2 OPERATIONS ON SETS

- Union of Sets
- Intersection of Sets
- Difference of Sets
- Distributive Laws
- De Morgan's Laws
- Cardinal Number of Sets

Introduction

This chapter deals with various operations on sets and the results of those operations. Later, some important laws, namely, distributive law and De Morgan's law, have been stated.

Union of Sets

The union of sets A and B is a set of all the elements that are either in set A or in set B or in both sets A and B. It is represented by the symbol $\cup$.

If x is an element in $A \cup B$, then $x \in A$ or $x \in B$ or x belongs to A as well as B.

Example 1: If $A = \{x \mid x = 2a, a \le 5, a \in N\}$,
$B = \{x \mid x = 3a, a \le 5, a \in N\}$, and
$C = \{x \mid x = 6a, a \le 5, a \in N\}$, find $A \cup B \cup C$.

In Roster form,
$A = \{2, 4, 6, 8, 10\}$, $B = \{3, 6, 9, 12, 15\}$, and
$C = \{6, 12, 18, 24, 30\}$

The union of the sets consists all the elements in the sets without repetition of any.
$A \cup B = \{2, 3, 4, 6, 8, 9, 10, 12, 15\}$
$B \cup C = \{3, 6, 9, 12, 15, 18, 24, 30\}$
$A \cup C = \{2, 4, 6, 8, 10, 12, 18, 24, 30\}$
$A \cup B \cup C = \{2, 3, 4, 6, 8, 9, 10, 12, 15, 18, 24, 30\}$

Union Facts

- $A \cup B = B \cup A$
- $(A \cup B) \cup C = A \cup (B \cup C)$
- $A \cup A = A$
- $A \cup \phi = A$
- $A \cup \xi = \xi$
- $A \cup A' = \xi$
- If $A \subset B$, then $A \cup B = B$.
- If $A \cup B = \phi$, then $A = \phi$ and $B = \phi$.

Try this!
If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$ find $A \cup B$.

Intersection of Sets

The intersection of sets A and B is a set containing all the elements that are common to set A and set B. It is represented by the symbol $\cap$.

If x is an element in $A \cap B$, then x belongs to A as well as B.

Example 2: If $A = \{x \mid x = \frac{30}{a}, a \in N \text{ and } x \in N\}$,

$B = \{x \mid x = \frac{45}{a}, a \in N, x \in N\}$, and

$C = \{x \mid x = \frac{90}{a}, a \in N, x \in N\}$, find $A \cap B \cap C$.

In Roster form,

$A = \{\underline{1}, 2, \underline{3}, \underline{5}, 6, 10, \underline{15}, 30\}$

$B = \{\underline{1}, \underline{3}, \underline{5}, 9, \underline{15}, 45\}$

$C = \{\underline{1}, 2, \underline{3}, \underline{5}, 6, 9, 10, \underline{15}, 18, 30, 45, 90\}$

The intersection of the given sets will contain the underlined common factors.

$$A \cap B \cap C = \{1, 3, 5, 15\}$$

Intersection Facts

- $A \cap B = B \cap A$
- $(A \cap B) \cap C = A \cap (B \cap C)$
- $A \cap A = A$
- $A \cap \phi = \phi$
- $A \cap \xi = A$
- $A \cap A' = \phi$
- If $A \subset B$, then $A \cap B = A$.
- If $A \cap B = \phi$, then A and B are disjoint sets.

Try this!
If $A = \{1, 2, 5, 8\}$ and $B = \{5, 6, 7, 8\}$ find $A \cap B$.

Difference of Sets

The difference of set A and set B is a set containing all those elements in set A that do not belong to set B.

If x is an element in $A - B$, then $x \in A$, but $x \notin B$.

Example 3: If $A = \{x \mid x = \frac{18}{a}, a \in N \text{ and } x \in N\}$ and $B = \{x \mid x = \frac{24}{a}, a \in N \text{ and } x \in N\}$, find $A - B$ and $B - A$.

In Roster form, $A = \{1, 2, 3, 6, 9, 18\}$ and $B = \{1, 2, 3, 4, 6, 8, 12, 24\}$

The difference of A and B will have all the elements in A that are not in B.

As $A \cap B = \{1, 2, 3, 6\}$, $A - B = \{9, 18\}$.

The difference of B and A will have all the elements in B that are not in A.

As $B \cap A = \{1, 2, 3, 6\}$, $B - A = \{4, 8, 12, 24\}$.

