3

Exponents

In this chapter, we will revise what you have learnt about exponents in your previous class and discuss fractional indices.

You know that $x \times x = x^2$, which is read as *x* squared or *x* raised to the power 2 or *x* to the power 2. Here, *x* is the base and 2 is the exponent or index.

> An exponent (or index) is a number written to the right and a little above the base. It indicates the number of times the base occurs in a product.

Examples (i) In x^4 $(= x \times x \times x \times x)$, read as *x* to the power 4, the exponent is 4.

(ii) In x^m $(= x \times x \times x \times x \times \ldots m \text{ times})$, read as *x* to the power *m*, the exponent is *m*.

Reciprocal of a power

$x \div x = 1$, that is, $x \times \frac{1}{x} = 1$; $\frac{1}{x}$ is called the reciprocal of *x*, and is written as x^{-1}.

Similarly, $\frac{1}{x^2}$ is the reciprocal of x^2 and is written as x^{-2}.

$\frac{1}{x^3}$ is the reciprocal of x^3 and is written as x^{-3}.

In general, $$\frac{1}{x^m} = x^{-m}, \quad \frac{1}{x^{-m}} = x^m$$

because $x^m \times x^{-m} = x^{m-m} = x^0 = 1$.

Some laws of indices

(i) $x^m \times x^n = x^{m+n}$ and $x^m \times x^n \times x^p = x^{m+n+p}$

(ii) $x^m \div x^n = x^{m-n}$

(iii) $(x^m)^n = x^{mn}$

(iv) $(x \times y)^m = x^m \times y^m$

(v) $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$

(vi) If $x^m = x^n$ then $m = n$ so long as *x* is a positive number other than 1.

Some important results

(i) $1 = a^m \div a^m = a^{m-m} = a^0$, so $a^0 = 1$. (Remember: $a \neq 0$)

(ii) $\{(a^m)^n\}^p = (a^{mn})^p = a^{mnp}$, where a is a nonzero number and m, n and p are integers.

(iii) If n is an even integer, $(-1)^n = (-1)^{2m} = \{(-1)^2\}^m = \{(-1)\times(-1)\}^m = 1^m = 1$.

If n is an odd integer, $(-1)^n = (-1)^{2m+1} = (-1)\times(-1)^{2m} = -1\times 1 = -1$.

Fractional indices

$x^m \times x^n = x^{m+n}$, so $x^{1/2} \times x^{1/2} = x^{1/2+1/2} = x$.

But $x^{1/2} \times x^{1/2} = (x^{1/2})^2$, so $(x^{1/2})^2 = x$.

Taking the square root of both sides,

$$x^{1/2} = \sqrt{x}.$$

Again, $x^{1/3} \times x^{1/3} \times x^{1/3} = x^{1/3+1/3+1/3} = x$.

In other words, $(x^{1/3})^3 = x$.

Taking the cube root of both sides,

$$x^{1/3} = \sqrt[3]{x}.$$

Similarly, $(x^{1/4})^4 = x$, so $x^{1/4} = \sqrt[4]{x}$.

In general, $(x^{1/n})^n = x$ or $x^{1/n} = \sqrt[n]{x}$

Now, $x^{2/3} \times x^{2/3} \times x^{2/3} = x^{2/3+2/3+2/3} = x^2$.

In other words, $(x^{2/3})^3 = x^2$.

Taking the cube root of both sides,

$$x^{2/3} = (x^2)^{1/3}.$$

Also, $(x^{1/3})^2 = x^{1/3} \times x^{1/3} = x^{1/3+1/3} = x^{2/3}$.

So, $x^{2/3} = (x^2)^{1/3} = (x^{1/3})^2$.

In general, $x^{m/n} = (x^m)^{1/n} = (x^{1/n})^m$

Solved Examples

EXAMPLE 1 **Evaluate each of the following expressions.**

(i) $(-8)^0$ **(ii)** $(6+4+1)^0$ **(iii)** $6^0 + 4^0 + 1^0$ **(iv)** $(6x)^0, x \neq 0$

(v) $6x^0, x \neq 0$ **(vi)** $(7^0)^{-2}$ **(vii)** $[(x^4)^0]^{-3}, x \neq 0$ **(viii)** $[(x^{-5})^0]^2, x \neq 0$

Solution

(i) $(-8)^0 = 1$. $[\because x^0 = 1, x \neq 0]$

(ii) $(6+4+1)^0 = 11^0 = 1$.

(iii) $6^0 + 4^0 + 1^0 = 1 + 1 + 1 = 3$.

(iv) $(6x)^0 = 6^0 \times x^0$ $[\because (xy)^m = x^m y^m]$

$= 1 \times 1 = 1$.

