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Rational Numbers

Any number which can be expressed in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$, is called a rational number. The set of rational numbers is denoted by Q.

$$\therefore \quad Q = \left\{ \frac{p}{q} : p, q \in I \text{ and } q \neq 0 \right\}.$$

Obviously, fractions are rational numbers. For example, $\frac{2}{3}, \frac{5}{7}, \frac{-6}{11}, \frac{-23}{35}$ are rational numbers.

The natural numbers and integers are also rational numbers because $7 = \frac{7}{1}$, $-25 = \frac{-25}{1}$, and so on.

Any terminating or recurring decimal can be written as a rational number. For example, $1.79 = \frac{179}{100}$ and $0.\dot{2} = \frac{2}{9}$ are rational numbers.

Numbers such as $\sqrt{2}, \sqrt{3}$ and π cannot be expressed as rational numbers. They are called irrationals.

Properties of rational numbers

1. The sum of two rational numbers is always a rational number.

Example $\frac{2}{7}$ and $\frac{13}{9}$ are rational numbers.

Their sum $= \frac{2}{7} + \frac{13}{9} = \frac{18+91}{63} = \frac{109}{63}$ is a rational number.

2. Two rational numbers can be added in any order.

Example $\frac{3}{5} + \frac{11}{17} = \frac{11}{17} + \frac{3}{5}$ because $\frac{3}{5} + \frac{11}{17} = \frac{51+55}{85} = \frac{106}{85}$,

$\frac{11}{17} + \frac{3}{5} = \frac{55+51}{85} = \frac{106}{85}$.

3. While adding three rational numbers, they can be grouped in any order.

Example Consider three rational numbers $\frac{1}{2}, \frac{3}{7}$ and $\frac{5}{9}$.

$$\left(\frac{1}{2} + \frac{3}{7}\right) + \frac{5}{9} = \left(\frac{7+6}{14}\right) + \frac{5}{9} = \frac{13}{14} + \frac{5}{9} = \frac{117+70}{126} = \frac{187}{126},$$

$$\frac{1}{2}+\left(\frac{3}{7}+\frac{5}{9}\right)=\frac{1}{2}+\left(\frac{27+35}{63}\right)=\frac{1}{2}+\frac{62}{63}=\frac{63+124}{126}=\frac{187}{126}.$$

$$\therefore \quad \left(\frac{1}{2}+\frac{3}{7}\right)+\frac{5}{9}=\frac{1}{2}+\left(\frac{3}{7}+\frac{5}{9}\right).$$

4. 0 is a rational number such that the sum of any rational number and 0 is the rational number itself.

Example $\frac{7}{9}+0=\frac{7}{9}, \quad 0+\frac{7}{9}=\frac{7}{9}.$

$$\therefore \quad \frac{7}{9}+0=0+\frac{7}{9}=\frac{7}{9}.$$

5. For every rational number $\frac{p}{q}$, there is a rational number $\frac{-p}{q}$ such that $\frac{p}{q}+\left(\frac{-p}{q}\right)=0=\left(\frac{-p}{q}\right)+\frac{p}{q}.$

Example $\frac{3}{5}+\frac{-3}{5}=0=\frac{-3}{5}+\frac{3}{5}.$

6. The difference of two rational numbers is also a rational number.

Example $\frac{9}{11}$ and $\frac{3}{5}$ are rational numbers.

$\frac{9}{11}-\frac{3}{5}=\frac{45-33}{55}=\frac{12}{55}$ is also a rational number.

7. The product of two rational numbers is always a rational number.

Example $\frac{2}{9}$ and $\frac{11}{13}$ are rational numbers and their product $=\frac{2}{9}\times\frac{11}{13}=\frac{22}{117}$ is also a rational number.

