22 QUADRATIC EQUATIONS

- Quadratic Equation
- Solving and Framing Quadratic Equation

Introduction

A simple linear equation is an algebraic statement of equality that involves only one variable, the degree of which is not more than 1. Solving a simple linear equation gives us the one and only value of the variable that satisfies the equation.

> A quadratic equation is an algebraic statement of equality that involves only one variable, the degree of which is not more than 2. Solving a quadratic equation gives us two values of the variable, both of which satisfy the equation.

Quadratic equations are of two forms.

1. In the **pure form** of quadratic equations, the degree of the variable is always 2. Examples of pure quadratic equations are:

$$x^2 = 25,\ 8x^2 = 392$$

2. In the **standard form** of quadratic equations, the degree of the variable is 2 as well as 1. Examples of the standard form of quadratic equations are:

$$x^2 - 4x - 32 = 0,\ 6x^2 + x - 12 = 0$$

Solving Quadratic Equations

Example 1: Solve $6x^2 = 384$

$\Rightarrow \quad x^2 = \frac{384}{6} = 64$

$\Rightarrow \quad x^2 - 64 = 0$

$\Rightarrow \quad (x + 8)(x - 8) = 0$

Thus, either $x + 8 = 0$, or $x - 8 = 0$

when $x + 8 = 0 \Rightarrow x = -8$

when $x - 8 = 0 \Rightarrow x = 8$

Thus, the roots of $6x^2 = 384$ are ± 8.

Example 2: Solve $16y^2 = 25$

$\Rightarrow \quad y^2 = \frac{25}{16}$

$\Rightarrow \quad y^2 - \frac{25}{16} = 0$

$\Rightarrow \quad \left(y + \frac{5}{4}\right)\left(y - \frac{5}{4}\right) = 0$

Thus, either $y + \frac{5}{4} = 0$, or $y - \frac{5}{4} = 0$

when $y + \frac{5}{4} = 0 \Rightarrow y = -\frac{5}{4} = -1\frac{1}{4}$

when $y - \frac{5}{4} = 0 \Rightarrow y = \frac{5}{4} = 1\frac{1}{4}$

Thus, the roots of $16y^2 = 25$ are $\pm 1\frac{1}{4}$.

Example 3: Solve $2x^2 + 9x = 35$

Step 1: Move all terms to LHS leaving 0 on the RHS to convert the given equation to the standard form.

$\Rightarrow \quad 2x^2 + 9x - 35 = 0$

Step 2: Factorise the quadratic trinomial on LHS.

$\Rightarrow \quad 2x^2 + 14x - 5x - 35 = 0$

$\Rightarrow \quad 2x(x + 7) - 5(x + 7) = 0$

$\Rightarrow \quad (2x - 5)(x + 7) = 0$

Step 3: Treat each factor as a simple linear equation to solve and find the roots of the quadratic equation.

$$2x - 5 = 0 \quad \text{or} \quad x + 7 = 0$$
$$\Rightarrow \quad 2x = 5 \quad \text{or} \Rightarrow \quad x = -7$$
$$\Rightarrow \quad x = \frac{5}{2} \quad \text{or} \quad x = -7$$

Thus, the roots of $2x^2 + 9x = 35$ are $\frac{5}{2}$ and -7.

CHECK: When $x = \frac{5}{2} = 2.5$

$$(2 \times 2.5 \times 2.5) + (9 \times 2.5) = 35$$
$$\Rightarrow \quad 12.5 + 22.5 = 35$$
$$\Rightarrow \quad 35 = 35$$

When $x = -7$

$$(2 \times -7 \times -7) + (9 \times -7) = 35$$
$$\Rightarrow \quad 98 - 63 = 35$$
$$\Rightarrow \quad 35 = 35$$

The roots obtained satisfy the given quadratic equation.

Example 4: Solve $x^2 - x - 30 = 0$

$$\Rightarrow \quad x^2 - 6x + 5x - 30 = 0$$
$$\Rightarrow \quad x(x-6) + 5(x-6) = 0$$
$$\Rightarrow \quad (x+5)(x-6) = 0$$
$$\Rightarrow \quad x + 5 = 0 \quad \text{or} \quad x - 6 = 0$$
$$\Rightarrow \quad x = -5 \quad \text{or} \quad x = 6$$

Thus, the roots of $x^2 - x - 30 = 0$ are -5 and 6.