(v) $6x^0 = 6 \times 1 = 6$.

(vi) $(7^0)^{-2} = 7^{0\times(-2)}$ $[\because (x^m)^n = x^{mn}]$
$= 7^0 = 1.$

(vii) $[(x^4)^0]^{-3} = (x^4)^{0\times(-3)} = (x^4)^0 = x^{4\times 0} = x^0 = 1.$

(viii) $[(x^{-5})^0]^2 = (x^{-5})^{0\times 2} = (x^{-5})^0 = x^{(-5)\times 0} = x^0 = 1.$

EXAMPLE 2 **Simplify and express the result with positive index.**

(i) $(7^3)^2$ (ii) $[(5)^{-3}]^6$ (iii) $(2^8)^{-2}$ (iv) $[(3)^{-7}]^{-3}$ (v) $[(-x)^{-4}]^{-3}$ (vi) $[2a^{-3}]^{-3}$

Solution

(i) $(7^3)^2 = 7^{3\times 2} = 7^6.$

(ii) $[(5)^{-3}]^6 = (5)^{-3\times 6} = (5)^{-18} = \frac{1}{5^{18}}.$ $\left[\because x^{-m} = \frac{1}{x^m}\right]$

(iii) $(2^8)^{-2} = 2^{8\times(-2)} = 2^{-16} = \frac{1}{2^{16}}.$

(iv) $[(3)^{-7}]^{-3} = (3)^{(-7)\times(-3)} = 3^{21}.$

(v) $[(-x)^{-4}]^{-3} = (-x)^{(-4)\times(-3)} = (-x)^{12} = [(-1)\times x]^{12} = (-1)^{12}\times x^{12}$
$= 1\times x^{12} = x^{12}.$ $[\because (-1)^n = 1$ if n is even$]$

(vi) $[2a^{-3}]^{-3} = (2)^{-3}\times(a^{-3})^{-3} = \frac{1}{2^3}\times a^{(-3)\times(-3)} = \frac{1}{2^3}a^9.$

EXAMPLE 3 **Simplify (i) $[(3^{-2})^3]^{-4}$, (ii) $\left[\frac{8^3}{4^2}\right]^4$.**

Solution

(i) $[(3^{-2})^3]^{-4} = [3^{-2\times 3}]^{-4} = (3^{-6})^{-4} = 3^{(-6)\times(-4)} = 3^{24}.$

(ii) $\left[\frac{8^3}{4^2}\right]^4 = \left[\frac{(2^3)^3}{(2^2)^2}\right]^4 = \left[\frac{2^{3\times 3}}{2^{2\times 2}}\right]^4 = \left(\frac{2^9}{2^4}\right)^4 = (2^{9-4})^4$ $\left[\because \frac{x^m}{x^n} = x^{m-n}\right]$
$= (2^5)^4 = 2^{5\times 4} = 2^{20}.$

EXAMPLE 4 **Simplify the following.**

(i) $\frac{8x^5y^7}{12x^9y^4}$ (ii) $\left(\frac{2a^{-3}}{3b^2}\right)^2$ (iii) $\left(\frac{-5x^3}{2y^{-4}}\right)^{-3}$

Solution

(i) $\frac{8x^5y^7}{12x^9y^4} = \frac{8}{12}\times\frac{x^5}{x^9}\times\frac{y^7}{y^4} = \frac{2}{3}\times\frac{1}{x^{9-5}}\times y^{7-4} = \frac{2y^3}{3x^4}.$

(ii) $\left(\frac{2a^{-3}}{3b^2}\right)^2 = \frac{(2a^{-3})^2}{(3b^2)^2}$ $\left[\because \left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}\right]$

$= \frac{(2)^2\cdot(a^{-3})^2}{3^2\cdot(b^2)^2} = \frac{4}{9}\times\frac{a^{-3\times 2}}{b^{2\times 2}} = \frac{4}{9}\times\frac{a^{-6}}{b^4} = \frac{4}{9a^6b^4}.$

(iii) $\left(\frac{-5x^3}{2y^{-4}}\right)^{-3} = \frac{(-5)^{-3}\cdot(x^3)^{-3}}{(2)^{-3}\cdot(y^{-4})^{-3}} = \frac{-\frac{1}{5^3}x^{3\times(-3)}}{\frac{1}{2^3}y^{-4\times(-3)}} = \frac{-8\times x^{-9}}{125y^{12}} = -\frac{8}{125x^9y^{12}}.$

EXAMPLE 5 **Express the following in positive indices only.**

(i) $\frac{2^{-3} \times 3^4 \times (5^2)^{-2}}{5^{-6} \times 2^2 \times 3^6}$ (ii) $\frac{l^{-6} m^9 n^{-10} p^{-4}}{l^{-7} m^6 n^{-8} p^{-2}}$

