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Special Products and Expansions

Special Products

You can find the products of certain types of algebraic expressions directly without actually carrying out the multiplication. These products called special products, come in handy while simplifying expressions or solving equations. Some of the special products are as follows.

1. $(x+a)(x+b) = x^2 + (a+b)x + ab$

Proof $(x+a)(x+b) = x(x+b) + a(x+b) = x^2 + xb + ax + ab$
$= x^2 + bx + ax + ab = x^2 + (b+a)x + ab = x^2 + (a+b)x + ab.$

Examples (i) $(x+2)(x+3) = x^2 + (2+3)x + 2\times 3 = x^2 + 5x + 6.$

(ii) $(m+3n)(m+7n) = m^2 + (3n+7n)m + 3n\times 7n = m^2 + 10mn + 21n^2.$

(iii) $(4x^2+5y^2)(4x^2+9y^2) = (4x^2)^2 + (5y^2+9y^2)\times 4x^2 + 5y^2\times 9y^2$
$= 16x^4 + 56x^2y^2 + 45y^4.$

(iv) $(3ab+2xy)(3ab+7xy) = (3ab)^2 + (2xy+7xy)\times 3ab + 2xy\times 7xy$
$= 9a^2b^2 + 27abxy + 14x^2y^2.$

2. $(x+a)(x-b) = x^2 + (a-b)x - ab$

Proof $(x+a)(x-b) = x(x-b) + a(x-b) = x^2 - xb + ax - ab$
$= x^2 - bx + ax - ab = x^2 + ax - bx - ab = x^2 + (a-b)x - ab.$

Alternative method

From 1, $(x+a)(x+b) = x^2 + (a+b)x + ab.$

Substituting $-b$ for b, $(x+a)[x+(-b)] = x^2[a+(-b)]x + a\times(-b).$

$\therefore \quad (x+a)(x-b) = x^2 + (a-b)x - ab.$

Examples (i) $(x+9)(x-7) = x^2 + (9-7)x - 9\times 7 = x^2 + 2x - 63.$

(ii) $(l+3m)(l-5m) = l^2 + (3m-5m)l - 3m\times 5m = l^2 - 2lm - 15m^2.$

(iii) $(2c^2+4d^2)(2c^2-d^2) = (2c^2)^2 + (4d^2-d^2)\times 2c^2 - 4d^2\times d^2$
$= 4c^4 + 6c^2d^2 - 4d^4.$

3. $(x-a)(x+b) = x^2 - (a-b)x - ab$

Proof $(x-a)(x+b) = (x-a)x + (x-a)b = x^2 - ax + xb - ab$
$= x^2 - ax + bx - ab = x^2 - (a-b)x - ab.$

Alternative method

From 1, $(x+a)(x+b) = x^2 + (a+b)x + ab$.

Substituting $-a$ with a, $[x+(-a)](x+b) = x^2 + (-a+b)x + (-a)\times b$

or $(x-a)(x+b) = x^2 - (a-b)x - ab$.

Examples (i) $(x-9)(x+7) = x^2 - (9-7)x - 9\times 7 = x^2 - 2x - 63$.

(ii) $(l^2 - 6mn)(l^2 + 9mn) = l^2 - (6mn - 9mn)l^2 - 6mn \times 9mn$

$= l^2 + 3l^2mn - 54m^2n^2$.

4. $(x-a)(x-b) = x^2 - (a+b)x + ab$

Proof $(x-a)(x-b) = x(x-b) - a(x-b) = x^2 - xb - ax + ab$

$= x^2 - bx - ax + ab = x^2 - ax - bx + ab$

$= x^2 - (a+b)x + ab$.

Alternative method

From 1, $(x+a)(x+b) = x^2 + (a+b)x + ab$.

Substituting $-a$ for a and $-b$ for b.

$[x+(-a)][x+(-b)] = x^2 + [(-a)+(-b)]x + (-a)\times(-b)$

or $(x-a)(x-b) = x^2 - (a+b)x + ab$.

Examples (i) $(a-5)(a-7) = a^2 - (5+7)a + 5\times 7 = a^2 - 12a + 35$.

(ii) $(m-n)(m-6n) = m^2 - (n+6n)m + n\times 6n = m^2 - 7mn + 6n^2$.

(iii) $(4ab - 3cd)(4ab - 5cd) = (4ab)^2 - (3cd + 5cd)\times 4ab + 3cd \times 5cd$

$= 16a^2b^2 - 8cd \times 4ab + 15c^2d^2$

$= 16a^2b^2 - 32abcd + 15c^2d^2$.

Solved Examples

EXAMPLE 1 **Multiply each of the following using a special product.**

(i) $(a+0.1)(a+0.2)$ **(ii)** $\left(\frac{x}{2}+3\right)\left(\frac{x}{2}+6\right)$

(iii) $\left(\frac{m}{5}+\frac{n}{3}\right)\left(\frac{m}{5}+\frac{n}{6}\right)$ **(iv)** $(2p^2 + 0.2qr)(2p^2 + 0.3qr)$

Solution In each case, we can use $(x+a)(x+b) = x^2 + (a+b)x + ab$.

(i) $(a+0.1)(a+0.2) = a^2 + (0.1+0.2)a + 0.1\times 0.2 = a^2 + 0.3a + 0.02$.

(ii) $\left(\frac{x}{2}+3\right)\left(\frac{x}{2}+6\right) = \left(\frac{x}{2}\right)^2 + (3+6)\times\frac{x}{2} + 3\times 6 = \frac{x^2}{4} + \frac{9}{2}x + 18$.

(iii) $\left(\frac{m}{5}+\frac{n}{3}\right)\left(\frac{m}{5}+\frac{n}{6}\right) = \left(\frac{m}{5}\right)^2 + \left(\frac{n}{3}+\frac{n}{6}\right)\times\frac{m}{5} + \frac{n}{3}\times\frac{n}{6} = \frac{m^2}{25} + \left(\frac{1}{3}+\frac{1}{6}\right)\times\frac{mn}{5} + \frac{n^2}{18}$

$= \frac{m^2}{25} + \frac{1}{2}\times\frac{mn}{5} + \frac{n^2}{18} = \frac{m^2}{25} + \frac{mn}{10} + \frac{n^2}{18}$.

(iv) $(2p^2+0.2qr)(2p^2+0.3qr) = (2p^2)^2+(0.2qr+0.3qr)\times 2p^2+0.2qr\times 0.3qr$
$= 4p^4+0.5qr\times 2p^2+0.06q^2r^2$
$= 4p^4+2\times 0.5p^2qr+0.06q^2r^2$
$= 4p^4+p^2qr+0.06q^2r^2.$

EXAMPLE 2 **Simplify each of the following using a special product.**

(i) $(2l+0.7)(2l-0.3)$ (ii) $\left(\frac{xy}{2}+0.4q\right)\left(\frac{xy}{2}-0.3q\right)$ (iii) $(0.1ab+0.7cd)(0.1ab-0.8cd)$

Solution In each case, we can use $(x+a)(x-b)=x^2+(a-b)x-ab$.

(i) $(2l+0.7)(2l-0.3) = (2l)^2+(0.7-0.3)\times 2l-0.7\times 0.3$
$= 4l^2+0.4\times 2l-0.21 = 4l^2+0.8l-0.21.$

(ii) $\left(\frac{xy}{2}+0.4q\right)\left(\frac{xy}{2}-0.3q\right) = \left(\frac{xy}{2}\right)^2+(0.4q-0.3q)\times\frac{xy}{2}-0.4q\times 0.3q$
$= \frac{x^2y^2}{4}+0.1q\times\frac{xy}{2}-0.12q^2$
$= \frac{x^2y^2}{4}+0.05qxy-0.12q^2.$

(iii) $(0.1ab+0.7cd)(0.1ab-0.8cd) = (0.1ab)^2+(0.7cd-0.8cd)\times 0.1ab-0.7cd\times 0.8cd$
$= 0.01a^2b^2+\{-(0.1cd)\times(0.1ab)\}-0.7\times 0.8c^2d^2$
$= 0.01a^2b^2-0.01abcd-0.56c^2d^2.$

EXAMPLE 3 **Use a special product to multiply:**

(i) $(2xy-0.3m)(2xy+0.1m)$ (ii) $\left(4l^2-\frac{mn}{3}\right)\left(4l^2+\frac{2mn}{5}\right)$

Solution Here, we can use $(x-a)(x+b)=x^2-(a-b)x-ab$.

