16 SPECIAL PRODUCTS AND EXPANSIONS

- Special Products
- Expansions

Special Products

In the previous class we used geometry to show that:

$$(a + b)(a - b) = a^2 - b^2$$

Multiplying the two binomials we have

$$(a + b)(a - b) = a^2 - ab + ab - b^2 = a^2 - b^2$$

Thus, we observe a pattern that emerges in the **special product** of the two binomials as:

$$(\text{term } 1 + \text{term } 2)(\text{term } 1 - \text{term } 2) = (\text{term } 1)^2 - (\text{term } 2)^2$$

Example 1: Apply the identity $(a + b)(a - b) = a^2 - b^2$ to find the value of 9015×8985.

$$9015 \times 8985 = (9000 + 15)(9000 - 15) = 9000^2 - 15^2 = 80999775$$

Let us now study special products involving three terms. Let x, a, and b be three terms in two binomials. Then we have

1. $(x + a)(x + b) = x^2 + xb + ax + ab = x^2 + x(a + b) + ab$
2. $(x - a)(x - b) = x^2 - bx - ax + ab = x^2 - x(a + b) + ab$
3. $(x + a)(x - b) = x^2 - bx + ax - ab = x^2 + x(a - b) - ab$
4. $(x - a)(x + b) = x^2 + bx - ax - ab = x^2 + x(b - a) - ab$

Example 2: Find the product of $(x + 3)(x + 8)$

Applying $(x + a)(x + b) = x^2 + x(a + b) + ab$

$$(x + 3)(x + 8) = x^2 + x(3 + 8) + (3 \times 8) = x^2 + 11x + 24$$

Example 3: Find the product of $\left(\frac{p}{5} + \frac{1}{2}\right)\left(\frac{p}{5} - \frac{1}{3}\right)$

Let $\frac{p}{5} = x$, $\frac{1}{2} = a$, and $\frac{1}{3} = b$. Then we observe a pattern resembling $(x + a)(x - b)$ and we can write the product without having to actually multiply the binomials.

Thus, $\left(\frac{p}{5} + \frac{1}{2}\right)\left(\frac{p}{5} - \frac{1}{3}\right)$

$$= \left(\frac{p}{5}\right)^2 + \frac{p}{5}\left(\frac{1}{2} - \frac{1}{3}\right) - \frac{1}{2} \times \frac{1}{3} = \frac{p^2}{25} + \frac{p}{30} - \frac{1}{6}$$

Try this!

$$\left(\frac{2x}{3} + \frac{y}{5}\right)\left(\frac{2x}{3} - \frac{y}{5}\right)$$

Example 4: $(3x^3 + 10)(3x^3 + 15)$

$$= (3x^3)^2 + 3x^3(10 + 15) + 10 \times 15$$

[applying $(x + a)(x + b) = x^2 + x(a + b) + ab$]

$$= 9x^6 + 75x^3 + 150$$

Example 5: Find the continued product of $(x - 1)(x + 1)(x^2 + 1)(x^4 + 1)(x^8 + 1)$

$$= (x^2 - 1)(x^2 + 1)(x^4 + 1)(x^8 + 1)$$

[applying $(a - b)(a + b) = a^2 - b^2$]

$$= (x^4 - 1)(x^4 + 1)(x^8 + 1)$$

$$= (x^8 - 1)(x^8 + 1) = x^{16} - 1$$

Try this!

1. Apply the identity $a^2 - b^2 = (a + b)(a - b)$ to find the value of $90^2 - 80^2$.
2. Find the product of $(7x + 3)(7x + 10)$.

