Chapter 12

EXPONENTS

You already know that $3 \times 3 \times 3 \times 3 \times 3$ can be written as 3^5. Here 3 is the base and 5 is the exponent or index. 3^5 is read as '3 raised to the power 5' or '3 to the power 5' or simply '3 power 5'. In general, we have :

If a is any real number and n is a natural number, then $a^n = a \times a \times a....$ *n times* *where a is called the* ***base****, n is called the* ***exponent*** *or* ***index*** *and* a^n *is the* ***exponential form****.* a^n *is read as 'a raised to the power n' or 'a to the power n' or simply 'a power n'.*

For zero power, we have : $a^0 = 1$ (where $a \neq 0$).

For example :

(i) $7^0 = 1$ (ii) $\left(-\frac{2}{3}\right)^0 = 1$ (iii) $(\sqrt{7})^0 = 1$

For negative powers, we have :

$$a^{-n} = \frac{1}{a^n} \text{ and } \frac{1}{a^{-n}} = a^n \text{ (where } a \neq 0).$$

For example :

(i) $5^{-3} = \frac{1}{5^3} = \frac{1}{125}$ (ii) $(-3)^{-4} = \frac{1}{(-3)^4} = \frac{1}{81}$

(iii) $\frac{1}{2^{-5}} = 2^5 = 32$ (iv) $\frac{1}{(-7)^{-3}} = (-7)^3 = -243$

For fractional indices, remember that :

$$\sqrt[n]{a} = a^{\frac{1}{n}} \text{ and } \sqrt[n]{a^m} = (a^m)^{\frac{1}{n}} = a^{\frac{m}{n}}$$

For example :

(i) $\sqrt{3} = 3^{\frac{1}{2}}$ (ii) $\sqrt[3]{8} = 8^{\frac{1}{3}}$ (iii) $\sqrt[4]{5^3} = 5^{\frac{3}{4}}$

LAWS OF EXPONENTS

If a and b are any two real numbers and m and n are any two integers, then the following results hold :

1. $a^m \times a^n = a^{m+n}$
2. $a^m \div a^n = a^{m-n}$ $(a \neq 0)$
3. $(a^m)^n = a^{mn}$
4. $(ab)^m = a^m b^m$
5. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ $(b \neq 0)$
6. $a^n = a^m \Rightarrow n = m$, *provided* $a > 0$ *and* $a \neq 1$

Remark

If a is any real number and n is a natural number, then

$$(-a)^n = (-1 \times a)^n = (-1)^n a^n = \begin{cases} a^n \text{ if } n \text{ is even} \\ -a^n \text{ if } n \text{ is odd} \end{cases}$$

Example 1. Use the laws of exponents to simplify the following :

(i) $[(2^3)^4]^5$ (ii) $[3^6 \div 3^4]^3$ (iii) $(2^4)^3 \times 2 \times 3^0$

(iv) $(81)^{-1} \times 3^5$ (v) $\left(\frac{2}{3}\right)^0 + \left(\frac{2}{3}\right)^{-2}$ (vi) $(3^{-2} \times 5^3)^4$.

Solution. (i) $[(2^3)^4]^5 = [2^{3 \times 4}]^5 = [2^{12}]^5 = 2^{12 \times 5} = 2^{60}$.

(ii) $[3^6 \div 3^4]^3 = [3^{6-4}]^3 = [3^2]^3 = 3^{2 \times 3} = 3^6$.

(iii) $(2^4)^3 \times 2 \times 3^0 = 2^{12} \times 2^1 \times 1 = 2^{12+1} = 2^{13}$.

(iv) $(81)^{-1} \times 3^5 = (3^4)^{-1} \times 3^5 = 3^{-4} \times 3^5 = 3^{-4+5} = 3^1 = 3$.

(v) $\left(\frac{2}{3}\right)^0 + \left(\frac{2}{3}\right)^{-2} = 1 + \frac{1}{\left(\frac{2}{3}\right)^2} = 1 + \frac{1}{\frac{2^2}{3^2}} = 1 + \frac{1}{\frac{4}{9}} = 1 + \frac{9}{4} = \frac{13}{4}$.

(vi) $(3^{-2} \times 5^3)^4 = (3^{-2})^4 \times (5^3)^4 = 3^{-2 \times 4} \times 5^{3 \times 4} = 3^{-8} \times 5^{12}$.

Example 2. Simplify the following :

(i) $\frac{(-2)^5 \times (3^5)^2}{12 \times 3^7}$ (ii) $\frac{(2^3 \times 3^4)^3 \times (-5)^3}{60 \times (-2)^5}$ (iii) $\frac{(2^{-4})^2 \times 2^{-5}}{2^{-6}}$.

