28 QUADRILATERALS

- Quadrilaterals
- Types of Quadrilaterals
- Properties of Quadrilaterals

Introduction

Four non-collinear co-planar points A, B, C, and D joined by four line segments form quadrilateral ABCD. The eight elements of a quadrilateral are its four sides and four angles.

In Figure 28.1, A, B, C, and D are the vertices of quadrilateral ABCD. ∠1, ∠2, ∠3, and ∠4 are its four interior angles. AB, BC, CD, and DA are its four sides. ∠5, ∠6, ∠7, and ∠8 are its four exterior angles. AC and BD, which connect the opposite vertices, are known as its diagonals.

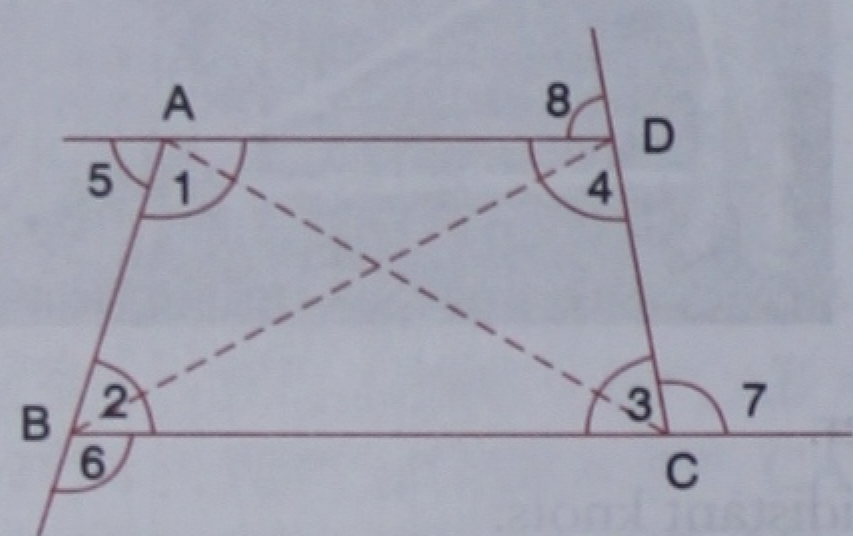


Fig. 28.1

Sum of Angles of a Quadrilateral

1. *To prove:* The sum of the interior angles of a quadrilateral is 360°.

   Proof: Draw a quadrilateral ABCD and connect its opposite vertices B and D (Figure 28.2).

   In ΔABD $\angle 1 + \angle 2 + \angle 3 = 180°$ (sum of interior angles)

   In ΔBCD $\angle 4 + \angle 5 + \angle 6 = 180°$ (sum of interior angles)

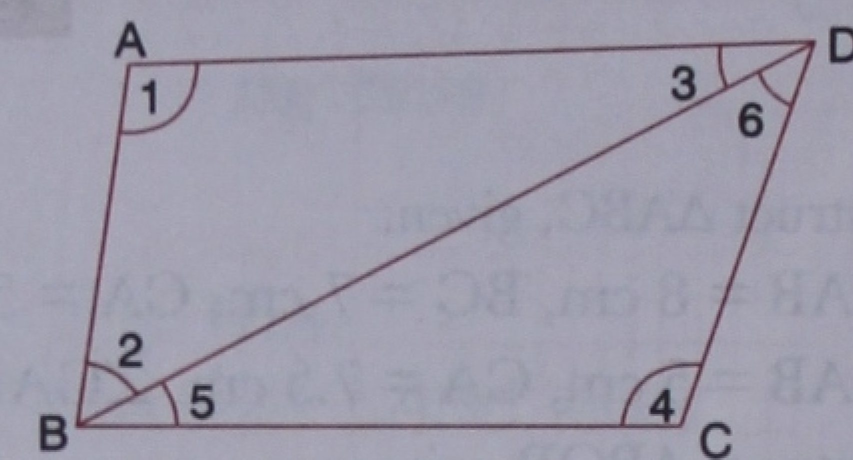


Fig. 28.2

   ⇒ In quadrilateral ABCD,

   $\angle 1 + (\angle 2 + \angle 5) + \angle 4 + (\angle 3 + \angle 6)$

   $= 180° + 180°$ (adding equations)

   ⇒ Sum of interior angles of quadrilateral ABCD = 360° Q.E.D

2. *To prove:* The sum of the exterior angles of a quadrilateral is 360°.

   Proof: Extend sides AB, BC, CD, and DA of quadrilateral ABCD to form exterior angles, ∠1, ∠2, ∠3, ∠4 respectively (Figure 28.3).

   Now $\angle 1 + \angle 5 = 180°$ (linear pair)

   ⇒ $\angle 1 = 180° - \angle 5$

   Similarly $\angle 2 = 180° - \angle 6$, $\angle 3 = 180° - \angle 7$

   and $\angle 4 = 180° - \angle 8$

   ⇒ $\angle 1 + \angle 2 + \angle 3 + \angle 4$

   $= 180° - \angle 5 + 180° - \angle 6 + 180° - \angle 7 + 180° - \angle 8$

   (adding the four equations)

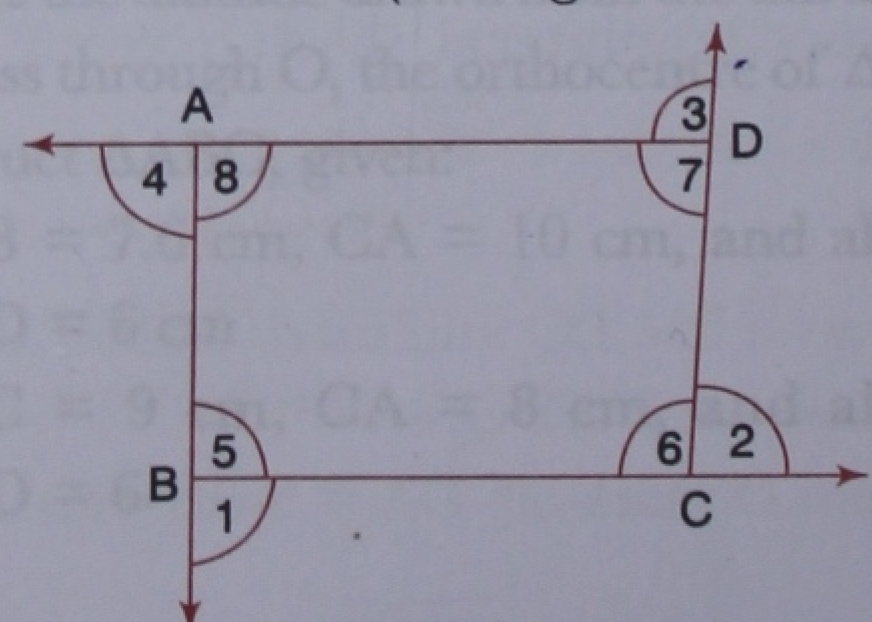


Fig. 28.3

$\Rightarrow \angle 1 + \angle 2 + \angle 3 + \angle 4$

$= 720° - (\angle 5 + \angle 6 + \angle 7 + \angle 8)$

$\Rightarrow$ Sum of exterior angles

$= 720°$ – Sum of interior angles

$\Rightarrow$ Sum of exterior angles

$= 720° - 360° = 360°$ Q.E.D

Alternatively,

$\angle 1 + \angle 5 = 180°$
$\angle 2 + \angle 6 = 180°$
$\angle 3 + \angle 7 = 180°$
$\angle 4 + \angle 8 = 180°$ } all linear pairs

$\angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5 + \angle 6 + \angle 7 + \angle 8 = 720°$

But $\angle 5 + \angle 6 + \angle 7 + \angle 8 = 360°$ (sum of interior angles)