Exercise 16.1

1. Write the products of the following binomials by applying the concept and patterns of special products.
   (i) $(2x + 3y)(2x + 5y)$
   (ii) $(5x + 7y)(5x + 8y)$
   (iii) $(3x + 7)(3x + 9)$
   (iv) $(2x^2 + 0.1)(2x^2 + 0.4)$
   (v) $\left(3x^3 + \frac{1}{3}\right)(3x^3 + 6)$
   (vi) $(4x - 2y)(4x - 3y)$
   (vii) $(5x - 4y)(5x - 5y)$
   (viii) $(2x - 3)(2x - 8)$
   (ix) $(6x - 2.5)(6x - 0.2)$
   (x) $\left(7x^2 - \frac{2}{3}\right)\left(7x^2 - \frac{1}{3}\right)$
   (xi) $(3x + 5y)(3x - 4y)$
   (xii) $(4x + 2y)(4x - y)$
   (xiii) $(5x + 5)(5x - 3)$
   (xiv) $(3x + 5.5)(3x - 0.5)$
   (xv) $\left(2x^3 + \frac{3}{5}\right)\left(2x^3 - \frac{2}{5}\right)$
   (xvi) $(10x - 4y)(10x + 5y)$
   (xvii) $(7x - 3y)(7x + 7y)$
   (xviii) $(9x - 10)(9x + 12)$
   (xix) $(6x - 0.4)(6x + 1)$
   (xx) $\left(12x^2 - \frac{5}{6}\right)(12x^2 + 1)$
   (xxi) $(3x + 4y)(3x - 4y)$
   (xxii) $(7x + 5)(7x - 5)$
   (xxiii) $(x^2 + y^2)(x^2 - y^2)$
   (xxiv) $(2x + 1.1)(2x - 1.1)$
   (xxv) $\left(6x^2 + \frac{2}{5}\right)\left(6x^2 - \frac{2}{5}\right)$
   (xxvi) $\left(\frac{p^2}{3} + 1.1\right)\left(\frac{p^2}{3} + 0.9\right)$
   (xxvii) $\left(\frac{4x}{y} - \frac{1}{8}\right)\left(\frac{4x}{y} - \frac{3}{8}\right)$
   (xxviii) $\left(\frac{7y^2}{z} + \frac{7}{8}\right)\left(\frac{7y^2}{z} - \frac{5}{8}\right)$
   (xxix) $(2xy - 1.4)(2xy + 2)$
   (xxx) $(x^2y^2 + 2z^2)(x^2y^2 - 2z^2)$
2. Apply the identity $(a + b)(a - b) = a^2 - b^2$ to find the value of the following:
   (i) 99×101 (ii) 508×492
   (iii) 188×212 (iv) 307×293
   (v) 980×1020 (vi) $140^2 - 130^2$
   (vii) $291^2 - 289^2$ (viii) $347^2 - 342^2$
   (ix) $1010^2 - 1007^2$ (x) $1739^2 - 1639^2$
3. Find the continued product of the following:
   (i) $(2x + 2)(2x - 2)(4x^2 + 4)$
   (ii) $(3x - y)(3x + y)(9x^2 + y^2)$
   (iii) $(x^2 + 2)(x^2 - 2)(x^4 + 4)$

Challenge
1. $(1 - y)(1 + y)(1 + y^2)(1 + y^4)(1 + y^8)$
2. $(1 - y)(1 + y)(1 + y^2)(1 + y^4)$

Expansions

The number 121 is written in exponential form as 11^2, which can be written in expanded form as 11×11.

Similarly, the algebraic expression $(a + b)^2$ is written in expanded form as:

$$(a + b)(a + b) = a^2 + 2ab + b^2$$

In the previous class we used geometry to show that:

$$(a + b)^2 = a^2 + 2ab + b^2$$

and $$(a - b)^2 = a^2 - 2ab + b^2$$

Observe the pattern of both expansions:

$(\text{term 1} + \text{term 2})^2$
$= (\text{term 1})^2 + (2 \times \text{term 1} \times \text{term 2}) + (\text{term 2})^2$

and $(\text{term 1} - \text{term 2})^2$
$= (\text{term 1})^2 - (2 \times \text{term 1} \times \text{term 2}) + (\text{term 2})^2$

Example 6: Solve $\left(3x + \frac{y}{6}\right)^2$

Let $3x = a$ and $\frac{y}{6} = b$. Observe that we obtain a pattern resembling $(a + b)^2$ and we can expand the expression without having to multiply the binomial with itself.

