Chapter 15

SIMPLIFICATION OF ALGEBRAIC FRACTIONS

You have worked on numerical fractions like $\frac{3}{12}$, $\frac{5}{7}$, $-\frac{4}{9}$, $1\frac{3}{4}$ and algebraic fractions like $\frac{x}{3}$, $\frac{2y+3}{7}$, $\frac{x+2y-1}{5}$, $\frac{5}{2x-1}$ etc. Working with algebraic fractions often involves the concept of H.C.F. and L.C.M. of polynomials (with integral coefficients). In this chapter, we shall introduce the idea of H.C.F. and L.C.M. of polynomials and strengthen the concept of simplification of algebraic expressions.

H.C.F. AND L.C.M. OF POLYNOMIALS

H.C.F. of two or more polynomials (with integral coefficients) is the highest common factor of the given polynomials.

In particular,

H.C.F. of two or more monomials = (H.C.F. of their numerical coefficients) × (H.C.F. *of their literal coefficients)*

H.C.F. of literal coefficients = product of each common literal raised to the lowest power.

Remark

Here it is understood that the numerical coefficients of the monomials (under consideration) are integers and the powers of the literals involved in the monomials are positive integers.

L.C.M. of two or more polynomials (with integral coefficients) is the smallest common multiple of the given polynomials.

In particular,

L.C.M. of two or more monomials = (L.C.M. of their numerical coefficients) × *(L.C.M. of their literal coefficients)*

L.C.M. of literal coefficients = product of each literal raised to the highest power.

Example 1. Find H.C.F. and L.C.M. of the monomials $24x^3y^3$ and $32x^2y^4z$.

Solution. H.C.F. of numerical coefficients = H.C.F. of 24 and 32 = 8

H.C.F. of literal coefficients = product of each common literal raised to the lowest power

$= x^2y^3$

$\therefore$ H.C.F. of the given monomials = $8 \times x^2y^3 = 8x^2y^3$

L.C.M. of numerical coefficients = L.C.M. of 24 and 32 = 96

L.C.M. of literal coefficients = product of each literal raised to the highest power

$= x^3y^4z$

$\therefore$ L.C.M. of the given monomials = $96 \times x^3y^4z = 96x^3y^4z$.

Example 2. Find H.C.F. and L.C.M. of the monomials $12a^3b^2$, $16ab^3c^2$ and $20a^2b^2c$.

Solution. H.C.F. of numerical coefficients = H.C.F. of 12, 16 and 20 = 4

H.C.F. of literal coefficients = product of each common literal raised to the lowest power

$= ab^2$

$\therefore$ H.C.F. of the given monomials = $4 \times ab^2 = 4ab^2$

L.C.M. of numerical coefficients = L.C.M. of 12, 16 and 20 = 240

L.C.M. of literal coefficients = product of each literal raised to the highest power

$= a^3b^3c^2$

$\therefore$ L.C.M. of the given monomials = $240 \times a^3b^3c^2 = 240a^3b^3c^2$.

Method to find H.C.F. and L.C.M. of polynomials

To find H.C.F. and L.C.M. of given polynomials—factorise each polynomial, then

(i) H.C.F. = (H.C.F. of numerical coefficients) × (each common factor raised to the lowest power)

(ii) L.C.M. = (L.C.M. of numerical coefficients) × (each factor raised to the highest power)

Example 3. Find the H.C.F. and L.C.M. of the polynomials $4x^2 + 4xy$, $6x^2 - 6y^2$.

Solution. $4x^2 + 4xy = 4x(x + y)$

Factorise given polynomials

and $6x^2 - 6y^2 = 6(x^2 - y^2) = 6(x + y)(x - y)$

H.C.F. of numerical coefficients = H.C.F. of 4 and 6 = 2

$\therefore$ H.C.F. of the given polynomials = $2(x + y)$

L.C.M. of numerical coefficients = L.C.M. of 4 and 6 = 12

$\therefore$ L.C.M. of the given polynomials = $12x(x + y)(x - y)$.

Example 4. Find the H.C.F. and L.C.M. of $5a^2 + 15a$, $a^2 + 6a + 9$ and $2a^2 - 18$.

