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Quadratic Equations

A quadratic equation in x is an equation in which the highest power of x is 2. The standard form of a quadratic equation in x is $ax^2+bx+c=0$, where a, b and c are constants and $a \neq 0$.

Examples $3x^2+5x=2$, $-6x^2=7x+9$ and $3x^2-25=0$ are quadratic equations in x.

Solution of a quadratic equation

In general, a quadratic equation has two solutions, which are also called roots. The two solutions may be equal.

Let the roots of the equation $ax^2+bx+c=0$ be $x=\alpha$ and $x=\beta$. Then each of the values of x must satisfy the equation, that is, $a\alpha^2+b\alpha+c=0$ and $a\beta^2+b\beta+c=0$.

Solving a quadratic equation

To solve a quadratic equation, we first factorize it and then use the principle of zero product, which states: If the product of two expressions (or numbers) is zero then at least one of the expressions must be zero. We can express this symbolically as follows.

If $ab=0$ then $a=0$ or $b=0$ or both $a=0$ and $b=0$.

Example If $(x-1)(x-2)=0$ then by the principle of zero product, either $x-1=0$ or $x-2=0$.

If $x-1=0$ then $x=1$. If $x-2=0$ then $x=2$. $\therefore$ $x=1$ or 2.

Now we can write the steps for solving a quadratic equation.

Steps
1. Write the equation in the standard form (that is, in the form $ax^2+bx+c=0$).
2. Factorize the quadratic expression on the LHS.
3. Set each factor equal to zero.
4. Solve each of the resulting linear equations.

Solved Examples

EXAMPLE 1 **Solve the equation $x^2+4x+3=0$.**

Solution To begin with, let us factorize the LHS of the given equation with 0 on RHS.

$x^2 + 4x + 3 = 0$ or $x^2 + x + 3x + 3 = 0$

or $x(x+1) + 3(x+1) = 0$ or $(x+1)(x+3) = 0$.

By the zero-product principle, either $x + 1 = 0$, which gives $x = -1$

or $x + 3 = 0$, which gives $x = -3$.

Therefore, the roots (or solutions) of the equation are –1 and –3.

Verification

Let us substitute the two values of x in the given equation one by one.

When $x = -1$, $x^2 + 4x + 3 = (-1)^2 + 4 \times (-1) + 3 = 1 - 4 + 3 = 0$, which is true.

When $x = -3$, $x^2 + 4x + 3 = (-3)^2 + 4 \times (-3) + 3 = 9 - 12 + 3 = 0$, which is also true.

Hence, the solutions $x = -1$ and $x = -3$ are correct.

EXAMPLE 2 **Solve the quadratic equation $8m^2 - 10m - 7 = 0$.**

Solution Given, $8m^2 - 10m - 7 = 0$ or $8m^2 - 14m + 4m - 7 = 0$

or $2m(4m - 7) + 1(4m - 7) = 0$ or $(4m - 7)(2m + 1) = 0$.

By the zero-product principle,

either $4m - 7 = 0$, i.e., $4m = 7$, i.e., $m = \frac{7}{4}$

or $2m + 1 = 0$, i.e., $2m = -1$, i.e., $m = \frac{-1}{2}$.

Hence, the solutions $= \frac{7}{4}$ and $\frac{-1}{2}$.

EXAMPLE 3 **Solve the equation $4x^2 - 12x + 9 = 0$.**

Solution Given, $4x^2 - 12x + 9 = 0$ or $4x^2 - 6x - 6x + 9 = 0$

or $2x(2x - 3) - 3(2x - 3) = 0$ or $(2x - 3)(2x - 3) = 0$

By the zero-product rule,

either $2x - 3 = 0$ or $2x - 3 = 0$.

$\therefore$ either $x = \frac{3}{2}$ or $x = \frac{3}{2}$.

Hence, the solutions are $\frac{3}{2}$ and $\frac{3}{2}$, which are equal.

EXAMPLE 4 **Solve $4p^2 - 81 = 0$.**

Solution Given, $4p^2 - 81 = 0$ or $(2p)^2 - (9)^2 = 0$ or $(2p + 9)(2p - 9) = 0$.