8. Two rational numbers can be multiplied in any order.

Example $\frac{3}{17}$ and $\frac{15}{19}$ are two rational numbers. $\frac{3}{17}\times\frac{15}{19}=\frac{45}{323}, \frac{15}{19}\times\frac{3}{17}=\frac{45}{323}.$

So, $\frac{3}{17}\times\frac{15}{19}=\frac{15}{19}\times\frac{3}{17}.$

9. While multiplying three (or more) rational numbers, they can be grouped in any order.

Example Consider the rationals $\frac{1}{2}, \frac{3}{5}$ and $\frac{-7}{11}.$

$$\left(\frac{1}{2}\times\frac{3}{5}\right)\times\left(\frac{-7}{11}\right)=\frac{3}{10}\times\left(\frac{-7}{11}\right)=\frac{-21}{110}$$

$$\text{and } \frac{1}{2}\times\left\{\frac{3}{5}\times\left(\frac{-7}{11}\right)\right\}=\frac{1}{2}\times\left(\frac{-21}{55}\right)=\frac{-21}{110}.$$

$$\therefore \quad \left(\frac{1}{2}\times\frac{3}{5}\right)\times\left(\frac{-7}{11}\right)=\frac{1}{2}\times\left\{\frac{3}{5}\times\left(\frac{-7}{11}\right)\right\}.$$

10. For any rational number $\frac{p}{q}$, we have a rational number 1 such that $\frac{p}{q} \times 1 = 1 \times \frac{p}{q} = \frac{p}{q}$.

Example Consider the rational $\frac{11}{13}$.

$$\frac{11}{13} \times 1 = \frac{11}{13}, \quad 1 \times \frac{11}{13} = \frac{11}{13}. \qquad \therefore \quad \frac{11}{13} \times 1 = 1 \times \frac{11}{13} = \frac{11}{13}.$$

11. For any rational number $\frac{p}{q}$, there is a rational number $\frac{q}{p}$ such that $\frac{p}{q} \times \frac{q}{p} = \frac{q}{p} \times \frac{p}{q} = 1$.

Example $\frac{3}{5}$ and $\frac{5}{3}$ are rationals such that $\frac{3}{5} \times \frac{5}{3} = \frac{5}{3} \times \frac{3}{5} = 1$.

12. If $\frac{p}{q}$ and $\frac{r}{s}$ be two rationals such that $\frac{r}{s} \neq 0$ then $\frac{p}{q} \div \frac{r}{s}$ is also a rational number.

Example $\frac{2}{7}$ and $\frac{3}{19}$ are two rationals.

$\frac{2}{7} \div \frac{3}{19} = \frac{2}{7} \times \frac{19}{3} = \frac{38}{21}$ is also a rational number.

13. If x and y be two rational numbers such that $x < y$ then $\frac{x+y}{2}$ is a rational number between x and y, that is, $x < \frac{x+y}{2} < y$.

Example A rational number between $\frac{1}{7}$ and $\frac{1}{3}$ is $\frac{\frac{1}{7}+\frac{1}{3}}{2} = \frac{10}{42} = \frac{5}{21}$.

$\therefore$ the rational $\frac{5}{21}$ lies between $\frac{1}{7}\left(=\frac{3}{21}\right)$ and $\frac{1}{3}\left(=\frac{7}{21}\right)$.

Irrational numbers

A number $\sqrt{a}$ (square root of a) is called an irrational number if a is positive and a is not the square of a rational number.

Examples (i) $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{\frac{7}{5}}, \frac{\sqrt{6}}{3}$ are positive irrational numbers.

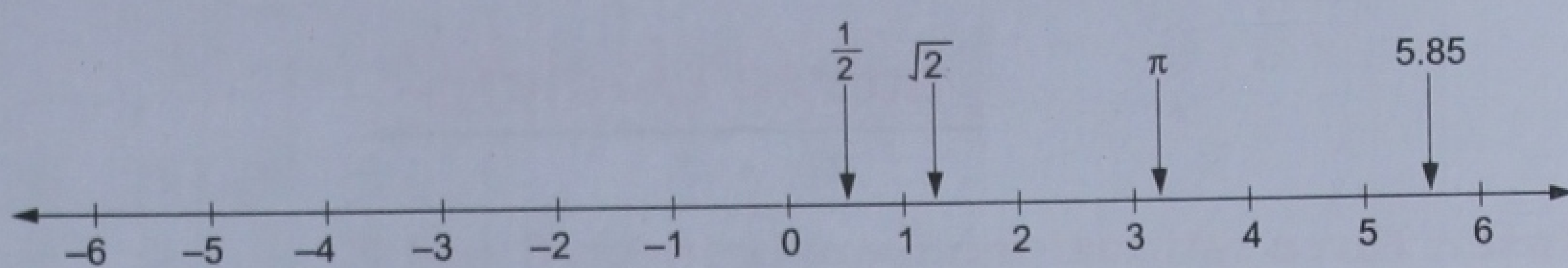
Similarly, $-\sqrt{2}, -\sqrt{3}, \frac{-\sqrt{6}}{5}$, etc., are negative irrational numbers.