$\therefore \angle 1 + \angle 2 + \angle 3 + \angle 4 + 360° = 720°$

or $\angle 1 + \angle 2 + \angle 3 + \angle 4 = 360°$

or Sum of exterior angles = 360°

Types of Quadrilaterals and Their Properties

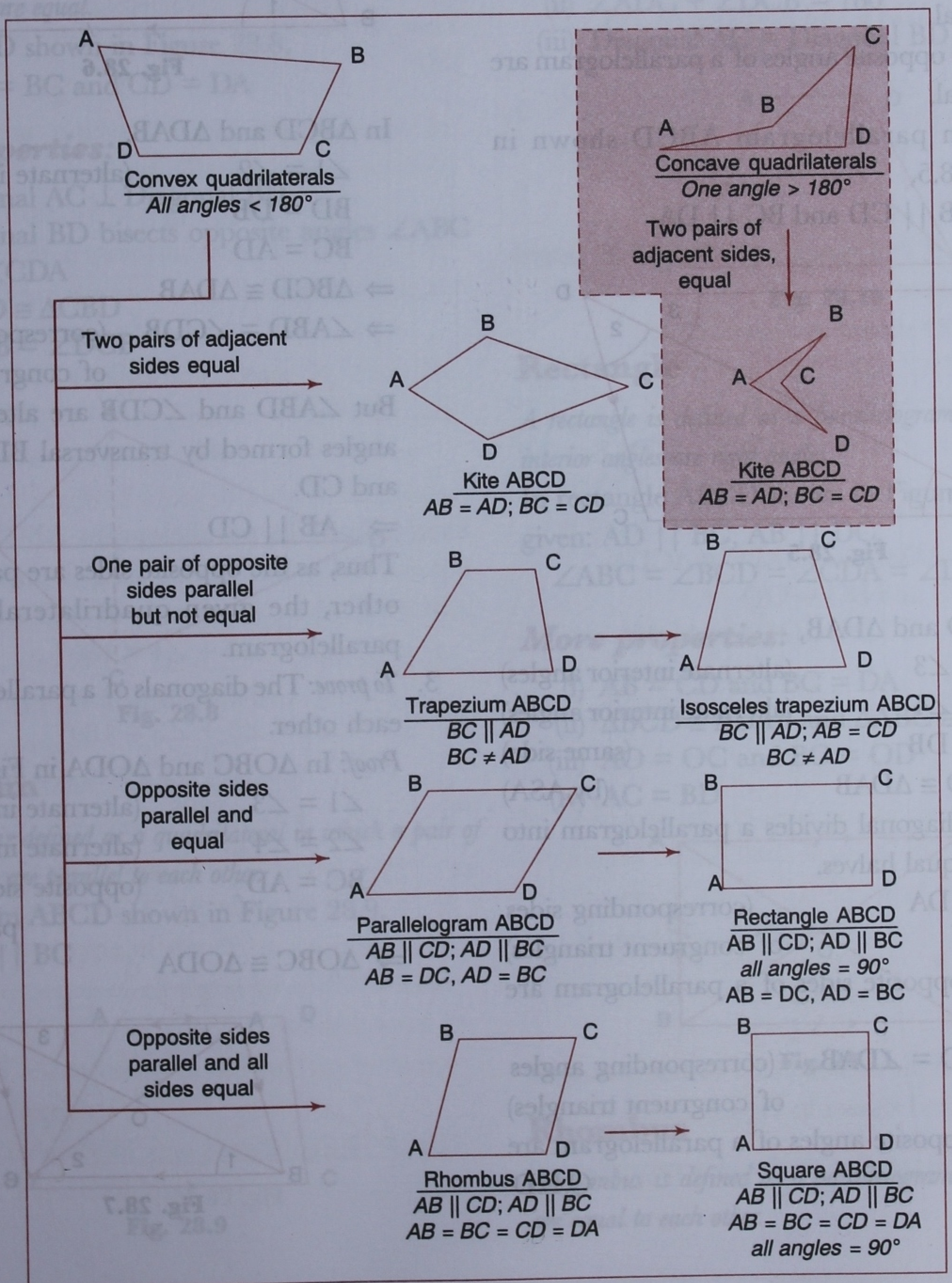


Fig. 28.4

Additional Properties of Some Quadrilaterals

From the properties that define a quadrilateral, we can derive some additional properties.

Parallelogram

A parallelogram is defined as a quadrilateral in which the opposite sides are parallel.

1. *To prove:*
   (i) The diagonal divides a parallelogram into two equal halves.
   (ii) The opposite sides of a parallelogram are equal.
   (iii) The opposite angles of a parallelogram are equal.

   Proof: In parallelogram ABCD shown in Figure 28.5,
   given: AB || CD and BC || DA

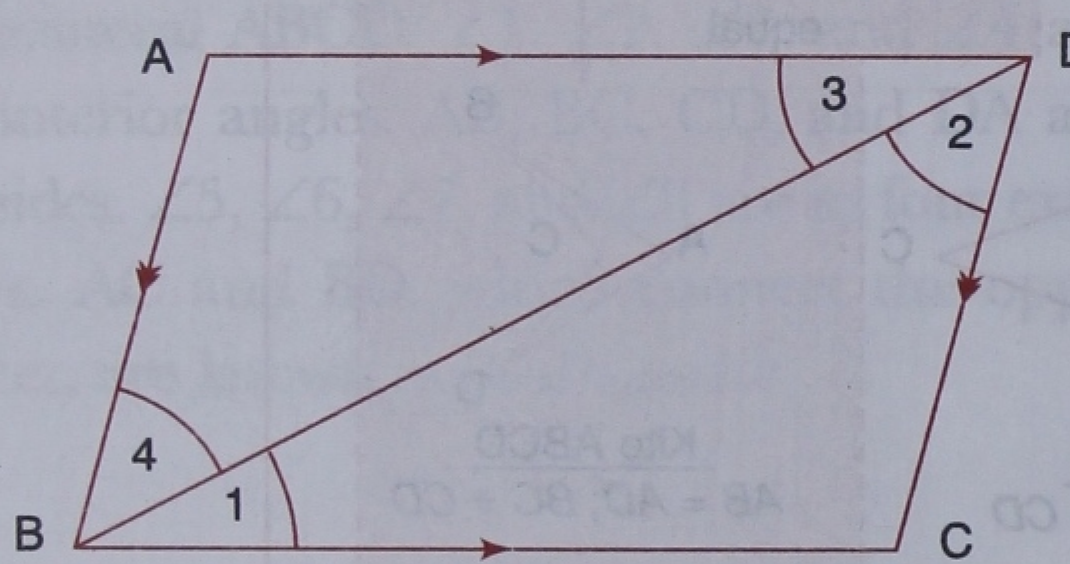


Fig. 28.5

In ΔBCD and ΔDAB,

$\angle 1 = \angle 3$ (alternate interior angles)

$\angle 2 = \angle 4$ (alternate interior angles)

BD = DB (same side)

$\Rightarrow \Delta BCD \cong \Delta DAB$ (by ASA)

$\Rightarrow$ The diagonal divides a parallelogram into two equal halves.

BC = DA (corresponding sides of congruent triangles)

$\Rightarrow$ The opposite sides of a parallelogram are equal.

$\angle BCD = \angle DAB$ (corresponding angles of congruent triangles)

$\Rightarrow$ The opposite angles of a parallelogram are equal. Q.E.D

2. *To prove:* If one pair of opposite sides in a quadrilateral are given to be parallel and equal, the quadrilateral is a parallelogram.

   Proof: In quadrilateral ABCD shown in Figure 28.6,
   given: BC || AD and BC = AD

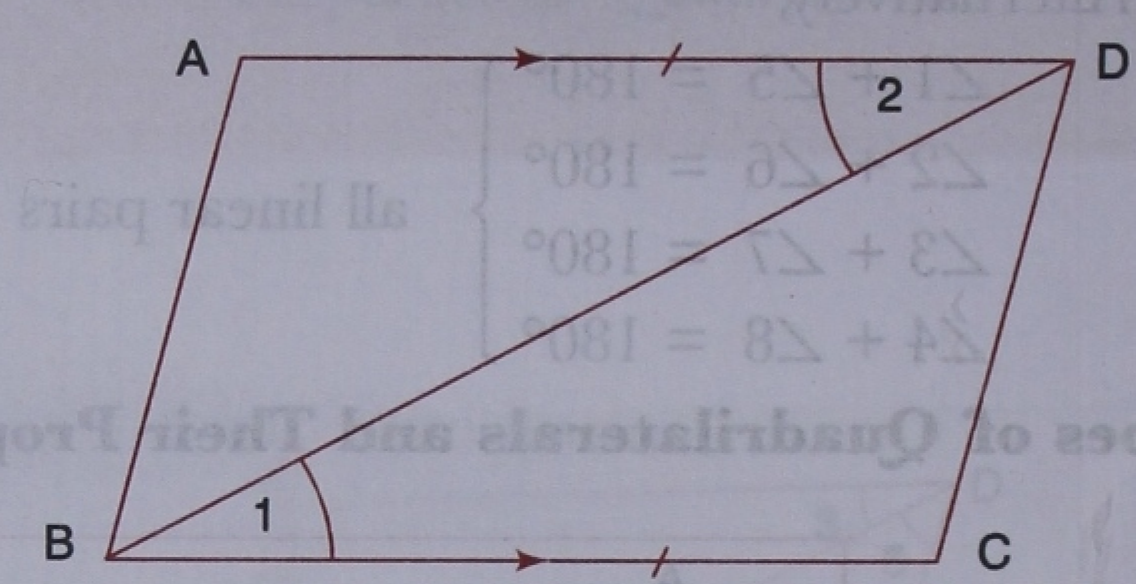


Fig. 28.6

In ΔBCD and ΔDAB,

$\angle 1 = \angle 2$ (alternate interior angles)

BD = DB (same side)

BC = AD (given)

$\Rightarrow \Delta BCD \cong \Delta DAB$ (by SAS)

$\Rightarrow \angle ABD = \angle CDB$ (corresponding angles of congruent triangles)

But $\angle ABD$ and $\angle CDB$ are alternate interior angles formed by transversal BD on lines AB and CD.

$\Rightarrow$ AB || CD

Thus, as the opposite sides are parallel to each other, the given quadrilateral ABCD is a parallelogram. Q.E.D

3. *To prove:* The diagonals of a parallelogram bisect each other.

