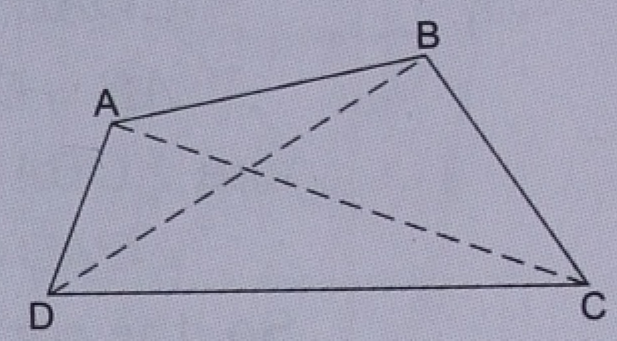
4

Quadrilaterals

Quadrilateral

A quadrilateral is a polygon with four sides. The quadrilateral *ABCD* shown in the figure has:



Four sides *AB*, *BC*, *CD* and *DA*

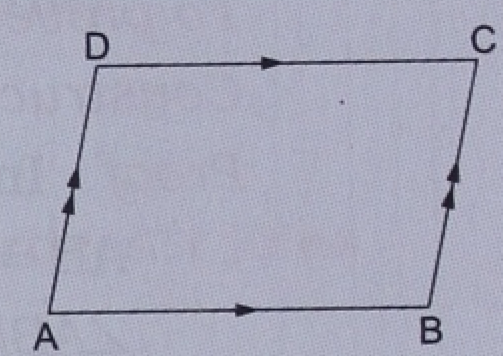
Four vertices *A*, *B*, *C* and *D*

Four angles $\angle A$ (or $\angle DAB$), $\angle B$ (or $\angle ABC$), $\angle C$ (or $\angle BCD$) and $\angle D$ (or $\angle CDA$)

Two diagonals *AC* and *BD*.

Parallelogram

A quadrilateral is called a parallelogram if its opposite sides are parallel. In the figure, *ABCD* is a parallelogram, in which *AB* is parallel to *DC* and *AD* is parallel to *BC*. A parallelogram has some special properties, which we will now study.

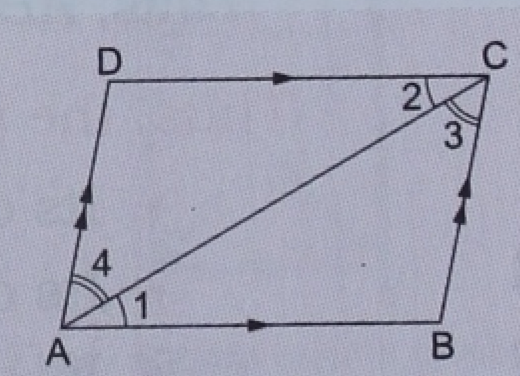


THEOREM 1 **(i) The opposite sides of a parallelogram are equal.**

(ii) The opposite angles of a parallelogram are equal.

(iii) Each diagonal bisects a parallelogram into two congruent triangles.

Given *ABCD* is a parallelogram in which $AB \parallel DC$ and $AD \parallel BC$.



To Prove (i) $AB = DC$ and $BC = AD$,

(ii) $\angle A = \angle C$ and $\angle B = \angle D$,

(iii) $\triangle ABC \cong \triangle CDA$ and $\triangle ABD \cong \triangle CDB$.

Construction Join the points *A* and *C*.

Proof In $\triangle ABC$ and $\triangle CDA$,

$\angle 1 = \angle 2$ ($\because$ $AB \parallel DC$, alternate angles are equal),

$AC = AC$ (common),

and $\angle 3 = \angle 4$ ($\because$ $BC \parallel AD$, alternate angles are equal).

$\therefore$ $\triangle ABC \cong \triangle CDA$ (A-S-A condition of congruency).

So, the corresponding parts of the triangles are equal.

$\therefore$ $AB = DC$ and $BC = AD$. (Proved)

Also, $\angle B = \angle D$.

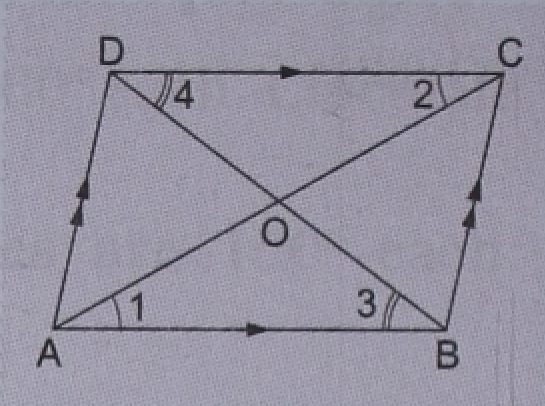
We have $\angle 1 = \angle 2$ and $\angle 4 = \angle 3$. So, $\angle 1 + \angle 4 = \angle 2 + \angle 3 \Rightarrow \angle A = \angle C$.