But numbers such as $\sqrt{4}, \sqrt{9}, \sqrt{\frac{16}{25}}$ are not irrational because $4 = 2^2, 9 = 3^2, \frac{16}{25} = \left(\frac{4}{5}\right)^2$, that is, $4, 9, \frac{16}{25}$ are squares of rational numbers.

(ii) The solutions of the equation $x^2 = k$ are irrational numbers if k is not a perfect square.

Real numbers

The set of real numbers is the collection of rational numbers and irrational numbers.



Real number line

Properties of real numbers

1. The sum, difference and product of two real numbers are real numbers.
2. The division of a real number by a nonzero real number gives a real number.
3. Every real number has a negative real number. 0 is its own negative number.
4. The sum, difference, product and quotient of a rational number and an irrational number are irrational.

Example 2 is rational and $\sqrt{3}$ is irrational, but $2+\sqrt{3}$, $2-\sqrt{3}$, $2\sqrt{3}$ and $\frac{2}{\sqrt{3}}$ are all irrationals.

5. The sum, difference, product and quotient of two irrational numbers need not be irrational.

Example $(2+\sqrt{3})+(2-\sqrt{3}) = 4$, which is not irrational.
$(\sqrt{3}+5)-(\sqrt{3}-5) = 10$, which is not irrational.
$(3+\sqrt{2})(3-\sqrt{2}) = 9-2 = 7$, which is not irrational.
$\frac{9+3\sqrt{3}}{3+\sqrt{3}} = \frac{3(3+\sqrt{3})}{3+\sqrt{3}} = 3$, which is not irrational.

6. Given two real numbers a and b, $\sqrt{a}\times\sqrt{b} = \sqrt{ab}$, $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$ and $(\sqrt{a})^2 = \sqrt{a}\times\sqrt{a} = \sqrt{a\times a} = a$.

Rationalization

$3+\sqrt{5}$ is an irrational number. Let us multiply it by another irrational number $3-\sqrt{5}$.

$(3+\sqrt{5})(3-\sqrt{5}) = 3^2-(\sqrt{5})^2 = 9-5 = 4$, which is a rational number.

This process is called rationalization. We say that $3-\sqrt{5}$ is the rationalizing factor of $3+\sqrt{5}$. Similarly, $3+\sqrt{5}$ is the rationalizing factor of $3-\sqrt{5}$.

Note If $a+\sqrt{b}$ is an irrational number then $a-\sqrt{b}$ is the rationalizing factor of $a+\sqrt{b}$. Similarly, $a+\sqrt{b}$ is the rationalizing factor of $a-\sqrt{b}$. $a+\sqrt{b}$ and $a-\sqrt{b}$ are said to be conjugate to each other.

EXAMPLE **Rationalize the denominator of $\frac{5}{6-\sqrt{3}}$.**

Solution The given expression $= \frac{5}{6-\sqrt{3}} = \frac{5}{6-\sqrt{3}} \times \frac{6+\sqrt{3}}{6+\sqrt{3}} = \frac{5(6+\sqrt{3})}{(6+\sqrt{3})(6-\sqrt{3})}$

$= \frac{30+5\sqrt{3}}{6^2-(\sqrt{3})^2} = \frac{30+5\sqrt{3}}{36-3} = \frac{30+5\sqrt{3}}{33} = \frac{10}{11} + \frac{5}{33}\sqrt{3}.$

Solved Examples

EXAMPLE 1 **Insert three rational numbers between 6 and 7.**

Solution A rational number between 6 and 7 is $\frac{6+7}{2} = \frac{13}{2}$.