Example 5: Solve $\frac{7x}{x^2 + 10} = \frac{6}{x+1}$

$$\Rightarrow \quad 7x(x+1) = 6(x^2 + 10)$$
$$\Rightarrow \quad 7x^2 + 7x = 6x^2 + 60$$
$$\Rightarrow \quad 7x^2 - 6x^2 + 7x - 60 = 0$$
$$\Rightarrow \quad x^2 + 7x - 60 = 0$$
$$\Rightarrow \quad x^2 + 12x - 5x - 60 = 0$$
$$\Rightarrow \quad x(x+12) - 5(x+12) = 0$$
$$\Rightarrow \quad (x-5)(x+12) = 0$$
$$\Rightarrow \quad x - 5 = 0 \quad \text{or} \quad x + 12 = 0$$
$$\Rightarrow \quad x = 5 \quad \text{or} \quad x = -12$$

Thus, the roots of $\frac{7x}{x^2+10} = \frac{6}{x+1}$ are 5 and -12.

Example 6: Solve $\frac{x+3}{3x-5} = \frac{2x-5}{x-1}$

$$\Rightarrow \quad (x+3)(x-1) = (2x-5)(3x-5)$$
$$\Rightarrow \quad x^2 - x + 3x - 3 = 6x^2 - 10x - 15x + 25$$
$$\Rightarrow \quad x^2 + 2x - 3 = 6x^2 - 25x + 25$$
$$\Rightarrow \quad 6x^2 - 25x + 25 - x^2 - 2x + 3 = 0$$
$$\Rightarrow \quad 5x^2 - 27x + 28 = 0$$
$$\Rightarrow \quad 5x^2 - 20x - 7x + 28 = 0$$
$$\Rightarrow \quad 5x(x-4) - 7(x-4) = 0$$
$$\Rightarrow \quad (5x-7)(x-4) = 0$$
$$\Rightarrow \quad 5x - 7 = 0 \quad \text{or} \quad x - 4 = 0$$
$$\Rightarrow \quad 5x = 7 \quad \text{or} \quad x = 4$$
$$\Rightarrow \quad x = \frac{7}{5} \quad \text{or} \quad x = 4$$

Thus, the roots of $\frac{x+3}{3x-5} = \frac{2x-5}{x-1}$ are 4 and $\frac{7}{5}$.

Some equations can be reduced to the standard form of quadratic equations by simple substitution.

Example 7: Solve $(a-2)^2 - 6 = a - 2$

Let $a - 2 = x$, then we have

$$x^2 - 6 = x$$
$$\Rightarrow \quad x^2 - x - 6 = 0$$
$$\Rightarrow \quad x^2 - 3x + 2x - 6 = 0$$
$$\Rightarrow \quad x(x-3) + 2(x-3) = 0$$
$$\Rightarrow \quad (x+2)(x-3) = 0$$
$$\Rightarrow \quad x + 2 = 0 \text{ or } x - 3 = 0$$
$$\Rightarrow \quad x = -2 \text{ or } x = 3$$

As $a - 2 = x$,

$$\Rightarrow \quad a - 2 = -2 \text{ or } a - 2 = 3$$
$$\Rightarrow \quad a = 0 \text{ or } a = 5$$

Example 8: Solve $a^4 - 25a^2 + 144 = 0$

Let $a^2 = x$, then we have

$$x^2 - 25x + 144 = 0$$
$$\Rightarrow \quad x^2 - 9x - 16x + 144 = 0$$
$$\Rightarrow \quad x(x-9) - 16(x-9) = 0$$
$$\Rightarrow \quad (x-16)(x-9) = 0$$
$$\Rightarrow \quad x - 16 = 0 \text{ or } x - 9 = 0$$
$$\Rightarrow \quad x = 16 \text{ or } x = 9$$

As $a^2 = x$,

$$\Rightarrow \quad a^2 = 16 \quad \text{or} \quad a^2 = 9$$
$$\Rightarrow \quad a = \pm 4 \quad \text{or} \quad a = \pm 3$$

Try this!

1. Solve $16x^2 - 144 = 0$
2. $x^2 - 14x + 49 = 0$

Framing a Quadratic Equation

A quadratic equation is solved by expressing it as a product of factors being equal to 0. As a reverse process, given the roots, we can frame a quadratic equation as shown in the following examples.

Example 9: Frame the quadratic equation whose roots are 2 and –4.

Given $x = 2$ or $x = -4$

$\Rightarrow$ $x - 2 = 0$ or $x + 4 = 0$

Multiplying the two equations, we have

$$(x - 2)(x + 4) = 0$$

$$\Rightarrow \quad x^2 + 4x - 2x - 8 = 0$$

$$\Rightarrow \quad x^2 + 2x - 8 = 0$$

Example 10: Frame the quadratic equation whose roots are $-\frac{3}{5}$ and $\frac{2}{7}$.