So, $\angle A = \angle C$ and $\angle B = \angle D$. (Proved)

Now, $\triangle ABC \cong \triangle CDA$ (proved already). Similarly, $\triangle ABD \cong \triangle CDB$.

Hence, each diagonal bisects the parallelogram into two congruent parts. (Proved)

THEOREM 2 **The diagonals of a parallelogram bisect each other.**



Given $ABCD$ is a parallelogram in which $AB \parallel DC$, $AD \parallel BC$ and the diagonals AC and BD intersect at the point O.

To prove $OA = OC$ and $OB = OD$.

Proof In $\triangle OAB$ and $\triangle OCD$,

$\angle OAB = \angle OCD$ ($AB \parallel DC$ and alternate angles are equal)

$AB = DC$ (Opposite sides of a parallelogram are equal)

and $\angle OBA = \angle ODC$ ($AB \parallel DC$ and alternate angles are equal).

$\therefore$ $\triangle OAB \cong \triangle OCD$ (A-S-A conditon of congruency).

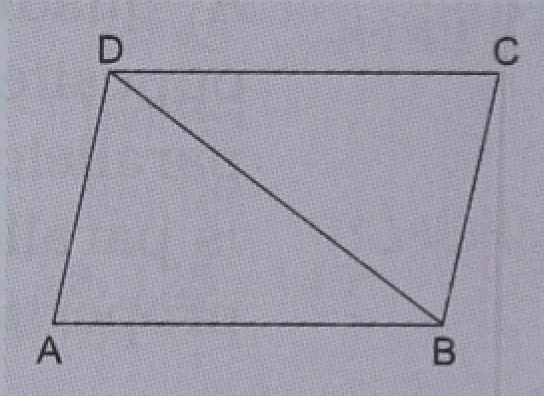
So, the corresponding sides of $\triangle OAB$ and $\triangle OCD$ are equal.

$\therefore$ $OA = OC$ and $OB = OD$. (Proved)

THEOREM 3 **If a pair of opposite sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram.**

Given $ABCD$ is a quadrilateral in which $AB = DC$ and $AB \parallel DC$.

To prove $ABCD$ is a parallelogram.



Construction Join the points B and D.

Proof In $\triangle ABD$ and $\triangle CDB$,

$AB = DC$ (Given)

$\angle ABD = \angle CDB$ ($\because$ $AB \parallel DC$, alternate angles are equal)

and $BD = DB$ (Common side).

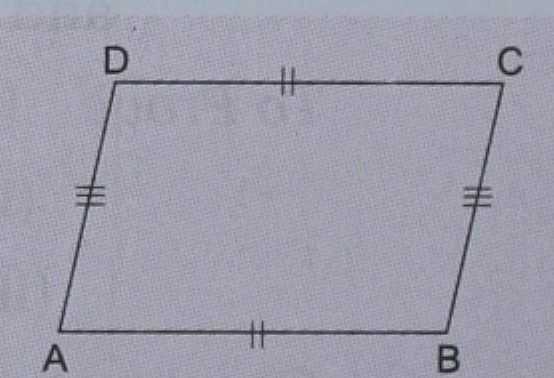
$\therefore$ $\triangle ABD \cong \triangle CDB$ (S-A-S condition of congruency).

So, the corresponding parts of these triangles are equal.

$\therefore$ $\angle ADB = \angle CBD$, but these are alternate angles. So, $AD \parallel BC$.

Thus, $AB \parallel DC$ and $AD \parallel BC$. Hence, $ABCD$ is a parallelogram. (Proved)

Thus, the properties of a parallelogram are:

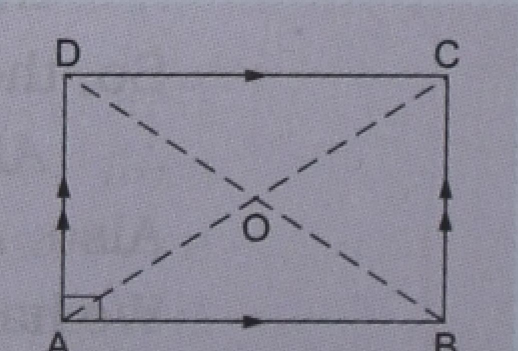


$\angle A + \angle D = 180°$, $\angle D + \angle C = 180°$
$\angle C + \angle B = 180°$, $\angle B + \angle A = 180°$

- Its opposite sides are equal.
- Its opposite angles are equal.
- Its diagonals bisect each other.
- Each of its diagonals divides it into two congruent triangles.
- Its adjacent angles are supplementary because they are co-interior angles and the opposite sides are parallel.