Solution. (i) $\frac{(-2)^5 \times (3^5)^2}{12 \times 3^7} = \frac{(-2^5) \times 3^{5 \times 2}}{2 \times 2 \times 3 \times 3^7} = -\frac{2^5 \times 3^{10}}{2^2 \times 3^{7+1}}$

$= -2^{5-2} \times 3^{10-8} = -2^3 \times 3^2 = -8 \times 9 = -72$.

(ii) $\frac{(2^3 \times 3^4)^3 \times (-5)^3}{60 \times (-2)^5} = \frac{(2^3)^3 \times (3^4)^3 \times (-5^3)}{2 \times 2 \times 3 \times 5 \times (-2^5)} = \frac{-2^9 \times 3^{12} \times 5^3}{-2^2 \times 3 \times 5 \times 2^5}$

$= 2^{9-2-5} \times 3^{12-1} \times 5^{3-1} = 2^2 \times 3^{11} \times 5^2$.

(iii) $\frac{(2^{-4})^2 \times 2^{-5}}{2^{-6}} = \frac{2^{-8} \times 2^{-5}}{2^{-6}} = 2^{-8+(-5)-(-6)} = 2^{-7} = \frac{1}{2^7} = \frac{1}{128}$.

Example 3. Simplify and write in the exponential form :

$2^3 \times 3^2 + (-11)^2 + 2^{-5} \div 2^{-8} - \left(-\frac{2}{5}\right)^0$

Solution. $2^3 \times 3^2 + (-11)^2 + 2^{-5} \div 2^{-8} - \left(-\frac{2}{5}\right)^0$

$= 8 \times 9 + 11^2 + 2^{-5-(-8)} - 1 = 72 + 121 + 2^3 - 1$

$= 72 + 121 + 8 - 1 = 200$

$= 2 \times 2 \times 2 \times 5 \times 5 = 2^3 \times 5^2$.

Example 4. Simplify the following :

(i) $\frac{3x^4y^3}{18x^3y^5}$ (ii) $\left(\frac{-2x^2}{y^3}\right)^3$ (iii) $\frac{5^{n+2} - 5^{n+1}}{5^{n+3}}$.

Solution. (i) $\frac{3x^4y^3}{18x^3y^5} = \frac{3}{18} \cdot \frac{x^4}{x^3} \cdot \frac{y^3}{y^5} = \frac{1}{6} x^{4-3} \cdot y^{3-5} = \frac{1}{6} xy^{-2} = \frac{x}{6y^2}$.

(ii) $\left(\frac{-2x^2}{y^3}\right)^3 = (-2)^3 \cdot \frac{(x^2)^3}{(y^3)^3} = -\frac{8x^6}{y^9}$.

(iii) $\frac{5^{n+2} - 5^{n+1}}{5^{n+3}} = \frac{5^n.5^2 - 5^n.5^1}{5^n.5^3} = \frac{5^n(5^2 - 5^1)}{5^n.5^3} = \frac{25-5}{125} = \frac{20}{125} = \frac{4}{25}$.

Example 5. Simplify and express the following in positive indices only :

(i) $\dfrac{3^{-5} \times 5^{-7} \times (-2)^3}{3^4 \times 5^{-2} \times (-2)^{-3}}$ (ii) $(4^{-2} \times 3^{-3})^2 \div 6^{-4}$ (iii) $\dfrac{a^{-4} \times b^{-7} \times c^{-3} \times d^3}{a^{-7} \times b^{-9} \times c^3 \times d^3}$.

Solution. (i) $\dfrac{3^{-5} \times 5^{-7} \times (-2)^3}{3^4 \times 5^{-2} \times (-2)^{-3}} = 3^{-5-4} \times 5^{-7-(-2)} \times (-2)^{3-(-3)}$

$$= 3^{-9} \times 5^{-5} \times (-2)^6 = \frac{2^6}{3^9 \times 5^5}.$$

(ii) $(4^{-2} \times 3^{-3})^2 \div 6^{-4} = \dfrac{(4^{-2})^2 \times (3^{-3})^2}{6^{-4}} = \dfrac{4^{-4} \times 3^{-6}}{(2 \times 3)^{-4}} = \dfrac{(2^2)^{-4} \times 3^{-6}}{2^{-4} \times 3^{-4}}$

$$= 2^{-8-(-4)} \times 3^{-6-(-4)} = 2^{-4} \times 3^{-2} = \frac{1}{2^4 \times 3^2}.$$

