17 FACTORISATION

- Factorisation
- Factorising the Difference of Two Squares
- Factorising Quadratic Trinomials

Factorisation of Algebraic Expressions

Factorisation of an algebraic expression means finding two or more expressions which when multiplied give the original expression as the product.

Thus, factorisation is the opposite of finding the product, as illustrated by the following examples.

Finding the product	Factorisation
1. Multiplication by a monomial $3x^2y(3x + 2xy + y)$ $= 9x^3y + 6x^3y^2 + 3x^2y^2$	$9x^3y + 6x^3y^2 + 3x^2y^2$ $= 3x^2y(3x + 2xy + y)$
2. Multiplication of binomials $(3x + y)(x + 2y)$ $= 3x^2 + 7xy + 2y^2$	$3x^2 + 7xy + 2y^2$ $= (3x + y)(x + 2y)$
3. Special products $(2x + 3y)(2x - 3y)$ $= 4x^2 - 9y^2$	$4x^2 - 9y^2$ $= (2x + 3y)(2x - 3y)$
4. $(5x + y)(5x + y)$ $= 25x^2 + 10xy + y^2$	$25x^2 + 10xy + y^2$ $= (5x + y)(5x + y)$
5. $(x - 3y)(x - 3y)$ $= x^2 - 6xy + 9y^2$	$x^2 - 6xy + 9y^2$ $= (x - 3y)(x - 3y)$

In class VII we factorised expressions by taking out the HCF as the common factor, by the grouping method and by expressing the product as a difference of squares, as recapitulated in the following examples.

***Method I:* Factorisation by taking out the common factor**

Example 1: Factorise $2x^4y^2 - 8x^4y^3 + 6x^3y^3 - 4x^3y^2$

Find the HCF of all the terms in the expression

$$2x^4y^2 = 2\times x\times x\times x\times x\times y\times y$$

$$8x^4y^3 = 2\times 2\times 2\times x\times x\times x\times x\times y\times y\times y$$

$$6x^3y^3 = 2\times 3\times x\times x\times x\times y\times y\times y$$

$$4x^3y^2 = 2\times 2\times x\times x\times x\times y\times y$$

Thus, HCF of all the terms in the polynomial

$$= 2\times x\times x\times x\times y\times y = 2x^3y^2$$

Dividing the polynomial by the HCF, we get

$$\frac{2x^4y^2}{2x^3y^2} - \frac{8x^4y^3}{2x^3y^2} + \frac{6x^3y^3}{2x^3y^2} - \frac{4x^3y^2}{2x^3y^2}$$

$$= x - 4xy + 3y - 2$$

Thus, $2x^4y^2 - 8x^4y^3 + 6x^3y^3 - 4x^3y^2$

$$= 2x^3y^2(x - 4xy + 3y - 2)$$

***Method II:* Factorisation by the grouping method**

Example 2: Factorise $18xy - 9y + 30x - 15$

$= 3(6xy - 3y + 10x - 5)$ (taking 3 as a common factor)

$= 3(6xy + 10x - 3y - 5)$ (grouping terms with common factor)

$= 3\{2x(3y + 5) - 1(3y + 5)\}$

$= 3(3y + 5)(2x - 1)$ [taking $(3y + 5)$ as common factor]

Example 3: Factorise $2x^5 + 9x^2 - 3x^3 + 3x - 6x^4 - 9$

$= 2x^5 - 6x^4 - 3x^3 + 9x^2 + 3x - 9$ (rearranging terms in decreasing powers of variable)

$= 2x^4 (x - 3) - 3x^2 (x - 3) + 3(x - 3)$
(taking common factors from pairs)
$= (x - 3)(2x^4 - 3x^2 + 3)$
[taking $(x - 3)$ as common factor]

Example 4: Factorise $2x^3 + 6xy + 3x^2 + 4x^2y + 8x + 12$

The terms can be rearranged in decreasing powers of x in more than one way.

$$2x^3 + 3x^2 + 4x^2y + 8x + 6xy + 12$$

Taking common factors from pairs we get

$$x^2(2x + 3) + 4x(xy + 2) + 6(xy + 2)$$

But this grouping does not help as the factors left are different. The terms have to be rearranged in pairs in such a way that after dividing each pair by the common factors, the same factor is left.

Rearranging as:

$$2x^3 + 3x^2 + 4x^2y + 6xy + 8x + 12$$
$$= x^2(2x + 3) + 2xy(2x + 3) + 4(2x + 3)$$
$$= (2x + 3)(x^2 + 2xy + 4)$$

> Observe that the factorisation of the first pair as $x^2(2x + 3)$ gave an indication on how the other terms would be grouped such that after each group was divided by a common factor, $(2x + 3)$ would be left as the other factor.