Rectangle

A parallelogram is called a rectangle if one of its angles is a right angle. In the figure, $ABCD$ is a parallelogram in which $\angle A = 90°$. Thus, $ABCD$ is a rectangle. A rectangle has all the properties of a parallelogram. In addition, it has the following special properties.



Rectangle

PROPERTY 1 **All the interior angles of a rectangle are right angles.**

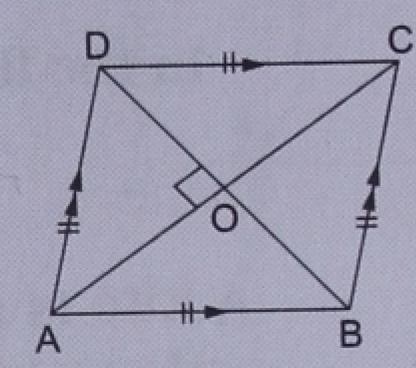
In the figure, $AB \parallel DC \Rightarrow \angle A + \angle D = 180°$. But $\angle A = 90°$, so $\angle D = 90°$.
Also, the opposite angles are equal. So, $\angle C = \angle A = 90°$ and $\angle B = \angle D = 90°$.
Thus, $\angle A = \angle B = \angle C = \angle D = 90°$.

PROPERTY 2 **The diagonals of a rectangle are equal in length.**

In the figure above, $\triangle ABC \cong \triangle BAD$, because $AB = BA$, $\angle ABC = \angle BAD$ $(= 90°)$ and $BC = AD$.
So, diagonal AC = diagonal BD.

Rhombus

A parallelogram is called a rhombus if two adjacent sides are equal. In the figure, $ABCD$ is a parallelogram in which $AB = AD$. So, it is a rhombus. A rhombus has all the properties of a parallelogram. It also has the following additional properties.



Rhombus

PROPERTY 1 **All the sides of a rhombus are equal in length.**

In the figure, $AB = BC = CD = DA$.

PROPERTY 2 **The two diagonals of a rhombus are perpendicular to each other.**

In the figure, $AC \perp BD$.

PROPERTY 3 **Each diagonal of a rhombus bisects the angles at the two vertices it joins.**

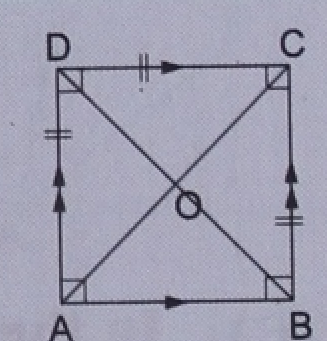
In the figure, AC bisects $\angle A$ and $\angle C$ and BD bisects $\angle B$ and $\angle D$.

PROPERTY 4 **The diagonals of a rhombus form four congruent triangles.**

In the figure, $\triangle OAB \cong \triangle OBC \cong \triangle OCD \cong \triangle ODA$.

Square

If two adjacent sides of a rectangle are equal, it is called a square. In the figure, $ABCD$ is a square. A square has all the properties of a rectangle and the following additional property.



Every square is a rhombus.

PROPERTY **All the sides of a square are equal in length.**

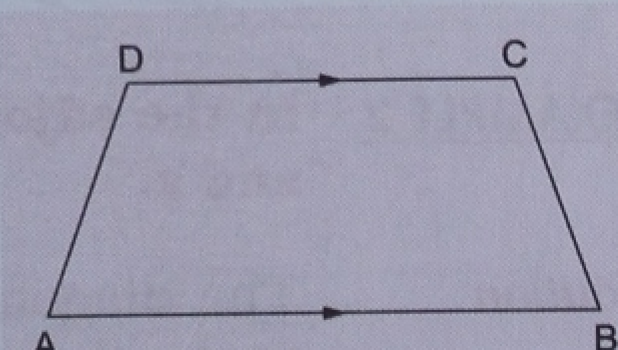
Thus, every square is a rhombus and it has all the properties of a rhombus.

$\therefore$ $AC \perp BD$, AC bisects $\angle A$ and $\angle C$, and BD bisects $\angle B$ and $\angle D$ and $\triangle OAB \cong \triangle OBC \cong \triangle OCD \cong \triangle ODA$.

Trapezium

A quadrilateral in which one pair of sides is parallel, is called a trapezium. In the figure, $ABCD$ is a trapezium in which $AB \parallel DC$ but AD is not parallel to BC. AD and BC are called oblique sides.

