7 Indices [Exponents]

7.1 INTRODUCTION

If m is a positive integer, then $a \times a \times a \times a$ ----- upto m terms, is written as a^m; where '*a*' is called the **base** and '*m*' is called the **power** (or *exponent* or *index*).

a^m is read as '***a power m***' or '***a raised to the power m***'.

Thus : (i) $a \times a \times a \times$ ---- upto 10 terms = a^{10} [*a raised to the power* 10]

(ii) $2 \times 2 \times 2 \times$ ---- upto 7 terms = 2^7 [2 *raised to the power* 7] and so on.

7.2 LAWS OF INDICES

Ist Law (Product Law) : $a^m \times a^n = a^{m+n}$

e.g. (i) $a^7 \times a^4 = a^{7+4} = a^{11}$ (ii) $a^3 \times a^{-6} = a^{3-6} = a^{-3}$ and so on.

2nd Law (Quotient Law) : $\frac{a^m}{a^n} = a^{m-n}$

e.g. (i) $\frac{a^7}{a^4} = a^{7-4} = a^3$ (ii) $\frac{a^3}{a^6} = a^{3-6} = a^{-3}$ and so on.

3rd Law (Power Law) : $(a^m)^n = a^{mn}$

e.g. (i) $(a^3)^4 = a^{12}$ (ii) $(a^{-2})^5 = a^{-10}$ and so on.

7.3 HANDLING POSITIVE, FRACTIONAL, NEGATIVE AND ZERO INDICES

1. $(a \times b)^m = a^m \times b^m$ **and** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

e.g. (i) $(2 \times 3)^5 = 2^5 \times 3^5$ (ii) $\left(\frac{2}{3}\right)^5 = \frac{2^5}{3^5}$ and so on.

2. If a ≠ 0 and n is a positive integer, then $\sqrt[n]{a} = a^{1/n}$

e.g. $\sqrt[3]{a} = a^{1/3}$; $\sqrt[4]{a} = a^{1/4}$; $\sqrt[8]{a} = a^{1/8}$ and so on.

Also, $\sqrt{a} = a^{1/2}$ *i.e.* $\sqrt{2} = 2^{1/2}$, $\sqrt{10} = 10^{1/2}$ and so on.

3. $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ **; where $a \neq 0$.**

e.g. $a^{\frac{4}{5}} = \sqrt[5]{a^4}$; $5^{\frac{2}{3}} = \sqrt[3]{5^2}$ and so on.

Conversely : $\sqrt[n]{a^m} = a^{m/n}$ *i.e.* $\sqrt[3]{a^5} = a^{5/3}$, $\sqrt[5]{3^8} = 3^{8/5}$ and so on.

4. For any non-zero number a,

$$a^n = \frac{1}{a^{-n}} \text{ and } a^{-n} = \frac{1}{a^n}$$

e.g. $a^7 = \frac{1}{a^{-7}}$; $a^{-3} = \frac{1}{a^3}$; $a^4 = \frac{1}{a^{-4}}$ and so on.

5. Any non-zero number raised to the power zero is always equal to unity (*i.e.* 1).

e.g. $a^0 = 1$; $5^0 = 1$; $2^0 = 1$ and so on.

$(-a)^m = a^m$; if m is an even number.

$(-a)^m = -a^m$; if m is an odd number.

e.g. $(-2)^4 = 2^4$; $(-2)^5 = -2^5$ and so on.