Factorising the Difference of Two Squares

> As $(a + b)(a - b) = a^2 - b^2$,
> $a^2 - b^2 = (a + b)(a - b)$

Example 5: Factorise $\frac{8x^2p}{27z^2} - \frac{18y^2p}{12}$

$$\frac{8x^2p}{27z^2} - \frac{18y^2p}{12} = \frac{2p}{3}\left(\frac{4x^2}{9z^2} - \frac{9y^2}{4}\right)$$

(taking $\frac{2p}{3}$ as common factor)

As $\frac{4x^2}{9z^2} = \left(\frac{2x}{3z}\right)^2$ and $\frac{9y^2}{4} = \left(\frac{3y}{2}\right)^2$,

$$\frac{2p}{3}\left(\frac{4x^2}{9z^2} - \frac{9y^2}{4}\right) = \frac{2p}{3}\left(\frac{2x}{3z} + \frac{3y}{2}\right)\left(\frac{2x}{3z} - \frac{3y}{2}\right)$$

Example 6: Factorise $81x^4 - 16$

$$81x^4 - 16 = (9x^2)^2 - 4^2$$
$$= (9x^2 + 4)(9x^2 - 4)$$
$$= (9x^2 + 4)\{(3x)^2 - 2^2\}$$
$$= (9x^2 + 4)(3x + 2)(3x - 2)$$

Try this!

1. Factorise $7x^2 + 8xy$
2. Factorise $144x^4 - 121$

Exercise 17.1

1. Factorise the following:

(i) $2x^2 + 6xy$

(ii) $6x^2y^2 - 9y^2x$

(iii) $24x^2y^2 - 36x^2y$

(iv) $12y^2x^2 + 15x^3y + 6x^4$

(v) $7.5x^2y^2 - 4.5y^3x$

(vi) $1.2y^3x^2 - 2.4y^2x + 3.6y$

(vii) $\frac{2x^3y}{3} - \frac{2x^2y^2}{3}$

(viii) $\frac{x^3}{6} + \frac{x^2y}{2} - \frac{y^2x}{8}$

(ix) $\frac{4x^3y^2}{3} + \frac{2x^3y}{21} - 2x^2y^2$

(x) $\frac{21x^3}{y^2} + \frac{6x^2}{y^2} - \frac{9x^3}{y}$

(xi) $\frac{x^3y^3}{6} - \frac{x^3y^2}{4} - 4y^3x^2$

(xii) $\frac{5x^2}{9y^2z} - \frac{5x^2}{6y^2z^2} + \frac{15x^2}{12y^2}$

(xiii) $3x^2(y^2 + 4) - 2y^2(y^2 + 4)$

(xiv) $5y^3(x^2-5) - 2y^2(x^2-5) + 3y(x^2-5)$

(xv) $x^2y(x-z) + y^2x(z-x)$

2. Factorise the following:
   (i) $6xy - 15x + 8y - 20$
   (ii) $10xy + 5x - 14y - 7$
   (iii) $15x^2 - 10xy + 12x - 8y$
   (iv) $6y - 3xy + 6x - 3x^2$
   (v) $63xy - 72y - 48 + 42x$
   (vi) $48x^4yz - 16y^3x^2z + 120x^2yz - 40y^3z$
   (vii) $9xy - 3x + 3y - 1$
   (viii) $10x^3 + 5x^2 + 2x + 1$
   (ix) $9y^3x^2 + 18x^3y^2 + 21x^2y^2 + 42x^3y$
   (x) $50y^3x + 30y^3 - 40y^2x - 24y^2$
   (xi) $98x^3y + 343x^2y - 14x^2y^2 - 49y^2x$
   (xii) $3x^3y^2 - 9x^3y + 3x^2y^2 - 9x^2y$
   (xiii) $x^3 + xy + 2x - 5x^2 - 5y - 10$
   (xiv) $2x^2y^2 + 4x^2 + 3y^2x - 4y^2 + 6x - 8$

3. Factorise the following:
   (i) $x^2 - 16$
   (ii) $9 - 4y^2$
   (iii) $25x^2 - 36y^2$
   (iv) $49x^4 - 121y^2$
   (v) $12x^2y^2 - 300$
   (vi) $245x^5 - 80y^2x$
   (vii) $288x^3y^3 - 18xy$
   (viii) $12x^2y^2 - 75y^4$
   (ix) $16x^4 - 1$
   (x) $256x^4 - y^4$
   (xi) $81x^4 - \dfrac{1}{81x^4}$
   (xii) $256y^8 - 1$