(i) $(2xy-0.3m)(2xy+0.1m) = (2xy)^2-(0.3m-0.1m)\times 2xy-0.3m\times 0.1m$
$= 4x^2y^2-(0.2m)\times 2xy-0.03m^2$
$= 4x^2y^2-0.4mxy-0.03m^2.$

(ii) $\left(4l^2-\frac{mn}{3}\right)\left(4l^2+\frac{2mn}{5}\right) = (4l^2)^2-\left(\frac{mn}{3}-\frac{2mn}{5}\right)\times 4l^2-\frac{mn}{3}\times\frac{2mn}{5}$
$= 16l^4-\left(\frac{1}{3}-\frac{2}{5}\right)\times 4l^2mn-\frac{2}{15}m^2n^2$
$= 16l^4-\left(-\frac{1}{15}\right)\times 4l^2mn-\frac{2}{15}m^2n^2$
$\left[\because \frac{1}{3}-\frac{2}{5}=\frac{5-6}{15}=\frac{-1}{15}\right]$
$= 16l^4+\frac{4}{15}l^2mn-\frac{2}{15}m^2n^2.$

EXAMPLE 4 **Use a special product to find the following.**

(i) $(x-0.6m)(x-0.4m)$ (ii) $\left(\frac{1}{3}lx-\frac{1}{4}my\right)\left(\frac{1}{3}lx-\frac{3}{4}my\right)$ (iii) $(4x-3y)(5x-3y)$

Solution Here, we can use $(x-a)(x-b) = x^2 - (a+b)x + ab$.

(i) $(x-0.6m)(x-0.4m) = x^2 - (0.6m+0.4m)x + 0.6m \times 0.4m$

$= x^2 - (0.6+0.4)mx + (0.6\times 0.4)m^2 = x^2 - mx + 0.24m^2$.

(ii) $\left(\frac{1}{3}lx - \frac{1}{4}my\right)\left(\frac{1}{3}lx - \frac{3}{4}my\right) = \left(\frac{1}{3}lx\right)^2 - \left(\frac{1}{4}my + \frac{3}{4}my\right)\times\frac{1}{3}lx + \frac{1}{4}my \times \frac{3}{4}my$

$= \frac{1}{9}l^2x^2 - \left(\frac{1}{4}+\frac{3}{4}\right)my \times \frac{1}{3}lx + \left(\frac{1}{4}\times\frac{3}{4}\right)m^2y^2$

$= \frac{1}{9}l^2x^2 - \frac{1}{3}lmxy + \frac{3}{16}m^2y^2$.

(iii) $(4x-3y)(5x-3y) = \{-(3y-4x)\}\{-(3y-5x)\}$

$= (3y-4x)(3y-5x) = (3y)^2 - (4x+5x)3y + 4x\cdot 5x$

$= 9y^2 - 9x\cdot 3y + 20x^2 = 9y^2 - 27xy + 20x^2$.

EXAMPLE 5 **Find the following products using distributive law.**

(i) $(2x+7)(3x-8)$ **(ii) $(3a-4b)(4a+5b)$**

(iii) $(3x^2-4yz)(5x^2-9yz)$ **(iv) $(m-1)(2m+3)(3m+7)$**

Solution (i) $(2x+7)(3x-8) = 2x(3x-8) + 7(3x-8) = 6x^2 - 16x + 21x - 56 = 6x^2 + 5x - 56$.

(ii) $(3a-4b)(4a+5b) = 3a(4a+5b) - 4b(4a+5b)$

$= 12a^2 + 15ab - 16ab - 20b^2 = 12a^2 - ab - 20b^2$.

(iii) $(3x^2-4yz)(5x^2-9yz) = 3x^2(5x^2-9yz) - 4yz(5x^2-9yz)$

$= 15x^4 - 27x^2yz - 20x^2yz + 36y^2z^2$

$= 15x^4 - 47x^2yz + 36y^2z^2$.

(iv) $(m-1)(2m+3)(3m+7) = [(m-1)(2m+3)](3m+7)$

$= [m(2m+3) - 1(2m+3)](3m+7)$

$= (2m^2+3m-2m-3)(3m+7) = (2m^2+m-3)(3m+7)$

$= 2m^2(3m+7) + m(3m+7) - 3(3m+7)$

$= 6m^3 + 14m^2 + 3m^2 + 7m - 9m - 21$

$= 6m^3 + 17m^2 - 2m - 21$.

Remember These

1. $(x+a)(x+b) = x^2 + (a+b)x + ab$
2. $(x+a)(x-b) = x^2 + (a-b)x - ab$
3. $(x-a)(x+b) = x^2 - (a-b)x - ab$
4. $(x-a)(x-b) = x^2 - (a+b)x + ab$
5. Product by distributive law: $(x+a)(y+b) = x(y+b) + a(y+b) = xy + xb + ay + ab$

EXERCISE 4A

Use special products to find each of the following.

1. (i) $(p+1)(p+9)$ (ii) $(2m+3)(2m+5)$ (iii) $(3x+0.2)(3x+0.4)$

(iv) $(2a^2+3b^2)(2a^2+9b^2)$ (v) $\left(2xy+\frac{1}{7}z\right)\left(2xy+\frac{1}{8}z\right)$ (vi) $(r^2+0.2t)(r^2+0.3t)$

(vii) $\left(\frac{x}{6}+\frac{yz}{3}\right)\left(\frac{x}{6}+\frac{2yz}{3}\right)$

2. (i) $(x+2)(x-1)$ (ii) $(2m+4)(2m-3)$ (iii) $(5a+0.3)(5a-0.1)$

(iv) $(4a+3b)(4a-b)$ (v) $\left(2ab+\frac{1}{3}cd\right)\left(2ab-\frac{1}{5}cd\right)$

(vi) $(5mn+0.4pq)(5mn-0.3pq)$

3. (i) $(x-3)(x+1)$ (ii) $(x-15)(x+7)$ (iii) $(a-0.3)(a+0.25)$

(iv) $(2l-3m)(2l+m)$ (v) $\left(\frac{3x^2}{5}-\frac{2yz}{7}\right)\left(\frac{3x^2}{5}+\frac{5yz}{9}\right)$

(vi) $(0.4xy-0.21mn)(0.4xy+0.2mn)$

4. (i) $(m-2)(m-3)$ (ii) $(2x-5)(2x-7)$ (iii) $(3p-q)(5p-q)$

(iv) $\left(\frac{a}{2}-3\right)\left(\frac{a}{2}-9\right)$ (v) $(0.2x-3yz)(0.2x-7yz)$

(vi) $\left(\frac{1}{3}ab-\frac{1}{6}cd\right)\left(\frac{1}{3}ab-\frac{5}{6}cd\right)$ (vii) $(3l-0.8)(3l-0.2)$

5. Multiply the following using distributive law.

(i) $(2a^2+3b^2)(5a^2-6b^2)$ (ii) $(2xy-3z)(3xy+5z)$ (iii) $(a+2)(2a+3)(3a-1)$

ANSWERS

1. (i) $p^2+10p+9$ (ii) $4m^2+16m+15$ (iii) $9x^2+1.8x+0.08$ (iv) $4a^4+24a^2b^2+27b^4$

(v) $4x^2y^2+\frac{15}{28}xyz+\frac{1}{56}z^2$ (vi) $r^4+0.5r^2t+0.06t^2$ (vii) $\frac{x^2}{36}+\frac{1}{6}xyz+\frac{2}{9}y^2z^2$

2. (i) x^2+x-2 (ii) $4m^2+2m-12$ (iii) $25a^2+a-0.03$ (iv) $16a^2+8ab-3b^2$

(v) $4a^2b^2+\frac{4}{15}abcd-\frac{1}{15}c^2d^2$ (vi) $25m^2n^2+0.5mnpq-0.12p^2q^2$

3. (i) x^2-2x-3 (ii) $x^2-8x-105$ (iii) $a^2-0.05a-0.75$ (iv) $4l^2-4lm-3m^2$

(v) $\frac{9}{25}x^4+\frac{17}{105}x^2yz-\frac{10}{63}y^2z^2$ (vi) $0.16x^2y^2-0.004xymn-0.042m^2n^2$

4. (i) m^2-5m+6 (ii) $4x^2-24x+35$ (iii) $15p^2-8pq+q^2$ (iv) $\frac{a^2}{4}-6a+27$

(v) $0.04x^2-2xyz+21y^2z^2$ (vi) $\frac{1}{9}a^2b^2-\frac{1}{3}abcd+\frac{5}{36}c^2d^2$ (vii) $9l^2-3l+0.16$

5. (i) $10a^4+3a^2b^2-18b^4$ (ii) $6x^2y^2+xyz-15z^2$ (iii) $6a^3+19a^2+11a-6$

Product of sum and difference of two terms

Let us take two terms a and b and find the product of their sum $(a+b)$ and difference $(a-b)$.

$(a+b)(a-b)=(a+b)a-(a+b)b=a^2+ba-ab-b^2=a^2+ab-ab-b^2=a^2-b^2.$

Thus, $$(a+b)(a-b)=a^2-b^2$$

In words, the product of the sum and the difference of two terms is equal to the difference of their squares.