$AB \parallel DC$, so $\angle A + \angle D = 180°$ and $\angle B + \angle C = 180°$.

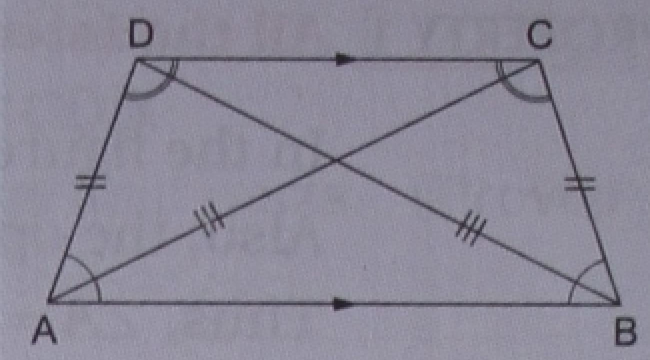


Trapezium

Isosceles trapezium

If the oblique sides of a trapezium are equal, it is called an isosceles trapezium.

Here, $AB \parallel DC$ and $AD = BC$.



Isosceles trapezium

PROPERTY 1 **The base angles of an isosceles trapezium are equal.**

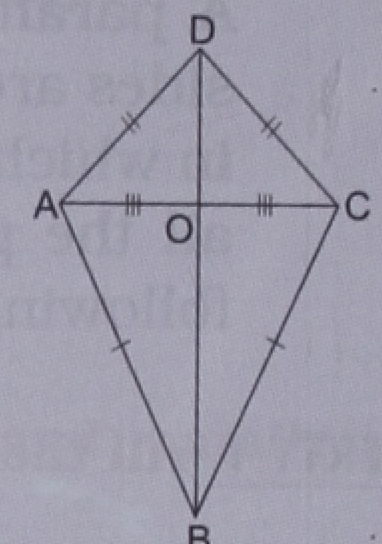
In the figure, $\angle A = \angle B$. Also, $\angle C = \angle D$.

PROPERTY 2 **The diagonals are equal.**

In the figure, $AC = BD$.

Kite

A kite is a quadrilateral in which the adjacent sides on either side of a diagonal are equal. In the figure, $ABCD$ is a kite in which $AD = DC$ and $AB = BC$. A kite has the following properties.



Kite

PROPERTY 1 **The diagonals of a kite are perpendicular to each other.**

In the figure, $AC \perp BD$.

PROPERTY 2 **The diagonal on either side of which the adjacent sides are equal, is bisected by the other diagonal.**

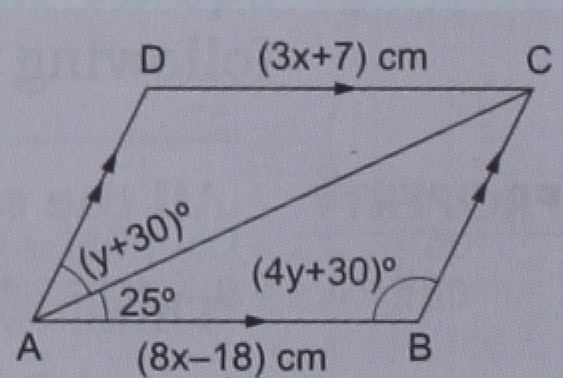
In the figure, the diagonal BD bisects the diagonal AC, hence, $OA = OC$.

The kite $ABCD$ also has the following properties.

(i) $\angle A = \angle C$ (ii) BD bisects $\angle ABC$ and $\angle ADC$.

Solved Examples

EXAMPLE 1 **In the figure, *ABCD* is a parallelogram. Find *x* and *y*.**



Solution The opposite sides of a parallelogram are equal.

$\therefore \quad AB = DC \Rightarrow 8x - 18 = 3x + 7 \Rightarrow 5x = 25 \Rightarrow x = 5.$

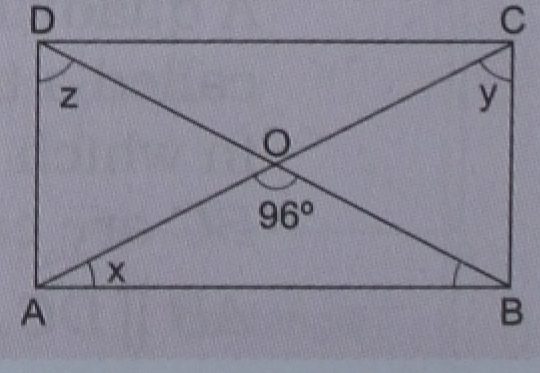
$AD \parallel BC \Rightarrow \angle A + \angle B = 180°$ as these are cointerior angles.