$\therefore \quad 6 < \frac{13}{2} < 7.$

A rational number between 6 and $\frac{13}{2} = \frac{6+\frac{13}{2}}{2} = \frac{\frac{12+13}{2}}{2} = \frac{\frac{25}{2}}{2} = \frac{25}{4}$.

A rational number between $\frac{13}{2}$ and $7 = \frac{\frac{13}{2}+7}{2} = \frac{\frac{13+14}{2}}{2} = \frac{27}{4}$.

$\therefore \quad 6 < \frac{25}{4} < \frac{13}{2} < \frac{27}{4} < 7.$

Hence, the required numbers are $\frac{25}{4}, \frac{13}{2}$ and $\frac{27}{4}$.

Note There are innumerable rational numbers between any two rational numbers.

EXAMPLE 2 **Simplify the following.**

(i) $7\sqrt{18} - 9\sqrt{8} + 12\sqrt{32} - 3\sqrt{50}$ **(ii) $7\sqrt{3}(\sqrt{27} + 5\sqrt{3}) - 9\sqrt{3}(\sqrt{48} - 2\sqrt{75})$**

Solution (i) The given expression $= 7\sqrt{9 \times 2} - 9\sqrt{4 \times 2} + 12\sqrt{16 \times 2} - 3\sqrt{25 \times 2}$

$= 7 \times \sqrt{9} \times \sqrt{2} - 9 \times \sqrt{4} \times \sqrt{2} + 12 \times \sqrt{16} \times \sqrt{2} - 3 \times \sqrt{25} \times \sqrt{2}$

$= 7 \times \sqrt{3 \times 3} \times \sqrt{2} - 9 \times \sqrt{2 \times 2} \times \sqrt{2} + 12 \times \sqrt{4 \times 4} \times \sqrt{2} - 3 \times \sqrt{5 \times 5} \times \sqrt{2}$

$= 7 \times 3 \times \sqrt{2} - 9 \times 2 \times \sqrt{2} + 12 \times 4 \times \sqrt{2} - 3 \times 5 \times \sqrt{2}$

$= 21\sqrt{2} - 18\sqrt{2} + 48\sqrt{2} - 15\sqrt{2} = 36\sqrt{2}.$

(ii) The given expression $= 7\sqrt{3}(\sqrt{9 \times 3} + 5\sqrt{3}) - 9\sqrt{3}(\sqrt{16 \times 3} - 2\sqrt{25 \times 3})$

$= 7\sqrt{3}(\sqrt{9} \times \sqrt{3} + 5\sqrt{3}) - 9\sqrt{3}(\sqrt{16} \times \sqrt{3} - 2 \times \sqrt{25} \times \sqrt{3})$

$= 7\sqrt{3}(\sqrt{3 \times 3} \times \sqrt{3} + 5\sqrt{3}) - 9\sqrt{3}(\sqrt{4 \times 4} \times \sqrt{3} - 2 \times \sqrt{5 \times 5} \times \sqrt{3})$

$= 7\sqrt{3}(3\sqrt{3} + 5\sqrt{3}) - 9\sqrt{3}(4\sqrt{3} - 2 \times 5\sqrt{3})$

$= 7\sqrt{3} \times 8\sqrt{3} + 9\sqrt{3} \times 6\sqrt{3}$

$= 7 \times 8 \times 3 + 9 \times 6 \times 3 = 168 + 162 = 330.$

EXAMPLE 3 **Rationalize the denominator of each of the following.**

(i) $\frac{7}{2\sqrt{3}}$ **(ii) $\frac{7}{5+\sqrt{11}}$** **(iii) $\frac{9+\sqrt{6}}{9-\sqrt{6}}$** **(iv) $\frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}-\sqrt{5}}$**

Solution (i) $\frac{7}{2\sqrt{3}} = \frac{7}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{7\sqrt{3}}{2 \times 3} = \frac{7}{6}\sqrt{3}.$

(ii) $\frac{7}{5+\sqrt{11}} = \frac{7}{5+\sqrt{11}} \times \frac{5-\sqrt{11}}{5-\sqrt{11}} = \frac{7(5-\sqrt{11})}{(5+\sqrt{11})(5-\sqrt{11})}$