7.4 SIMPLIFICATION OF EXPRESSIONS

1 Evaluate : (i) $27^{-1/3}$ (ii) $9^{\frac{3}{2}} - 3\,(5)^0 - \left(\frac{1}{81}\right)^{-\frac{1}{2}}$ (iii) $\left(\frac{64}{125}\right)^{\frac{-2}{3}} \div \frac{1}{\left(\frac{256}{625}\right)^{\frac{1}{4}}} \times \frac{\sqrt{25}}{\sqrt[3]{64}}$

Solution :

(i) $27^{-1/3} = (3^3)^{-\frac{1}{3}} = 3^{3 \times -\frac{1}{3}} = 3^{-1} = \frac{1}{3}$ **Ans.**

(ii) $= (3^2)^{\frac{3}{2}} - 3 \times 1 - (81)^{\frac{1}{2}}$ $\left[\because 5^0 = 1 \text{ and } \left(\frac{1}{81}\right)^{-\frac{1}{2}} = (81)^{\frac{1}{2}}\right]$

$= 3^3 - 3 - 9^{2 \times \frac{1}{2}} = 27 - 3 - 9 = 15$ **Ans.**

(iii) $= \left(\frac{125}{64}\right)^{\frac{2}{3}} \div \left(\frac{625}{256}\right)^{\frac{1}{4}} \times \frac{5}{\sqrt[3]{4^3}}$

$= \left[\left(\frac{5}{4}\right)^3\right]^{\frac{2}{3}} \times \left(\frac{256}{625}\right)^{\frac{1}{4}} \times \frac{5}{4} = \left(\frac{5}{4}\right)^2 \times \frac{4}{5} \times \frac{5}{4}$ $\left[\because \left(\frac{256}{625}\right)^{\frac{1}{4}} = \left(\frac{4}{5}\right)^{4 \times \frac{1}{4}} = \frac{4}{5}\right]$

$= \frac{25}{16} = 1\frac{9}{16}$ **Ans.**

2 Simplify :

(i) $(27)^{\frac{4}{3}} + (32)^{0\cdot 8} + (0\cdot 8)^{-1}$ (ii) $27^{-\frac{1}{3}}\left(27^{\frac{1}{3}} - 27^{\frac{2}{3}}\right)$

(iii) $\left[5\left(8^{\frac{1}{3}} + 27^{\frac{1}{3}}\right)^3\right]^{\frac{1}{4}}$

Solution :

(i) $= (3^3)^{\frac{4}{3}} + (2^5)^{\frac{8}{10}} + \left(\frac{8}{10}\right)^{-1}$

$= 3^4 + 2^4 + \frac{10}{8} = 81 + 16 + 1{\cdot}25 = 98{\cdot}25$ **Ans.**

(ii) $= (3^3)^{-\frac{1}{3}}\left[(3^3)^{\frac{1}{3}} - (3^3)^{\frac{2}{3}}\right]$

$= 3^{-1}(3^1 - 3^2) = \frac{1}{3}(3 - 9) = \frac{1}{3} \times -6 = -2$ **Ans.**

(iii) $= \left[5\left(2^{3\times\frac{1}{3}} + 3^{3\times\frac{1}{3}}\right)^3\right]^{\frac{1}{4}}$

$= [5(2+3)^3]^{\frac{1}{4}} = (5\times5^3)^{\frac{1}{4}} = (5^4)^{\frac{1}{4}} = 5$ **Ans.**

3 Given : $1176 = 2^p . 3^q . 7^r$, find :
(i) the numerical values of p, q and r. (ii) the value of $2^p . 3^q . 7^{-r}$ as a fraction.

Solution :

(i) $1176 = 2^p . 3^q . 7^r$

$\Rightarrow 2^3 \times 3^1 \times 7^2 = 2^p . 3^q . 7^r$ $[1176 = 2 \times 2 \times 2 \times 3 \times 7 \times 7]$

$\Rightarrow p = 3, q = 1$ and $r = 2$ **Ans.**

(ii) $2^p . 3^q . 7^{-r} = 2^3 . 3^1 . 7^{-2}$

$= \frac{8\times3}{7^2} = \frac{24}{49}$ **Ans.**

4 Simplify : (i) $\frac{3^{a+2} - 3^{a+1}}{4\times3^a - 3^a}$ (ii) $\left(\frac{a^m}{a^n}\right)^{m+n} \cdot \left(\frac{a^n}{a^l}\right)^{n+l} \cdot \left(\frac{a^l}{a^m}\right)^{l+m}$