Solution. Factorising the given polynomials, we get

$$5a^2 + 15a = 5a(a + 3),$$

$$a^2 + 6a + 9 = (a)^2 + 2 \times a \times 3 + (3)^2 = (a + 3)^2$$

and $$2a^2 - 18 = 2(a^2 - 9) = 2(a + 3)(a - 3)$$

H.C.F. of numerical coefficients = H.C.F. of 5, 1 and 2 = 1

$\therefore$ H.C.F. of the given polynomials = $1 \times (a + 3) = a + 3$

L.C.M. of numerical coefficients = L.C.M. of 5, 1 and 2 = 10

$\therefore$ L.C.M. of the given polynomials = $10a(a + 3)^2(a - 3)$.

Example 5. Find the H.C.F. and L.C.M. of $12x^2 - 75$, $4x^2 - 20x + 25$ and $6x^2 - 13x - 5$.

Solution. Factorising the given polynomials, we get

$$12x^2 - 75 = 3(4x^2 - 25) = 3(2x + 5)(2x - 5),$$

$$4x^2 - 20x + 25 = (2x)^2 - 2 \times 2x \times 5 + (5)^2 = (2x - 5)^2$$

and $6x^2 - 13x - 5 = 6x^2 + 2x - 15x - 5$

We need two integers whose sum is -13 and product is $6 \times (-5)$ *i.e.* -30.

By trial, $2 + (-15) = -13$ and $2 \times (-15) = -30$]

$$= 2x(3x + 1) - 5(3x + 1)$$

$$= (3x + 1)(2x - 5).$$

H.C.F. of numerical coefficients = H.C.F. of 3, 1 and 1 = 1

$\therefore$ H.C.F. of the given polynomials $= 1 \times (2x - 5)$

$= 2x - 5$

L.C.M. of numerical coefficients = L.C.M. of 3, 1 and 1 = 3

$\therefore$ L.C.M. of the given polynomials $= 3(2x + 5)(2x - 5)^2(3x + 1)$.

Exercise 15.1

Find the H.C.F. and L.C.M. of the following (1 to 4) monomials :

1. (*i*) $6x^2y$ and $4xy^2$ (*ii*) $12a^3b^2$ and $18a^2b^5$
2. (*i*) $12x^3y^3$ and $28xy^2z^2$ (*ii*) $48a^2bc$ and $56ab^2c^2d$
3. (*i*) $2m^2n^2$, $3mn^3$ and $5m^3n$ (*ii*) $6a^2b^3$, $15ab^4$ and $12a^4b^2$
4. (*i*) $9p^2q^2r$, $36pq^3r^2$ and $12p^3q^3r^3$ (*ii*) $10x^3y^3z^4$, $15x^2y^2$ and $20yz^5$

Find the H.C.F. and L.C.M. of the following (5 to 8) polynomials :

5. (*i*) $x^2 + 3xy$ and $x^2 - 9y^2$ (*ii*) $9x^2 - 16y^2$ and $15x^2 - 20xy$
6. (*i*) $4a^2 - 25$ and $4a^2 - 20a + 25$ (*ii*) $6x^2 + 12xy$ and $4x^2 - 16y^2$
7. (*i*) $x^2 + 6x + 9$ and $2x^2 + 7x + 3$ (*ii*) $4a^2 + 4a + 1$ and $6a^2 + 7a + 2$
8. (*i*) $3x^2 + 6x$, $x^2 + 5x + 6$ and $2x^2 + 8x + 8$
   (*ii*) $4x^2 - 36$, $2x^2 - 12x + 18$ and $2x^2 + x - 21$

module 2

ALGEBRAIC FRACTIONS

Fractions involving polynomials either in numerator or denominator (or both) are called ***algebraic fractions.***

For example :

(*i*) $\frac{2x}{7}$, $\frac{3x-2}{5}$, $\frac{x+2y-7}{11}$, $\frac{2x^2+5}{3}$ etc. are algebraic fractions with integral denominators.

(*ii*) $\frac{a}{b}$, $-\frac{3}{x}$, $\frac{1}{x+2}$, $\frac{2x+3}{x-5}$, $\frac{x^2-1}{2x+1}$, $\frac{3x^2+5}{x^2+4}$ etc. are algebraic fractions involving variables in the denominators.

*Such fractions are **meaningful** only when denominator is not zero.*

Thus, $\frac{a}{b}$ is meaningful when $b \neq 0$,

$-\frac{3}{x}$ is meaningful when $x \neq 0$,

$\frac{1}{x+2}$ is meaningful when $x + 2 \neq 0$ *i.e.* when $x \neq -2$,

$\frac{2x+3}{x-5}$ is meaningful when $x - 5 \neq 0$ *i.e.* when $x \neq 5$,

$\frac{x^2-1}{2x+1}$ is meaningful when $2x + 1 \neq 0$ *i.e.* when $x \neq -\frac{1}{2}$.