$\Rightarrow y + 30 + 25 + 4y + 30 = 180$

$\Rightarrow 5y = 180 - 85 = 95.$

$\therefore \quad y = \dfrac{95}{5} = 19$. Thus, $x = 5$ and $y = 19$.

EXAMPLE 2 **In the adjoining figure, *ABCD* is a rectangle. Find *x*, *y* and *z*.**



Soution The diagonals of a rectangle are equal and they bisect each other. $\therefore \quad OA = OB = OC = OD.$

In $\triangle OAB$, $OA = OB \Rightarrow \angle OBA = \angle OAB = x$.

The sum of the angles of $\triangle OAB = 180°$

$\Rightarrow \quad x + x + 96° = 180° \Rightarrow 2x = 180° - 96° = 84° \Rightarrow x = 42°$.

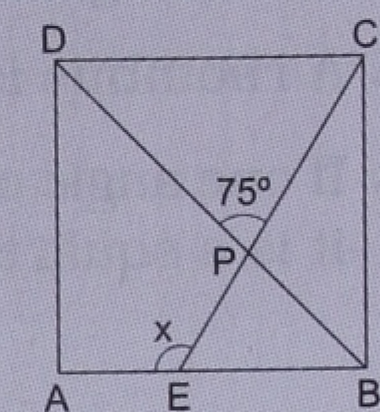
$\therefore \quad \angle OBA = 42°$ but $\angle ABC = 90° \Rightarrow \angle OBC = \angle ABC - \angle OBA = 90° - 42° = 48°$.

But $OB = OC \Rightarrow \angle OBC = \angle OCB = y \Rightarrow y = 48°$.

Again, $\angle OAD = \angle DAB - \angle OAB = 90° - x = 90° - 42° = 48°$.

In $\triangle OAD$, $OA = OD \Rightarrow \angle ODA = \angle OAD \Rightarrow z = 48°$.

EXAMPLE 3 **In the adjoining figure, *ABCD* is a square. Find *x*.**



Solution The diagonal BD of the square $ABCD$ bisects $\angle ABC$.

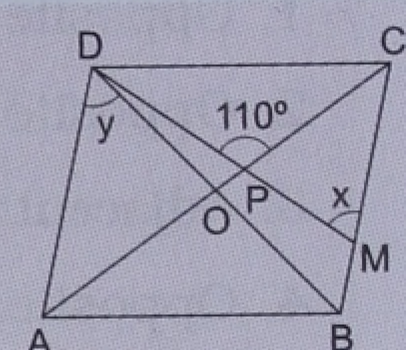
$\therefore \quad \angle PBE = \frac{1}{2}\angle ABC = \frac{1}{2} \times 90° = 45°$.

Also, $\angle BPE =$ opposite $\angle DPC = 75°$.

In $\triangle PEB$, exterior $\angle AEP = \angle BPE + \angle PBE$.

$\therefore \quad x = 75° + 45° = 120°$.

EXAMPLE 4 **In the adjoining figure, *ABCD* is a rhombus and $\angle BCD = 80°$. Find *x* and *y*.**



Solution $ABCD$ is a rhombus. So, AC bisects $\angle BCD$.

$\therefore \quad \angle PCM = \frac{1}{2}\angle BCD = \frac{1}{2} \times 80° = 40°$.

In $\triangle PCM$, exterior $\angle DPC = x + \angle PCM \Rightarrow 110° = x + 40° \Rightarrow x = 110° - 40° = 70°$.

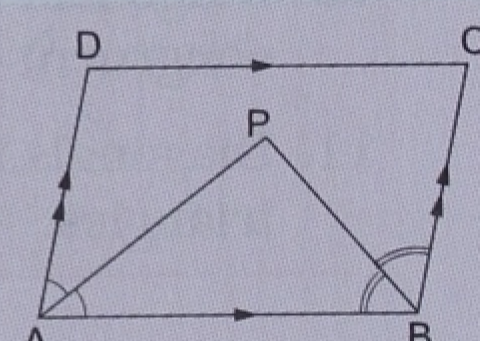
Now, $AD \parallel BC$ ($ABCD$ being a rhombus) $\Rightarrow \angle BCD + \angle ADC = 180°$

$\Rightarrow \quad 80° + \angle ADC = 180° \Rightarrow \angle ADC = 180° - 80° = 100°$.

$ABCD$ being a rhombus, the diagonal BD bisects $\angle ADC$

$\Rightarrow \quad \angle ADB = \frac{1}{2} \times \angle ADC \Rightarrow y = \frac{1}{2} \times 100° = 50°$.