$= \frac{35-7\sqrt{11}}{5^2-(\sqrt{11})^2} = \frac{35-7\sqrt{11}}{25-11} = \frac{35-7\sqrt{11}}{14} = \frac{5}{2} - \frac{1}{2}\sqrt{11}.$

(iii) $\frac{9+\sqrt{6}}{9-\sqrt{6}} = \frac{9+\sqrt{6}}{9-\sqrt{6}} \times \frac{9+\sqrt{6}}{9+\sqrt{6}} = \frac{(9+\sqrt{6})^2}{(9-\sqrt{6})(9+\sqrt{6})}$

$= \frac{81+18\sqrt{6}+6}{9^2-6} = \frac{87+18\sqrt{6}}{75} = \frac{29}{25} + \frac{6}{25}\sqrt{6}.$

(iv) $\frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}-\sqrt{5}} = \frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}-\sqrt{5}} \times \frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}+\sqrt{5}} = \frac{(\sqrt{7}+\sqrt{5})^2}{(\sqrt{7}+\sqrt{5})(\sqrt{7}-\sqrt{5})}$

$= \frac{7+5+2\sqrt{35}}{7-5} = \frac{12+2\sqrt{35}}{2} = 6+\sqrt{35}.$

EXAMPLE 4 **Arrange the following real numbers in ascending order.**

$2\sqrt{3},\ 3\sqrt{2},\ \sqrt{6},\ 5\sqrt{7},\ 9$

Solution First, we express all the numbers in the form $\sqrt{a}$.

$2\sqrt{3} = \sqrt{4} \times \sqrt{3} = \sqrt{4 \times 3} = \sqrt{12}$; $3\sqrt{2} = \sqrt{9} \times \sqrt{2} = \sqrt{9 \times 2} = \sqrt{18}$;

$\sqrt{6} = \sqrt{6}$; $5\sqrt{7} = \sqrt{25} \times \sqrt{7} = \sqrt{25 \times 7} = \sqrt{175}$; $9 = \sqrt{9 \times 9} = \sqrt{81}$.

So, the given numbers are $\sqrt{12}, \sqrt{18}, \sqrt{6}, \sqrt{175}$ and $\sqrt{81}$.

Clearly, $6 < 12 < 18 < 81 < 175$.

$\therefore \quad \sqrt{6} < \sqrt{12} < \sqrt{18} < \sqrt{81} < \sqrt{175} \Rightarrow \sqrt{6} < 2\sqrt{3} < 3\sqrt{2} < 9 < 5\sqrt{7}.$

Hence, the numbers in ascending order are $\sqrt{6}, 2\sqrt{3}, 3\sqrt{2}, 9$ and $5\sqrt{7}$.

EXAMPLE 5 **Insert two irrational numbers between $\sqrt{5}$ and $\sqrt{11}$.**

Solution $(\sqrt{5})^2 = 5$ and $(\sqrt{11})^2 = 11$.

Obviously, $5 < 6 < 7 < 11 \Rightarrow \sqrt{5} < \sqrt{6} < \sqrt{7} < \sqrt{11}$.

Hence, two irrational numbers between $\sqrt{5}$ and $\sqrt{11}$ are $\sqrt{6}$ and $\sqrt{7}$.

Note There are innumerable irrational numbers between two irrational numbers.

Remember These

1. A rational number is of the form $\frac{p}{q}$, where p, q are integers and $q \neq 0$.
2. $\sqrt{a}$ is irrational if a is positive and not the square of a rational.
3. $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$; $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$; $(\sqrt{a})^2 = \sqrt{a} \times \sqrt{a} = \sqrt{a \times a} = a$.
4. $(a+\sqrt{b})(a-\sqrt{b}) = a^2 - (\sqrt{b})^2 = a^2 - b$.
5. $a+\sqrt{b}$ and $a-\sqrt{b}$ are rationalizing factors of each other.