This means either $2p + 9 = 0$ or $2p - 9 = 0$.

Therefore, either $p = -\frac{9}{2}$ or $p = \frac{9}{2}$.

Hence, the roots of the equation $= \frac{-9}{2}, \frac{9}{2}$.

EXAMPLE 5 **Solve the equation $(x - 3)(x + 5) = 9$.**

Solution Here, $(x - 3)(x + 5) = 9$ or $x^2 + (5 - 3)x - 5 \times 3 = 9$

$[\because (x + a)(x - b) = x^2 + (a - b)x - ab]$

or $x^2+2x-15=9$ or $x^2+2x-15-9=0$ or $x^2+2x-24=0$

or $x^2+6x-4x-24=0$ or $x(x+6)-4(x+6)=0$ or $(x+6)(x-4)=0$.

This means either $x+6=0$, that is, $x=-6$

or $x-4=0$, that is, $x=4$.

Hence, $x=-6$ or $x=4$. So, the solutions of the equation are –6 and 4.

EXAMPLE 6 **Solve the equation $x-\frac{40}{x}=3, x\neq 0$.**

Solution Multiplying both sides of the given equation by x,

$$x\left(x-\frac{40}{x}\right)=3x \quad \text{or} \quad x^2-40=3x \quad \text{or} \quad x^2-3x-40=0$$

or $x^2-8x+5x-40=0$ or $x(x-8)+5(x-8)=0$ or $(x-8)(x+5)=0$.

Hence, either $x-8=0$, which gives $x=8$

or $x+5=0$, which gives $x=-5$.

Therefore, $x=8$ or $x=-5$. So, the solutions are $x=8, -5$.

EXAMPLE 7 **Solve the equation $\frac{3}{x-2}+\frac{8}{x+3}=2$.**

Solution Multiplying both sides of the given equation by $(x-2)(x+3)$, the LCM of the denominators,

$$(x-2)(x+3)\left[\frac{3}{x-2}+\frac{8}{x+3}\right]=2(x-2)(x+3)$$

or $3(x+3)+8(x-2)=2\{x^2+(3-2)x-2\times 3\}$

or $3x+9+8x-16=2(x^2+x-6)$ or $2x^2+2x-12=11x-7$

or $2x^2-9x-5=0$ or $2x^2-10x+x-5=0$

or $2x(x-5)+1(x-5)=0$ or $(x-5)(2x+1)=0$.

Hence, either $x-5=0$, which gives $x=5$

or $2x+1=0$, which gives $x=-\frac{1}{2}$.

Therefore, $x=5$ or $x=-\frac{1}{2}$. Hence, the solutions are $x=5, -\frac{1}{2}$.

EXAMPLE 8 **Solve the equation $\frac{x+3}{x+2}=\frac{3x-7}{2x-3}$.**

Solution Given, $\frac{x+3}{x+2}=\frac{3x-7}{2x-3}$ or $(x+3)(2x-3)=(x+2)(3x-7)$

or $x(2x-3)+3(2x-3)=x(3x-7)+2(3x-7)$

or $2x^2-3x+6x-9=3x^2-7x+6x-14$

or $2x^2+3x-9=3x^2-x-14$

or $3x^2-2x^2-x-3x-14+9=0$

or $x^2-4x-5=0$ or $x^2-5x+x-5=0$

or $x(x-5)+1(x-5)=0$ or $(x-5)(x+1)=0$.

Hence, either $x-5=0$, which gives $x=5$ or $x+1=0$, which gives $x=-1$.

Therefore, the solutions of the equation are $x=5, -1$.

Remember These

1. A quadratic equation is of the form $ax^2 + bx + c = 0, a \neq 0$.
2. To solve a quadratic equation, (i) factorize the LHS, (ii) set each factor equal to zero, and (iii) solve the resulting linear equations.

EXERCISE 11

Solve the following equations.