EXAMPLE 5 **In the adjoining figure, the bisectors of two consecutive angles $\angle A$ and $\angle B$ of the parallelogram *ABCD* meet at the point *P*. Prove that $\angle APB = 90°$.**



Solution PA and PB are the bisectors of $\angle A$ and $\angle B$ respectively.

$\therefore \quad \angle PAB = \frac{1}{2}\angle A$ and $\angle PBA = \frac{1}{2}\angle B$.

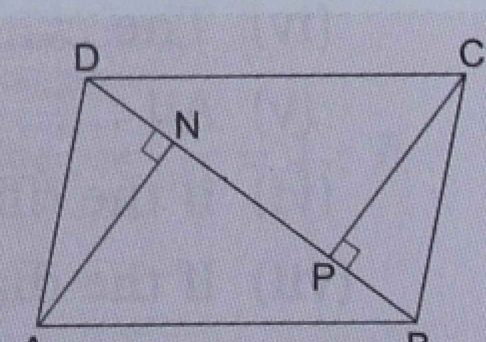
Now, $AD \parallel BC \Rightarrow \angle A + \angle B = 180° \Rightarrow \frac{1}{2}(\angle A + \angle B) = 90°$

$\Rightarrow \quad \frac{1}{2}\angle A + \frac{1}{2}\angle B = 90° \Rightarrow \angle PAB + \angle PBA = 90°$. ... (1)

In $\triangle PAB$, the sum of angles $= 180° \Rightarrow \angle PAB + \angle PBA + \angle APB = 180°$

$\Rightarrow \quad 90° + \angle APB = 180° \Rightarrow \angle APB = 180° - 90° = 90°$.

EXAMPLE 6 **In the adjoining figure, *ABCD* is a parallelogram. Prove that (i) $\triangle ADN \cong \triangle CBP$ (ii) $AN = CP$.**



Solution In $\triangle ADN$ and $\triangle CBP$, $\angle AND = \angle CPB$ $(= 90°)$,

$\angle ADN =$ alternate $\angle CBP$ $(\because AD \parallel BC)$

and $AD = BC$ (opposite sides of a parallelogram).

$\therefore \quad \triangle ADN \cong \triangle CBP$ (A-A-S conditions of congruence).

So, the corresponding sides are equal. Hence, $AN = CP$.

Remember These

1. A parallelogram is a rectangle if one of its angles is a right angle.
2. A rectangle is a parallelogram with equal diagonals.
3. A rhombus is a parallelogram with equal adjacent sides.
4. If one angle of a parallelogram is a right angle and two equal adjacent sides are equal then it is a square. A square is both a rectangle and a rhombus.

Important properties of some quadrilaterals

Property	Parallelogram	Rectangle	Square	Rhombus
1. Opposite sides are parallel.	Yes	Yes	Yes	Yes
2. Opposite sides are equal.	Yes	Yes	Yes	Yes
3. Adjacent sides are equal.			Yes	Yes
4. Opposite angles are equal.	Yes	Yes	Yes	Yes
5. Each interior angle is 90°.		Yes	Yes	
6. Diagonals are equal.		Yes	Yes	
7. Diagonals bisect each other.	Yes	Yes	Yes	Yes
8. Diagonals are ⊥ to each other.			Yes	Yes
9. Each diagonal bisects the angles through which it passes.			Yes	Yes
10. Diagonals form 2 pairs of congruent triangles.	Yes	Yes	Yes	Yes
11. Diagonals form 4 congruent triangles.			Yes	Yes

EXERCISE 4

1. Fill in the blanks.
   (i) Two consecutive angles of a parallelogram are ....
   (ii) The opposite angles of a parallelogram are ....
   (iii) Each diagonal ...... a parallelogram.
   (iv) The ...... sides of a parallelogram are equal.
   (v) All ...... of a rhombus are equal.
   (vi) If the diagonals of a parallelogram are equal, it is a ....
   (vii) If the diagonals of a quadrilateral bisect each other, it is a ....
   (viii) The diagonals of a ...... are equal.

(ix) The diagonals of a ...... bisect each other.

(x) The diagonals of a ...... are perpendicular to each other.

(xi) Each diagonal of a ...... bisects the angles through which it passes.

(xii) If the two adjacent angles of a parallelogram are in the ratio 3 : 7, the largest angle is ....

2. Which of the following statements are true?

(i) Every rhombus is a parallelogram.

(ii) Every square is a rectangle.

(iii) Every square is a rhombus.

(iv) Every rhombus is a square.

(v) Every rectangle is a rhombus.