This can also be expressed as,

(first term + second term) × (first term – second term) = (first term)^2 – (second term)^2

Examples (i) $(m+3)(m-3)=(m)^2-(3)^2=m^2-9.$

(ii) $(7+2x)(7-2x)=(7)^2-(2x)^2=(7)^2-2^2\times x^2=49-4x^2.$

(iii) $(3a^2-8b^2)(3a^2+8b^2)=(3a^2)^2-(8b^2)^2=3^2\times(a^2)^2-(8)^2\times(b^2)^2$
$=9a^4-64b^4.$

Solved Examples

EXAMPLE 1 **Use a special product to multiply the following.**

(i) $(5+3x^2y)(5-3x^2y)$ **(ii)** $\left(7m+\frac{n}{6}\right)\left(7m-\frac{n}{6}\right)$ **(iii)** $\left(\frac{4p}{7q}+\frac{5x}{3y}\right)\left(\frac{4p}{7q}-\frac{5x}{3y}\right)$

(iv) $(0.2a+0.3b)(0.2a-0.3b)$

Solution We will use $(a+b)(a-b)=a^2-b^2$ in each case.

(i) $(5+3x^2y)(5-3x^2y)=(5)^2-(3x^2y)^2=5^2-3^2\times(x^2)^2\times y^2=25-9x^4y^2.$

(ii) $\left(7m+\frac{n}{6}\right)\left(7m-\frac{n}{6}\right)=(7m)^2-\left(\frac{n}{6}\right)^2=7^2\times m^2-\frac{n^2}{6^2}=49m^2-\frac{n^2}{36}.$

(iii) $\left(\frac{4p}{7q}+\frac{5x}{3y}\right)\left(\frac{4p}{7q}-\frac{5x}{3y}\right)=\left(\frac{4p}{7q}\right)^2-\left(\frac{5x}{3y}\right)^2=\frac{(4p)^2}{(7q)^2}-\frac{(5x)^2}{(3y)^2}=\frac{16p^2}{49q^2}-\frac{25x^2}{9y^2}.$

(iv) $(0.2a+0.3b)(0.2a-0.3b)=(0.2a)^2-(0.3b)^2$
$=(0.2)^2\times a^2-(0.3)^2\times b^2=0.04a^2-0.09b^2.$

EXAMPLE 2 **Find the following by using a special product.**

(i) $(-ax+by)(ax+by)$ **(ii)** $(2x+4y)(x-2y)$ **(iii)** $\left(\frac{z^2}{9}-\frac{x^3}{3}\right)\left(\frac{z^2}{3}+x^3\right)$

Solution In each case, we will use $(a+b)(a-b)=a^2-b^2$.

(i) $(-ax+by)(ax+by)=-(ax-by)(ax+by)$
$=-\{(ax)^2-(by)^2\}=-(a^2x^2-b^2y^2)=b^2y^2-a^2x^2.$

(ii) $2x+4y=2(x+2y).$

$\therefore$ $(2x+4y)(x-2y)=2(x+2y)(x-2y)$
$=2\{(x)^2-(2y)^2\}=2(x^2-4y^2)=2x^2-8y^2.$

(iii) $\frac{z^2}{9}-\frac{x^3}{3}=\frac{1}{3}\left(\frac{z^2}{3}-x^3\right)$.

$$\therefore \left(\frac{z^2}{9}-\frac{x^3}{3}\right)\left(\frac{z^2}{3}+x^3\right)=\frac{1}{3}\left(\frac{z^2}{3}-x^3\right)\left(\frac{z^2}{3}+x^3\right)$$

$$=\frac{1}{3}\left\{\left(\frac{z^2}{3}\right)^2-(x^3)^2\right\}=\frac{1}{3}\left(\frac{z^4}{9}-x^6\right).$$

EXAMPLE 3 **Find the product $(5-3x)(5+3x)(25+9x^2)$.**

Solution $(5-3x)(5+3x)=5^2-(3x)^2$, using $(a-b)(a+b)=a^2-b^2$.

$$=25-9x^2.$$

$$\therefore (5-3x)(5+3x)(25+9x^2)=(25-9x^2)(25+9x^2)$$

$$=(25)^2-(9x^2)^2=625-81x^4.$$

EXAMPLE 4 **Use a special product to work out the following.**

(i) 98×102 **(ii) $3\frac{1}{4}\times 2\frac{3}{4}$** **(iii) 2.04×1.96**

Solution (i) $98=100-2$ and $102=100+2$

$$\therefore 98\times 102=(100-2)\times(100+2)=(100)^2-(2)^2=10000-4=9996.$$

(ii) Here, $3\frac{1}{4}\times 2\frac{3}{4}=\left(3+\frac{1}{4}\right)\times\left(3-\frac{1}{4}\right)=(3)^2-\left(\frac{1}{4}\right)^2=9-\frac{1}{16}=8\frac{15}{16}$.

(iii) $2.04\times 1.96=(2+.04)(2-.04)=2^2-(.04)^2=(4-.0016)=3.9984.$

EXERCISE 4B

Find the following products.

1. (i) $(a+5)(a-5)$ (ii) $(3-5x)(3+5x)$ (iii) $\left(\frac{p}{2}+3\right)\left(\frac{p}{2}-3\right)$

   (iv) $\left(2ab+\frac{3}{2}\right)\left(2ab-\frac{3}{2}\right)$ (v) $(2l+3m)(2l-3m)$

2. (i) $(x+0.2)(x-0.2)$ (ii) $(0.1a-0.3b)(0.1a+0.3b)$

3. (i) $\left(\frac{1}{3x}+9\right)\left(\frac{1}{3x}-9\right)$ (ii) $\left(x+\frac{4}{m}\right)\left(x-\frac{4}{m}\right)$ (iii) $\left(\frac{5}{a}+\frac{7}{2}b\right)\left(\frac{5}{a}-\frac{7}{2}b\right)$

   (iv) $\left(\frac{x}{3a}-\frac{y}{4b}\right)\left(\frac{x}{3a}+\frac{y}{4b}\right)$ (v) $\left(-\frac{2}{5y}+\frac{1}{6x}\right)\left(\frac{1}{6x}+\frac{2}{5y}\right)$

4. (i) $(14a^2b-6cd)(7a^2b+3cd)$ (ii) $\left(\frac{5x}{6}+\frac{2y}{9}\right)\left(\frac{5x}{2}-\frac{2y}{3}\right)$

5. $\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)\left(x^2+\frac{1}{x^2}\right)$

6. (i) 201×199 (ii) 52×48 (iii) 79×81 (iv) 403×397

7. (i) $8\frac{1}{3} \times 7\frac{2}{3}$ (ii) $20\frac{1}{2} \times 19\frac{1}{2}$ (iii) 9.8×10.2

ANSWERS

1. (i) $a^2 - 25$ (ii) $9 - 25x^2$ (iii) $\frac{p^2}{4} - 9$ (iv) $4a^2b^2 - \frac{9}{4}$ (v) $4l^2 - 9m^2$

2. (i) $x^2 - 0.04$ (ii) $0.01a^2 - 0.09b^2$

3. (i) $\frac{1}{9x^2} - 81$ (ii) $x^2 - \frac{16}{m^2}$ (iii) $\frac{25}{a^2} - \frac{49}{4}b^2$ (iv) $\frac{x^2}{9a^2} - \frac{y^2}{16b^2}$ (v) $\frac{1}{36x^2} - \frac{4}{25y^2}$

4. (i) $98a^4b^2 - 18c^2d^2$ (ii) $\frac{25x^2}{12} - \frac{4y^2}{27}$ **5.** $x^4 - \frac{1}{x^4}$

6. (i) 39999 (ii) 2496 (iii) 6399 (iv) 159991

7. (i) $63\frac{8}{9}$ (ii) $399\frac{3}{4}$ (iii) 99.96

Expansions

The product of an algebraic expression multiplied by itself (any number of times) when expressed as a polynomial is called an expansion. In fact, the term 'expansion' is used for the process of such multiplication. In this section, we will look at some expansions and their corollaries. A corollary is a result or relation that follows from another result or relation that is known.

Squares of binomials

Let a and b be two terms. We will find the expansions of the squares of the sum and difference of these two terms. In other words, we will find the expansions of $(a+b)^2$ and $(a-b)^2$.

1. $(a+b)^2 = a^2 + 2ab + b^2$

Proof $(a+b)^2 = (a+b)(a+b) = a(a+b) + b(a+b) = a^2 + ab + ba + b^2$
$= a^2 + ab + ab + b^2 = a^2 + 2ab + b^2.$

We can express this as follows.