(iii) $\dfrac{a^{-4} \times b^{-7} \times c^{-3} \times d^3}{a^{-7} \times b^{-9} \times c^3 \times d^3} = a^{-4-(-7)} \times b^{-7-(-9)} \times c^{-3-3} \times d^{3-3}$

$$= a^3 \times b^2 \times c^{-6} \times d^0 = \frac{a^3 \times b^2 \times 1}{c^6} = \frac{a^3b^2}{c^6}.$$

Example 6. Simplify : $\dfrac{5^{n+2} - 6 \times 5^{n+1}}{13 \times 5^n - 2 \times 5^{n+1}}$

Solution. $\dfrac{5^{n+2} - 6 \times 5^{n+1}}{13 \times 5^n - 2 \times 5^{n+1}} = \dfrac{5^n \times 5^2 - 6 \times 5^n \times 5^1}{13 \times 5^n - 2 \times 5^n \times 5^1} = \dfrac{5^n(5^2 - 6 \times 5)}{5^n(13 - 2 \times 5)} = \dfrac{25 - 30}{13 - 10} = -\dfrac{5}{3}.$

Example 7. Show that $\left(\dfrac{x^m}{x^n}\right)^{m+n} \cdot \left(\dfrac{x^n}{x^l}\right)^{n+l} \cdot \left(\dfrac{x^l}{x^m}\right)^{l+m} = 1$

Solution. $\left(\dfrac{x^m}{x^n}\right)^{m+n} \cdot \left(\dfrac{x^n}{x^l}\right)^{n+l} \cdot \left(\dfrac{x^l}{x^m}\right)^{l+m} = (x^{m-n})^{m+n} \cdot (x^{n-l})^{n+l} \cdot (x^{l-m})^{l+m}$

$$= x^{m^2-n^2} \cdot x^{n^2-l^2} \cdot x^{l^2-m^2} = x^{m^2-n^2+n^2-l^2+l^2-m^2}$$

$$= x^0 = 1.$$

Example 8. Simplify the following :

(i) $(27)^{2/3}$ (ii) $(125)^{-1/3}$ (iii) $\left(\dfrac{16}{81}\right)^{-3/4}$.

Solution. (i) $(27)^{2/3} = (3^3)^{2/3} = 3^{3 \times \frac{2}{3}} = 3^2 = 9.$

(ii) $(125)^{-1/3} = (5^3)^{-1/3} = 5^{3 \times \left(-\frac{1}{3}\right)} = 5^{-1} = \dfrac{1}{5^1} = \dfrac{1}{5}.$

(iii) $\left(\dfrac{16}{81}\right)^{-3/4} = \left(\dfrac{2^4}{3^4}\right)^{-3/4} = \left(\left(\dfrac{2}{3}\right)^4\right)^{-3/4} = \left(\dfrac{2}{3}\right)^{4 \times \left(-\frac{3}{4}\right)} = \left(\dfrac{2}{3}\right)^{-3} = \dfrac{2^{-3}}{3^{-3}} = \dfrac{3^3}{2^3} = \dfrac{27}{8}.$

Example 9. Simplify : $7^0 + \sqrt[5]{32} + (27)^{-2/3} + \sqrt[3]{3^6} - \left(\dfrac{16}{25}\right)^{-1/2}$.

Solution. $7^0 + \sqrt[5]{32} + (27)^{-2/3} + \sqrt[3]{3^6} - \left(\dfrac{16}{25}\right)^{-1/2}$

$$= 1 + (2^5)^{\frac{1}{5}} + (3^3)^{-\frac{2}{3}} + (3^6)^{\frac{1}{3}} - \left(\left(\frac{4}{5}\right)^2\right)^{-\frac{1}{2}}$$

$$= 1 + 2^{5\times\frac{1}{5}} + 3^{3\times\left(-\frac{2}{3}\right)} + 3^{6\times\frac{1}{3}} - \left(\frac{4}{5}\right)^{2\times\left(-\frac{1}{2}\right)}$$

$$= 1 + 2^1 + 3^{-2} + 3^2 - \left(\frac{4}{5}\right)^{-1}$$

$$= 1 + 2 + \frac{1}{3^2} + 9 - \frac{1}{\left(\frac{4}{5}\right)^1} = 1 + 2 + \frac{1}{9} + 9 - \frac{1}{\frac{4}{5}}$$

$$= 12 + \frac{1}{9} - \frac{5}{4} = \frac{432 + 4 - 45}{36} = \frac{391}{36} = 10\frac{31}{36}.$$

Example 10. Find n so that $2^{11} \div 2^5 = 2^{-3} \times 2^{2n-1}$.

