[Opposite sides of parallelograms]

Now applying intercept theorem, we find that the transversal AB makes equal intercepts AE = EB on three parallel lines AD // EF // BC; therefore another transversal GH will also make equal intercepts on these parallel lines *i.e.* **GP = PH**.

Hence Proved

10 Use the information, given in the adjoining figure, to show that : AB = AC.



Solution :

Since PB, AD and QC are perpendiculars to the same line BC, they are parallel to each other *i.e.* PB // AD // QC.

Since, PB // AD // QC and PQ is a transversal making equal intercepts *i.e.* PA = AQ; therefore the other transversal BC will also make equal intercepts *i.e.* BD = CD.

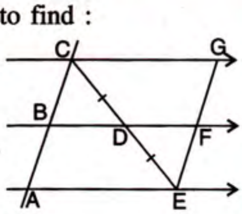
Now in Δ ABD and Δ ACD,

(i)	BD = CD	[Proved above]
(ii)	AD = AD	[Common]
(iii)	∠ADB = ∠ADC = 90°	[As, AD ⊥ BC]
$\therefore$	Δ ABD ≡ Δ ACD	[By SAS]
$\Rightarrow$	**AB = AC**	[By C.P.C.T.C]

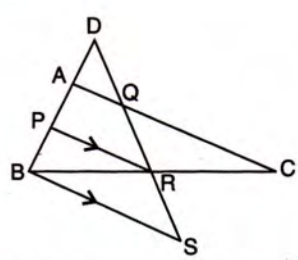
Hence Proved.

EXERCISE 12(B)

1. Use the following figure to find :
   (i) BC, if AB = 7·2 cm.
   (ii) GE, if FE = 4 cm.
   (iii) AE, if BD = 4·1 cm.
   (iv) DF, if CG = 11 cm.



2. In the figure, given below, 2AD = AB, P is mid-point of AB, Q is mid-point of DR and PR // BS. Prove that :
   (i) AQ // BS,
   (ii) DS = 3 RS.



3. The side AC of a triangle ABC is produced to point E so that $CE = \frac{1}{2}$ AC. D is the mid-point of BC and ED produced meets AB at

F. Lines through D and C are drawn parallel to AB which meet AC at point P and EF at point R respectively. Prove that :

(i) 3DF = EF (ii) 4CR = AB.

4. In triangle ABC, the medians BP and CQ are produced upto points M and N respectively such that BP = PM and CQ = QN. Prove that :
   (i) M, A and N are collinear.
   (ii) A is the mid-point of MN.

5. In triangle ABC, angle B is obtuse. D and E are mid-points of sides AB and BC respectively and F is a point on side AC such that EF is parallel to AB. Show that BEFD is a parallelogram.

6. In parallelogram ABCD, E and F are mid-points of the sides AB and CD respectively. The line segments AF and BF meet the line segments ED and EC at points G and H respectively. Prove that:
   (i) triangles HEB and FHC are congruent;
   (ii) GEHF is a parallelogram.

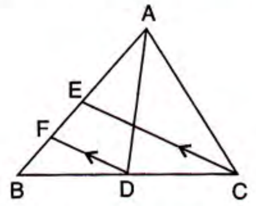
7. In triangle ABC, D and E are points on side AB such that AD = DE = EB. Through D and E, lines are drawn parallel to BC which meet side AC at points F and G respectively. Through F and G, lines are drawn parallel to AB which meet side BC at points M and N respectively. Prove that : BM = MN = NC.

8. In triangle ABC; M is mid-point of AB, N is mid-point of AC and D is any point in base BC. Use Intercept Theorem to show that MN bisects AD.

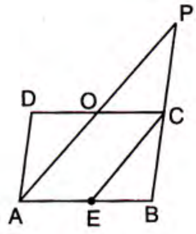
9. If the quadrilateral formed by joining the mid-points of the adjacent sides of quadrilateral ABCD is a rectangle, show that the diagonals AC and BD intersect at right angle.

10. In triangle ABC; D and E are mid-points of the sides AB and AC respectively. Through E, a straight line is drawn parallel to AB to meet BC at F. Prove that BDEF is a parallelogram. If AB = 16 cm, AC = 12 cm and BC = 18 cm, find the perimeter of the parallelogram BDEF.

11. In the given figure, AD and CE are medians and DF//CE. Prove that : FB = $\frac{1}{4}$AB.



12. In parallelogram ABCD, E is the mid-point of AB and AP is parallel to EC which meets DC at point O and BC produced at P. Prove that :



   (i) BP = 2AD
   (ii) O is mid-point of AP.

13. In trapezium ABCD, sides AB and DC are parallel to each other. E is mid-point of AD and F is mid-point of BC.

    Prove that : AB + DC = 2EF

> Join BE and produce to meet CD produced at point P.
> Δ PDE ≅ Δ BAE by ASA which gives
> BE = EP and AB = PD.
> Now apply mid-point theorem for Δ BPC.

14. In Δ ABC, AD is the median and DE is parallel to BA, where E is a point in AC. Prove that BE is also a median.