*An algebraic fraction is said to be in its **simplest form** or **lowest terms** if the numerator and denominator have no common factor (except 1).*

Reducing an algebraic fraction to its lowest terms

To reduce an algebraic fraction to its lowest terms, proceed as under :

(i) Factorise the polynomials in numerator and denominator.

(ii) Cancel out the common factors of numerator and denominator.

Example 1. Reduce the following algebraic fractions to lowest terms :

(i) $\frac{6pq^2}{9pqr}$ (ii) $\frac{3x}{6x - 9x^2}$ (iii) $\frac{x-2}{x^2-4}$.

Solution. (i) $\frac{6pq^2}{9pqr} = \frac{2.\cancel{3}.\cancel{p}.\cancel{q}.q}{3.\cancel{3}.\cancel{p}.\cancel{q}.r} = \frac{2q}{3r}$.

(ii) $\frac{3x}{6x-9x^2} = \frac{3x}{3x(2-3x)} = \frac{1}{2-3x}$.

(iii) $\frac{x-2}{x^2-4} = \frac{x-2}{(x+2)(x-2)} = \frac{1}{x+2}$.

Factorise denominator

Example 2. Reduce the following algebraic fractions to their lowest terms :

(i) $\frac{x^2-y^2}{x^3+x^2y}$ (ii) $\frac{2x^2+5x-3}{x^2-9}$ (iii) $\frac{12x^2-3}{10x^2+9x-7}$

Solution. (i) $\frac{x^2-y^2}{x^3+x^2y} = \frac{(x+y)(x-y)}{x^2(x+y)} = \frac{x-y}{x^2}$

(ii) $2x^2 + 5x - 3 = 2x^2 + 6x - x - 3$

$= 2x(x + 3) - 1(x + 3)$

$= (x + 3)(2x - 1)$

and $x^2 - 9 = (x + 3)(x - 3)$

$\therefore \frac{2x^2+5x-3}{x^2-9} = \frac{(x+3)(2x-1)}{(x+3)(x-3)} = \frac{2x-1}{x-3}$.

Factorise numerator and denominator

(iii) $12x^2 - 3 = 3(4x^2 - 1) = 3(2x + 1)(2x - 1)$ and

$10x^2 + 9x - 7 = 10x^2 + 14x - 5x - 7 = 2x(5x + 7) - 1(5x + 7)$

$= (5x + 7)(2x - 1)$

$\therefore \frac{12x^2-3}{10x^2+9x-7} = \frac{3(2x+1)(2x-1)}{(5x+7)(2x-1)} = \frac{3(2x+1)}{5x+7}$.

Simplification of algebraic fractions

The usual rules of simplification of Arithmetic are applicable for algebraic fractions also.

Example 3. Simplify the following algebraic expressions :

(i) $\frac{5x^2y^3}{3z}$ of $\left(\frac{2xy^2}{5z^2} \div \frac{3xy}{2z}\right)$ (ii) $\frac{p^2+2p}{p-4} \div \frac{p^2-4}{2p-8}$.

Solution. (i) $\frac{5x^2y^3}{3z}$ of $\left(\frac{2xy^2}{5z^2} \div \frac{3xy}{2z}\right) = \frac{5x^2y^3}{3z}$ of $\left(\frac{2xy^2}{5z^2} \times \frac{2z}{3xy}\right)$

$$= \frac{5x^2y^3}{3z} \text{ of } \frac{4y}{15z} = \frac{5x^2y^3}{3z} \times \frac{4y}{15z} = \frac{4x^2y^4}{9z^2}.$$

(ii) $\frac{p^2+2p}{p-4} \div \frac{p^2-4}{2p-8} = \frac{p^2+2p}{p-4} \times \frac{2p-8}{p^2-4}$

$$= \frac{p\ (p+2)}{p-4} \times \frac{2\ (p-4)}{(p-2)\ (p+2)} = \frac{2p}{p-2}.$$

Example 4. Simplify : $\frac{2x^2-5x-3}{x^2-x-6} \div \frac{2x^2-3x-2}{3x^2-12}$.