Solution. Given $2^{11} \div 2^5 = 2^{-3} \times 2^{2n-1}$

$\Rightarrow \quad 2^{11-5} = 2^{-3+2n-1}$

$\Rightarrow \quad 2^6 = 2^{2n-4} \Rightarrow 6 = 2n - 4$ | $a^n = a^m \Rightarrow n = m$

$\Rightarrow \quad 6 + 4 = 2n \Rightarrow 2n = 10 \Rightarrow n = 5$

Hence, $n = 5$.

Example 11. Determine $(8x)^x$ if $9^{x+2} = 240 + 9^x$.

Solution. Given $9^{x+2} = 240 + 9^x$

$\Rightarrow \quad 9^x \times 9^2 - 9^x = 240 \Rightarrow 9^x(81 - 1) = 240$

$\Rightarrow \quad 80 \times 9^x = 240 \Rightarrow 9^x = \frac{240}{80} = 3$

$\Rightarrow \quad (3^2)^x = 3 \Rightarrow 3^{2x} = 3^1 \Rightarrow 2x = 1$ | $a^n = a^m \Rightarrow n = m$

$\Rightarrow \quad x = \frac{1}{2}$

$\therefore \quad (8x)^x = \left(8 \times \frac{1}{2}\right)^{\frac{1}{2}} = 4^{1/2} = (2^2)^{1/2} = 2^{2\times\frac{1}{2}} = 2^1 = 2.$

Exercise 12

1. Find the value of the following :

(*i*) $2^3 \times (-3)^2$ (*ii*) $(2^3)^2$ (*iii*) $(3^5 \div 3^3)^2$

(*iv*) $(2^2 \times 3^{-1})^3$ (*v*) $\left(1\frac{1}{4}\right)^{-1}$ (*vi*) $8^{-2} \times 2^7$

(*vii*) $5^{-1} + 5^0 + 5^1$ (*viii*) $\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^{-1} + \left(\frac{1}{2}\right)^{-2}$.

2. Find the value of :

(*i*) $7^4 \times 7^8 \div (7^5)^2$ (*ii*) $5^0 \times 8^3 \times 4^{-2}$ (*iii*) $((-2)^3)^{-1}$

(*iv*) $\frac{2^{-3}}{5^{-3}}$ (*v*) $\frac{3^{-5} \times 3^7}{3^{-2}}$ (*vi*) $(3^{-4} \div 3^{-3})^3$.

3. Find the value of :

(*i*) $(3^{-2} - 3^{-3}) \times (3^0 + 3^{-1})$ (*ii*) $\frac{(2^3 \times 3^2)^3 \times 6^{-2}}{(-2)^3 \times 3^3}$ (*iii*) $\frac{3^4 \times 3^{-2} \times 5^2}{120 \times (-6)^2}$.

4. Simplify and write the following in exponential form :

(i) $((-2)^3)^2 + 5^{-3} \div 5^{-5} - \left(-\frac{1}{2}\right)^0$

(ii) $3^{-5} \times 3^2 \div 3^{-6} + (2^2 \times 3)^2 + \left(\frac{2}{3}\right)^{-1} + 2^{-1} + \left(\frac{1}{19}\right)^{-1}$.

5. Simplify the following and write each with positive exponent :

(i) $\frac{5^{-3} \times 7^4 \times 11^{-4}}{5^{-5} \times 7^6 \times 11^3}$ (ii) $\frac{3^{-4} \times 3^2 \times 5^{-3}}{3^{-6} \times 5^{-4} \times 5^7}$

6. Simplify the following :

(i) $5x^9y^4 \div x^5y^{-2}$ (ii) $(a^3b^{-2})^{-5}$

(iii) $\left(\frac{-3xz^2}{2y^2}\right)^{-3}$ (iv) $2g^2\left(g^3 - g + \frac{1}{g} - \frac{1}{g^3}\right)$.

7. Find the value of :

(i) $(125)^{2/3}$ (ii) $(32)^{-2/5}$ (iii) $\left(\frac{8}{27}\right)^{-2/3}$

(iv) $(16)^{-3/4}$ (v) $\left(\frac{81}{16}\right)^{-1/4}$ (vi) $\left(\frac{27}{8}\right)^{-4/3}$.

8. Simplify the following :

(i) $(8x^3)^{1/3}$ (ii) $(27p^{-3})^{2/3}$ (iii) $\left(-3x^{\frac{1}{4}}y^{-\frac{3}{4}}\right)^4$

(iv) $(32p^{10}q^{-15})^{1/5}$ (v) $\sqrt[3]{x^9y^{-6}z^{12}}$ (vi) $\sqrt[4]{(p^4q^{-12})^3}$.