Difference Fact

$A - B \neq B - A$

Try this!
If $A = \{1, 3, 10, 15\}$
$B = \{2, 4, 2, 8, 10\}$,
find $A - B$ and $B - A$.

Laws of Union and Intersection of Sets

Distributive Laws

Consider $A = \{a, b, c, d, e\}$
$B = \{c, d, e, f, g\}$
$C = \{a, c, e, g, h\}$

(i) $B \cap C = \{c, e, g\}$,

$A \cup (B \cap C) = \{a, b, c, d, e, g\}$, (1)

$A \cup B = \{a, b, c, d, e, f, g\}$ and

$A \cup C = \{a, b, c, d, e, g, h\}$

Thus $(A \cup B) \cap (A \cup C) = \{a, b, c, d, e, g\}$ (2)

Combining (1) and (2), we obtain

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(ii) $B \cup C = \{a, c, d, e, f, g, h\}$,

$A \cap (B \cup C) = \{a, c, d, e\}$ (1)

$A \cap B = \{c, d, e\}$ and $A \cap C = \{a, c, e\}$

Thus $(A \cap B) \cup (A \cap C) = \{a, c, d, e\}$ (2)

Combining (1) and (2), we obtain

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

De Morgan's Laws

Consider $A = \{a, b, c, d, e\}$
$B = \{c, d, e, f, g\}$
$C = \{a, c, e, g, h\}$
$\xi = \{a, b, c, d, e, f, g, h\}$

Then $A' = \{f, g, h\}$, $B' = \{a, b, h\}$, and $C' = \{b, d, f\}$

The complement of unions is the intersection of complements.

(i) $A \cup B = \{a, b, c, d, e, f, g\}$ and
$(A \cup B)' = \{h\}$ (1)
$A' \cap B' = \{h\}$ (2)

Thus $(A \cup B)' = A' \cap B'$

$A \cup B \cup C = \{a, b, c, d, e, f, g, h\}$ and
$(A \cup B \cup C)' = \phi$ (1)
$A' \cap B' \cap C' = \phi$ (2)

Thus $(A \cup B \cup C)' = A' \cap B' \cap C'$

The complement of intersections is the union of complements.

(ii) $A \cap B = \{c, d, e\}$ and
$(A \cap B)' = \{a, b, f, g, h\}$ (1)
$A' \cup B' = \{a, b, f, g, h\}$ (2)

Thus $(A \cap B)' = A' \cup B'$

$A \cap B \cap C = \{c, e\}$ and
$(A \cap B \cap C)' = (a, b, d, f, g, h\}$ (1)
$A' \cup B' \cup C' = \{a, b, d, f, g, h\}$ (2)

Thus $(A \cap B \cap C)' = A' \cup B' \cup C'$

Cardinal Number of Union and Intersection of Sets

1. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Example 4: Let set $A = \{3, 4, 5, 6, 7\}$
set $B = \{1, 2, 3, 4\}$
$n(A) = 5, n(B) = 4$

Thus, we observe that the sum of cardinal numbers of sets A and B = $n(A) + n(B) = 5 + 4 = 9$

The number of common elements or $n(A \cap B) = 2$

Thus, $n(A) + n(B) - n(A \cap B) = 5 + 4 - 2 = 7$
$= n(A \cup B)$

CHECK: $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$
$n(A \cup B) = 7$

2. $n(A \cap B) = n(A) + n(B) - n(A \cup B)$

Example 5: Let set $A = \{5, 6, 7, 8, 9, 10\}$
set $B = \{7, 8, 9, 10, 11\}$
$n(A) = 6, n(B) = 5.$

Thus, we observe that the sum of cardinal numbers of sets A and B = $n(A) + n(B) = 6 + 5 = 11$

The numbers of elements in the union of set A and set B

or $n(A \cup B) = 7$
$[\because A \cup B = \{5, 6, 7, 8, 9, 10, 11\}]$

Thus, $n(A) + n(B) - n(A \cup B) = 6 + 5 - 7 = 4$
$= n(A \cap B)$

CHECK: $A \cap B = \{7, 8, 9, 10\}$
$n(A \cap B) = 4$

Remember
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $(A \cup B)' = A' \cap B'$
- $(A \cup B \cup C)' = A' \cap B' \cap C'$
- $(A \cap B)' = A' \cup B'$
- $(A \cap B \cap C)' = A' \cup B' \cup C'$
- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A \cap B) = n(A) + n(B) - n(A \cup B)$