Given $x = -\frac{3}{5}$ or $x = \frac{2}{7}$

$\Rightarrow$ $5x = -3$ or $7x = 2$

$\Rightarrow$ $5x + 3 = 0$ or $7x - 2 = 0$

Multiplying the two equations, we have

$$(5x + 3)(7x - 2) = 0$$

$$\Rightarrow \quad 35x^2 - 10x + 21x - 6 = 0$$

$$\Rightarrow \quad 35x^2 + 11x - 6 = 0$$

Try this!

Frame the quadratic equation that would have the following roots:

(i) 6 and 8 (ii) $-\frac{2}{3}$ and $\frac{1}{3}$

Exercise 22.1

1. How many roots do each of the following equations have?
   (i) $3x - 2 = 7$ (ii) $25x - 9 = 0$
   (iii) $25x^2 - 9 = 0$ (iv) $\frac{x}{16} = 1$
   (v) $\frac{x^2}{16} = 1$ (vi) $x + 4x - 3 = 0$
   (vii) $x^2 + 4x - 5 = 0$ (viii) $x^2 + 3x = 0$
2. Solve the following quadratic equations.
   (i) $x^2 = 81$
   (ii) $2x^2 = 450$
   (iii) $4x^2 = 64$
   (iv) $5x^2 - 245 = 0$
   (v) $6x^2 - 726 = 0$
   (vi) $(x + 7)(x - 7) = 0$
   (vii) $(2x + 1)(3x - 2) = 0$
   (viii) $x(2x - 1) = 0$
   (ix) $x^2 - x - 6 = 0$
   (x) $x^2 + 5x + 4 = 0$
   (xi) $x^2 - 7x + 10 = 0$
   (xii) $x^2 - 4x - 21 = 0$
   (xiii) $x^2 + 2x - 35 = 0$
   (xiv) $x^2 - 10x - 24 = 0$
   (xv) $2x^2 - 5x + 2 = 0$
   (xvi) $5x^2 - 16x + 3 = 0$
   (xvii) $3x^2 + 7x - 6 = 0$
   (xviii) $4x^2 + 17x - 15 = 0$
   (xix) $5x^2 = 18 - 27x$
   (xx) $8x^2 + 1 = 6x$
   (xxi) $15x^2 + 14x + 3 = 0$
   (xxii) $15x^2 = x + 6$
   (xxiii) $21x^2 = 23x - 6$
   (xxiv) $3x + 10 = \frac{8}{x}$
   (xxv) $6(x^2 + 1) = 13x$

(xxvi) $5x - 2 = \frac{3}{5x}$

(xxvii) $\frac{20 - 3x}{x + 3} = 8x$

(xxviii) $\frac{5x}{x+9} = \frac{2}{x-5}$

(xxix) $\frac{5x+7}{3x-1} = 8x$

(xxx) $\frac{2x}{x^2+9} = \frac{3}{2x+3}$

(xxxi) $\frac{3}{x-1} = \frac{4x}{x^2+4}$

(xxxii) $\frac{x}{x^2-1} = -\frac{2}{4x+1}$

(xxxiii) $\frac{5x-20}{4x+8} = \frac{x-1}{x+2}$

3. Solve the following by reducing to the standard form of quadratic equations.
   (i) $(4 + a)^2 + 15 = 8(4 + a)$
   (ii) $(2a - 1)^2 + 35 = 12(2a - 1)$
   (iii) $(3a + 5)^2 + 16 = 30a + 50$
   (iv) $a^4 - 29a^2 + 100 = 0$
   (v) $a^6 - 35a^3 + 216 = 0$
   (vi) $\frac{6}{a^2} - \frac{5}{a} + 1 = 0$ (*Hint*: Let $\frac{1}{a} = x$)
4. Frame quadratic equations that would have the following roots.
   (i) 2 and 3
   (ii) 2 and –2
   (iii) 4 and –5
   (iv) –3 and 7
   (v) $\frac{1}{2}$ and 4
   (vi) 3 and $\frac{1}{3}$
   (vii) $\frac{3}{5}$ and $\frac{1}{2}$
   (viii) $-\frac{4}{5}$ and $\frac{1}{4}$
   (ix) $\frac{3}{8}$ and $-\frac{1}{3}$
   (x) $-\frac{2}{5}$ and $-\frac{3}{7}$

Revision Exercise

1. Solve $9x^2 = 729$
2. Solve $25y^2 = 49$
3. Solve $4x^2 + 52x + 25 = 0$
4. Solve $x^4 - 22x^2 + 120 = 0$
5. Solve: $\frac{10x+14}{6x-2} = 16x$