9. Express the following in positive indices only :

(i) $(xy^{-1})^{-2}$ (ii) $\frac{a^2 \times b^{-2} \times c}{a^{-3} \times b^5 \times c}$

(iii) $(3x^{-3}y^{-1})^3 \div x^{-4}y^2z^{-1}$ (iv) $(x^{-2}y)^{1/2}(xy^{-3})^{1/3}$.

10. Show that : $(x^{a+b})^{a-b} \times (x^{b+c})^{b-c} \times (x^{c+a})^{c-a} = 1$.

11. Simplify the following :

(i) $\frac{x^{2n+3} \times (x^2)^{n-1}}{x^{3n-5}}$ (ii) $\frac{5^{2(n+6)} \times (25)^{2n-7}}{(125)^{2n}}$ (iii) $\frac{x^{m+n} \times x^{n+l} \times x^{l+m}}{(x^m \times x^n \times x^l)^2}$.

12. Simplify the following :

(i) $9^{5/2} - 3 \times 5^0 - \left(\frac{1}{81}\right)^{-1/2}$ (ii) $\left(\frac{1}{4}\right)^{-2} - 3 \times 8^{2/3} \times 4^0 + \left(\frac{9}{16}\right)^{-1/2}$

(iii) $(64)^{2/3} - \sqrt[3]{125} + \frac{1}{2^{-5}} + (27)^{-2/3} \times \left(\frac{25}{9}\right)^{-1/2}$.

13. Show that : $\sqrt{x^{a-b}} \times \sqrt{x^{b-c}} \times \sqrt{x^{c-a}} = 1$.

14. Show that :

(i) $\left(\frac{x^m}{x^n}\right)^{\frac{1}{mn}} \left(\frac{x^n}{x^l}\right)^{\frac{1}{nl}} \left(\frac{x^l}{x^m}\right)^{\frac{1}{lm}} = 1$ (ii) $\frac{1}{1+x^{m-n}} + \frac{1}{1+x^{n-m}} = 1$.

15. (i) If $2^n = \sqrt[3]{8} \div (2^3)^{2/3}$, find n.   (ii) If $\frac{9^n \times 3^5 \times (27)^3}{3 \times (81)^4} = 27$, find n.

(iii) If $5^{2x-1} = \frac{1}{(125)^{x-3}}$, find x.   (iv) If $\left(\frac{a}{b}\right)^{3x+2} = \left(\frac{b}{a}\right)^{2-x}$, find x.

Summary

- If a is any real number and n is a natural number, then $a^n = a \times a \times a \ldots n$ times where a is called the **base**, n is called the **exponent** or **index**.

  In particular, $a^0 = 1$ and $a^{-n} = \frac{1}{a^n}$, $\frac{1}{a^{-n}} = a^n$ (where $a \neq 0$).

- For fractional indices : $\sqrt[n]{a} = a^{\frac{1}{n}}$ and $\sqrt[n]{a^m} = (a^m)^{\frac{1}{n}} = a^{\frac{m}{n}}$ $(a > 0)$.
- **Laws of exponents**

  If a and b are any two real numbers and m and n are any two integers, then
  - $a^m \times a^n = a^{m+n}$
  - $a^m \div a^n = a^{m-n}$ $(a \neq 0)$
  - $(a^m)^n = a^{mn}$
  - $(ab)^m = a^m b^m$
  - $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ $(b \neq 0)$
  - $a^n = a^m \Rightarrow n = m$, provided $a > 0$ and $a \neq 1$.

Check Your Progress

1. Find the value of :

   (i) $3^0 + 3^{-1} + 3^{-2} + 3^{-3} + \sqrt[5]{(32)^2}$ (ii) $(81)^{3/4} - \left(\frac{1}{32}\right)^{-2/5} - (8)^{1/3} \times \left(\frac{1}{2}\right)^{-1} \times 5^0$.

2. Simplify : $7^{-20} - 7^{-21}$.
3. Simplify the following :

   (i) $y^{2-a} \times y^{a-2}$ (ii) $\left(\frac{1}{3x}\right)^2 (2x)^3 (x^{-1})^0$ (iii) $\frac{a^{-1} + b^{-1}}{(ab)^{-1}}$.

4. Prove that $(a + b)^{-1} (a^{-1} + b^{-1}) = (ab)^{-1}$.
5. Show that $\left(\frac{x^m}{x^n}\right)^l \times \left(\frac{x^n}{x^l}\right)^m \times \left(\frac{x^l}{x^m}\right)^n = 1$.
6. If $\frac{2^{-n} \times 8^{2n+1} \times 16^{2n}}{4^{3n}} = \frac{1}{16}$, find n.