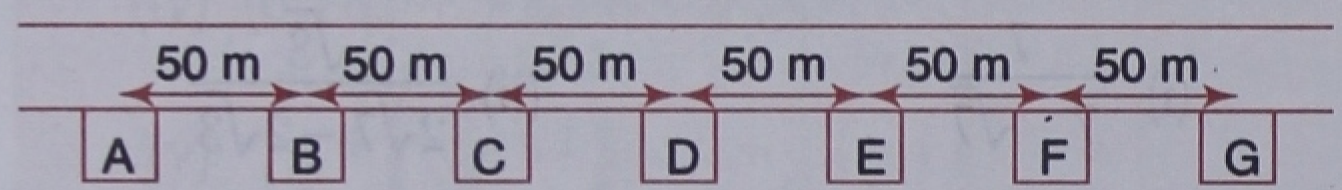
5 DIRECTED NUMBERS

- Operations Involving Directed Numbers

Introduction

Seven boys A, B, C, D, E, F, and G live in houses, 50 m from each other, on the same side of a straight road.



G's house is 300 m from A's house and F's house is 200 m from B's house. The distance of A's as well as G's house is 150 m from D's house. We can say that A stays 150 m away to D's left and G stays 150 m away to D's right.

Thus, to describe a location, along with the **distance**, the **direction** is important too. As we see from the diagram above, if D's house is considered as the **reference point**, E, F, and G's houses lie to the right and A, B, and C's houses lie to the left. If the right direction is denoted by a positive sign and the left direction by a negative sign, then from D's house,

A = – 150 m, B = – 100 m, C = – 50 m and
E = + 50 m, F = + 100 m, and G = + 150 m

If B's house is the reference point, then from B's house,

A = – 50 m, C = + 50 m, D = + 100 m,
E = + 150 m, F = + 200 m, and G = + 250 m

Each distance, along with its direction, in the above example, is known as a **directed number**. The **reference point** in each case is known as the **zero point**.

> **Remember**
> **Integers** are directed numbers, where the reference point is always the digit 0 on the number line.

Example 1: The maximum temperature recorded in the third week of May 2004 in a place is given in the following table:

Day of the Week	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Maximum temp. in °C	35	39	37	40	36	34	38

Taking the average maximum temperature during the week as the zero point, represent the temperature of each day of the week in °C above or below average.

The average maximum temperature of the week

$$= \frac{35 + 39 + 37 + 40 + 36 + 34 + 38}{7}$$

$$= \frac{259}{7} = 37\ °C = \text{zero point}$$

Day of the Week	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Above average in °C	–	+2	0	+3	–	–	+1
Below average in °C	–2	–	0	–	–1	–3	–

Operations Involving Directed Numbers

1. Positive directed numbers are numerically greater than negative directed numbers.
2. The zero point is greater than all negative numbers and less than all positive numbers.
3. Every directed number (other than 0) has another number of opposite sign, known as its additive inverse.

Addition

1. The sum of two directed numbers of the same sign is the sum of the absolute numbers with the same sign as the addends.
2. The sum of two directed numbers of different signs is the difference of the absolute numbers with sign of the addend with the greater numerical value.

Example 2: (–71) + (–44) = –115

Example 3: (+105) + (+27) = +132

Example 4: (+85) + (–99) = –14

Example 5: (+261) + (–205) = +56

Try this!

1. (+183) + (–85) =
2. (–364) + (–36) =

Subtraction

Subtraction of a directed number involves addition of its opposite or additive inverse.

Example 6: (+109) – (+54) = (+109) + (–54) = +55

Example 7: (+64) – (–46) = (+64) + (+46) = +110

Example 8: (–219) – (+29) = (–219) + (–29)
= –248

Example 9: (–102) – (–88) = (–102) + (+88)
= –14

Try this!

1. (–183) – (–85) =
2. (–364) – (–36) =

Multiplication

1. The product of two directed numbers of the same sign is positive.
2. The product of two directed numbers of different signs is negative.

Example 10: (–36) × (–5) = +180

Example 11: (+42) × (+6) = +252

Example 12: (–212) × (+3) = –636

Example 13: (+3) × (–81) = –243

Division

1. The quotient of two directed numbers of the same sign is positive.
2. The quotient of two directed numbers of different signs is negative.