EXERCISE 6

1. Identify the irrational numbers in the following.

(i) $\frac{5}{-7}, \frac{0}{-127}, \sqrt{2}, 9, \sqrt{\frac{4}{9}}$ (ii) $1.35, \frac{-2}{109}, \sqrt{5}+7, \frac{9\sqrt{18}}{\sqrt{2}}$

(iii) $2\sqrt{12}-4\sqrt{3}, -1, \frac{1}{2\sqrt{11}-11\sqrt{2}}, \sqrt{\frac{36}{225}}$

(iv) $5\sqrt{3}+\sqrt{12}-\sqrt{147}, 0, 0.\dot{3}, \frac{\sqrt{48}}{\sqrt{3}}, 2+\sqrt{5}$

2. Insert a rational number between

(i) $\frac{2}{3}$ and $\frac{1}{9}$ (ii) $1\frac{3}{4}$ and 2

3. Insert

(i) two rational numbers between $\frac{5}{6}$ and $\frac{6}{5}$, (ii) three rational numbers between $\frac{3}{7}$ and $\frac{1}{11}$.

4. Evaluate the following.

(i) $12\sqrt{18}+\sqrt{32}$ (ii) $3\sqrt{125}-13\sqrt{5}$

(iii) $12\sqrt{18}\times 5\sqrt{32}$ (iv) $3\sqrt{2}(6\sqrt{8}+15\sqrt{18}-3\sqrt{50})$

(v) $28\sqrt{32}+5\sqrt{50}+16\sqrt{18}-29\sqrt{8}$ (vi) $\sqrt{3}(3\sqrt{48}+9\sqrt{27})-2\sqrt{3}(2\sqrt{75}-\sqrt{300})$

5. Rationalize the denominator of each of the following.

(i) $\frac{8}{3\sqrt{5}}$ (ii) $\frac{5}{4+\sqrt{7}}$ (iii) $\frac{3-\sqrt{5}}{3+\sqrt{5}}$ (iv) $\frac{\sqrt{17}-\sqrt{11}}{\sqrt{17}+\sqrt{11}}$

(v) $\frac{3+\sqrt{5}}{5-\sqrt{3}}$ (vi) $\frac{24\sqrt{5}}{2\sqrt{7}-1}$

6. Arrange the following in ascending order.

(i) $\sqrt{3}, 4, \sqrt{15}, 2\sqrt{2}$ (ii) $2\sqrt{3}, 5, 2\sqrt{15}, 3\sqrt{7}, 2\sqrt{6}$

(iii) $3\sqrt{11}, 5\sqrt{7}, 15\sqrt{2}, 9\sqrt{3}, 11\sqrt{5}$

7. Insert two irrational numbers between

(i) $\sqrt{2}$ and $\sqrt{7}$ (ii) $\sqrt{5}$ and $2\sqrt{2}$

ANSWERS

1. (i) $\sqrt{2}$ (ii) $\sqrt{5}+7$ (iii) $\frac{1}{2\sqrt{11}-11\sqrt{2}}$ (iv) $2+\sqrt{5}$ **2.** (i) $\frac{7}{18}$ (ii) $\frac{15}{8}$ **3.** (i) $\frac{37}{40}, \frac{61}{60}$ (ii) $\frac{53}{154}, \frac{20}{77}, \frac{27}{154}$

4. (i) $40\sqrt{2}$ (ii) $2\sqrt{5}$ (iii) 1440 (iv) 252 (v) $127\sqrt{2}$ (vi) 117

5. (i) $\frac{8}{15}\sqrt{5}$ (ii) $\frac{20}{9}-\frac{5}{9}\sqrt{7}$ (iii) $\frac{7}{2}-\frac{3}{2}\sqrt{5}$ (iv) $\frac{14}{3}-\frac{1}{3}\sqrt{187}$ (v) $\frac{15}{22}+\frac{3}{22}\sqrt{3}+\frac{5}{22}\sqrt{5}+\frac{1}{22}\sqrt{15}$
(vi) $\frac{16}{9}\sqrt{35}+\frac{8}{9}\sqrt{5}$

6. (i) $\sqrt{3}, 2\sqrt{2}, \sqrt{15}, 4$ (ii) $2\sqrt{3}, 2\sqrt{6}, 5, 2\sqrt{15}, 3\sqrt{7}$ (iii) $3\sqrt{11}, 5\sqrt{7}, 9\sqrt{3}, 15\sqrt{2}, 11\sqrt{5}$

7. (i) $\sqrt{3}, \sqrt{5}$ (ii) $\sqrt{6}, \sqrt{7}$

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