Thus, $\left(3x + \frac{y}{6}\right)^2 = (\text{term 1})^2 + (2 \times \text{term 1} \times \text{term 2}) + (\text{term 2})^2$

$$= (3x)^2 + \left(2 \times 3x \times \frac{y}{6}\right) + \left(\frac{y}{6}\right)^2$$

$$= 9x^2 + xy + \frac{y^2}{36}$$

Example 7: What should be added to $49x^2 - 74xy + 25y^2$ to make it a perfect square?

The expanded form of a perfect square in the form $(\text{term 1} - \text{term 2})^2$ would be:

$(\text{term 1})^2 - (2 \times \text{term 1} \times \text{term 2}) + (\text{term 2})^2$.

In the expression $49x^2 - 74xy + 25y^2$, the first and the last terms are squares of $7x$ and $5y$ respectively, but if $7x - 5y$ were to be squared, the middle term of the expanded form would be $-(2 \times 7x \times 5y) = -70xy$, whereas in the given expression the middle term is $-74xy$.

Thus, if $-70xy - (-74xy) = +4xy$ were to be added to $49x^2 - 74xy + 25y^2$, the sum $49x^2 - 70xy + 25y^2$ would be a perfect square.

Example 8: Expand $(a + b - c)^2$.

If $(a + b)$ is considered as a single term then the expression can be written as:

$$\{(a + b) - c\}^2 = (a + b)^2 - \{2 \times (a + b) \times c\} + c^2$$
$$= a^2 + 2ab + b^2 - (2ac + 2bc) + c^2$$
$$= a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$$
$$= a^2 + b^2 + c^2 + 2(ab - ac - bc)$$

Example 9: Given that $x + \frac{1}{x} = 4$, find the value of $x^4 + \frac{1}{x^4}$.

Squaring both the left hand side and right hand side of the given statement does not change the equation.

So, $\left(x + \frac{1}{x}\right)^2 = 4^2$

$$\Rightarrow x^2 + \left(2 \times x \times \frac{1}{x}\right) + \frac{1^2}{x^2} = 16$$

$$\Rightarrow x^2 + \frac{1}{x^2} = 16 - 2 = 14$$

$$\Rightarrow \left(x^2 + \frac{1}{x^2}\right)^2 = 14^2$$

$$\Rightarrow x^4 + 2 + \frac{1}{x^4} = 196$$

$$\Rightarrow x^4 + \frac{1}{x^4} = 196 - 2 = 194$$

Example 10: Given $x^2 + \frac{1}{x^2} = 51$, find the value of $x - \frac{1}{x}$.

Adding 2 to $x^2 + \frac{1}{x^2}$ gives us $x^2 + 2 + \frac{1}{x^2}$ which is the expanded form of $\left(x + \frac{1}{x}\right)^2$.

Subtracting 2 from $x^2 + \frac{1}{x^2}$ gives us $x^2 - 2 + \frac{1}{x^2}$ which is the expanded form of $\left(x - \frac{1}{x}\right)^2$.

Thus, subtracting 2 from both sides of the given statement,

$$x^2 - 2 + \frac{1}{x^2} = 51 - 2$$

$$\Rightarrow \left(x - \frac{1}{x}\right)^2 = 49 \Rightarrow x - \frac{1}{x} = \sqrt{49} = 7$$

Example 11: Given that $x^2 + y^2 = 74$ and $xy = 35$, find $x + y$ and $x - y$.

$$(x + y)^2 = x^2 + 2xy + y^2$$
$$\Rightarrow (x + y)^2 = x^2 + y^2 + 2xy$$
$$\Rightarrow (x + y)^2 = 74 + (2 \times 35)$$
$$\Rightarrow (x + y)^2 = 144$$
$$\Rightarrow x + y = \sqrt{144} = 12$$
$$(x - y)^2 = x^2 - 2xy + y^2$$

$\Rightarrow \quad (x-y)^2 = x^2 + y^2 - 2xy$

$\Rightarrow \quad (x-y)^2 = 74 - (2 \times 35)$

$\Rightarrow \quad (x-y)^2 = 4$

$\Rightarrow \quad x - y = \sqrt{4} = 2$

Having learnt and applied the two expansions $(a+b)^2$ and $(a-b)^2$, let us study some more expansions.