   Proof: In ΔOBC and ΔODA in Figure 28.7,

$\angle 1 = \angle 3$ (alternate interior angles)

$\angle 2 = \angle 4$ (alternate interior angles)

BC = AD (opposite sides of a parallelogram)

$\Rightarrow \Delta OBC \cong \Delta ODA$ (by ASA)

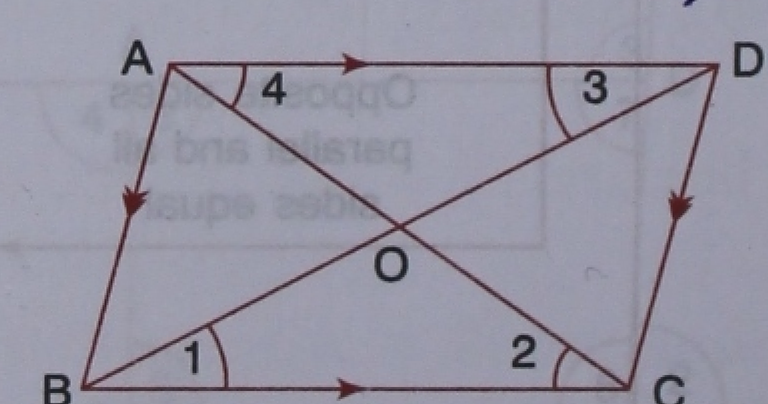


Fig. 28.7

$\Rightarrow$ OB = OD and AO = CO (corresponding sides of congruent triangles)

Thus, O is the mid-point of diagonals AC and BD or the diagonals of a parallelogram bisect each other.

Q.E.D

Similarly, more properties can be derived for other types of quadrilaterals.

Kite

A kite is defined as a quadrilateral in which two pairs of adjacent sides are equal.

In kite ABCD shown in Figure 28.8,

given: AB = BC and CD = DA

More properties:

(i) Diagonal AC $\perp$ Diagonal BD

(ii) Diagonal BD bisects opposite angles $\angle$ABC and $\angle$CDA

(iii) $\Delta ABD \cong \Delta CBD$

(iv) $\angle$DAB = $\angle$DCB

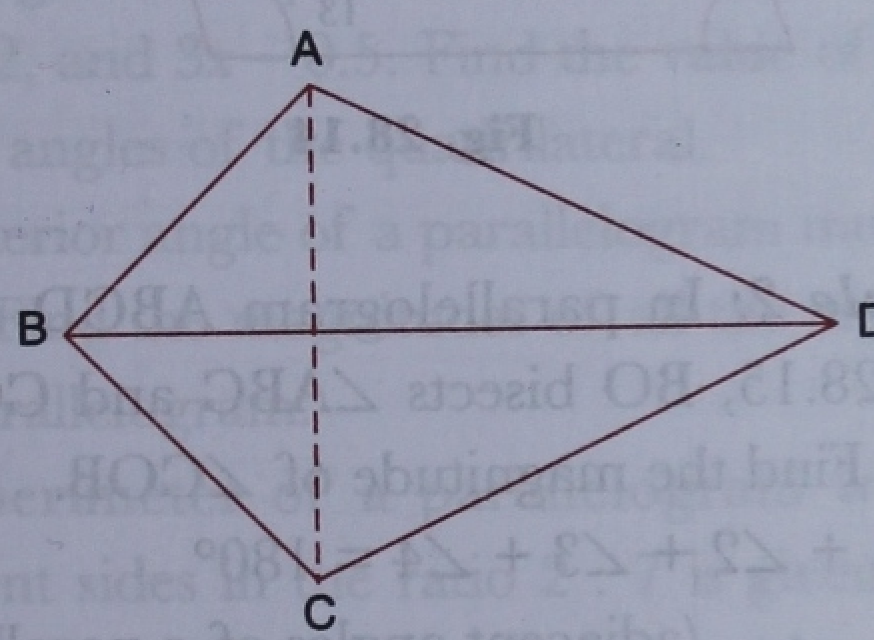


Fig. 28.8

Trapezium

A trapezium is defined as a quadrilateral in which a pair of opposite sides are parallel to each other.

In trapezium ABCD shown in Figure 28.9,

given: AD || BC

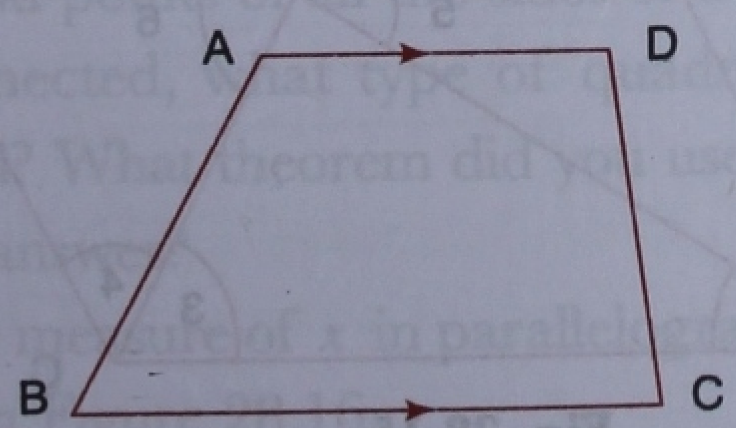


Fig. 28.9

More properties:

(i) $\angle$DAB + $\angle$ABC = 180°

(ii) $\angle$ADC + $\angle$DCB = 180°

Isosceles Trapezium

An isosceles trapezium is defined as a trapezium in which the non-parallel sides are equal to each other.

In trapezium ABCD shown in Figure 28.10,

given: AD || BC and AB = DC

More properties:

(i) $\angle$DAB + $\angle$ABC = 180°

(ii) $\angle$ADC + $\angle$DCB = 180°

(iii) Diagonal AC = Diagonal BD

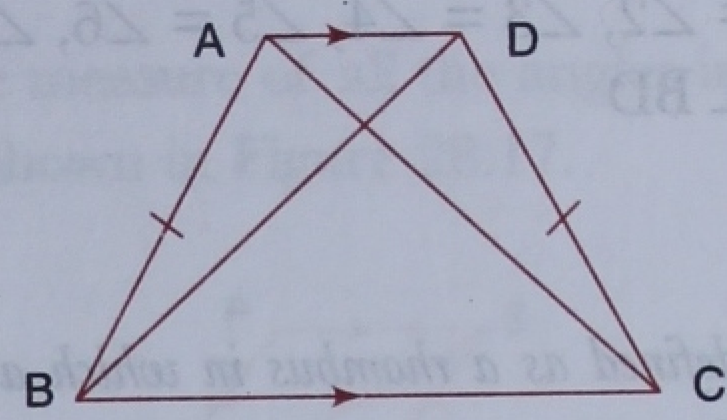


Fig. 28.10

Rectangle

A rectangle is defined as a parallelogram in which all the interior angles are right angles.

In rectangle ABCD shown in Figure 28.11,

given: AD || BC, AB || DC,

$\angle$ABC = $\angle$BCD = $\angle$CDA = $\angle$DAB = 90°

More properties:

(i) AB = CD and BC = DA

(ii) $\Delta BCD \cong \Delta DAB$ and $\Delta ABC \cong \Delta CDA$

(iii) AO = OC and BO = OD

(iv) AC = BD

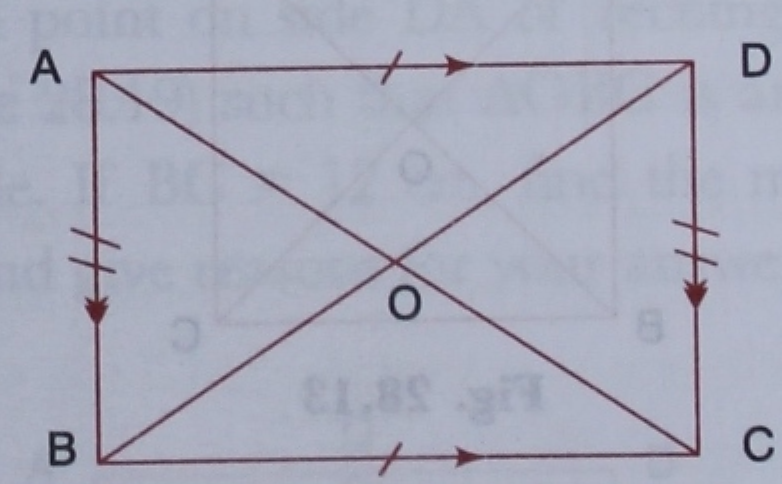


Fig. 28.11

Rhombus

A rhombus is defined as a parallelogram in which all sides are equal to each other.