Factorising Quadratic Trinomials

Case I: Factorising quadratic trinomials that are perfect squares

1. $(a+b)^2 = a^2 + 2ab + b^2$
   $= (\text{term } 1)^2 + (2 \times \text{term } 1 \times \text{term } 2) + (\text{term } 2)^2$
   $= (\text{term } 1 + \text{term } 2)\,(\text{term } 1 + \text{term } 2)$

2. $(a-b)^2 = a^2 - 2ab + b^2$
   $= (\text{term } 1)^2 - (2 \times \text{term } 1 \times \text{term } 2)\ (\text{term } 2)^2$
   $= (\text{term } 1 - \text{term } 2)\,(\text{term } 1 - \text{term } 2)$

Example 7: Factorise $18x^3 + 48x^2y + 32y^2x$

$= 2x(9x^2 + 24xy + 16y^2)$ (taking $2x$ as common factor)

$= 2x(3x+4y)(3x+4y)$ [as $a^2 + 2ab + b^2 = (a+b)(a+b)$]

Example 8: Factorise $4x^2 - 28xy + 49y^2 - 16z^2$

$= (4x^2 - 28xy + 49y^2) - 16z^2$ (separating perfect square trinomial)

$= (2x - 7y)^2 - (4z)^2$ [as $a^2 - 2ab + b^2 = (a-b)^2$]

$= (2x - 7y + 4z)(2x - 7y - 4z)$ [as $a^2 - b^2 = (a+b)(a-b)$]

Case II: Factorising quadratic trinomials that are not perfect squares

Consider the following expansion:

$(2x+3y)(x+2y) = 2x^2 + 4xy + 3xy + 6y^2$
$= 2x^2 + 7xy + 6y^2$

The middle term of the quadratic trinomial $7xy$ is the sum of $4xy$ and $3xy$. Factorising being the reverse of multiplication, $2x^2 + 7xy + 6y^2$ would involve splitting up $7xy$ into $4xy$ and $3xy$ such that the terms could be grouped and common factors obtained. But how do we know that $7xy$ should be split as $4xy$ and $3xy$ but not as $5xy$ and $2xy$?

The pattern that indicates how products are formed will help in splitting the middle term.

1. Multiply the numerical coefficients of the first and the last terms.
   $2 \times 6 = 12$
2. By trial and error, think of two factors of the product 12 that add up to the sum 7.
   $3 \times 4 = 12$ and $3 + 4 = 7$

Thus, $2x^2 + 7xy + 6y^2$

$= 2x^2 + 4xy + 3xy + 6y^2$

(here $3xy$ has been grouped with $6y^2$ so as to get a numerical factor in common)

$= 2x(x + 2y) + 3y(x + 2y)$

$= (2x + 3y)(x + 2y)$

Example 9: Factorise $10x^2 - 21xy + 9y^2$

Here the product of the first and last numerical coefficients is $9 \times 10 = 90$, a positive integer.

The numerical coefficient of the middle term is a negative integer, -21.

The product indicates that factors must be either both positive or both negative. The sum suggests that the factors will be both negative.

$+90 = -15 \times -6$ and $-15 + (-6) = -21$

Thus, $10x^2 - 21xy + 9y^2$

$= 10x^2 - 15xy - 6xy + 9y^2$

$= 5x(2x - 3y) - 3y(2x - 3y)$

$= (5x - 3y)(2x - 3y)$

Example 10: Factorise $2x^2 + 15xy - 8y^2$

Product $= -8 \times 2 = -16$, a negative integer

$\Rightarrow$ opposite signs of factors

Sum $= +15$, a positive integer

$\Rightarrow$ absolute value of +ve factor > absolute value of −ve factor

$-16 = +16 \times -1$ and $+16 + (-1) = +15$

Thus, $2x^2 + 15xy - 8y^2$

$= 2x^2 - xy + 16xy - 8y^2$

$= x(2x - y) + 8y(2x - y)$

$= (x + 8y)(2x - y)$

Try this!