Example 14: (+544) ÷ (+8) = +68

Example 15: (–623) ÷ (–7) = +89

Example 16: (+264) ÷ (–11) = –24

Example 17: (–306) ÷ (+9) = –34

Try this!

1. (–23) × (+11) =
2. (–23) × (–9) =
3. (–99) ÷ (+11) =
4. (–99) ÷ (–9) =

Simplification

1. When there is a (+) sign before a grouping symbol or bracket, the bracket may be removed without changing the sign of any number within it.

2. When there is a (–) sign before a bracket, the bracket may be removed, changing the sign of all the numbers within it.
3. When there is a directed number before a bracket, the bracket may be removed, multiplying the directed number with all the numbers within.
4. The order of operations involving different grouping symbols is as follows:

$\overline{\quad}$	vinculum	— operate first
( )	parentheses	— operate second
{ }	braces	— operate third
[ ]	square brackets	— operate fourth

5. The order of operations in simplification is as follows:

B — first operate within grouping symbols, or brackets.
O — operate on 'of' between two directed numbers by multiplying
D — divide
M — multiply
A — add
S — subtract

Example 18: Simplify $(+1) \times (-1) + (+1) \div (-1) - (-1) + (-1) \div (-1) \times (-1) + (-1)$

$= (+1) \times (-1) + (-1) - (-1) + (+1) \times (-1) + (-1)$ (first divide)

$= (-1) + (-1) - (-1) + (-1) + (-1)$ (then multiply)

$= (-2) - (-1) + (-2)$ (then add)

$= (-4) - (-1)$ (finally subtract)

$= (-4) + (+1) = -3$

Example 19: Simplify:

$\{7 \text{ of } 6 \div 2 - 4(8 \times 12 \div 3 + \overline{-3 \text{ of } +6 - 2})\} \div (-3)$

$= \{7 \text{ of } 6 \div 2 - 4(8 \times 12 \div 3 + \overline{-18 - 2})\} \div (-3)$

$= \{7 \text{ of } 6 \div 2 - 4(8 \times 12 \div 3 - 20)\} \div (-3)$

$= \{7 \text{ of } 6 \div 2 - 4(8 \times 4 - 20)\} \div (-3)$

$= \{7 \text{ of } 6 \div 2 - 4(32 - 20)\} \div (-3)$

$= \{7 \text{ of } 6 \div 2 - 4 \times 12\} \div (-3)$

$= \{42 \div 2 - 48\} \div (-3)$

$= \{21 - 48\} \div (-3)$

$= -27 \div (-3) = +9$

Try this!

1. $24 \div (-8) + 3 \times (-2)$

Exercise 5.1

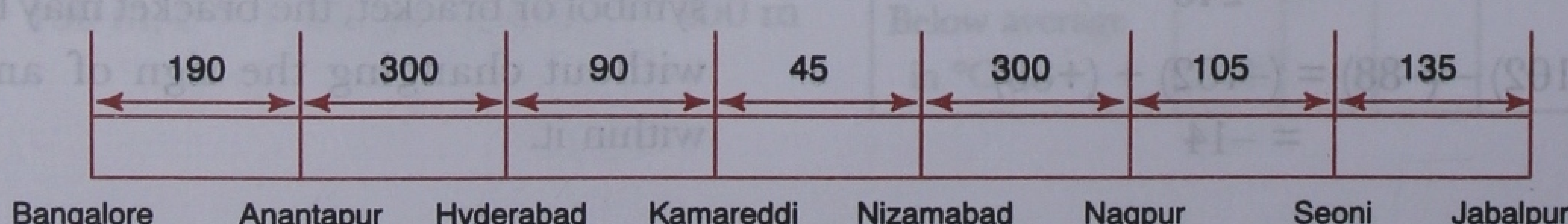
1. The height, in metres, of some mountain peaks are given in the following table:

Etna	Fujiyama	K2	Kiliminjaro	Mauna Kea	Everest	Kanchenjunga	Matterhorn
3263	3776	8611	5895	4205	8848	8598	4474

If 'higher than' and 'shorter than' are represented by (+) and (–) signs, respectively, write the directed numbers to describe the following:

(i) the height of Mt Everest if the height of K2 is the zero point
(ii) the height of Mt Etna if the height of Kanchenjunga is the point of reference
(iii) the height of Fujiyama if the height of Mauna Kea is the zero point
(iv) the height of Matterhorn if the height of Kiliminjaro is the point of reference
(v) the height of Mt Etna if the height of Mt Everest is the zero point

2. A straight long road links Bangalore to Jabalpur. The distances, in km, between sóme towns along this road are given below:



If Jabalpur, which is to the North of Bangalore, is described to be +1165 km from Bangalore, describe the following distances using directed numbers.