(Sum of two terms)^2 = (first term)^2 + 2 × (first term) × (second term) + (second term)^2

Examples (i) $(a+4)^2 = a^2 + 2 \times a \times 4 + (4)^2 = a^2 + 8a + 16.$

(ii) $(1+3m)^2 = 1^2 + 2\times 1\times 3m + (3m)^2 = 1 + 6m + 9m^2$.

(iii) $\left(\frac{3}{4}x + 2y\right)^2 = \left(\frac{3}{4}x\right)^2 + 2\times\frac{3}{4}x\times 2y + (2y)^2 = \frac{9}{16}x^2 + 3xy + 4y^2$.

(iv) $(\sqrt{3}l + 2m)^2 = (\sqrt{3}\,l)^2 + 2\times\sqrt{3}\,l\times 2m + (2m)^2 = 3l^2 + 4\sqrt{3}\,lm + 4m^2$.

Corollaries

Two corollaries follow from the expansion we have just discussed.

(i) $(a+b)^2 - 2ab = a^2 + 2ab + b^2 - 2ab = a^2 + b^2$.

$$\therefore\ a^2 + b^2 = (a+b)^2 - 2ab$$

(ii) $(a+b)^2 - (a^2+b^2) = a^2 + 2ab + b^2 - a^2 - b^2 = 2ab$.

$$\therefore\ 2ab = (a+b)^2 - (a^2+b^2)$$

EXAMPLE **If $a+b=7$ and $ab=10$, find a^2+b^2.**

Solution $a^2 + b^2 = (a+b)^2 - 2ab = (7)^2 - 2\times 10 = 49 - 20 = 29$.

EXAMPLE **If $a+b=6$ and $a^2+b^2=20$, find ab.**

Solution $2ab = (a+b)^2 - (a^2+b^2) = (6)^2 - 20 = 36 - 20 = 16$.

$\therefore\ ab = \frac{1}{2}\times 16 = 8$.

2. $(a-b)^2 = a^2 - 2ab + b^2$

Proof $(a-b)^2 = (a-b)(a-b) = a(a-b) - b(a-b) = a^2 - ab - ba + b^2$

$= a^2 - ab - ab + b^2 = a^2 - 2ab + b^2$.

We can express this as follows.

(Difference of two terms)^2 = (first term)^2 − 2 × (first term) × (second term) + (second term)^2

Examples (i) $(m-2)^2 = (m)^2 - 2\times m\times 2 + (2)^2 = m^2 - 4m + 4$.

(ii) $\left(3 - \frac{4}{5}a\right)^2 = (3)^2 - 2\times 3\times\frac{4}{5}a + \left(\frac{4}{5}a\right)^2 = 9 - \frac{24}{5}a + \frac{16}{25}a^2$.

(iii) $\left(\frac{2}{3}x - \frac{3}{2}y\right)^2 = \left(\frac{2}{3}x\right)^2 - 2\times\frac{2}{3}x\times\frac{3}{2}y + \left(\frac{3}{2}y\right)^2 = \frac{4}{9}x^2 - 2xy + \frac{9}{4}y^2$.

Corollaries

Several corollaries follow from the two expansions we have discussed.

(i) $(a-b)^2 + 2ab = a^2 - 2ab + b^2 + 2ab = a^2 + b^2$.

$$\therefore\ a^2 + b^2 = (a-b)^2 + 2ab$$

(ii) $a^2+b^2-(a-b)^2=a^2+b^2-(a^2-2ab+b^2)=a^2+b^2-a^2+2ab-b^2=2ab.$

$$\therefore\ 2ab=a^2+b^2-(a-b)^2$$

(iii) $(a-b)^2+4ab=a^2-2ab+b^2+4ab=a^2+2ab+b^2=(a+b)^2.$

$$\therefore\ (a+b)^2=(a-b)^2+4ab$$

(iv) $(a+b)^2-4ab=a^2+2ab+b^2-4ab=a^2-2ab+b^2=(a-b)^2.$

$$\therefore\ (a-b)^2=(a+b)^2-4ab$$

(v) $\frac{1}{4}[(a+b)^2-(a-b)^2]=\frac{1}{4}[(a^2+2ab+b^2)-(a^2-2ab+b^2)]$

$=\frac{1}{4}[a^2+2ab+b^2-a^2+2ab-b^2]=\frac{1}{4}\times 4ab=ab.$

$$\therefore\ ab=\frac{1}{4}[(a+b)^2-(a-b)^2]$$

(vi) $\frac{1}{2}[(a+b)^2+(a-b)^2]=\frac{1}{2}[(a^2+2ab+b^2)+(a^2-2ab+b^2)]$

$=\frac{1}{2}[2a^2+2b^2]=\frac{1}{2}\cdot 2[a^2+b^2]=a^2+b^2.$

$$\therefore\ a^2+b^2=\frac{1}{2}[(a+b)^2-(a-b)^2]$$

EXAMPLE **Find the value of a^2+b^2 when $a-b=7$ and $ab=18$.**

Solution $a^2+b^2=(a-b)^2+2ab=7^2+2\times 18=49+36=85.$

EXAMPLE **Find the value of ab when $a-b=6$ and $a^2+b^2=40$.**

Solution $2ab=a^2+b^2-(a-b)^2=40-(6)^2=40-36=4.$

$\therefore\ ab=2.$

EXAMPLE **Find the value of $a+b$ when $a-b=8$ and $ab=9$.**

Solution $(a+b)^2=(a-b)^2+4ab=(8)^2+4\times 9=64+36=100.$

$\therefore\ a+b=\sqrt{100}=\pm 10.$

EXAMPLE **Find the value of $a-b$ when $a+b=7$ and $ab=6$.**

Solution $(a-b)^2=(a+b)^2-4ab=(7)^2-4\times 6=49-24=25.$

$\therefore\ a-b=\sqrt{25}=\pm 5.$

EXAMPLE **Find the value of ab when $a+b=8$ and $a-b=2$.**

Solution $ab=\frac{1}{4}[(a+b)^2-(a-b)^2]=\frac{1}{4}[(8)^2-(2)^2]=\frac{1}{4}[64-4]=\frac{1}{4}\times 60=15.$

Perfect square trinomials

A trinomial which is the square of a binomial is called a perfect square trinomial. Two examples of such trinomials are $a^2 + 2ab + b^2$ [$= (a+b)^2$] and $a^2 - 2ab + b^2$ [$= (a-b)^2$].

In a perfect square trinomial,

- two terms are perfect squares
- the third term (which may be positive or negative) is twice the product of the square roots of the other two terms.

Examples (i) Consider the expression $4a^2 + 4a + 1$.

Here, $4a^2 = (2a)^2, 1 = (1)^2$ and $4a = 2 \times 2a \times 1$ = twice the product of the square roots of the other two terms.

Thus, this is a perfect square trinomial that can be expressed as the square of a binomial as follows.

$4a^2 + 4a + 1 = (2a)^2 + 2 \times 2a \times 1 + (1)^2 = (2a+1)^2.$

(ii) In the expression $9x^2 - 24x + 16$, we observe that $9x^2 = (3x)^2$, $16 = (4)^2$ and $24x = 2 \times 3x \times 4$.

Thus, $9x^2 - 24x + 16 = (3x)^2 - 2 \times 3x \times 4 + (4)^2 = (3x-4)^2$.

(iii) Take the expression $8x^2 - 24xy + 18y^2$.

$8x^2 - 24xy + 18y^2 = 2(4x^2 - 12xy + 9y^2)$

$= 2[(2x)^2 - 2 \times 2x \times 3y + (3y)^2] = 2(2x - 3y)^2.$

Solved Examples

EXAMPLE 1 **Find the square of each of the following binomials.**

(i) $2m + \frac{1}{2m}$ **(ii)** $\frac{x}{4} + \frac{2y}{3}$ **(iii)** $\sqrt{5}\left(\frac{ab}{6} + \frac{c^2}{5}\right)$

Solution We will use $(a+b)^2 = a^2 + 2ab + b^2$ in each case.

(i) $\left(2m + \frac{1}{2m}\right)^2 = (2m)^2 + 2 \times 2m \times \frac{1}{2m} + \left(\frac{1}{2m}\right)^2 = 4m^2 + 2 + \frac{1}{4m^2}$.

(ii) $\left(\frac{x}{4} + \frac{2y}{3}\right)^2 = \left(\frac{x}{4}\right)^2 + 2 \times \frac{x}{4} \times \frac{2y}{3} + \left(\frac{2y}{3}\right)^2 = \frac{x^2}{16} + \frac{xy}{3} + \frac{4y^2}{9}$.