Factorise $18x^2 + 57xy - 21y^2$

Example 11: Factorise $12x^2 - 7xy - 10y^2$

Product $= 12 \times -10 = -120$, a negative integer

$\Rightarrow$ opposite signs of factors

Sum $= -7$, a negative integer

$\Rightarrow$ absolute value of −ve factor > absolute value of +ve factor

$-120 = -15 \times +8$ and $-15 + (+8) = -7$

Thus, $12x^2 - 7xy - 10y^2$

$= 12x^2 + 8xy - 15xy - 10y^2$

$= 4x(3x + 2y) - 5y(3x + 2y)$

$= (4x - 5y)(3x + 2y)$

Example 12: Factorise $2x^3y + 12x^2y - 54xy$

$2x^3y + 12x^2y - 54xy$

$= 2xy(x^2 + 6x - 27)$

$= 2xy(x^2 - 3x + 9x - 27)$

(as $-3 \times 9 = -27$ and $-3 + 9 = +6$)

$= 2xy\{x(x - 3) + 9(x - 3)\}$

$= 2xy(x + 9)(x - 3)$

Example 13: Factorise $(x + y)^2 - 7(x + y) + 12$

Let $x + y$ be represented by a, then the expression becomes:

$a^2 - 7a + 12$

$= a^2 - 4a - 3a + 12$

(as $-4 \times -3 = +12$ and $-4 + (-3) = -7$)

$= a(a - 4) - 3(a - 4) = (a - 3)(a - 4)$

Now substituting $a = x + y$

$(x + y)^2 - 7(x + y) + 12 = (x + y - 3)(x + y - 4)$

Remember to split the middle term in such a way that it satisfies both the signs and magnitudes of the product and sum.

Try this!

Factorise $6(x + y)^2 + 13(x + y) + 6$

Exercise 17.2

1. Factorise the following:

(i) $9x^2 + 24x + 16$

(ii) $4x^2 - 20x + 25$

(iii) $x^2 + 8xy + 16y^2$

(iv) $4x^2 - 24xy + 36y^2$

(v) $9x^2 - 42xy + 49y^2$

(vi) $x^4 + 4x^2y^2 + 4y^4$

(vii) $9x^6 - 12x^3y^3 + 4y^6$

(viii) $64x^4y^2 + 144x^2y + 81$

(ix) $121 - 66y^2x + 9y^4x^2$

(x) $144x^4z^2 - 48y^3x^2z + 4y^6$

(xi) $24x^2 + 120xy + 150y^2$

(xii) $108x^4 - 144x^2 + 48$

(xiii) $12x^3 + 36x^2y + 27y^2x$

(xiv) $36x^4y^2 - 72x^2y^2 + 36y^2$

(xv) $6x^3y + 24x^2y^2 + 24y^3x$

(xvi) $45x^4y - 120x^3y^2 + 80y^3x^2$
(xvii) $x^2 - 2xy + y^2 - z^2$
(xviii) $5x^2 + 5y^2 - 5z^2 + 10xy$
(xix) $36x^2 + 24xy + 4y^2 - 25z^2$
(xx) $49x^4 + 9y^2 - 49z^2 - 42x^2y$

2. Factorise the following:
(i) $x^2 + 6x + 5$
(ii) $x^2 - 5x + 6$
(iii) $x^2 + 10x + 21$
(iv) $x^2 - 8x + 15$
(v) $1 + 5x + 6x^2$
(vi) $9 - 18x + 8x^2$
(vii) $x^2 - 3x - 4$
(viii) $x^2 - 2x - 15$
(ix) $x^2 + 6x - 16$
(x) $x^2 + 2x - 35$
(xi) $9 + 3x - 12x^2$
(xii) $25 - 20x - 12x^2$
(xiii) $6x^2 + 16xy + 8y^2$
(xiv) $10x^2 - 26xy + 12y^2$
(xv) $16x^2 - 22xy - 3y^2$
(xvi) $4x^2 - 22xy - 12y^2$
(xvii) $14x^2 - 33xy - 5y^2$
(xviii) $5x^2 - 23xy - 42y^2$
(xix) $24x^2 + 34xy - 10y^2$
(xx) $5x^2 + 35x + 60$
(xxi) $6x^2 - 66x + 180$
(xxii) $4 - 8x - 60x^2$
(xxiii) $32 + 8x - 60x^2$
(xxiv) $6x^3 + 26x^2y + 28y^2x$
(xxv) $60x^2y - 66y^2x + 18y^3$
(xxvi) $60x^2y^2 + 68y^3x - 16y^4$
(xxvii) $45x^4y + 6x^3y^2 - 72y^3x^2$

Challenge

1. $30 - 17(x - y) - 21(x - y)^2$
2. $5(x - y)^2 - 38(x - y) + 21$
3. $4 - 7(x - y) - 15(x - y)^2$