(i) the distance of Hyderabad from Bangalore
(ii) the distance of Nagpur from Jabalpur
(iii) the distance of Kamareddi from Seoni
(iv) the distance of Nizamabad from Anantapur
(v) the distance of Bangalore from Jabalpur

3. Every floor of a 104-storey skyscraper is 4 m high. If a balloon rises 3 m every second along its outer wall, how long will it take to rise from the 36th floor to the 96th floor?

4. Fill in the boxes with the correct directed numbers to complete the following statements.

(i) $(+3) + (-7) = (-7) + \square$

(ii) $(+8) + \{(-6) + (-3)\}$
$= \{(+8) + \square\} + (-3)$

(iii) $\{(-5) + (+8)\} + (-7) = (-5) + \{(-7) + \square\}$

(iv) $(-11) \times (+4) = (+4) \times \square$

(v) $(-13) \times \{(+2) + (-3)\}$
$= \{(-13) \times (+2)\} + \{(-13) \times \square\}$

(vi) $\{(-6) - (+7)\} \times (-1)$
$= \{(-6) \times (-1)\} - \{\square \times (+7)\}$

5. Write the opposite, or the additive inverse, of the following directed numbers:

(i) +17 (ii) –21
(iii) +154 (iv) –261
(v) –3126

6. Write the reciprocal of the following directed numbers:

(i) +8 (ii) –9
(iii) –21 (iv) +18
(v) +184

7. Add or subtract the following directed numbers:

(i) $(+218) + (+36)$
(ii) $(+314) + (-117)$
(iii) $(-105) + (+79)$
(iv) $(-168) + (-35)$
(v) $(+63) + (-27) + (-18)$
(vi) $(-21) + (+14) + (-42)$
(vii) $(+310) - (+130)$
(viii) $(+217) - (-128)$
(ix) $(-246) - (+14)$
(x) $(-129) - (-116)$

8. Multiply or divide the following directed numbers:

(i) $(+13) \times (+4)$ (ii) $(+21) \times (-6)$
(iii) $(-14) \times (+3)$ (iv) $(-17) \times (-5)$
(v) $(-12) \times (+3) \times (-1)$
(vi) $(+16) \times (-10) \times (-1)$
(vii) $(+104) \div (+8)$ (viii) $(+168) \div (-12)$
(ix) $(-294) \div (+14)$ (x) $(-468) \div (-9)$

9. Simplify the following expressions:

(i) $(+7) - (-3) + (+4) \times (-3) \div (+3)$ of (-2)
(ii) (-3) of $(-5) \div (-3) \times (-2) + (-5) - (-2) \div (+2)$
(iii) $(-7) + (-8) - (-3)$ of $(-6) \div (+2) - (-4) \times (-4) \div (+2)$
(iv) $(+24) \div (-3)$ of $(+4) - (-25) \times (-6) \div (-3 + (-15) \div (-3) \times (-10))$
(v) (-3) of $(-8) \div (-6) - (-8 + 4 - 3)$
(vi) $(-5)[(-6) - \{-5 + (-2 + 1 - \overline{3 \quad 2})\}]$
(vii) $(-3)[(-8) - \{+7 - (4 - 5 - \overline{2 \quad 5 \quad 1})\}] \div (-11)$

Challenge

Simplify:

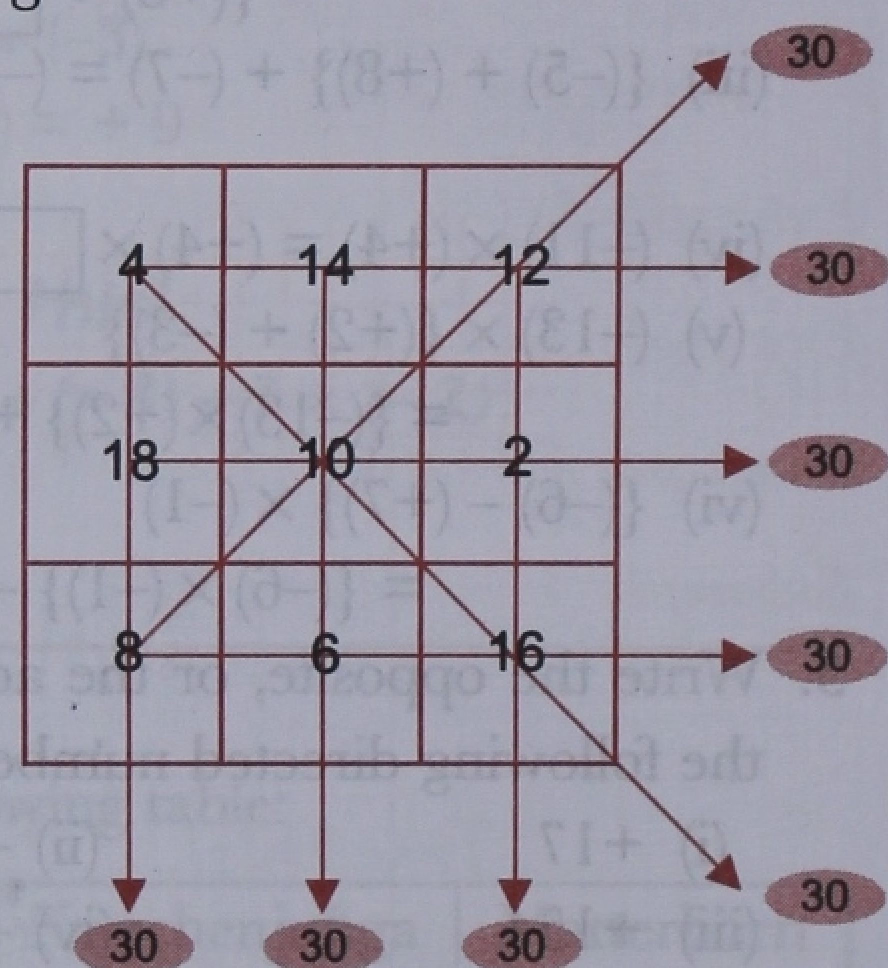
1. $(+8) \times (-3) \times (+2) \div [-1 - \{-3 + 8 - (6 - 2 - \overline{3 + 5 - 4})\}]$
2. $(+3)$ of $(-3) \div (+3) - [(+3) - \{(-3) - (-3 - \overline{3 - 3 + 3})\}]$
3. $(+32) \div (+2)$ of $(-4) \div [(-7)$ of $3 \div \{4 - 5(3 - \overline{4 \text{ of } 2 - 2 \text{ of } 5})\}]$
4. What is the result if a directed number is multiplied by the reciprocal of its additive inverse?

Revision Exercise

1. Write the opposite, or the additive inverse, of the following directed numbers:
   (i) +21 (ii) –27
   (iii) +245 (iv) –6213
2. Write the reciprocal of the following directed numbers:
   (i) +7 (ii) –15
   (iii) –46 (iv) +481
3. Add or subtract the following directed numbers:
   (i) (+812) + (+63) (ii) (+413) + (–711)
   (iii) (+36) + (–72) + (– 81) (iv) (–921) – (–611)
4. Multiply or divide the following directed numbers:
   (i) (+31) × (+8)
   (ii) (–21) × (4) × (–2)
   (iii) (+405) ÷ (–9)
   (iv) (–180) ÷ (–12)
5. Simplify the following expressions:
   (i) (–30) + (–8) ÷ (–4) × 2
   (ii) (–3) × (–6) ÷ (–2) + (–1)
   (iii) $56 \div (16 + \overline{4-6}) + (6-8)$
   (iv) $(7 + 6) \times [19 + \{(-15) + \overline{6-1}\}]$

3 x 3 magic square.