(vi) Every rhombus is a rectangle.

(vii) The diagonals of a rectangle bisect each other at right angles.

(viii) Each diagonal of a rhombus bisects the angle through which it passes.

(ix) The diagonals of all quadrilaterals bisect each other.

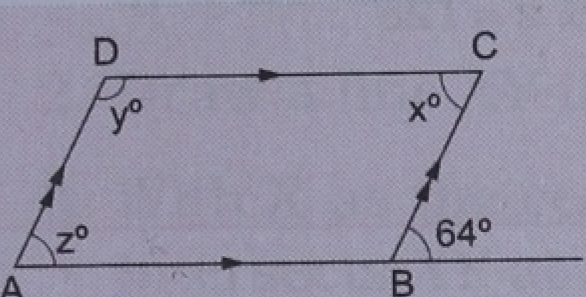
(x) If $AC = BD$ in a parallelogram $ABCD$ then $\angle ABC = 90°$.

3. Find the angles of the parallelogram $ABCD$ if

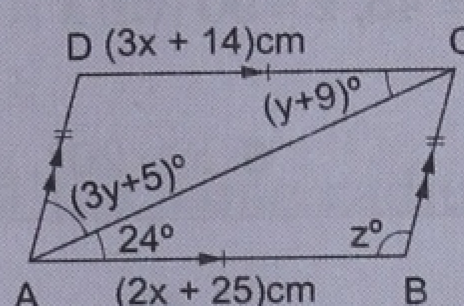
(i) $\angle A : \angle B = 2 : 7$ (ii) $\angle C = \frac{2}{3} \angle D$

4. Find x, y and z in each of the following figures.

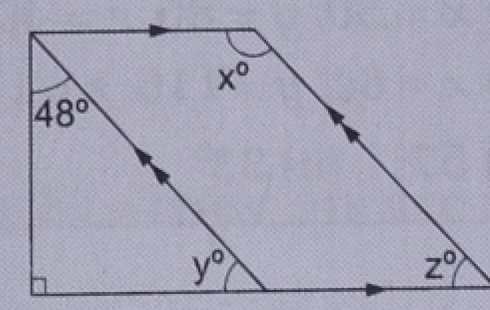
(i)



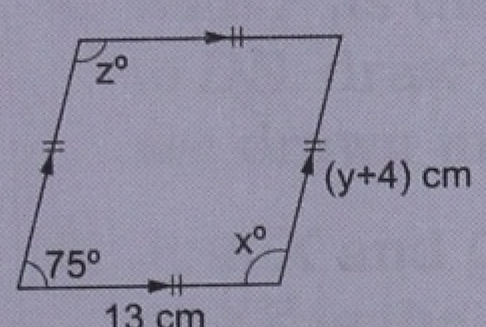
(ii)



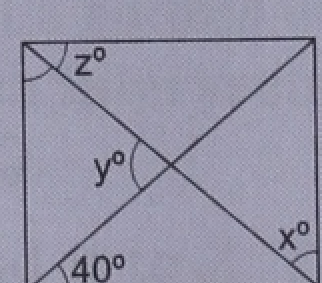
(iii)



(iv)

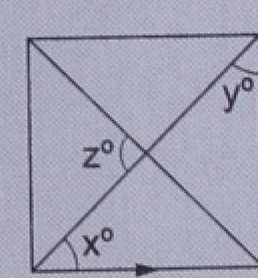


(v)



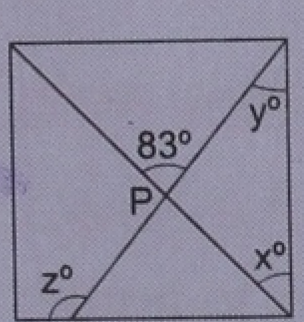
Rectangle

(vi)



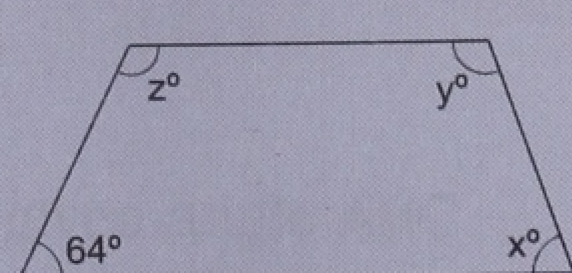
Square

(vii)



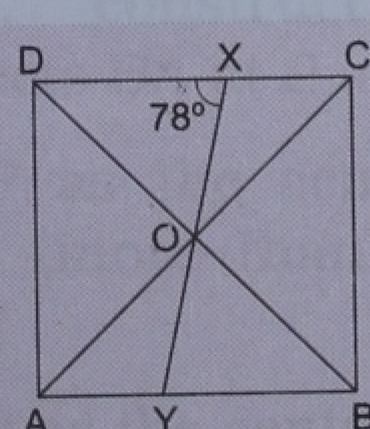
Square

(viii)