1. $(a+b)^3 = (a+b)(a+b)(a+b)$
$= (a^2 + 2ab + b^2)(a+b)$
$= a^3 + 2a^2b + ab^2 + a^2b + 2b^2a + b^3$
$= a^3 + 3a^2b + 3b^2a + b^3$
$= a^3 + 3ab(a+b) + b^3$

Pattern: (term 1 + term 2)^3
= (term 1)^3 + 3 × term 1 × term 2 (term 1 + term 2) + (term 2)^3
= sum of cubes of terms + three times the product of their sum and product

2. $(a-b)^3 = (a-b)(a-b)(a-b)$
$= (a^2 - 2ab + b^2)(a-b)$
$= a^3 - 2a^2b + b^2a - a^2b + 2b^2a - b^3$
$= a^3 - 3a^2b + 3b^2a - b^3$
$= a^3 - 3ab(a-b) - b^3$

Pattern: (term 1 – term 2)^3
= (term 1)^3 – 3 × term 1 × term 2 (term 1 – term 2) – (term 2)^3
= difference of cubes of terms – three times the product of their difference and product

3. $(a+b+c)^2 = (a+b+c)(a+b+c)$
$= a^2 + ab + ac + ba + b^2 + bc + ca + cb + c^2$
$= a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$
$= a^2 + b^2 + c^2 + 2(ab + ac + bc)$

Pattern: (term 1 + term 2 + term 3)^2
= (term 1)^2 + (term 2)^2 + (term 3)^2 + 2(term 1 × term 2 + term 1 × term 3 + term 2 × term 3)
= sum of squares of the three terms + two times the sum of their products in pairs.

Example 12: Expand $(2x + 3y)^3$

Following the pattern of the cube of the sum of two terms,

$$(2x+3y)^3 = (2x)^3 + (3y)^3 + \{3 \times 2x \times 3y\,(2x+3y)\}$$
$$= 8x^3 + 27y^3 + \{18xy\,(2x+3y)\}$$
$$= 8x^3 + 27y^3 + \{36x^2y + 54y^2x\}$$

Thus, $(2x+3y)^3 = 8x^3 + 36x^2y + 54y^2x + 27y^3$

Example 13: Expand $(5 - 6x)^3$

Following the pattern of the cube of the difference of two terms,

$$(5-6x)^3 = 5^3 - (6x)^3 - \{3 \times 5 \times 6x\,(5-6x)\}$$
$$= 125 - 216x^3 - \{90x\,(5-6x)\}$$
$$= 125 - 216x^3 - \{450x - 540x^2\}$$

Thus, $(5-6x)^3 = 125 - 450x + 540x^2 - 216x^3$

Example 14: Expand $(x + 2y + 3z)^2$

Following the pattern of the square of three terms,

$$(x+2y+3z)^2 = x^2 + (2y)^2 + (3z)^2 + 2(2xy + 3xz + 6yz)$$
$$= x^2 + 4y^2 + 9z^2 + 4xy + 6xz + 12yz$$

Thus, $(x+2y+3z)^2 = x^2 + 4y^2 + 9z^2 + 4xy + 6xz + 12yz$

Try this!
1. Expand $(17x + 7y)^2$.
2. Expand $(x + 2y + 5z)^2$.

Exercise 16.2

1. Expand the following:

(i) $(5x+3y)^2$ (ii) $(3x-y)^2$
(iii) $(7x+4y)^2$ (iv) $(4x-2y)^2$
(v) $(3x+6)^2$ (vi) $(10x-3)^2$
(vii) $(8+9y)^2$ (viii) $(11-2x)^2$
(ix) $\left(6x + \frac{y}{3}\right)^2$ (x) $\left(4x - \frac{y}{2}\right)^2$