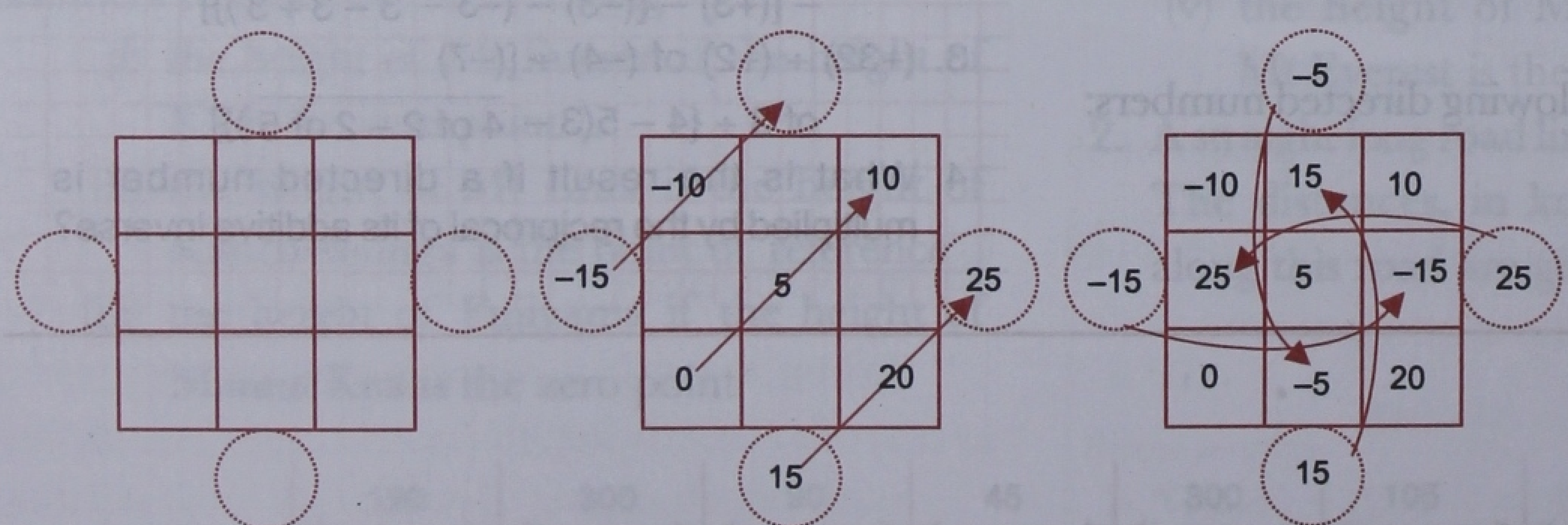
9 numbers have been arranged in a special manner in this 3 x 3 square grid. Add up the sets of three numbers horizontally, vertically or diagonally. All the sums are exactly 30. Is it magic, or simply mathematics ?!



Now observe that:
(i) One-third of 30 is 10 and the number 10 is in the centre of the magic square.
(ii) Of the remaining eight numbers, four are less than 10 and four are greater than 10.
(iii) If all the numbers are written in ascending or descending order, the difference between two numbers is the same. (2 < 4 < 6 < 8 < 10 < 12 < 14 < 16 < 18 ; common difference = 2)

Steps to construct a 3 x 3 magic square.

Let us construct a magic square in which the sets of three numbers add up to 15.
(i) Make a 3 x 3 grid and add 4 extra circles along the middle row and column. (**Step 1**)
(ii) The number in the centre of the magic square will be 15 ÷ 3 = 5. Let the common difference between two numbers be 5. Then the 9 numbers in order will be –15, –10, –5, 0, 5, 10, 15, 20, 25



Step 1 **Step 2** **Step 3**

–10	15	10
25	5	–15
0	–5	20

(iii) Write the numbers, in ascending or descending order, diagonally along the arrows as shown in **step 2**.

(iv) The numbers in the circles jump across to the opposite squares. (**Step 3**)