(iii) $\left[\sqrt{5}\left(\frac{ab}{6} + \frac{c^2}{5}\right)\right]^2 = (\sqrt{5})^2\left(\frac{ab}{6} + \frac{c^2}{5}\right)^2 = 5\left[\left(\frac{ab}{6}\right)^2 + 2 \times \frac{ab}{6} \times \frac{c^2}{5} + \left(\frac{c^2}{5}\right)^2\right]$

$= 5\left[\frac{a^2b^2}{36} + \frac{abc^2}{15} + \frac{c^4}{25}\right] = \frac{5}{36}a^2b^2 + \frac{abc^2}{3} + \frac{c^4}{5}.$

EXAMPLE 2 **Expand (i)** $\left(\frac{1}{\sqrt{3}}a - \frac{3}{2}b\right)^2$ **(ii)** $\left(\frac{m^2}{2n} - \frac{2n}{m^2}\right)^2$ **(iii)** $\left[\sqrt{3}\left(2p^2 - \frac{3}{4}q^3\right)^2\right]$.

Solution We will use $(a-b)^2 = a^2 - 2ab + b^2$.

(i) $\left(\frac{1}{\sqrt{3}}a-\frac{3}{2}b\right)^2=\left(\frac{1}{\sqrt{3}}a\right)^2-2\times\frac{1}{\sqrt{3}}a\times\frac{3}{2}b+\left(\frac{3}{2}b\right)^2$

$=\frac{1}{3}a^2-\frac{(\sqrt{3})^2}{\sqrt{3}}ab+\frac{9}{4}b^2=\frac{1}{3}a^2-\sqrt{3}ab+\frac{9}{4}b^2.$

(ii) $\left(\frac{m^2}{2n}-\frac{2n}{m^2}\right)^2=\left(\frac{m^2}{2n}\right)^2-2\times\frac{m^2}{2n}\times\frac{2n}{m^2}+\left(\frac{2n}{m^2}\right)^2=\frac{m^4}{4n^2}-2+\frac{4n^2}{m^4}.$

(iii) $\left[\sqrt{3}\left(2p^2-\frac{3}{4}q^3\right)\right]^2=(\sqrt{3})^2\left(2p^2-\frac{3}{4}q^3\right)^2$

$=3\left[(2p^2)^2-2\times 2p^2\times\frac{3}{4}q^3+\left(\frac{3}{4}q^3\right)^2\right]$

$=3\left[4p^4-3p^2q^3+\frac{9}{16}q^6\right]=12p^4-9p^2q^3+\frac{27}{16}q^6.$

EXAMPLE 3 **Use special expansions to find (i) 201^2 and (ii) $(49.9)^2$.**

Solution (i) $201=200+1.$

$\therefore\ 201^2=(200+1)^2=(200)^2+2\times 200\times 1+1^2$ $[\because (a+b)^2=a^2+2ab+b^2]$

$=40000+400+1=40401.$

(ii) $49.9=50-0.1.$

$\therefore\ (49.9)^2=(50-0.1)^2=50^2-2\times 50\times 0.1+(0.1)^2$ $[\because (a-b)^2=a^2-2ab+b^2]$

$=2500-100\times 0.1+0.01=2500-10+0.01=2490.01.$

EXAMPLE 4 **If $x^2+y^2=37$ and $xy=6$, find the values of (i) $x+y$ and (ii) $x-y$.**

Solution (i) $(x+y)^2=x^2+y^2+2xy=37+2\times 6=37+12=49.$

$\therefore\ x+y=\sqrt{49}=\pm 7.$

(ii) $(x-y)^2=x^2+y^2-2xy=37-2\times 6=37-12=25.$

$\therefore\ x-y=\sqrt{25}=\pm 5.$

EXAMPLE 5 **If $z+\frac{1}{z}=5$, find the values of (i) $z-\frac{1}{z}$, (ii) $z^2+\frac{1}{z^2}$ and $z^4+\frac{1}{z^4}$.**

Solution (i) $(a-b)^2=(a+b)^2-4ab.$

$\therefore\ \left(z-\frac{1}{z}\right)^2=\left(z+\frac{1}{z}\right)^2-4\times z\times\frac{1}{z}=(5)^2-4=21.$

$\therefore\ z-\frac{1}{z}=\pm\sqrt{21}.$

(ii) $a^2+b^2=(a+b)^2-2ab.$

$\therefore\ z^2+\frac{1}{z^2}=\left(z+\frac{1}{z}\right)^2-2\times z\times\frac{1}{z}=(5)^2-2=23.$

(iii) $\left(z^2+\frac{1}{z^2}\right)^2=z^4+2\times z^2\times\frac{1}{z^2}+\frac{1}{z^4}$

or $(23)^2 = z^4 + \frac{1}{z^4} + 2$. $\left[\because\ z^2 + \frac{1}{z^2} = 23 \text{ from (ii)}\right]$

$\therefore\quad z^4 + \frac{1}{z^4} = (23)^2 - 2 = 529 - 2 = 527.$

EXAMPLE 6 **If $m - \frac{1}{m} = 4$, find the values of (i) $m + \frac{1}{m}$, (ii) $m^2 + \frac{1}{m^2}$ and (iii) $m^4 + \frac{1}{m^4}$.**

Solution (i) $(a+b)^2 = (a-b)^2 + 4ab$.

$\therefore\quad \left(m + \frac{1}{m}\right)^2 = \left(m - \frac{1}{m}\right)^2 + 4 \times m \times \frac{1}{m} = (4)^2 + 4 = 20.$

$\therefore\quad m + \frac{1}{m} = \sqrt{20} = \sqrt{4 \times 5} = \pm 2\sqrt{5}.$

(ii) $a^2 + b^2 = (a-b)^2 + 2ab$.

$\therefore\quad m^2 + \frac{1}{m^2} = \left(m - \frac{1}{m}\right)^2 + 2 \times m \times \frac{1}{m} = (4)^2 + 2 = 18.$

(iii) $\left(m^2 + \frac{1}{m^2}\right)^2 = m^4 + 2 \times m^2 \times \frac{1}{m^2} + \frac{1}{m^4}$

or $(18)^2 = m^4 + \frac{1}{m^4} + 2$ or $324 = m^4 + \frac{1}{m^4} + 2$

or $m^4 + \frac{1}{m^4} = 324 - 2 = 322.$

EXAMPLE 7 **If $x^2 + \frac{1}{x^2} = 14$, find the values of (i) $x + \frac{1}{x}$ and (ii) $x - \frac{1}{x}$.**

Solution (i) $(a+b)^2 = a^2 + 2ab + b^2$.

$\therefore\quad \left(x + \frac{1}{x}\right)^2 = x^2 + 2 \times x \times \frac{1}{x} + \frac{1}{x^2} = x^2 + 2 + \frac{1}{x^2} = x^2 + \frac{1}{x^2} + 2 = 14 + 2 = 16.$

$\therefore\quad x + \frac{1}{x} = \sqrt{16} = \pm 4.$

(ii) $(a-b)^2 = a^2 - 2ab + b^2$.

$\therefore\quad \left(x - \frac{1}{x}\right)^2 = x^2 - 2 \times x \times \frac{1}{x} + \frac{1}{x^2} = x^2 + \frac{1}{x^2} - 2 = 14 - 2 = 12.$

$\therefore\quad x - \frac{1}{x} = \sqrt{12} = \sqrt{4 \times 3} = \pm 2\sqrt{3}.$

EXAMPLE 8 **Express each of the following trinomials as a perfect square if possible.**

(i) $4p^2 + \frac{12}{5}pq + \frac{9}{25}q^2$ (ii) $\frac{4}{9}x^2 - \frac{10}{9}xy + \frac{25}{36}y^2$ (iii) $25x^2 - 30xy + 36y^2$

Solution (i) $4p^2 = (2p)^2$, $\frac{9}{25}q^2 = \left(\frac{3}{5}q\right)^2$ and $\frac{12}{5}pq = 2 \times 2p \times \frac{3}{5}q$.

Thus, $4p^2 + \frac{12}{5}pq + \frac{9}{25}q^2 = (2p)^2 + 2 \times 2p \times \frac{3}{5}q + \left(\frac{3}{5}q\right)^2 = \left(2p + \frac{3}{5}q\right)^2.$

(ii) $\frac{4}{9}x^2 = \left(\frac{2}{3}x\right)^2, \frac{25}{36}y^2 = \left(\frac{5}{6}y\right)^2$ and $\frac{10}{9}xy = 2 \times \frac{2}{3}x \times \frac{5}{6}y$.

Thus, $\frac{4}{9}x^2 - \frac{10}{9}xy + \frac{25}{36}y^2 = \left(\frac{2}{3}x\right)^2 - 2 \times \frac{2}{3}x \times \frac{5}{6}y + \left(\frac{5}{6}y\right)^2 = \left(\frac{2}{3}x - \frac{5}{6}y\right)^2$.

(iii) The expression $25x^2 - 30xy + 36y^2$ is not a perfect square trinomial because $25x^2 = (5x)^2, 36y^2 = (6y)^2$ but $30xy \neq 2 \times 5x \times 6y$.