Solution :

(i) **The given expression** $= \frac{3^a.3^2 - 3^a.3^1}{4\times3^a - 3^a}$ $[3^a . 3^2 = 3^{a+2}]$

$= \frac{3^a(3^2 - 3^1)}{3^a(4-1)} = \frac{9-3}{3} = 2$ **Ans.**

(ii) **The given expression** $= (a^{m-n})^{m+n} . (a^{n-l})^{n+l} . (a^{l-m})^{l+m}$

$= a^{m^2-n^2} . a^{n^2-l^2} . a^{l^2-m^2}$

$= a^{m^2-n^2+n^2-l^2+l^2-m^2} = a^0 = 1$ **Ans.**

EXERCISE 7 (A)

1. Evaluate :

(i) $3^3 \times (243)^{-\frac{2}{3}} \times 9^{-\frac{1}{3}}$

(ii) $5^{-4} \times (125)^{\frac{5}{3}} \div (25)^{-\frac{1}{2}}$

(iii) $\left(\frac{27}{125}\right)^{\frac{2}{3}} \times \left(\frac{9}{25}\right)^{-\frac{3}{2}}$

(iv) $7^0 \times (25)^{-\frac{3}{2}} - 5^{-3}$

(v) $\left(\frac{16}{81}\right)^{-\frac{3}{4}} \times \left(\frac{49}{9}\right)^{\frac{3}{2}} \div \left(\frac{343}{216}\right)^{\frac{2}{3}}$

2. Simplify :

(i) $(8x^3 \div 125y^3)^{\frac{2}{3}}$

(ii) $(a+b)^{-1} \cdot (a^{-1} + b^{-1})$

(iii) $\dfrac{5^{n+3} - 6 \times 5^{n+1}}{9 \times 5^n - 5^n \times 2^2}$

(iv) $(3x^2)^{-3} \times (x^9)^{\frac{2}{3}}$

3. Evaluate :

(i) $\sqrt{\frac{1}{4}} + (0.01)^{-\frac{1}{2}} - (27)^{\frac{2}{3}}$

(ii) $\left(\frac{27}{8}\right)^{\frac{2}{3}} - \left(\frac{1}{4}\right)^{-2} + 5^0$

4. Simplify each of the following and express with positive index :

(i) $\left(\frac{3^{-4}}{2^{-8}}\right)^{\frac{1}{4}}$ (ii) $\left(\frac{27^{-3}}{9^{-3}}\right)^{\frac{1}{5}}$

(iii) $(32)^{-\frac{2}{5}} \div (125)^{-\frac{2}{3}}$

(iv) $[1 - \{1 - (1-n)^{-1}\}^{-1}]^{-1}$

5. If $2160 = 2^a \cdot 3^b \cdot 5^c$, find a, b and c. Hence calculate the value of $3^a \times 2^{-b} \times 5^{-c}$.

6. If $1960 = 2^a \cdot 5^b \cdot 7^c$, calculate the value of $2^{-a} \cdot 7^b \cdot 5^{-c}$.

7. Simplify :

(i) $\dfrac{8^{3a} \times 2^5 \times 2^{2a}}{4 \times 2^{11a} \times 2^{-2a}}$

(ii) $\dfrac{3 \times 27^{n+1} + 9 \times 3^{3n-1}}{8 \times 3^{3n} - 5 \times 27^n}$

8. Show that :

$$\left(\frac{a^m}{a^{-n}}\right)^{m-n} \times \left(\frac{a^n}{a^{-l}}\right)^{n-l} \times \left(\frac{a^l}{a^{-m}}\right)^{l-m} = 1$$

9. If $a = x^{m+n} \cdot y^l$; $b = x^{n+l} \cdot y^m$ and $c = x^{l+m} \cdot y^n$,

prove that : $a^{m-n} \cdot b^{n-l} \cdot c^{l-m} = 1$

10. Simplify : (i) $\left(\frac{x^a}{x^b}\right)^{a^2+ab+b^2} \times \left(\frac{x^b}{x^c}\right)^{b^2+bc+c^2} \times \left(\frac{x^c}{x^a}\right)^{c^2+ca+a^2}$