Simplification by Factorisation

Example 14: Simplify $\dfrac{36x^3y + 54x^2y - 90xy}{12x^2y + 60xy + 75y}$

$$= \frac{6xy(6x^2 + 9x - 15)}{3y(4x^2 + 20x + 25)}$$

$$= \frac{6xy(6x^2 + 15x - 6x - 15)}{3y(2x + 5)^2}$$

$$= \frac{6xy\{3x(2x + 5) - 3(2x + 5)\}}{3y(2x + 5)(2x + 5)}$$

$$= \frac{6xy(3x - 3)\cancel{(2x + 5)}}{3y(2x + 5)\cancel{(2x + 5)}}$$

$$= \frac{2x(3x - 3)}{2x + 5}$$

Example 15: Simplify $\dfrac{15(x - 3)^2 + 5(x - 3) - 20}{5x - 20}$

Let $x - 3$ be represented by y, then the expression becomes

$$\frac{15y^2 + 5y - 20}{5x - 20}$$

$$= \frac{15y^2 + 20y - 15y - 20}{5x - 20}$$

$$= \frac{5y(3y + 4) - 5(3y + 4)}{5x - 20}$$

$$= \frac{(3y + 4)(5y - 5)}{5x - 20}$$

Now substituting $y = x - 3$

$$= \frac{\{3(x - 3) + 4\}\{5(x - 3) - 5\}}{5x - 20}$$

$$= \frac{(3x - 9 + 4)(5x - 15 - 5)}{5x - 20}$$

$$= \frac{(3x - 5)\cancel{(5x - 20)}}{\cancel{5x - 20}}$$

$$= 3x - 5$$

Exercise 17.3

1. Simplify the following expressions.

(i) $\dfrac{4x^2+16x+16}{4x^2-16}$

(ii) $\dfrac{9x^2-24x+16}{9x^2-16}$

(iii) $\dfrac{36x^4y-64x^2y}{18x^4+48x^3+32x^2}$

(iv) $\dfrac{25x^4-9y^2}{75x^2z^2-45z^2y} \div \dfrac{10x^2z+6yz}{5z}$

(v) $\dfrac{x^2+2x-35}{x^2-25}$

(vi) $\dfrac{15x^2-xy-2y^2}{25x^2-4y^2}$

(vii) $\dfrac{(9x^2+24x+16)(9x^2-24x+16)}{9x^2-16}$

(viii) $\dfrac{(6x^2+13x-28)(6x^2+13x-28)}{(4x^2+28x+49)(9x^2-24x+16)}$

(ix) $\dfrac{5(x-2)^2+23(x-2)-10}{5x-12}$

(x) $\dfrac{6(x+3)^2+4(x+3)-16}{3x+5}$

Revision Exercise

1. Factorise the following :

(i) $16x^2-24xy$ (ii) $-5-10x+20y^2$

(iii) x^3-6x^2+x-6

(iv) $8y^3x^2+16x^3y^2+28x^2y^2$

2. Factorise the following :

(i) $4x^2-9y^2$ (ii) $64x^4-81y^2$

(iii) $216x^5-125xy^4$ (iv) $\dfrac{8x^3}{64}-\dfrac{2x}{49}$

3. Factorise the following :

(i) $3x^2+17xy-6y^2$

(ii) $5x^3y-19x^2y-30xy$

(iii) $7x^3y+28x^2y^2+28y^3x$

(iv) $15x^4y-65x^3y^2+50y^3x^2$

4. Simplify the following expressions :

(i) $\dfrac{4x^2-9}{2x+3}$ (ii) $\dfrac{18xy-24yz}{3x-4z}$

(iii) $\dfrac{x^3-3x^2+x-3}{x^2-9}$

(iv) $\dfrac{18x^3y^3-27x^2y^3+36x^3y^2}{18x^2y^2}$

5. Factorise the following :

(i) $\dfrac{6x^4}{y^2}(2x-3y)-\dfrac{4x^3}{y}(3y-2x)$

(ii) $5x^3+5x^2y+15x^2+20x+20y+60$

(iii) $3y^3x+3x^3y+15xy-6y^2x-6x^3-30x$

(iv) $\dfrac{x^3}{18}-\dfrac{x}{50}$ (v) $\dfrac{y^3x^2}{32}-\dfrac{z^2y}{18}$

(vi) $0.18x^4-0.08y^4$ (vii) $0.12y^3x^2-3z^2y$