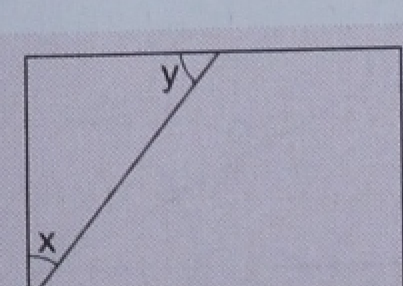
Isosceles Trapezium

5.



$ABCD$ is a square. Prove that $\triangle OAB$ is an isosceles triangle. Also, find (i) $\angle XOD$ (ii) $\angle XOC$.

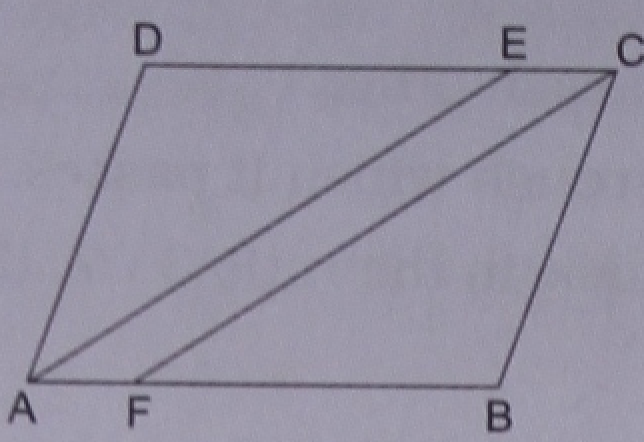
6.



Rectangle

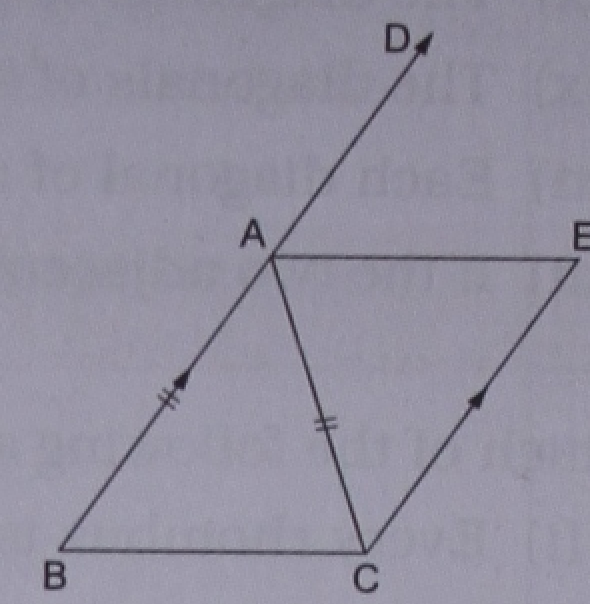
In the rectangle, $x : y = 2 : 7$. Find x and y.

7.



ABCD is a parallelogram and *AE* and *CF* bisect $\angle A$ and $\angle C$ respectively. Prove that $AE \parallel FC$.

8.



AE is the bisector of $\angle CAD$. Also, $BA \parallel CE$ and $AB = AC$. Prove that (i) $\angle EAC = \angle ACB$ (ii) *ABCE* is a parallelogram

ANSWERS

1. (i) supplementary (ii) equal (iii) bisects (iv) opposite (v) sides (vi) rectangle (vii) parallelogram (viii) rectangle and square (ix) parallelogram, rhombus, rectangle and square (x) rhombus and square (xi) rhombus and square (xii) 126°

2. (i), (ii), (iii), (viii), (x)

3. (i) $\angle A = \angle C = 40°$, $\angle B = \angle D = 140°$ (ii) $\angle A = \angle C = 72°$, $\angle B = \angle D = 108°$

4. (i) $x = 64$, $y = 116$, $z = 64$ (ii) $x = 6$, $y = 15$, $z = 106$ (iii) $x = 138$, $y = 42$, $z = 42$ (iv) $x = 105$, $y = 9$, $z = 105$ (v) $x = 50$, $y = 80$, $z = 40$ (vi) $x = 45$, $y = 45$, $z = 90$ (vii) $x = 45$, $y = 38$, $z = 128$ (viii) $x = 64$, $y = 116$, $z = 116$

5. (i) 57° (ii) 33°

6. $x = 20°$, $y = 70°$

❖