Remember These

1. (i) $(a+b)^2 = a^2 + 2ab + b^2$ (ii) $a^2 + b^2 = (a+b)^2 - 2ab$
2. (i) $(a-b)^2 = a^2 - 2ab + b^2$ (ii) $a^2 + b^2 = (a-b)^2 + 2ab$
3. (i) $(a+b)^2 = (a-b)^2 + 4ab$ (ii) $(a-b)^2 = (a+b)^2 - 4ab$

   (iii) $ab = \frac{1}{4}[(a+b)^2 - (a-b)^2]$ (iv) $a^2 + b^2 = \frac{1}{2}[(a+b)^2 + (a-b)^2]$

EXERCISE 4C

1. Find the following.

   (i) $(m+5)^2$ (ii) $(6x+7)^2$ (iii) $\left(\frac{5}{6}a + 3b\right)^2$ (iv) $(5a + \sqrt{6}b)^2$

   (v) $\left(6x + \frac{7y}{z}\right)^2$ (vi) $(a-4)^2$ (vii) $\left(3x - \frac{5}{2}\right)^2$ (viii) $\left(\frac{2}{5}x - \frac{4}{9}y\right)^2$

   (ix) $(2xy - 3pq^2)^2$ (x) $(3abx^2 - 4cdy^2)^2$

2. Find the squares of the following.

   (i) $-3x + 5y$ (ii) $4x^2 - \frac{1}{4x}$ (iii) $\sqrt{3}\left(\frac{2xy}{3} + \frac{5z}{x}\right)$ (iv) $\frac{\sqrt{3}}{2}x^2 - \frac{4}{\sqrt{3}}y^3$

   (v) $2 + \frac{x}{3}$ (vi) $\frac{1}{\sqrt{2}}(\sqrt{3}a - \sqrt{5}b)$ (vii) $\frac{3}{x} + \frac{4}{y}$ (viii) $\sqrt{7}\left(x - \frac{1}{\sqrt{2}}\right)$

   (ix) $x^2 + \frac{1}{x^2}$ (x) $x^2 - \frac{1}{x^2}$

3. Compute the following using the expansions $(a+b)^2 = a^2 + 2ab + b^2$ or $(a-b)^2 = a^2 - 2ab + b^2$.

   (i) $(54)^2$ (ii) $(192)^2$ (iii) $(1008)^2$ (iv) $(998)^2$

   (v) $(10.8)^2$ (vi) $(9.7)^2$ (vii) $(100.4)^2$

4. If $a + b = 6$ and $ab = 8$, evaluate (i) $a^2 + b^2$ and (ii) $a - b$.
5. If $a - b = 7$ and $ab = 8$, evaluate (i) $a^2 + b^2$ and (ii) $a + b$.
6. (a) If $p^2 + q^2 = 252$ and $p + q = 16$, evaluate pq.

   (b) If $x^2 + y^2 = 51$ and $x - y = 7$, evaluate xy.

7. Find the value of ab when $a+b=10$ and $a-b=6$.

8. If $x+\frac{1}{x}=3$, evaluate (i) $x-\frac{1}{x}$, (ii) $x^2+\frac{1}{x^2}$ and (iii) $x^4+\frac{1}{x^4}$.

9. If $3p+\frac{1}{3p}=2\sqrt{3}$, evaluate (i) $3p-\frac{1}{3p}$, (ii) $9p^2+\frac{1}{9p^2}$ and (iii) $81p^4+\frac{1}{81p^4}$.

10. If $d-\frac{1}{d}=6$, evaluate (i) $d+\frac{1}{d}$, (ii) $d^2+\frac{1}{d^2}$ and (iii) $d^4+\frac{1}{d^4}$.

11. If $2x-\frac{1}{2x}=1$, find (i) $2x+\frac{1}{2x}$, (ii) $4x^2+\frac{1}{4x^2}$ and (iii) $16x^4+\frac{1}{16x^4}$.

12. If $r^2+\frac{1}{r^2}=7$, evaluate (i) $r+\frac{1}{r}$ and (ii) $r-\frac{1}{r}$.

13. If $m^2+\frac{1}{m^2}=23$, evaluate (i) $m+\frac{1}{m}$ and (ii) $m-\frac{1}{m}$.

14. Express each of the following trinomials as a perfect square.

(i) $1+6b+9b^2$ (ii) $4x^2-12x+9$

(iii) $9x^2+24xy+16y^2$ (iv) $\frac{9x^2}{25}-\frac{3xy}{5}+\frac{y^2}{4}$

ANSWERS

1. (i) $m^2+10m+25$ (ii) $36x^2+84x+49$ (iii) $\frac{25}{36}a^2+5ab+9b^2$ (iv) $25a^2+10\sqrt{6}\,ab+6b^2$
(v) $36x^2+84\frac{xy}{z}+49\frac{y^2}{z^2}$ (vi) $a^2-8a+16$ (vii) $9x^2-15x+\frac{25}{4}$ (viii) $\frac{4}{25}x^2-\frac{16}{45}xy+\frac{16}{81}y^2$
(ix) $4x^2y^2-12xypq^2+9p^2q^4$ (x) $9a^2b^2x^4-24abcdx^2y^2+16c^2d^2y^4$

2. (i) $9x^2-30xy+25y^2$ (ii) $16x^4-2x+\frac{1}{16x^2}$ (iii) $3\left(\frac{4x^2y^2}{9}+\frac{20}{3}yz+\frac{25z^2}{x^2}\right)$ (iv) $\frac{3}{4}x^4-4x^2y^3+\frac{16}{3}y^6$
(v) $4+\frac{4}{3}x+\frac{x^2}{9}$ (vi) $\frac{1}{2}(3a^2-2\sqrt{15}ab+5b^2)$ (vii) $\frac{9}{x^2}+\frac{24}{xy}+\frac{16}{y^2}$ (viii) $7\left(x^2-\sqrt{2}\,x+\frac{1}{2}\right)$
(ix) $x^4+2+\frac{1}{x^4}$ (x) $x^4-2+\frac{1}{x^4}$

3. (i) 2916 (ii) 36864 (iii) 1016064 (iv) 996004 (v) 116.64 (vi) 94.09 (vii) 10080.16

4. (i) 20 (ii) ± 2 **5.** (i) 65 (ii) ± 9 **6.** (a) 2 (b) 1 **7.** 16

8. (i) $\pm\sqrt{5}$ (ii) 7 (iii) 47 **9.** (i) $\pm 2\sqrt{2}$ (ii) 10 (iii) 98

10. (i) $\pm 2\sqrt{10}$ (ii) 38 (iii) 1442 **11.** (i) $\pm\sqrt{5}$ (ii) 3 (iii) 7

12. (i) ± 3 (ii) $\pm\sqrt{5}$ **13.** (i) ± 5 (ii) $\pm\sqrt{21}$

14. (i) $(1+3b)^2$ (ii) $(2x-3)^2$ (iii) $(3x+4y)^2$ (iv) $\left(\frac{3x}{5}-\frac{y}{2}\right)^2$

Square of a trinomial

Let us find the square of the trinomial $a+b+c$.

$$(a+b+c)^2 = [(a+b)+c]^2 = (a+b)^2 + 2(a+b)c + c^2$$
$$= a^2 + 2ab + b^2 + 2ac + 2bc + c^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$
$$= a^2 + b^2 + c^2 + 2(ab + bc + ca).$$

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$$

Examples (i) $(2x+3y+4z)^2 = (2x)^2 + (3y)^2 + (4z)^2 + 2(2x \times 3y + 3y \times 4z + 4z \times 2x)$
$$= 4x^2 + 9y^2 + 16z^2 + 2(6xy + 12yz + 8zx)$$
$$= 4x^2 + 9y^2 + 16z^2 + 12xy + 24yz + 16zx.$$

(ii) $(m - 2n + p)^2 = [m + (-2n) + p]^2$
$$= m^2 + (-2n)^2 + p^2 + 2[m \times (-2n) + (-2n) \times p + p \times m]$$
$$= m^2 + 4n^2 + p^2 + 2[-2mn - 2np + pm]$$
$$= m^2 + 4n^2 + p^2 - 4mn - 4np + 2pm.$$

Corollaries

Two corollaries follow from this expansion.

(i) $(a+b+c)^2 - 2(ab+bc+ca) = a^2 + b^2 + c^2 + 2(ab+bc+ca) - 2(ab+bc+ca)$
$$= a^2 + b^2 + c^2.$$

$$\therefore\ a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+bc+ca)$$

(ii) $(a+b+c)^2 - (a^2+b^2+c^2) = a^2 + b^2 + c^2 + 2(ab+bc+ca) - a^2 - b^2 - c^2$
$$= 2(ab+bc+ca).$$

$$\therefore\ 2(ab+bc+ca) = (a+b+c)^2 - (a^2+b^2+c^2)$$

EXAMPLE **(i) If $a+b+c=15$ and $ab+bc+ca=50$, find $a^2+b^2+c^2$.**
(ii) If $a+b+c=21$ and $a^2+b^2+c^2=361$, find $ab+bc+ca$.