1. (i) $x^2 + 5x + 6 = 0$ (ii) $3x^2 + 17x + 24 = 0$ (iii) $x^2 - 7x + 12 = 0$ (iv) $2x^2 - 11x + 5 = 0$ (v) $x^2 + 8x - 20 = 0$ (vi) $6x^2 + 7x - 3 = 0$ (vii) $x^2 - x - 42 = 0$ (viii) $2x^2 - x - 3 = 0$
2. (i) $12x^2 - 6 = 9 + 8x$ (ii) $8x^2 + x = 6 - x$
3. (i) $\frac{2}{3}x^2 = 2 - \frac{1}{3}x$ (ii) $4x^2 + 13x + \frac{15}{2} = 0$ (iii) $\frac{x^2}{3} + x = -\frac{2}{3}$ (iv) $\frac{15}{2} - \frac{x^2}{2} = x$
4. (i) $x^2 - 2x + 1 = 0$ (ii) $9x^2 + 24x + 16 = 0$
5. (i) $a^2 - 4 = 0$ (ii) $2x^2 - 32 = 0$ (iii) $9x^2 - 625 = 0$
6. (i) $(x + 6)(x - 2) = -7$ (ii) $(y - 6)(y + 1) = -12$ (iii) $(2x + 3)(2x - 1) = -(3x + 1)$ (iv) $x^2 = 225 - (x + 3)^2$ (v) $x(x + 1) + (x + 2)(x + 3) = 42$
7. (i) $x - \frac{10}{x - 3} = 0$ (ii) $\frac{x}{8} = \frac{5}{3x - 2}$ (iii) $\frac{5x - 1}{3} = \frac{6}{x}$
8. (i) $\frac{15}{x^2} = 1 - \frac{2}{x}$ (ii) $\frac{6}{x} + \frac{3}{x + 2} = 2$ (iii) $3x - \frac{5}{x} = 14$
9. (i) $\frac{2}{x - 3} - \frac{6}{x - 8} = -1$ (ii) $\frac{14}{x - 6} - \frac{6}{x - 8} = \frac{1}{2}$ (iii) $\frac{4}{x - 2} + \frac{4}{x + 2} = \frac{5}{6}$
10. (i) $\frac{x + 2}{x + 3} = \frac{2x - 3}{3x - 7}$ (ii) $\frac{2x + 1}{5 - 3x} = \frac{3 + x}{1 - x}$
11. (i) $\frac{x + 1}{x - 2} + \frac{x + 11}{x + 3} = 4$ (ii) $\frac{x + 7}{x + 1} + \frac{2x + 6}{2x + 1} = 5$

ANSWERS

1. (i) –3, –2 (ii) $-\frac{8}{3}, -3$ (iii) 3, 4 (iv) $5, \frac{1}{2}$ (v) –10, 2 (vi) $-\frac{3}{2}, \frac{1}{3}$ (vii) 7, –6 (viii) $\frac{3}{2}, -1$

2. (i) $\frac{-5}{6}, \frac{3}{2}$ (ii) $-1, \frac{3}{4}$ **3.** (i) $\frac{3}{2}, -2$ (ii) $-\frac{3}{4}, \frac{-5}{2}$ (iii) –1, –2 (iv) –5, 3

4. (i) 1, 1 (ii) $\frac{-4}{3}, \frac{-4}{3}$

5. (i) 2, –2 (ii) 4, –4 (iii) $\frac{25}{3}, \frac{-25}{3}$

6. (i) 1, –5 (ii) 2, 3 (iii) $\frac{1}{4}, -2$ (iv) 9, –12 (v) 3, –6

7. (i) –2, 5 (ii) $4, \frac{-10}{3}$ (iii) $\frac{-9}{5}, 2$

8. (i) –3, 5 (ii) $4, \frac{-3}{2}$ (iii) $5, \frac{-1}{3}$

9. (i) 2, 13 (ii) 10, 20 (iii) $10, \frac{-2}{5}$

10. (i) 5, –1 (ii) 2, –7

11. (i) $5, \frac{-1}{2}$ (ii) $2, \frac{-2}{3}$

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Revision Exercise 7

1. Solve the following linear equations.

(i) $\frac{2x-6}{x-2} = \frac{2x+3}{x+2}$ (ii) $\frac{7x+1}{4} - 2x = 5 - \frac{x+7}{8}$

(iii) $\frac{2}{x-4} + \frac{1}{x+1} = \frac{1}{x^2-3x-4}, x \neq 4, -1$ (iv) $0.01x + 0.008(5000 - x) = 44$

2. In a panchayat election, the winning candidate received 1100 votes more than his only opponent. If 16,186 votes were cast in all, how many votes did each candidate receive?