In rhombus ABCD shown in Figure 28.12,
given: AD || BC, AB || DC,

AB = BC = CD = DA

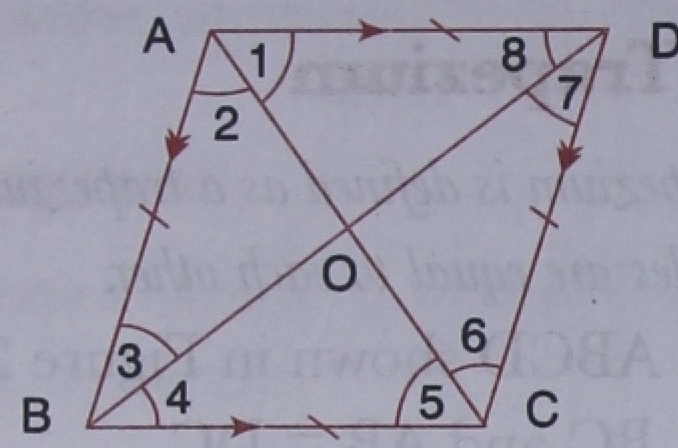


Fig. 28.12

More properties:

(i) ΔBCD ≅ ΔDAB and ΔABC ≅ ΔCDA
(ii) AO = OC and BO = OD
(iii) ∠1 = ∠2, ∠3 = ∠4, ∠5 = ∠6, ∠7 = ∠8
(iv) AC ⊥ BD

Square

A square is defined as a rhombus in which all the interior angles are right angles.

In square ABCD shown in Figure 28.13,
given: AB || DC, AD || BC,

AB = BC = CD = DA,

∠ABC = ∠BCD = ∠CDA = ∠DAB = 90°

More properties:

(i) ΔBCD ≅ ΔDAB and ΔABC ≅ ΔCDA
(ii) AO = OC and BO = OD
(iii) ΔOBC ≅ ΔOCD ≅ ΔODA ≅ ΔOAB
(iv) AC ⊥ BD
(v) AC = BD

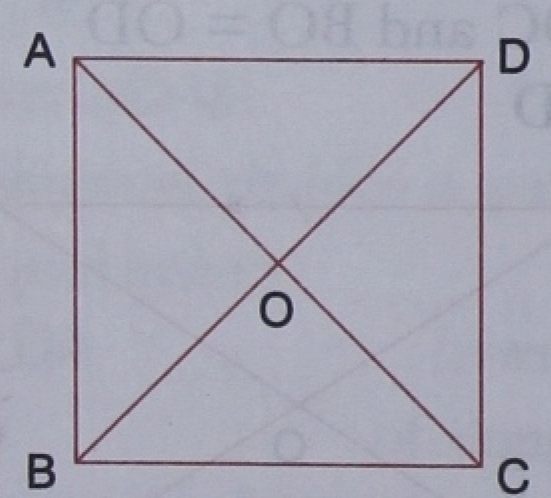


Fig. 28.13

Example 1: Find the measure of *x* and all the angles in the quadrilateral shown in Figure 28.14.

As the sum of the interior angles of a quadrilateral is 360°,

$$x + 2x - 10 + x + 5 + \frac{21x}{13} = 360°$$

$$\Rightarrow 4x + \frac{21x}{13} = 360° + 5 \Rightarrow \frac{52x + 21x}{13} = 365$$

$$\Rightarrow 73x = 365 \times 13 \Rightarrow x = \frac{4745}{73} = 65°$$

$$\Rightarrow 2x - 10 = (65 \times 2) - 10 = 120°$$

Similarly, more properties can be derived for other

$$x + 2x - 10 + x + 5 + \frac{21x}{13} = 360°$$

$$\Rightarrow x + 5 = 65 + 5 = 70°$$

$$\Rightarrow \frac{21x}{13} = \frac{21 \times 65}{13} = 105°$$

Thus, the angles of the given quadrilateral are 65°, 120°, 70°, and 105°.

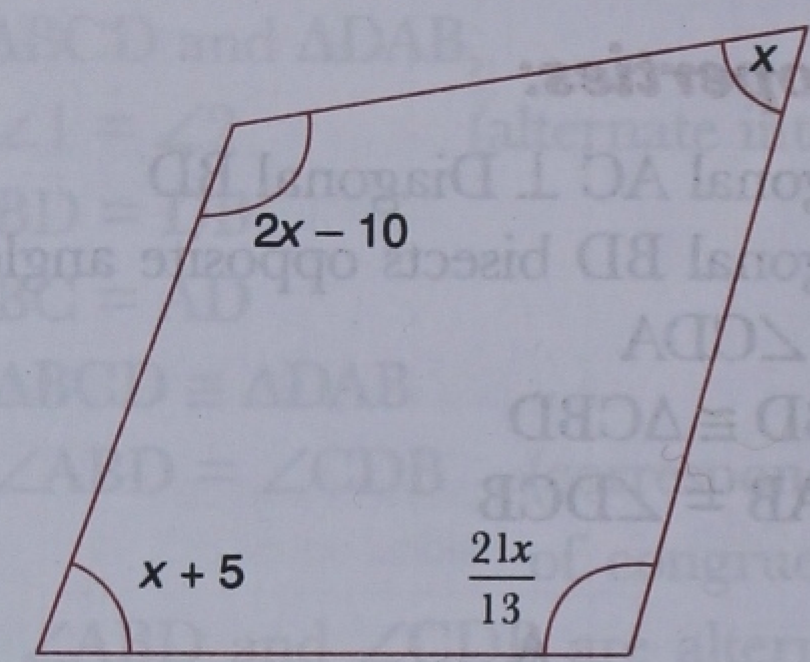


Fig. 28.14

Example 2: In parallelogram ABCD shown in Figure 28.15, BO bisects ∠ABC and CO bisects ∠BCD. Find the magnitude of ∠COB.

$$\angle 1 + \angle 2 + \angle 3 + \angle 4 = 180°$$

(adjacent angles of a parallelogram)

$$\Rightarrow \angle 2 + \angle 2 + \angle 3 + \angle 3 = 180°$$

(as ∠1 = ∠2 and ∠3 = ∠4 is given)

$$\Rightarrow 2(\angle 2 + \angle 3) = 180°$$

$$\Rightarrow \angle 2 + \angle 3 = \frac{180°}{2} = 90°$$

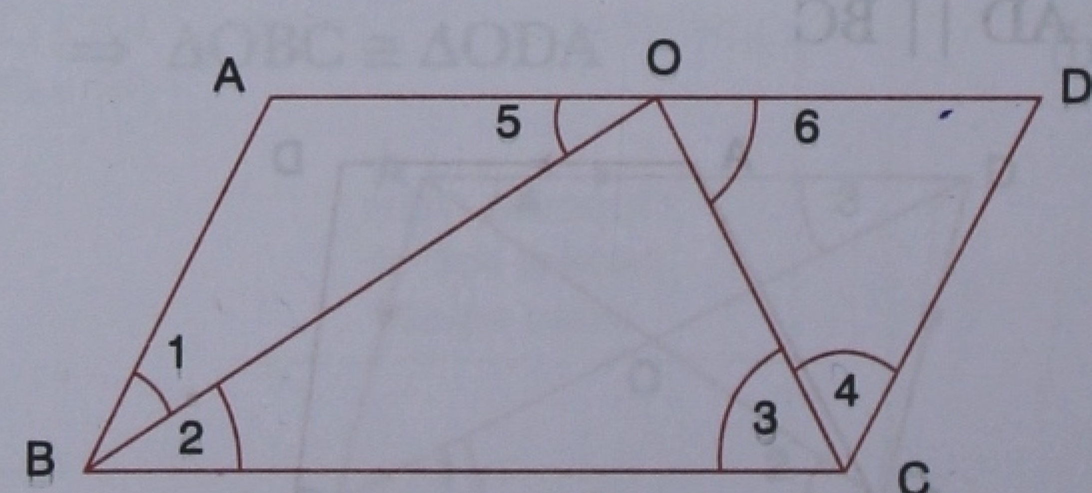


Fig. 28.15

Now $\angle COB + \angle 2 + \angle 3 = 180°$

(sum of angles in a triangle)

$\Rightarrow \quad \angle COB + 90° = 180°$

$\Rightarrow \quad \angle COB = 180° - 90° = 90°$

Try this!

If one angle of a quadrilateral is 97° and two other angles are equal to this angle, find the fourth angle.

Exercise 28.1

1. The angles of a quadrilateral are in the ratio 11 : 18 : 21 : 22. Find all the angles of the quadrilateral.
2. An angle in a quadrilateral measures 72°. If the other three angles are in the ratio 5 : 6 : 7, find all the angles of the quadrilateral.
3. The adjacent angles in a parallelogram are in the ratio 1 : 4. Find all the angles of the parallelogram.
4. The angles of a quadrilateral are $7x$, $10x$, $11x$, and $12x$. Find the value of x and all the angles of the quadrilateral.
5. The angles of a quadrilateral are $2x$, $2x + 15$, $4x - 12$, and $3x - 0.5$. Find the value of x and all the angles of the quadrilateral.
6. An exterior angle of a parallelogram measures 117°. Find the magnitudes of all the angles of the parallelogram.
7. The perimeter of a parallelogram with its adjacent sides in the ratio 2 : 7 is given to be 54 cm. Find the measure of the sides of the parallelogram.
8. In isosceles ΔABC where AB = AC, an altitude is drawn from vertex C to point X on side AB and another altitude is drawn from vertex B to point Y on side AC. If the altitudes intersect at point O, what type of quadrilateral is AXOY?
9. If the mid-points of all the sides of a kiteshape are connected, what type of quadrilateral is obtained? What theorem did you use to arrive at your answer?
10. Find the measure of x in parallelogram ABCD shown in Figure 28.16.