Solution (i) $a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ca) = (15)^2 - 2 \times 50 = 225 - 100 = 125.$

(ii) $2(ab+bc+ca) = (a+b+c)^2 - (a^2+b^2+c^2) = (21)^2 - 361 = 441 - 361 = 80.$

$\therefore\ ab+bc+ca = \frac{1}{2} \times 80 = 40.$

Cubes of binomials

Now, let us find the cubes of the sum and difference of two terms a and b. In other words, let us find $(a+b)^3$ and $(a-b)^3$.

1. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Proof $(a+b)^3 = (a+b)(a+b)^2 = (a+b)(a^2+2ab+b^2)$
$$= a(a^2+2ab+b^2) + b(a^2+2ab+b^2)$$
$$= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Examples (i) $(x+1)^3 = x^3 + 3 \times x^2 \times 1 + 3 \times x \times (1)^2 + (1)^3 = x^3 + 3x^2 + 3x + 1.$

(ii) $(2a+5b)^3 = (2a)^3 + 3\times(2a)^2\times 5b + 3\times 2a\times(5b)^2 + (5b)^3$
$= 8a^3 + 60a^2b + 150ab^2 + 125b^3$.

Corollaries

Two corollaries follow from this expansion.

(i) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3ab(a+b)$.

$\therefore\ (a+b)^3 = a^3 + b^3 + 3ab(a+b)$

(ii) $(a+b)^3 - 3ab(a+b) = a^3 + b^3 + 3ab(a+b) - 3ab(a+b) = a^3 + b^3$.

$\therefore\ a^3 + b^3 = (a+b)^3 - 3ab(a+b)$

EXAMPLE **If $a+b=3$ and $ab=2$, find a^3+b^3.**

Solution $a^3 + b^3 = (a+b)^3 - 3ab(a+b) = (3)^3 - 3\times 2\times 3 = 27 - 18 = 9$.

2. $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Proof $(a-b)^3 = (a-b)(a-b)^2 = (a-b)(a^2 - 2ab + b^2)$
$= a(a^2 - 2ab + b^2) - b(a^2 - 2ab + b^2)$
$= a^3 - 2a^2b + ab^2 - a^2b + 2ab^2 - b^3 = a^3 - 3a^2b + 3ab^2 - b^3$.

Examples (i) $(m-2)^3 = m^3 - 3\times m^2\times 2 + 3\times m\times 2^2 - 2^3 = m^3 - 6m^2 + 12m - 8$.

(ii) $(2y-x)^3 = (2y)^3 - 3\times(2y)^2\times x + 3\times 2y\times x^2 - x^3$
$= 8y^3 - 12xy^2 + 6x^2y - x^3$.

Corollaries

The corollaries that follow are:

(i) $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$

(ii) $(a-b)^3 + 3ab(a-b) = a^3 - b^3 - 3ab(a-b) + 3ab(a-b) = a^3 - b^3$.

$\therefore\ a^3 - b^3 = (a-b)^3 + 3ab(a-b)$

EXAMPLE **If $a-b=4$ and $ab=3$, find a^3-b^3.**

Solution $a^3 - b^3 = (a-b)^3 + 3ab(a-b) = 4^3 + 3\times 3\times 4 = 64 + 36 = 100$.

Solved Examples

EXAMPLE 1 **Expand $(3a-4b-5c)^2$.**

Solution $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$.

$\therefore\ (3a-4b-5c)^2 = [3a + (-4b) + (-5c)]^2$
$= (3a)^2 + (-4b)^2 + (-5c)^2 + 2[3a\times(-4b) + (-4b)\times(-5c) + (-5c)\times 3a]$
$= 9a^2 + 16b^2 + 25c^2 + 2(-12ab + 20bc - 15ca)$
$= 9a^2 + 16b^2 + 25c^2 - 24ab + 40bc - 30ca$.

EXAMPLE 2 **If $a^2+b^2+c^2=38$ and $ab+bc+ca=31$, find the value of $a+b+c$.**

Solution $(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)=38+2\times 31=38+62=100.$

$\therefore\quad a+b+c=\sqrt{100}=\pm 10.$

EXAMPLE 3 **Expand (i) $(3x+4y)^3$ and (ii) $\left(\frac{2}{x}-\frac{x}{3}\right)^3$.**

Solution (i) $(a+b)^3=a^3+3a^2b+3ab^2+b^3.$

$$\therefore\quad (3x+4y)^3=(3x)^3+3\cdot(3x)^2\cdot 4y+3\cdot 3x\cdot(4y)^2+(4y)^3$$
$$=27x^3+3\cdot 9x^2\cdot 4y+3\cdot 3x\cdot 16y^2+64y^3$$
$$=27x^3+108x^2y+144xy^2+64y^3.$$

(ii) We have $(a-b)^3=a^3-3a^2b+3ab^2-b^3.$

$$\therefore\quad \left(\frac{2}{x}-\frac{x}{3}\right)^3=\left(\frac{2}{x}\right)^3-3\cdot\left(\frac{2}{x}\right)^2\cdot\frac{x}{3}+3\cdot\frac{2}{x}\cdot\left(\frac{x}{3}\right)^2-\left(\frac{x}{3}\right)^3$$
$$=\frac{2^3}{x^3}-3\cdot\frac{2^2}{x^2}\cdot\frac{x}{3}+3\cdot\frac{2}{x}\cdot\frac{x^2}{3^2}-\frac{x^3}{3^3}=\frac{8}{x^3}-\frac{4}{x}+\frac{2}{3}x-\frac{x^3}{27}.$$

EXAMPLE 4 **If $2x+3y=10$ and $xy=4$, find the value of $8x^3+27y^3$.**

Solution $a^3+b^3=(a+b)^3-3ab(a+b).$

Also, $8x^3=(2x)^3$, $27y^3=(3y)^3$.

$$\therefore\quad 8x^3+27y^3=(2x)^3+(3y)^3=(2x+3y)^3-3\times 2x\times 3y(2x+3y)$$
$$=(2x+3y)^3-18xy(2x+3y)=10^3-18\times 4\times 10$$
$$=1000-720=280.$$

EXAMPLE 5 **If $2m-5n=3$ and $mn=2$, evaluate $8m^3-125n^3$.**

Solution $(a-b)^3=a^3-b^3-3ab(a-b).$

$$\therefore\quad (2m-5n)^3=(2m)^3-(5n)^3-3\times 2m\times 5n\times(2m-5n)$$

or $(3)^3=8m^3-125n^3-30\times 2\times 3$ or $8m^3-125n^3=27+180=207.$

EXAMPLE 6 **If $x+\frac{1}{x}=4$, find the value of $x^3+\frac{1}{x^3}$.**

Solution $(a+b)^3=a^3+b^3+3ab(a+b).$

$$\therefore\quad \left(x+\frac{1}{x}\right)^3=x^3+\frac{1}{x^3}+3x\times\frac{1}{x}\times\left(x+\frac{1}{x}\right)$$

or $(4)^3=x^3+\frac{1}{x^3}+3\times 4$ or $x^3+\frac{1}{x^3}=4^3-12=64-12=52.$

EXAMPLE 7 **If $a-\frac{1}{a}=3$, find the value of $a^3-\frac{1}{a^3}$.**

Solution $(a-b)^3=a^3-b^3-3ab(a-b).$

$$\therefore\quad \left(a-\frac{1}{a}\right)^3=a^3-\frac{1}{a^3}-3\times a\times\frac{1}{a}\times\left(a-\frac{1}{a}\right)$$

or $(3)^3=a^3-\frac{1}{a^3}-3\times 3$ or $a^3-\frac{1}{a^3}=27+9=36.$

EXAMPLE 8 **If $3m+\frac{1}{3m}=6$, find the value of $27m^3+\frac{1}{27m^3}$.**

solution $(a+b)^3=a^3+b^3+3ab(a+b)$.

$$\therefore \left(3m+\frac{1}{3m}\right)^3=(3m)^3+\left(\frac{1}{3m}\right)^3+3\times 3m\times\frac{1}{3m}\times\left(3m+\frac{1}{3m}\right)$$

or $(6)^3=27m^3+\frac{1}{27m^3}+3\times 6$ or $27m^3+\frac{1}{27m^3}=216-18=198.$

EXAMPLE 9 **If $2l-\frac{1}{2l}=5$, find the value of $8l^3-\frac{1}{8l^3}$.**

Solution $(a-b)^3=a^3-b^3-3ab(a-b)$.