(xi) $\left(\frac{x^2}{10}+5y^2\right)^2$ (xii) $\left(\frac{x}{12}-6y\right)^2$

(xiii) $(0.4x^2 + 0.5y^2)^2$ (xiv) $(0.3x^2 - 5y^2)^2$

(xv) $(1.5x^2y - 1.5y^2x)^2$

(xvi) $(2.5x^2y - 0.2y^2x)^2$

(xvii) $\left(6x^3+\frac{1}{12x^3}\right)^2$ (xviii) $\left(4x^2-\frac{1}{8x^2}\right)^2$

(xix) $\left(\frac{3x}{4}+\frac{2y}{3}\right)^2$ (xx) $\left(\frac{2x^2}{5}-\frac{5y^2}{6}\right)^2$

2. Expand the following:
   (i) $(3x + 3)^3$ (ii) $(4 - 2x)^3$
   (iii) $(2x + 3y + 4)^2$ (iv) $(4x + 2y)^3$
   (v) $(5x - 4y)^3$ (vi) $(2x - 2y + 2z)^2$
3. What should be added to the following algebraic expressions to make them perfect squares?
   (i) $64x^2 + 8xy + y^2$
   (ii) $36x^2 - 65xy + 25y^2$
   (iii) $100x^2 - 100xy + 100y^2$
   (iv) $16x^2 + 60x + 49$
   (v) $1.21x^2 + 4.6xy + 4y^2$
   (vi) $\frac{x^2}{4}-\frac{xy}{12}+\frac{y^2}{36}$
4. Fill in the boxes to complete the following algebraic statements.
   (i) $(2x + \square)^2 = 4x^2 + \square + 9y^2$
   (ii) $(x^2 - \square)^2 = x^4 - \square + 1$
   (iii) $(\square + 7y)^2 = 16x^2 + \square + \square$
   (iv) $(\square - 5)^2 = x^2y^2 - \square + \square$
   (v) $(3x + \square)^2 = \square + 6xy + \square$
   (vi) $(7x + \square)^2 = \square + 84xy + \square$
   (vii) $\left(\frac{x^2}{3}+\square\right)^2 = \square + \square + \frac{9y^4}{16}$
   (viii) $\left(\frac{2x^2}{5}+\square\right)^2 = \square - \square + 100y^4$
5. Write the expanded form of $(x + y + z)^2$.
6. Given that $x + y = 13$ and $xy = 40$, find the value of $x - y$.
7. Given that $x^2 + y^2 = 193$ and $xy = 84$, find the values of $x + y$ and $x - y$.
8. Given that $x - y = 7$ and $xy = 8$, find the value of $(x + y)^2$.
9. If $x+\frac{1}{x} = 5$, find the value of $x^2+\frac{1}{x^2}$.
10. If $x-\frac{1}{x} = 5$, find the value of $x^2+\frac{1}{x^2}$.
11. If $x+\frac{1}{x} = 3$, find the value of $x^4+\frac{1}{x^4}$.
12. If $x^2+\frac{1}{x^2} = 7$, find the value of $x+\frac{1}{x}$.
13. If $y^2+\frac{1}{y^2} = 6$, find the value of $y-\frac{1}{y}$.

Revision Exercise

1. Write the products of the following binomials by applying the concept and patterns of special products.
   (i) $(3x + 5y)(3x + 7y)$
   (ii) $(5x - 2)(5x - 7)$
   (iii) $(3x^2 + 0.5)(3x^2 + 0.7)$
   (iv) $(9x + 2)(9x - 1)$
2. Apply the identity $(a + b)(a - b) = a^2 - b^2$ to find the value of the following:
   (i) 199×102 (ii) 407×393
   (iii) $150^2 - 140^2$ (iv) $(743)^2 - (724)^2$
3. Expand the following:
   (i) $(9x + 7y)^2$ (ii) $(4x - 3y)^2$
   (iii) $\left(3x+\frac{2y}{5}\right)^2$ (iv) $\left(5x+\frac{3y}{7}\right)^2$
4. Expand the following:
   (i) $(5x + 2)^3$ (ii) $(11 - 7x)^3$
   (iii) $(3x + 5y + 9)^2$ (iv) $(7x - 3y + 3z)^2$
5. Given that $x - y =$ and $xy = 13$, find the value of $(x + y)^2$.