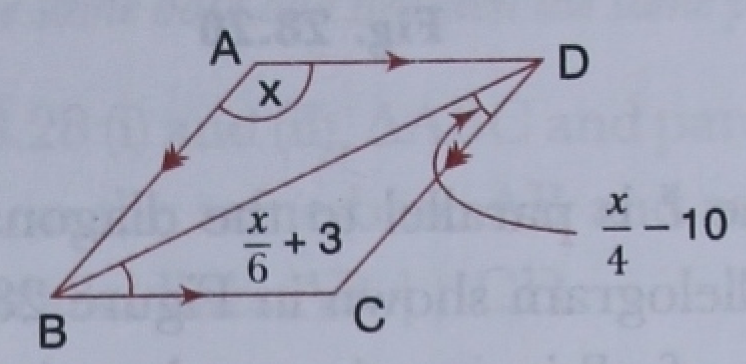


Fig. 28.16

11. Find the measure of all the angles in rhombus PQRS shown in Figure 28.17.

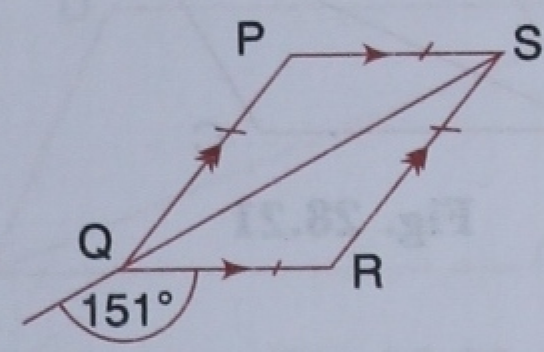


Fig. 28.17

12. Find the measure of x in rectangle ABCD shown in Figure 28.18.

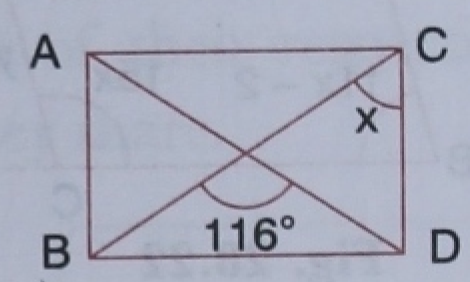


Fig. 28.18

13. O is a point on side DA of rectangle ABCD (Figure 28.19) such that ΔOBC is an isosceles triangle. If BC = 12 cm, find the measure of OD and give reasons for your answer.

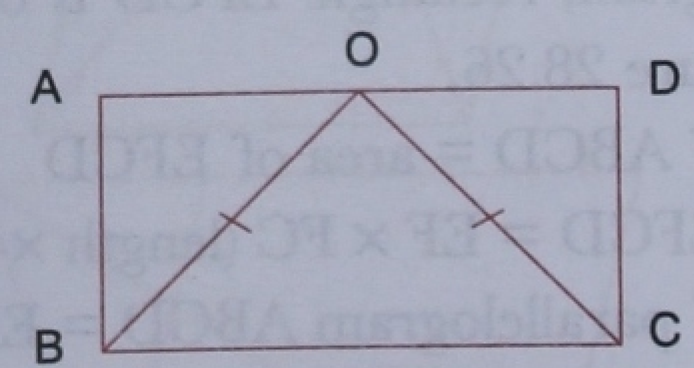


Fig. 28.19

14. In parallelogram ABCD shown in Figure 28.20, BO bisects $\angle ABC$. If $\angle AOB = 40°$, find all the angles of the parallelogram.

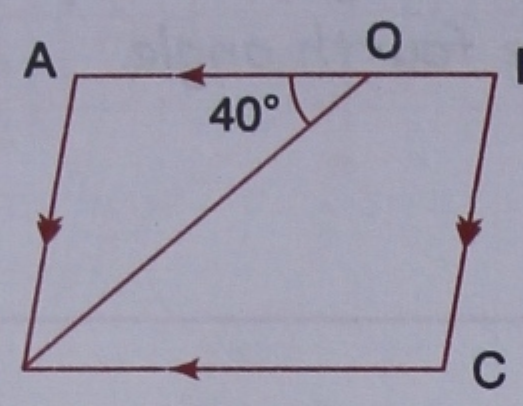


Fig. 28.20

15. If line ℓ is parallel to the diagonal BD in the parallelogram shown in Figure 28.21, find the value of all its interior angles, given $\angle CDA = 5x - 15$.

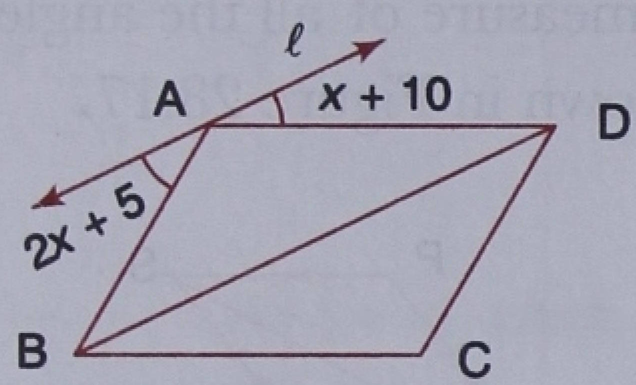


Fig. 28.21

16. DE is the altitude of parallelogram ABCD (Figure 28.22), such that $\angle CED = 90°$. Find the value of y.

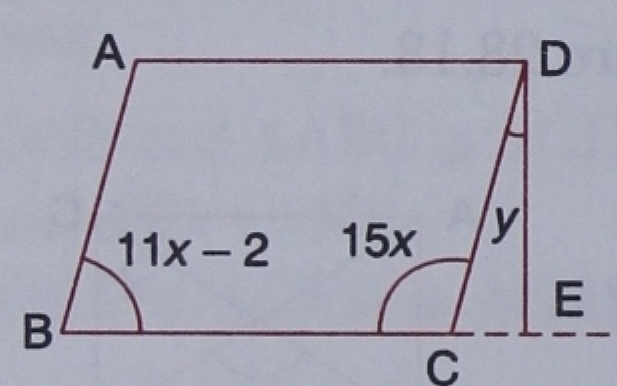


Fig. 28.22

17. If the opposite sides of a quadrilateral are equal, does the figure have to be a parallelogram? Why?

18. In parallelogram ABCD (Figure 28.23), BX = DY. Is AXCY a parallelogram too? Why?

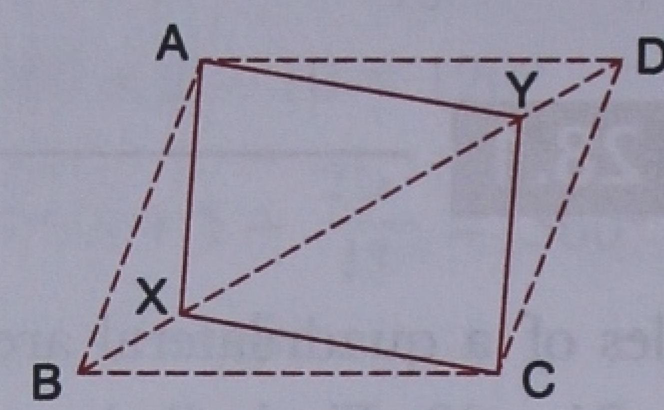


Fig. 28.23

19. In parallelogram ABCD (Figure 28.24), $AX \perp BD$ and $CY \perp DB$. If the measure of line segment BX = 3 cm, what is the measure of line segment DY and why?

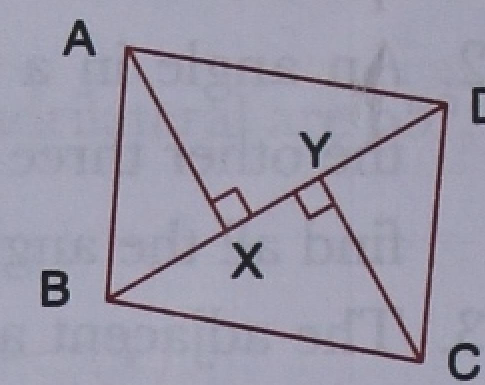


Fig. 28.24

20. In parallelogram ABCD (Figure 28.25) P is a point on AD that is joined to point O, the intersection of its two diagonals, and extended to intersect the opposite side BC at point Q. If line segment PO measures 6 cm, find the measure of OQ and give reasons for your answer.