$$\therefore \left(2l-\frac{1}{2l}\right)^3=(2l)^3-\left(\frac{1}{2l}\right)^3-3\times 2l\times\frac{1}{2l}\times\left(2l-\frac{1}{2l}\right)$$

or $(5)^3=8l^3-\frac{1}{8l^3}-3\times 5$ or $8l^3-\frac{1}{8l^3}=(5)^3+3\times 5=125+15=140.$

Remember These

1. (i) $(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)$
   (ii) $a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)$
   (iii) $ab+bc+ca=\frac{1}{2}[(a+b+c)^2-(a^2+b^2+c^2)]$
2. (i) $(a+b)^3=a^3+3a^2b+3ab^2+b^3=a^3+b^3+3ab(a+b)$
   (ii) $a^3+b^3=(a+b)^3-3ab(a+b)$
3. (i) $(a-b)^3=a^3-3a^2b+3ab^2-b^3=a^3-b^3-3ab(a-b)$
   (ii) $a^3-b^3=(a-b)^3+3ab(a-b)$

EXERCISE 4D

1. Expand the following.
   (i) $(1+x+y)^2$ (ii) $(x+2y+3z)^2$ (iii) $(2m-n-p)^2$ (iv) $\left(a-\frac{1}{a}+2\right)^2$
   (v) $(4a-5b+6c)^2$ (vi) $(-2a-3b-c)^2$
2. Expand the following.
   (i) $(x+3)^3$ (ii) $(3x+4y)^3$ (iii) $(-5m+n)^3$ (iv) $\left(3x+\frac{1}{3x}\right)^3$
   (v) $(m-1)^3$ (vi) $(3a-2b)^3$ (vii) $(7y-5x)^3$ (viii) $\left(5x-\frac{1}{5x}\right)^3$
3. if $x+y+z=10$ and $xy+yz+zx=40$, evaluate $x^2+y^2+z^2$.

4. If $a+b+c=9$ and $a^2+b^2+c^2=41$, evaluate $ab+bc+ca$.

5. If $m^2+n^2+p^2=58$ and $mn+np+pm=21$, find the value of $m+n+p$.

6. If $a+b=4$ and $ab=3$, find the value of a^3+b^3.

7. If $m-n=5$ and $mn=14$, find the value of m^3-n^3.

8. If $3a+4b=9$ and $ab=2$, evaluate $27a^3+64b^3$.

9. If $3x-2y=4$ and $xy=5$, find the value of $27x^3-8y^3$.

10. (a) If $m+\frac{1}{m}=3$, evaluate $m^3+\frac{1}{m^3}$.

(b) If $2p+\frac{1}{2p}=5$, evaluate $8p^3+\frac{1}{8p^3}$.

11. (a) If $x-\frac{1}{x}=2$, find the value of $x^3-\frac{1}{x^3}$.

(b) If $3n-\frac{1}{3n}=4$, find the value of $27n^3-\frac{1}{27n^3}$.

ANSWERS

1. (i) $1+x^2+y^2+2x+2xy+2y$ (ii) $x^2+4y^2+9z^2+4xy+12yz+6zx$

(iii) $4m^2+n^2+p^2-4mn+2np-4pm$ (iv) $a^2+\frac{1}{a^2}+4-2-\frac{4}{a}+4a$

(v) $16a^2+25b^2+36c^2-40ab-60bc+48ca$ (vi) $4a^2+9b^2+c^2+12ab+6bc+4ca$

2. (i) $x^3+9x^2+27x+27$ (ii) $27x^3+108x^2y+144xy^2+64y^3$

(iii) $-125m^3+75m^2n-15mn^2+n^3$ (iv) $27x^3+9x+\frac{1}{x}+\frac{1}{27x^3}$

(v) m^3-3m^2+3m-1 (vi) $27a^3-54a^2b+36ab^2-8b^3$

(vii) $343y^3-735y^2x+525yx^2-125x^3$ (viii) $125x^3-15x+\frac{3}{5x}-\frac{1}{125x^3}$

3. 20 **4.** 20 **5.** ± 10 **6.** 28

7. 335 **8.** 81 **9.** 424 **10.** (a) 18 (b) 110

11. (a) 14 (b) 76

❖

Revision Exercise 3

1. Evaluate the following.

(i) $(27)^{-1/3} + (27)^{-2/3}$

(ii) $(16)^{1/4} + (16)^{3/4} + (16)^{-1/4} + (16)^{-3/4}$

(iii) $4^0 + 4^{-1} + 4^{-2} + 4^{1/2} + 4^{-1/2}$

(iv) $\dfrac{\sqrt{64} + \sqrt{36}}{\sqrt{64+36}}$

(v) $\dfrac{(25)^{1/2}(64)^{1/3}(81)^{1/4}}{(25)^{1/2} + (64)^{1/3} + (81)^{1/4}}$

2. Simplify the expression $\sqrt[3]{\dfrac{125x^6}{216}} \times \left(\dfrac{5x}{3}\right)^{-2}$.

3. Prove:

(i) $\dfrac{1}{1+x^{b-a}+x^{c-a}} + \dfrac{1}{1+x^{a-b}+x^{c-b}} + \dfrac{1}{1+x^{b-c}+x^{a-c}} = 1$

(ii) $\left(\dfrac{x^a}{x^b}\right)^{1/ab}\left(\dfrac{x^b}{x^c}\right)^{1/bc}\left(\dfrac{x^c}{x^a}\right)^{1/ca} = 1$

4. Find each of the following.

(i) $\left(p^2 - \dfrac{2q}{3}\right)\left(p^2 - \dfrac{7q}{5}\right)$ (ii) $\left(7a + \dfrac{8b}{13}\right)\left(7a - \dfrac{13b}{16}\right)$

(iii) $(9m^2 - 11n^2)(13m^2 + 5n^2)$ (iv) $\left(\dfrac{3a}{4b} - \dfrac{5c}{6d}\right)\left(\dfrac{3a}{4b} + \dfrac{5c}{6d}\right)$

5. Expand each of the following.

(i) $\left(\dfrac{2a}{3} + \dfrac{3b}{2}\right)^2$ (ii) $(5x - 7yz)^2$ (iii) $(3a + 4b + 5c)^2$

(iv) $(2x - 3y - 7z)^2$ (v) $(2p + 3q)^3$ (vi) $(5a - 2)^3$

6. If $a^2 + b^2 = 13$ and $ab = 6$, find the values of (i) $a + b$ and (ii) $a - b$.

7. If $x + \dfrac{1}{x} = 6$, find the values of (i) $x^2 + \dfrac{1}{x^2}$ and (ii) $x^4 + \dfrac{1}{x^4}$.

8. If $m^2 + \dfrac{1}{m^2} = 98$, find the values of (i) $m + \dfrac{1}{m}$ and $m - \dfrac{1}{m}$.

9. If $a + b + c = 17$ and $ab + bc + ca = 90$, find $a^2 + b^2 + c^2$.

10. If $3m - \dfrac{1}{3m} = 7$, find the values of (i) $9m^2 + \dfrac{1}{9m^2}$ and (ii) $27m^3 - \dfrac{1}{27m^3}$.

11. If $2a + \dfrac{1}{2a} = 4$, find the values of (i) $4a^2 + \dfrac{1}{4a^2}$, (ii) $8a^3 + \dfrac{1}{8a^3}$ and (iii) $16a^4 + \dfrac{1}{16a^4}$.

ANSWERS

1. (i) $\frac{4}{9}$ (ii) $10\frac{5}{8}$ (iii) $3\frac{13}{16}$ (iv) $1\frac{2}{5}$ (v) 5

2. $\frac{3}{10}$

4. (i) $p^4 - \frac{31}{15}p^2q + \frac{14}{15}q^2$ (ii) $49a^2 - \frac{287}{208}ab - \frac{b^2}{2}$ (iii) $117m^4 - 98m^2n^2 - 55n^4$ (iv) $\frac{9a^2}{16b^2} - \frac{25c^2}{36d^2}$

5. (i) $\frac{4a^2}{9} + 2ab + \frac{9b^2}{4}$ (ii) $25x^2 - 70xyz + 49y^2z^2$ (iii) $9a^2 + 16b^2 + 25c^2 + 24ab + 40bc + 30ca$

(iv) $4x^2 + 9y^2 + 49z^2 - 12xy + 42yz - 28zx$ (v) $8p^3 + 36p^2q + 54pq^2 + 27q^3$

(vi) $125a^3 - 150a^2 + 60a - 8$

6. (i) ± 5 (ii) ± 1

7. (i) 34 (ii) 1154

8. (i) ± 10 (ii) $\pm 4\sqrt{6}$

9. 109

10. (i) 51 (ii) 364

11. (i) 14 (ii) 52 (iii) 194

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