3. Divide ₹ 6000 between two friends so that one gets four times the other.

4. Find the smallest integer three fourths of which exceeds 96.

5. Let x be an odd integer and $-7 \leq x \leq 7$. Represent the solution set of this inequation on a number line.

6. Solve the following simultaneous equations.

(i) $7x + 9y = 130, y - x = 2$

(ii) $11x - 7y + 13 = 0, \frac{y}{5} - \frac{x}{2} = 0$

(iii) $11(x-5) + 10(y-2) + 54 = 0, 7(2x-1) + 9(3y-1) = 25$

(iv) $100x + \frac{y}{2} = 201, 25x + y = 52$

(v) $\frac{1}{x} - \frac{1}{y} = \frac{1}{12}, \frac{2}{x} + \frac{3}{y} = \frac{17}{12}$

7. (i) The sum of two numbers is $\frac{7}{8}$ and their difference is $\frac{3}{8}$. Find the numbers.

(ii) The sum of the digits of a two-digit number is 14 and twice the digit in the unit place plus three times the digit in the tens place is 35. Find the number.

(iii) Two packets together weigh 84 kg. One of the packets weighs 20 kg more than the other. How much does each packet weigh?

8. (a) Plot the points on a graph paper and draw the line passing through them.

(i) (–2, 4) and (3, –1) (ii) (–4, –1) and (2, 2)

(b) Draw the graphs of (i) $y = 3 - x$ (ii) $2x - 5y = 1$.

(c) Solve graphically.

(i) $x - y = 3, 2x + y = 9$ (ii) $x + y + 2 = 0, 3x - 2y + 1 = 0$

9. Solve the following quadratic equations.

(i) $3x^2 - 5x + 2 = 0$ (ii) $5x^2 - 4x - 1 = 0$ (iii) $5x + \frac{5}{x} = 26$

(iv) $\frac{x}{2} + \frac{2}{x} = \frac{x}{8} + \frac{8}{x}$ (v) $\frac{x}{x+1} + \frac{x+1}{x} = \frac{25}{12}$ (vi) $\frac{x}{3} + \frac{3}{6-x} = \frac{2(6+x)}{15}$

(vii) $\frac{1}{x+7} + \frac{1}{x+3} = \frac{6}{5}$

10. Find the values of x and y.

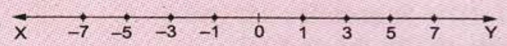
(i) $2x^2 + y^2 = 18, 3x^2 + 2y^2 = 35$ (ii) $x^2 + y^2 = 13, 3x^2 - 2y^2 = -6$

11. The sum of two numbers is 7. Five times the first number plus thrice the second number equals 29. Find the numbers.

ANSWERS

1. (i) $x = -6$ (ii) $x = -31$ (iii) $x = 1$ (iv) $x = 2000$

2. 8643 and 7543 **3.** ₹ 4800, ₹ 1200 **4.** 129

5. 

6. (i) $x = 7, y = 9$ (ii) $x = 2, y = 5$ (iii) $x = 1, y = 1$ (iv) $x = 2, y = 2$ (v) $x = 3, y = 4$

7. (i) $\frac{5}{8}, \frac{1}{4}$ (ii) 77 (iii) 52 kg, 32 kg **8.** (c) (i) $x = 4, y = 1$ (ii) $x = -1, y = -1$

9. (i) $1, \frac{2}{3}$ (ii) $-\frac{1}{5}, 1$ (iii) $5, \frac{1}{5}$ (iv) 4, –4 (v) –4, 3 (vi) 9, 1 (vii) $\frac{-19}{3}, -2$

10. (i) $x = \pm 1, y = \pm 4$ (ii) $x = \pm 2, y = \pm 3$ **11.** 4 and 3

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