(ii) $\left(\frac{x^a}{x^{-b}}\right)^{a^2-ab+b^2} \times \left(\frac{x^b}{x^{-c}}\right)^{b^2-bc+c^2} \times \left(\frac{x^c}{x^{-a}}\right)^{c^2-ca+a^2}$

7.5 USING LAWS OF EXPONENTS

5 Solve for x : (i) $9 \times 3^x = (27)^{2x-5}$ (ii) $\sqrt{\left(\frac{3}{5}\right)^{1-2x}} = 4\frac{17}{27}$

Solution :

(i) $9 \times 3^x = (27)^{2x-5} \Rightarrow 3^2 \times 3^x = (3^3)^{2x-5}$

$\Rightarrow 3^{2+x} = 3^{6x-15}$

$\Rightarrow 2 + x = 6x - 15 \Rightarrow x = \frac{17}{5} = 3\frac{2}{5}$ **Ans.**

(ii) $\Rightarrow \left[\left(\frac{3}{5}\right)^{1-2x}\right]^{\frac{1}{2}} = \frac{125}{27} \Rightarrow \left(\frac{3}{5}\right)^{\frac{1-2x}{2}} = \left(\frac{5}{3}\right)^3$

$\Rightarrow \left(\frac{3}{5}\right)^{\frac{1-2x}{2}} = \left(\frac{3}{5}\right)^{-3} \Rightarrow \frac{1-2x}{2} = -3 \Rightarrow x = 3{\cdot}5$ **Ans.**

6 Solve : $2^{2x+3} - 9 \times 2^x + 1 = 0$

Solution :

$2^{2x} \times 2^3 - 9 \times 2^x + 1 = 0$

$\Rightarrow 8y^2 - 9y + 1 = 0$ [Taking $2^x = y$]

$\Rightarrow 8y^2 - 8y - y + 1 = 0$

$\Rightarrow (8y - 1)(y - 1) = 0 \Rightarrow y = \frac{1}{8}$ or 1

When $y = \frac{1}{8} \Rightarrow 2^x = 2^{-3} \Rightarrow x = -3$ **Ans.**

When $y = 1 \Rightarrow 2^x = 2^0 \Rightarrow x = 0$ **Ans.**

7 Prove that : $\frac{1}{1+x^{b-a}+x^{c-a}} + \frac{1}{1+x^{a-b}+x^{c-b}} + \frac{1}{1+x^{b-c}+x^{a-c}} = 1$

Solution :

$$\text{L.H.S.} = \frac{1}{1+\frac{x^b}{x^a}+\frac{x^c}{x^a}} + \frac{1}{1+\frac{x^a}{x^b}+\frac{x^c}{x^b}} + \frac{1}{1+\frac{x^b}{x^c}+\frac{x^a}{x^c}}$$

$$= \frac{x^a}{x^a+x^b+x^c} + \frac{x^b}{x^b+x^a+x^c} + \frac{x^c}{x^c+x^b+x^a}$$

$$= \frac{x^a+x^b+x^c}{x^a+x^b+x^c} = 1 = \text{R.H.S.}$$

Hence Proved.