Solution. We have,

$$2x^2 - 5x - 3 = 2x^2 - 6x + x - 3$$
$$= 2x(x-3) + 1(x-3) = (x-3)\ (2x+1)$$
$$x^2 - x - 6 = x^2 - 3x + 2x - 6$$
$$= x(x-3) + 2(x-3) = (x-3)\ (x+2)$$
$$2x^2 - 3x - 2 = 2x^2 - 4x + x - 2$$
$$= 2x(x-2) + 1(x-2) = (x-2)\ (2x+1)$$

and $3x^2 - 12 = 3(x^2-4) = 3\ (x-2)\ (x+2)$

$$\therefore \frac{2x^2-5x-3}{x^2-x-6} \div \frac{2x^2-3x-2}{3x^2-12} = \frac{(x-3)\,(2x+1)}{(x-3)\,(x+2)} \div \frac{(x-2)\,(2x+1)}{3(x-2)\,(x+2)}$$

$$= \frac{2x+1}{x+2} \div \frac{2x+1}{3\,(x+2)} = \frac{2x+1}{x+2} \times \frac{3\,(x+2)}{2x+1}$$

$$= 3.$$

Example 5. Simplify the following :

(i) $\frac{x-3}{8} - \frac{2\,(2x+7)}{3} + \frac{11x-2}{6}$ (ii) $\frac{2x}{x+2} - \frac{x^2}{x^2-4}$.

Solution. (i) $\frac{x-3}{8} - \frac{2\,(2x+7)}{3} + \frac{11x-2}{6} = \frac{3\,(x-3) - 16\,(2x+7) + 4\,(11x-2)}{24}$

[L.C.M. of 8, 3 and 6 is 24]

$$= \frac{9x - 9 - 32x - 112 + 44x - 8}{24} = \frac{21x - 129}{24}.$$

(ii) $\frac{2x}{x+2} - \frac{x^2}{x^2-4} = \frac{2x}{x+2} - \frac{x^2}{(x+2)\,(x-2)} = \frac{2x\,(x-2) - x^2}{(x+2)\,(x-2)}$

[L.C.M. of denominators is $(x+2)\ (x-2)$]

$$= \frac{2x^2 - 4x - x^2}{(x+2)\,(x-2)} = \frac{x^2-4x}{(x+2)\,(x-2)} = \frac{x^2-4x}{x^2-4}.$$

Example 6. Simplify the following :

(i) $\frac{3}{a+b} - \frac{2}{a-b} + \frac{5a}{a^2-b^2}$ (ii) $\frac{2x}{x^2-4} + \frac{1}{x^2+3x+2}$.

Solution. (i) $\frac{3}{a+b} - \frac{2}{a-b} + \frac{5a}{a^2-b^2} = \frac{3}{a+b} - \frac{2}{a-b} + \frac{5a}{(a+b)(a-b)}$

$$= \frac{3(a-b) - 2(a+b) + 5a}{(a+b)(a-b)}$$

[L.C.M. of denominators is $(a+b)(a-b)$]

$$= \frac{3a - 3b - 2a - 2b + 5a}{(a+b)(a-b)} = \frac{6a-5b}{a^2-b^2}.$$

(ii) We have $x^2 - 4 = (x+2)(x-2)$ and

$x^2 + 3x + 2 = x^2 + 2x + x + 2 = x(x+2) + 1(x+2)$

$= (x+2)(x+1)$

∴ L.C.M. of denominators $= (x+2)(x-2)(x+1)$

$$\frac{2x}{x^2-4} + \frac{1}{x^2+3x+2} = \frac{2x}{(x+2)(x-2)} + \frac{1}{(x+2)(x+1)}$$

$$= \frac{2x(x+1) + (x-2)}{(x+2)(x-2)(x+1)} = \frac{2x^2 + 2x + x - 2}{(x+2)(x-2)(x+1)}$$

$$= \frac{2x^2 + 3x - 2}{(x+2)(x-2)(x+1)} = \frac{(2x-1)(x+2)}{(x+2)(x-2)(x+1)}$$

[∵ $2x^2 + 3x - 2 = 2x^2 + 4x - x - 2$

$= 2x(x+2) - 1(x+2) = (x+2)(2x-1)$]

$$= \frac{2x-1}{(x-2)(x+1)} = \frac{2x-1}{x^2-x-2}.$$

Exercise 15.2

Reduce the following (1 to 6) algebraic fractions to lowest terms :