Exercise 2.1

1. Given $A = \{x \mid x$ is a letter in the word MENSURATION$\}$ and
$B = \{x \mid x$ is a letter in the word ARITHMETIC$\}$, find

(i) $A \cup B$ (ii) $A \cap B$
(iii) $A - B$ (iv) $B - A$

2. Given $\xi = \{x \mid x = \frac{72}{a}, a \in N, x \in N\}$,
$A = \{x \mid x = \frac{12}{a}, a \in N, x \in N\}$,
$B = \{x \mid x = \frac{24}{a}, a \in N, x \in N\}$, and
$C = \{x \mid x = \frac{36}{a}, a \in N, x \in N\}$,
find the following:

(i) $A \cup B$ (ii) $A \cup C$

(iii) $B \cup C$ (iv) $A \cap B$
(v) $A \cap C$ (vi) $B \cap C$
(vii) $A - B$ (viii) $A - C$
(ix) $B - C$ (x) A'
(xi) B' (xii) C'
(xiii) $(A \cup B)'$ (xiv) $(A \cap B)'$
(xv) $(A - B)'$ (xvi) $A' \cup C'$
(xvii) $B' \cap C'$ (xviii) $(A' \cup B') \cap C'$
(xix) $A' \cup B' \cup C'$ (xx) $A' \cap B' \cap C'$

3. Given $A = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$,
$B = \{3, 6, 9, 12, 15, 18, 21\}$, and
$C = \{5, 10, 15, 20, 25\}$, show that
(i) $(A \cup B) \cup C = A \cup (B \cup C)$
(ii) $(A \cap B) \cap C = A \cap (B \cap C)$

4. If $\xi = \{x \mid 2 \leq x \leq 25, x \in N\}$ in the above question, show that
(i) $(A \cup B \cup C)' = A' \cap B' \cap C'$
(ii) $(A \cap B \cap C)' = A' \cup B' \cup C'$

5. Prove the two De Morgan's laws, given
$\xi = \{x \mid x = 2a, a \leq 15, a \in N\}$,
$A = \{x \mid x = 4a, a \leq 6, a \in N\}$, and
$B = \{x \mid x = 6a, a \leq 5, a \in N\}$

6. Given $\xi = \{x \mid x$ is a letter from the first 15 letters of the English alphabet$\}$,
$A = \{a, b, c, d, e, f, g, h\}$,
$B = \{h, i, j, k, l, m, n, o\}$, and
$C = \{a, c, e, g, i, k, m, o\}$, show that
(i) $A - (B \cup C) = (A - B) \cap (A - C)$
(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

7. Given that ξ contains 25 elements and subsets A and B contain 12 and 14 elements, respectively, with 4 elements common to A and B, find the following:
(i) $n(A')$ (ii) $n(B')$
(iii) $n(A \cap B)$ (iv) $n(A \cup B)$
(v) $n(A - B)$ (vi) $n(B - A)$
(vii) $n(A \cap B)'$ (viii) $n(A \cup B)'$
(ix) $n(A - B)'$ (x) $n(B - A)'$
(xi) $n(P(A \cap B))$ (xii) $n(P(A - B))$

Revision Exercise

1. Write the sets that represent $A \cap B$, $A \cup B$, $A - B$ and $B - A$ in each of the following.
(a) $A = \{x \mid x$ is a letter in the word EXTEMPORE$\}$
$B = \{x \mid x\}$ is a letter in the word ELOCUTION$\}$
(b) $A = \{3, 6, 9, 12, 15, 18, 21, 24\}$
$B = \{6, 12, 18, 24\}$
(c) $A = \{x \mid x$ is a prime number, $2 \leq x \leq 19\}$
$B = \{x \mid x$ is an odd number, $3 \leq x \leq 19\}$

2. Given that $\xi = \{c, e, g, i, k, l, o, p\}$
$A = \{x \mid x$ is a letter in the word PICKLE$\}$
$B = \{x \mid x$ is a letter in the word CLIP$\}$
$C = \{x \mid x$ is a letter in the word CLICK$\}$, find the following
(a) $A \cap B$ (b) $A \cup C$ (c) $B \cup C$ (d) $A \cap B$
(e) $B \cap C$ (f) $A \cap C$ (g) $A \cup (B \cap C)$
(h) $A \cap (B \cup C)$ (i) A' (j) B'
(k) C' (l) $B' \cup C'$ (m) $A' \cap C'$
(n) $A \cup (C - B)$ (o) $C \cap (A - B)$ (p) $(A \cap B \cap C)$