8 If $a = b^{2x}$, $b = c^{2y}$ and $c = a^{2z}$, show that $8xyz = 1$.

Solution :

$a = b^{2x}$ and $b = c^{2y} \Rightarrow a = (c^{2y})^{2x} = c^{4xy}$

Similarly, $a = c^{4xy}$ and $c = a^{2z} \Rightarrow a = (a^{2z})^{4xy} = a^{8xyz}$

Now, $a = a^{8xyz} \Rightarrow 8xyz = 1$

Alternative methods :

1. $c = a^{2z} = (b^{2x})^{2z} = (c^{2y})^{4xz} = c^{8xyz}$ i.e. $c = c^{8xyz} \Rightarrow$ **$1 = 8xyz$**
2. $b = c^{2y} = (a^{2z})^{2y} = (b^{2x})^{4yz} = b^{8xyz}$ i.e. $b = b^{8xyz} \Rightarrow$ **$1 = 8xyz$**

9 If $2^x = 4 \times 2^y$ and $9 \times 3^x = 3^{-y}$; find the values of x and y.

Solution :

$2^x = 4 \times 2^y \Rightarrow 2^x = 2^2 \times 2^y$

$\Rightarrow 2^x = 2^{2+y}$ and $x = 2 + y$ I

$9 \times 3^x = 3^{-y} \Rightarrow 3^2 \times 3^x = 3^{-y}$

$\Rightarrow 3^{2+x} = 3^{-y}$ and $2 + x = -y$ II

On solving equations I and II, we get :

$x = 0$ and $y = -2$ **Ans.**

EXERCISE 7 (B)

1. Solve for x :
   (i) $2^{2x+1} = 8$
   (ii) $2^{5x-1} = 4 \times 2^{3x+1}$
   (iii) $3^{4x+1} = (27)^{x+1}$
   (iv) $(49)^{x+4} = 7^2 \times (343)^{x+1}$

2. Find x, if :
   (i) $4^{2x} = \frac{1}{32}$ (ii) $\sqrt{2^{x+3}} = 16$
   (iii) $\left(\sqrt{\frac{3}{5}}\right)^{x+1} = \frac{125}{27}$ (iv) $\left(\sqrt[3]{\frac{2}{3}}\right)^{x-1} = \frac{27}{8}$

3. Solve :
   (i) $4^{x-2} - 2^{x+1} = 0$
   (ii) $3^{x^2} : 3^x = 9 : 1$

4. Solve :
   (i) $8 \times 2^{2x} + 4 \times 2^{x+1} = 1 + 2^x$
   (ii) $2^{2x} + 2^{x+2} - 4 \times 2^3 = 0$
   (iii) $\left(\sqrt{3}\right)^{x-3} = \left(\sqrt[4]{3}\right)^{x+1}$

5. Find the values of m and n if :
   $$4^{2m} = \left(\sqrt[3]{16}\right)^{-\frac{6}{n}} = \left(\sqrt{8}\right)^2$$

6. Solve for x and y, if :
   $\left(\sqrt{32}\right)^x \div 2^{y+1} = 1$ and $8^y - 16^{4-x/2} = 0$

7. Prove that :
   (i) $\left(\frac{x^a}{x^b}\right)^{a+b-c} \left(\frac{x^b}{x^c}\right)^{b+c-a} \left(\frac{x^c}{x^a}\right)^{c+a-b} = 1$
   (ii) $\frac{x^{a(b-c)}}{x^{b(a-c)}} \div \left(\frac{x^b}{x^a}\right)^c = 1.$

8. If $a^x = b$, $b^y = c$ and $c^z = a$, prove that : $xyz = 1$.

   $a^x = b \Rightarrow a^{xy} = b^y$ i.e. $a^{xy} = c$
   Now, $a^{xyz} = c^z$ i.e. $a^{xyz} = a \Rightarrow$ **$xyz = 1$**
   Alternative method :
   $a = c^z = (b^y)^z = b^{yz} \Rightarrow (a^x)^{yz} = a^{xyz}$
   i.e. $a = a^{xyz} \Rightarrow$ **$xyz = 1$**

9. If $a^x = b^y = c^z$ and $b^2 = ac$, prove that : $y = \frac{2xz}{x+z}$.

   Let $a^x = b^y = c^z = k$
   $\Rightarrow a = k^{\frac{1}{x}}$, $b = k^{\frac{1}{y}}$ and $c = k^{\frac{1}{z}}$
   $\therefore b^2 = ac \Rightarrow \left(k^{\frac{1}{y}}\right)^2 = \left(k^{\frac{1}{x}}\right) \cdot \left(k^{\frac{1}{z}}\right)$
   $\Rightarrow \frac{2}{y} = \frac{1}{x} + \frac{1}{z}$ and so on.