1. (i) $-\frac{4a}{6a^2}$ (ii) $\frac{16p^3q^2r}{40pq^3r^2}$

2. (i) $\frac{x(x-2)}{x^2(x^2-4)}$ (ii) $\frac{x^4-16x^2}{x^3(x+4)}$

3. (i) $\frac{3x}{6x-9x^2}$ (ii) $\frac{x^2-1}{x^2(x+1)}$

4. (i) $\frac{x^2+3x+2}{x^2+5x+6}$ (ii) $\frac{x^2y^2}{xy(x+y)}$

5. (i) $\frac{9xy}{3x^2-6xy}$ (ii) $\frac{3x^2-12}{x^2-3x-10}$

6. (i) $\frac{x^2y-y^3}{x^2y+2xy^2+y^3}$ (ii) $\frac{2x^2+7x+3}{3x^2+10x+3}$

Simplify the following (7 to 11) :

7. (i) $\frac{x^2-9}{x^2+3x} \times \frac{3x}{x-3}$ (ii) $\frac{x^2}{x^2-4} \times \frac{x^2+3x+2}{2x}$

8. (i) $\frac{3m-6}{m^2-m-6} \div \frac{m-2}{m+2}$ (ii) $\frac{x^2-5x}{3x-4y} \div \frac{x^2-25}{9x^2-16y^2}$

9. (i) $\frac{m}{5} - \frac{m-2}{3} + m$ (ii) $\left(\frac{x}{6} + \frac{2x}{3}\right) \div \left(x + \frac{2x-1}{3}\right)$

10. (ii) $\frac{2}{t+1} + \frac{3}{t+2}$ (ii) $\frac{m}{m^2-4} - \frac{1}{m-2}$

11. (i) $\frac{1}{x+y} - \frac{1}{x-y} + \frac{2x}{x^2-y^2}$ (ii) $\frac{1}{x^2-3x+2} - \frac{1}{x^2-5x+6}$

Summary

- H.C.F. of two or more polynomials (with integral coefficients) is the highest common factor of the given polynomials.
- L.C.M. of two or more polynomials (with integral coefficients) is the smallest common multiple of the given polynomials.
- To find H.C.F. and L.C.M. of given polynomials—factorise each polynomial, then
  - (*i*) H.C.F. = (H.C.F. of numerical coefficients) × (each common factor raised to the lowest power)
  - (*ii*) L.C.M. = (L.C.M. of numerical coefficients) × (each factor raised to the highest power)
- Fractions involving polynomials either in numerator or denominator (or both) are called algebraic fractions.
- Fractions involving variables in the denominator are meaningful only when denominator is not zero.
- An algebraic fraction is said to be in its simplest form or lowest terms if the numerator and denominator have no common factor (except 1).
- To reduce an algebraic fraction to its lowest terms, proceed as under :
  - (*i*) Factorise the polynomials in numerator and denominator.
  - (*ii*) Cancel out the common factors of numerator and denominator.
- The usual rules of simplification of Arithmetic are applicable for algebraic fractions also.

Check Your Progress

1. Find the H.C.F. and L.C.M. of the monomials $18x^3y^3z^2$, $24xy^4z^3$ and $30x^3yz^4$.

2. Find the H.C.F. and L.C.M. of the polynomials
$4x^2 - 8xy$, $6x^2 - 24y^2$, $2x^2 - 8xy + 8y^2$ and $2x^2 + 2xy - 12y^2$.

3. Reduce the following algebraic fractions to their lowest terms :

(*i*) $\dfrac{2a^3b - 2ab^3}{a^2 + 2ab + b^2}$ (*ii*) $\dfrac{3x^2 + 5xy - 2y^2}{4x^2 + 7xy - 2y^2}$.

Simplify the following (4 to 6) expressions :

4. (*i*) $\left(a + \dfrac{1}{b}\right) \div \left(b + \dfrac{1}{a}\right)$ (*ii*) $\dfrac{x^2 - 3x - 4}{x^2 - 3x} \div \dfrac{x^2 - 4x}{x + 3}$

5. (*i*) $\dfrac{x^2 + (p+1)x + p}{x^2 + 2px + p^2} \times \dfrac{p + x}{p + 1}$ (*ii*) $\dfrac{3x - x^2 - 2}{x^2 - 2x - 3} \div \dfrac{x^2 - 5x + 6}{6x - x^2 - 9}$

6. (*i*) $\dfrac{3}{x - y} - \dfrac{2(x - 2y)}{x^2 - y^2}$ (*ii*) $\dfrac{3}{2x^2 + x - 1} - \dfrac{2}{3x^2 + 2x - 1}$