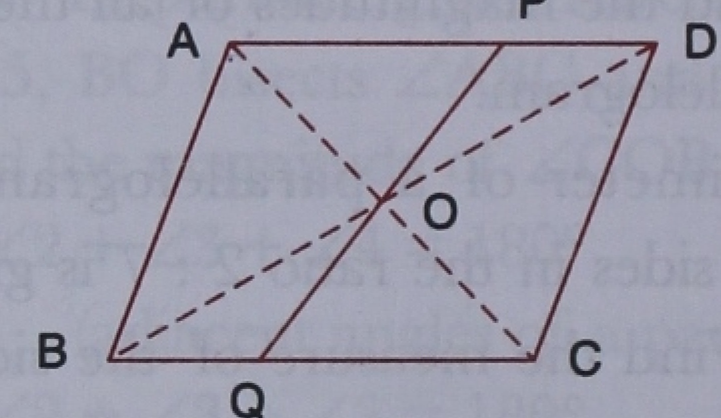


Fig. 28.25

Area Propositions on Parallelograms

If right-angled ΔDAE is cut off from parallelogram ABCD and its hypotenuse DA affixed to side BC of the parallelogram, rectangle EFCD is obtained as shown in Figure 28.26.

Thus, area of ABCD = area of EFCD

But area of EFCD = EF × FC (length × breadth)

Thus, area of parallelogram ABCD = EF × FC

But AB – AE + BF = EF (as AE = BF)

$\Rightarrow$ AB = EF

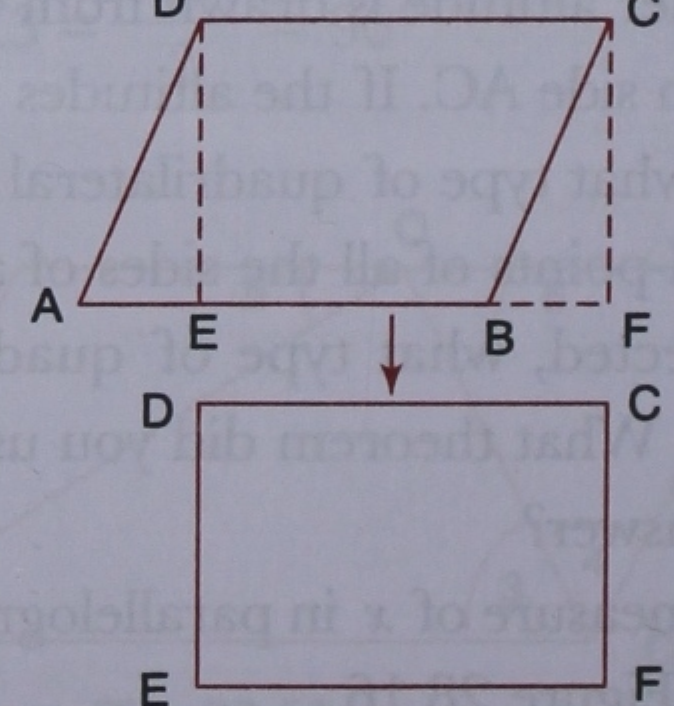


Fig. 28.26

Thus, area of parallelogram ABCD
= AB × FC or length × altitude

Case I: Parallelograms on the same base and between the same parallel lines are equal in area.

In Figures 28.27(i) and (ii), parallelograms ABCD and EBCF rest on the same base BC and are between the same parallel lines or BC || AF.

Area of parallelogram ABCD = BC × DH
Area of parallelogram BCFE = BC × EG
But EG = DH

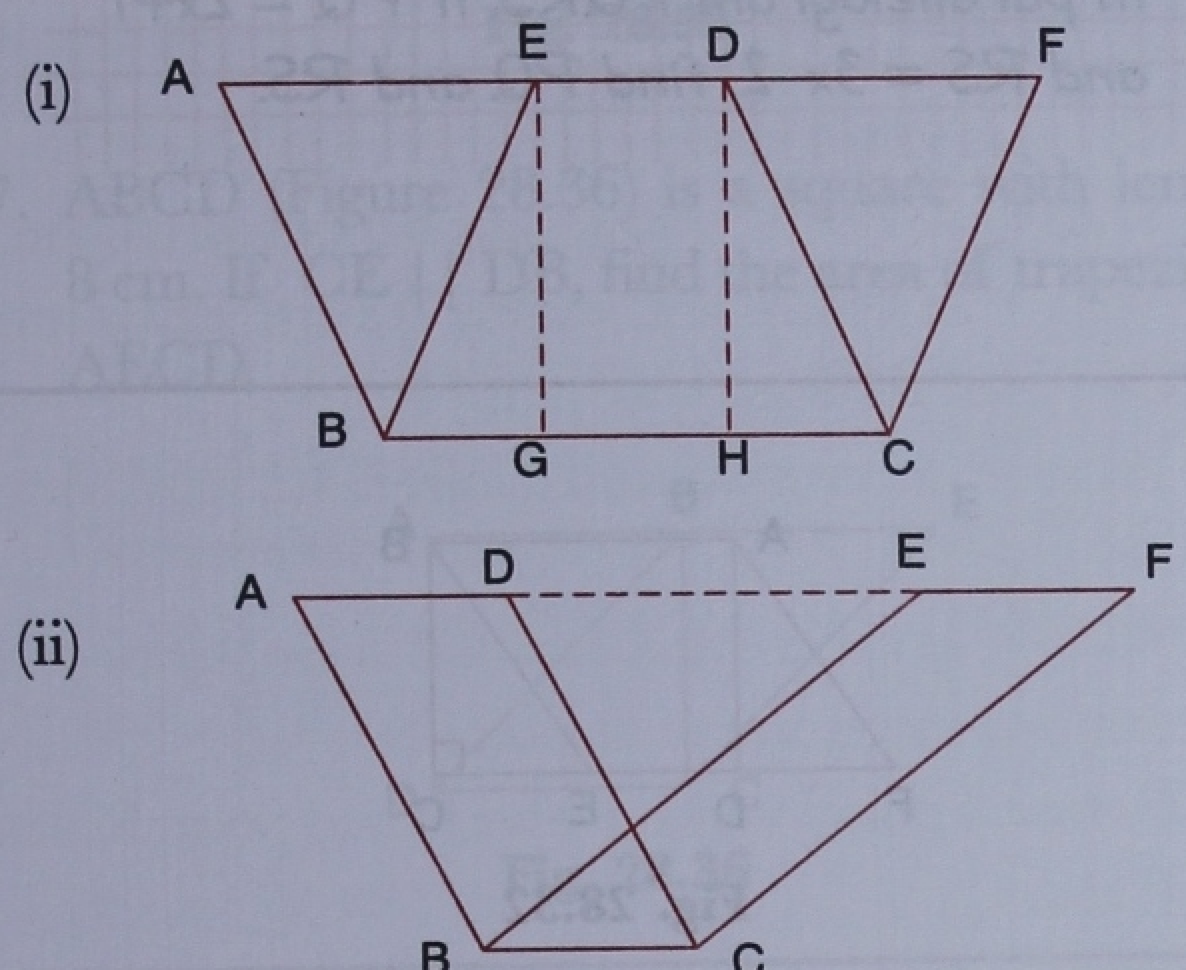


Fig. 28.27

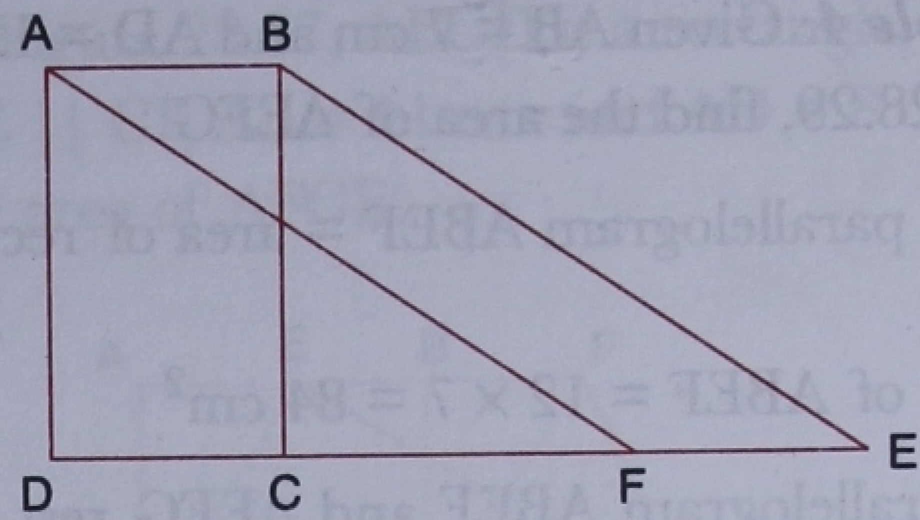
Thus, area of parallelogram ABCD = area of parallelogram EBCF. We have already demonstrated the proof of this theorem by 'finding' the area of parallelogram ABCD from the area of rectangle EFCD.

Notice that both ABCD and EFCD (Figure 28.26) are two parallelograms resting on the same base CD between the same parallel lines DC and AF.

Example 3: Given AB = 5 cm and AD = 9 cm in the figure below, find the area of parallelogram ABEF.

Rectangle ABCD and parallelogram ABEF rest on the same base AB between the same parallel lines AB || DE.