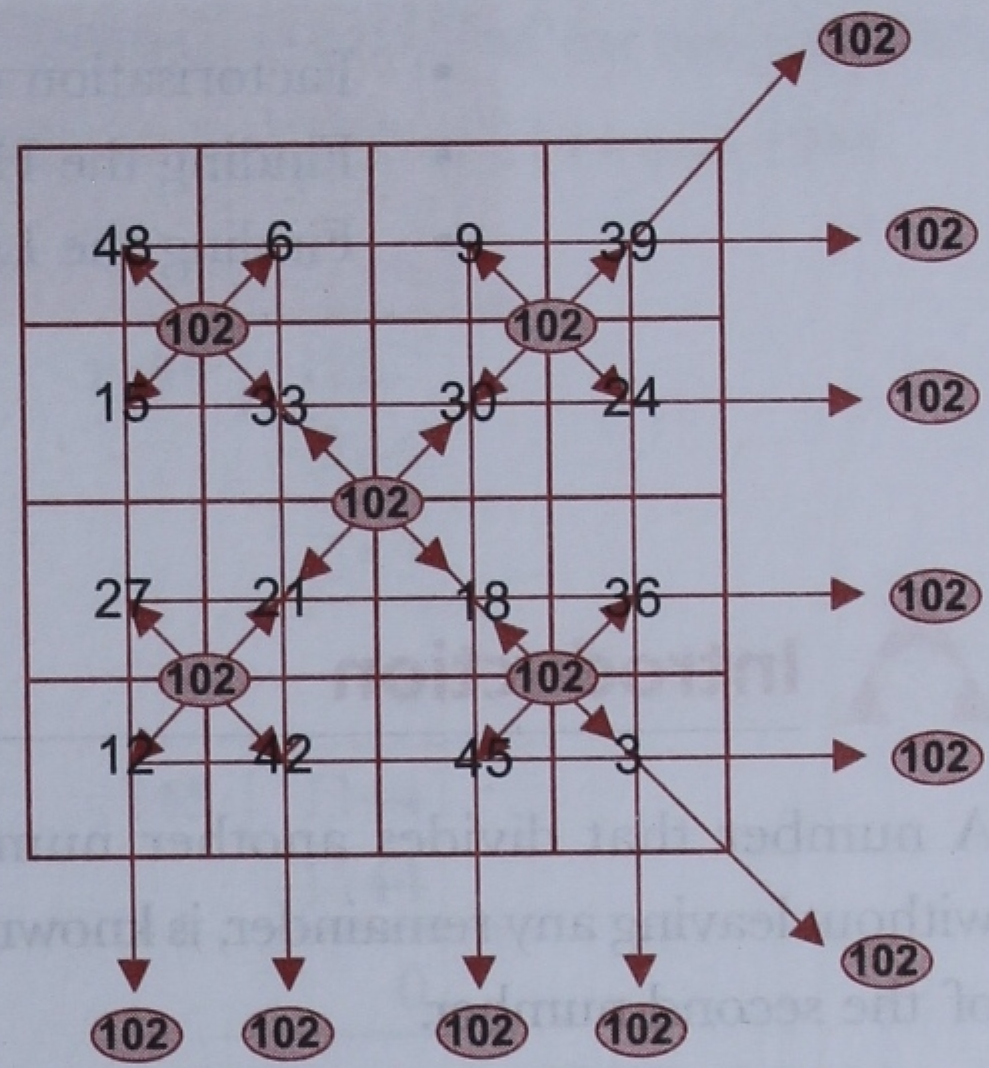
(v) The magic square is complete. Check if all the sums equal 15.

4 x 4 magic square.

16 numbers have been arranged in a special manner in this 4 x 4 square grid.

The sets of four numbers horizontally, vertically or diagonally add up to the same sum.

Look at the arrows inside the square. Even within the magic square sets of four numbers add up to result in the same sum. The sum of the four corner numbers is also the same.



Now observe that :

(i) If all the numbers are written in ascending or descending order, the difference between two numbers is the same (series of 16 numbers from 3 to 48, common difference being 3).

(ii) The sum of the smallest and the biggest number is 51 or half of the common sum 102.

(iii) The sum of the next-to smallest (6) and next-to biggest (45) is also 51 and so on.

(iv) The sum of the middle two numbers (24 and 27) is also 51.

Steps to construct a 4 x 4 magic square.

Let us construct a magic square with a series of 16 number with a common difference of 4 beginning with, say 10.

(i) The series in ascending order would be: 10, 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, 58, 62, 66, 70.

(ii) Write the numbers in the grid in order: **(Step 1)**

(iii) Reverse the order of the numbers written along the diagonals of the magic square. (**Step 2**)

(iv) The magic square is complete. Check if all the sums equal 160.

10	14	18	22
26	30	34	38
42	46	50	54
58	62	66	70

Step 1

70	14	18	58
26	50	46	38
42	34	30	54
22	62	66	10

Step 2

70	14	18	58
26	50	46	38
42	34	30	54
22	62	66	10