10. If $5^{-p} = 4^{-q} = 20^r$; show that :

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 0.$$

11. If $m \neq n$ and $(m + n)^{-1} (m^{-1} + n^{-1}) = m^x n^y$; show that : $x + y + 2 = 0$.

12. If $5^{x+1} = 25^{x-2}$; find the value of :

$$3^{x-3} \times 2^{3-x}.$$

13. If $4^{x+3} = 112 + 8 \times 4^x$; find $(18x)^{3x}$.

14. Solve for x :

(i) $4^{x-1} \times (0{\cdot}5)^{3-2x} = \left(\frac{1}{8}\right)^{-x}$.

(ii) $(a^{3x+5})^2 \cdot (a^x)^4 = a^{8x+12}$.

(iii) $(81)^{\frac{3}{4}} - \left(\frac{1}{32}\right)^{-\frac{2}{5}} + x\left(\frac{1}{2}\right)^{-1} \cdot 2^0 = 27$

(iv) $2^{3x+3} = 2^{3x+1} + 48$.

(v) $3(2^x + 1) - 2^{x+2} + 5 = 0$.

EXERCISE 7 (C)

1. Evaluate :

(i) $9^{\frac{5}{2}} - 3 \times 8^0 - \left(\frac{1}{81}\right)^{-\frac{1}{2}}$

(ii) $(64)^{\frac{2}{3}} - \sqrt[3]{125} - \frac{1}{2^{-5}} + (27)^{-\frac{2}{3}} \times \left(\frac{25}{9}\right)^{-\frac{1}{2}}$

(iii) $\left[\left(-\frac{2}{3}\right)^{-2}\right]^3 \times \left(\frac{1}{3}\right)^{-4} \times 3^{-1} \times \frac{1}{6}$

2. Simplify : $\dfrac{3 \times 9^{n+1} - 9 \times 3^{2n}}{3 \times 3^{2n+3} - 9^{n+1}}$.

3. Solve : $3^{x-1} \times 5^{2y-3} = 225$.

4. If $\left(\dfrac{a^{-1}b^2}{a^2b^{-4}}\right)^7 \div \left(\dfrac{a^3b^{-5}}{a^{-2}b^3}\right)^{-5} = a^x \cdot b^y$, find $x + y$.

5. If $3^{x+1} = 9^{x-3}$, find the value of 2^{1+x}.

6. If $2^x = 4^y = 8^z$ and $\dfrac{1}{2x} + \dfrac{1}{4y} + \dfrac{1}{8z} = 4$, find the value of x.

$2^x = 4^y = 8^z$

$\Rightarrow \quad 2^x = 2^{2y} = 2^{3z}$

$\Rightarrow \quad x = 2y = 3z$

$\Rightarrow \quad y = \dfrac{x}{2}$ and $z = \dfrac{x}{3}$.

7. If $\dfrac{9^n \cdot 3^2 \cdot 3^n - (27)^n}{(3^m \cdot 2)^3} = 3^{-3}$.

Show that : $m - n = 1$.

8. Solve for x : $(13)^{\sqrt{x}} = 4^4 - 3^4 - 6$.

9. If $3^{4x} = (81)^{-1}$ and $(10)^{\frac{1}{y}} = 0{\cdot}0001$, find the value of $2^{-x} \times 16^y$.

10. Solve : $3(2^x + 1) - 2^{x+2} + 5 = 0$.

11. If $(a^m)^n = a^m \cdot a^n$, find the value of : $m(n - 1) - (n - 1)$

12. If $m = \sqrt[3]{15}$ and $n = \sqrt[3]{14}$, find the value of $m - n - \dfrac{1}{m^2 + mn + n^2}$