Thus, area of ABCD = area of ABEF
Area of ABCD = length × breadth
= 9 × 5 = 45 cm^2
∴ Area of parallelogram ABEF = 45 cm^2



Case II: The area of a triangle is half that of a parallelogram on the same base and between the same parallel lines.

In Figure 28.28 (i) and (ii), ΔABC and parallelogram ABCD rest on the same base AB and are between the same parallel lines AB || CD.

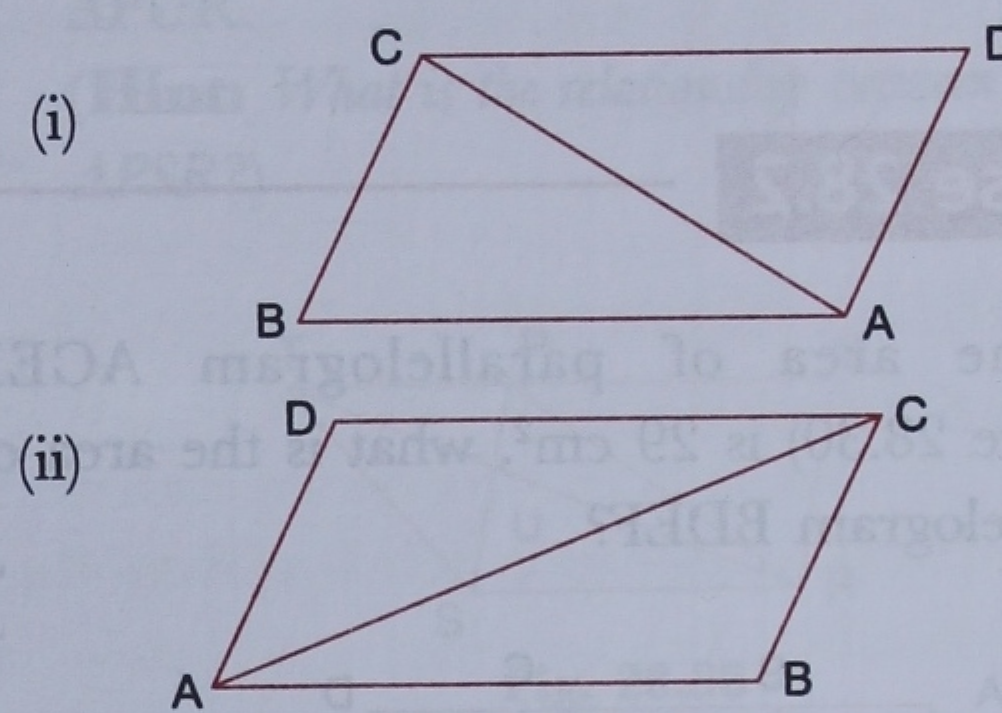
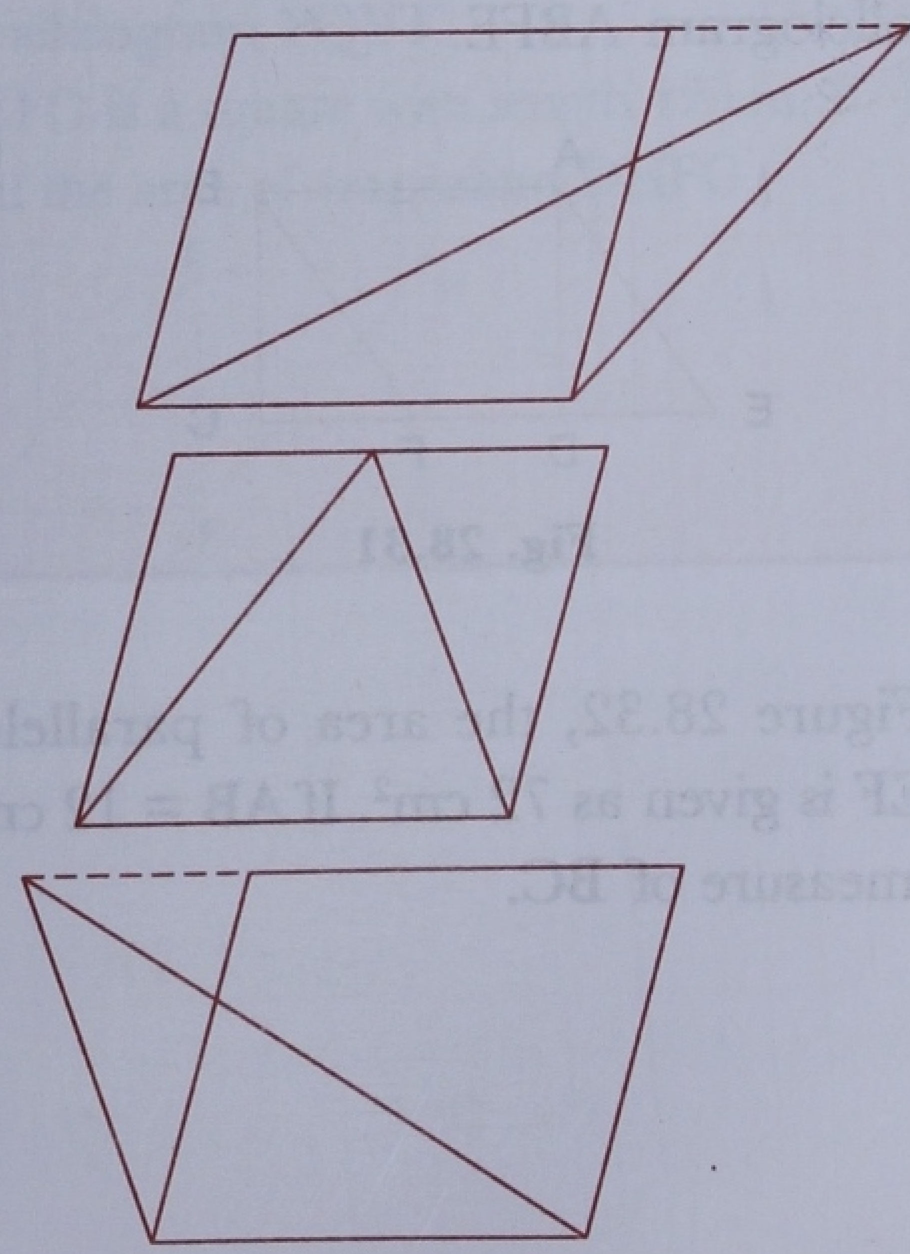


Fig. 28.28

Thus, area of ΔABC = $\frac{1}{2}$ area of parallelogram ABCD.

Similarly, the areas of the following triangles are also half the areas of their respective parallelograms whose base they share.



Example 4: Given AB = 7 cm and AD = 12 cm in Figure 28.29, find the area of ΔEFG.

Area of parallelogram ABEF = area of rectangle ABCD

⇒ Area of ABEF = 12 × 7 = 84 cm^2

Now parallelogram ABEF and ΔEFG rest on the same base EF between the same parallel lines AG || EF.

⇒ Area of ΔEFG = $\frac{1}{2}$ area of parallelogram ABEF

⇒ Area of ΔEFG $\frac{1}{2}$ 84 cm^2 = 42 cm^2

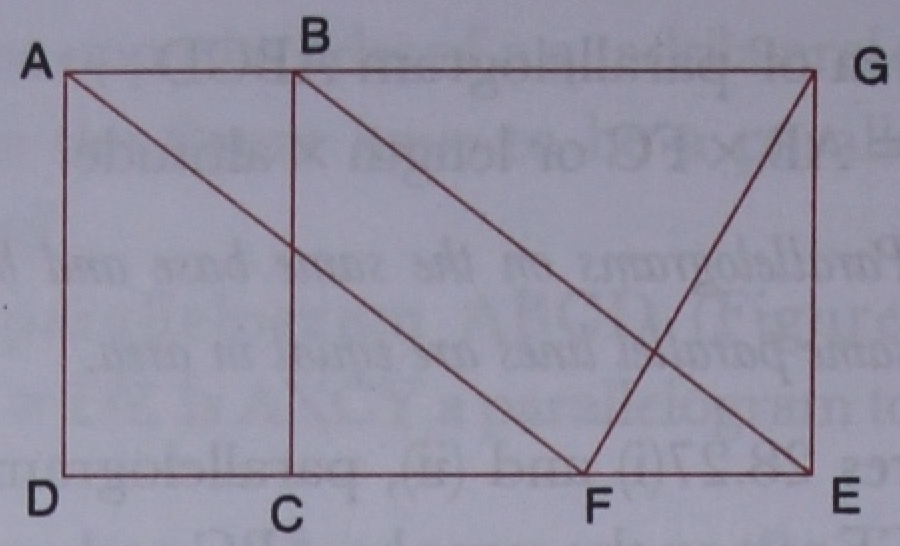


Fig. 28.29

Try this!

In parallelogram PQRS, if PQ = 2x+1 and RS = 3x–2, find PQ and RS.

Exercise 28.2

1. If the area of parallelogram ACEF (Figure 28.30) is 29 cm^2, what is the area of parallelogram BDEF?

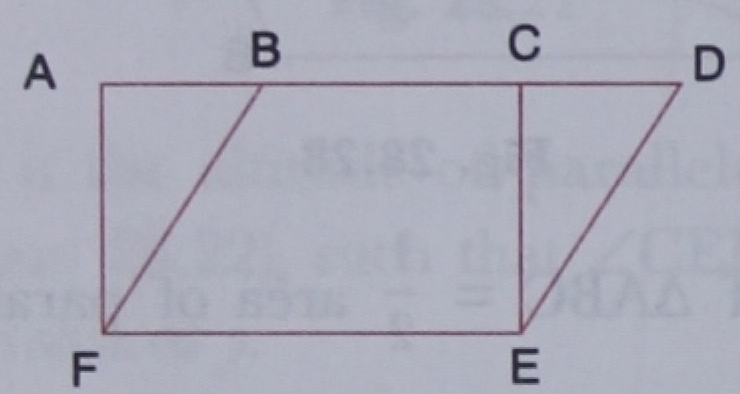


Fig. 28.30

2. In rectangle ABCD (Figure 28.31), AB = 12 cm and BC = 9 cm. Find the area of parallelogram ABFE.

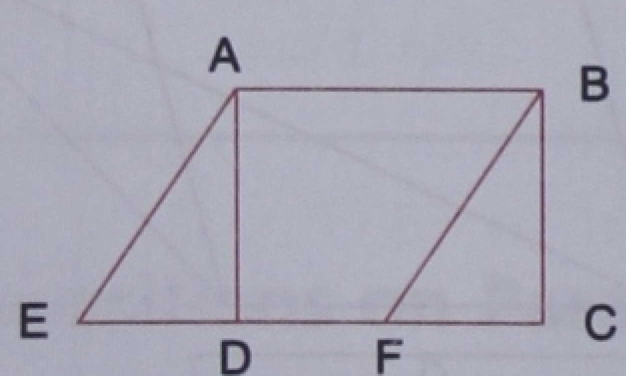


Fig. 28.31

3. In Figure 28.32, the area of parallelogram ABEF is given as 72 cm^2. If AB = 12 cm, find the measure of BC.

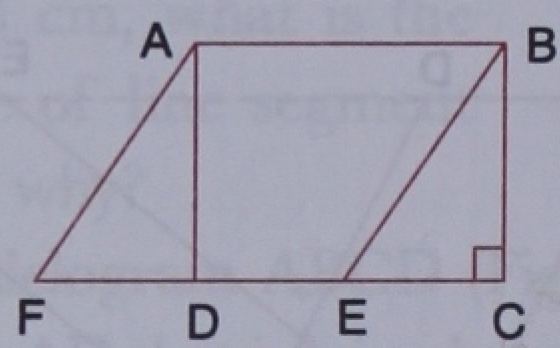


Fig. 28.32

4. If the area of rectangle ABCE is 34 cm^2 (Figure 28.33), find the area of:
   (i) Parallelogram ABDF
   (ii) Triangle BFD

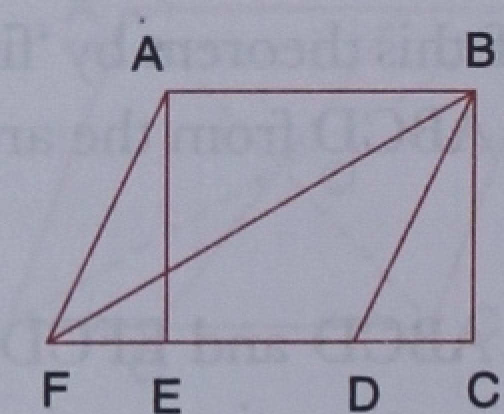


Fig. 28.33

5. In Figure 28.34, if the area of ΔABE = 13 cm^2, find the area of parallelogram ABCD.

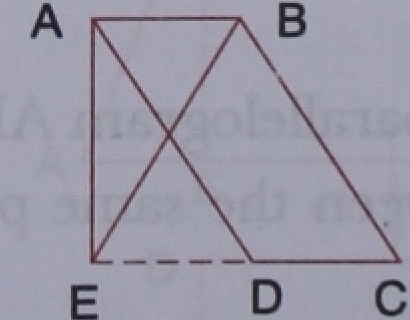


Fig. 28.34

6. Given PQRS is a rectangle (Figure 28.35), where PQ = 6 cm and QR = 8 cm, find:
   (i) Area of parallelogram PQST
   (ii) Area of ΔQRS
   (iii) Area of trapezium PQRT

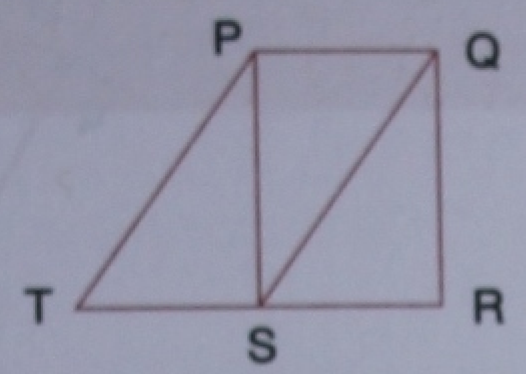


Fig. 28.35

7. ABCD (Figure 28.36) is a square with length 8 cm. If CE || DB, find the area of trapezium AECD.

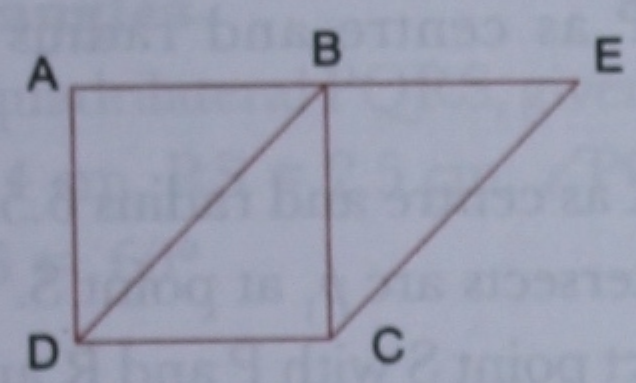


Fig. 28.36

8. In Figure 28.37, ABCD is a rectangle, DE || CF, AB = 11 cm. and AD = 8 cm. Find the area of ΔFGE.

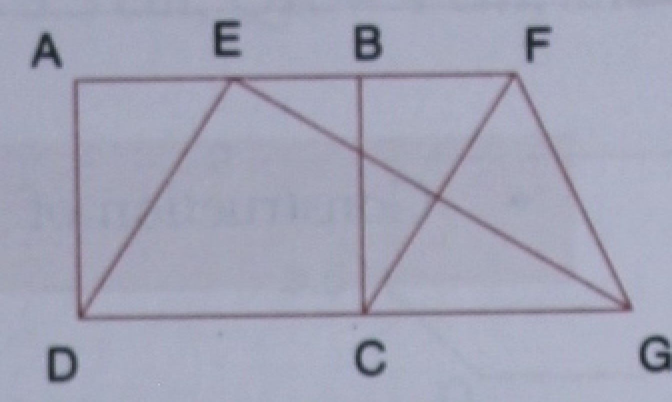


Fig. 28.37

9. In Figure 28.38, PQRS is a parallelogram. Find the relationship between the areas of ΔTUS and ΔPUR.
   (**Hint:** *What is the relationship between ΔTSR and ΔPSR?*)

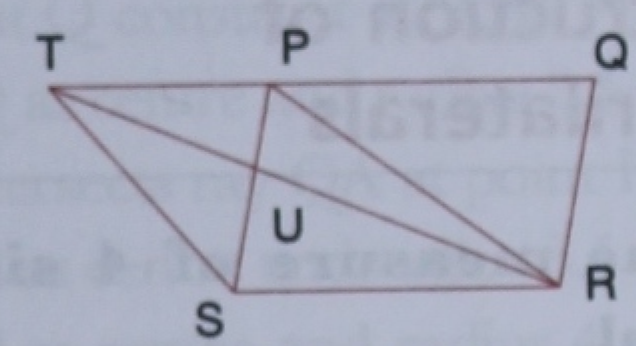
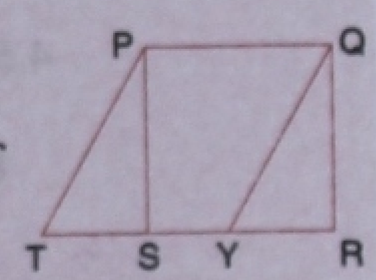


Fig. 28.38

Revision Exercise

1. The angles of a quadrilateral are in the ratio 12 : 14 : 16 : 18. Find all the angles of the quadrilateral.
2. An exterior angle of a parallelogram measures 125°. Find the magnitudes of all the angles of the parallelogram.
3. The perimeter of a parallelogram with its adjacent sides in the ratio 5 : 9 is given to be 112 cm. Find the measure of the sides of the parallelogram.
4. In rectangle PQRS, PQ = 15 cm, and QR = 12 cm. Find the area of parallelogram PQYT.



5. DEFG is a square with length 12 cm. If FH || GE, find the area of trapezium DHFG.