Solution (i) $\frac{2^{-3} \times 3^4 \times (5^2)^{-2}}{5^{-6} \times 2^2 \times 3^6} = \frac{2^{-3}}{2^2} \times \frac{3^4}{3^6} \times \frac{5^{2\times(-2)}}{5^{-6}} = 2^{-3-2} \times 3^{4-6} \times 5^{6-4}$

$= 2^{-5} \times 3^{-2} \times 5^2 = \frac{1}{2^5} \times \frac{1}{3^2} \times 5^2 = \frac{5^2}{2^5 \times 3^2}$.

(ii) $\frac{l^{-6} m^9 n^{-10} p^{-4}}{l^{-7} m^6 n^{-8} p^{-2}} = l^{7-6} m^{9-6} n^{-10+8} p^{-4+2} = l m^3 n^{-2} p^{-2} = \frac{lm^3}{n^2 p^2}$.

EXAMPLE 6 **Simplify:**

(i) $(8)^{1/3}$ (ii) $(-32)^{1/5}$ (iii) $(64)^{-2/3}$ (iv) $(-27)^{-4/3}$ (v) $\left(\frac{1}{8}\right)^{-2/3}$

Solution (i) $(8)^{\frac{1}{3}} = (2^3)^{\frac{1}{3}} = (2)^{3 \times \frac{1}{3}} = 2^1 = 2$.

(ii) $(-32)^{\frac{1}{5}} = [(-2)^5]^{\frac{1}{5}} = (-2)^{5 \times \frac{1}{5}} = (-2)^1 = -2$.

(iii) $(64)^{-\frac{2}{3}} = (4^3)^{-\frac{2}{3}} = 4^{3 \times \left(\frac{-2}{3}\right)} = 4^{-2} = \frac{1}{4^2} = \frac{1}{16}$.

(iv) $(-27)^{-\frac{4}{3}} = [(-3)^3]^{-\frac{4}{3}} = (-3)^{3 \times \left(-\frac{4}{3}\right)} = (-3)^{-4} = \frac{1}{(-3)^4} = \frac{1}{3^4} = \frac{1}{81}$.

(v) $\left(\frac{1}{8}\right)^{-\frac{2}{3}} = \left(\frac{1}{2^3}\right)^{\frac{-2}{3}} = (2^{-3})^{\frac{-2}{3}} = 2^{(-3) \times \left(\frac{-2}{3}\right)} = 2^2 = 4$.

EXAMPLE 7 **Simplify (i) $\sqrt[4]{16x^{12}}$, (ii) $(27a^9)^{2/3}$.**

Solution (i) $\sqrt[4]{16x^{12}} = (16x^{12})^{\frac{1}{4}} = (2^4 \times x^{12})^{\frac{1}{4}} = 2^{4 \times \frac{1}{4}} \times x^{12 \times \frac{1}{4}} = 2x^3$.

(ii) $(27a^9)^{\frac{2}{3}} = (3^3 \times a^9)^{\frac{2}{3}} = 3^{3 \times \frac{2}{3}} \times a^{9 \times \frac{2}{3}} = 3^2 \times a^6 = 9a^6$.

EXAMPLE 8 **Simplify the following expressions:**

(i) $8^{4/3} + 25^{3/2} - \left(\frac{1}{27}\right)^{-1/3}$ (ii) $3 \times \sqrt[3]{125} - 5 \times \sqrt[6]{64} + \sqrt[4]{256} + 2 \times \left(\frac{8}{27}\right)^{-1/3}$

Solution (i) $8^{\frac{4}{3}} + 25^{\frac{3}{2}} - \left(\frac{1}{27}\right)^{\frac{-1}{3}} = (2^3)^{\frac{4}{3}} + (5^2)^{\frac{3}{2}} - \left(\frac{1}{3^3}\right)^{\frac{-1}{3}} = 2^{3 \times \frac{4}{3}} + 5^{2 \times \frac{3}{2}} - 3^{(-3)\frac{-1}{3}}$

$= 2^4 + 5^3 - 3^{-3 \times \left(\frac{-1}{3}\right)} = 16 + 125 - 3 = 138$.

(ii) The given expression $= 3 \times (125)^{\frac{1}{3}} - 5 \times (64)^{\frac{1}{6}} + (256)^{\frac{1}{4}} + 2 \times \left(\frac{8}{27}\right)^{\frac{-1}{3}}$

$= 3 \times (5^3)^{\frac{1}{3}} - 5 \times (2^6)^{\frac{1}{6}} + (4^4)^{\frac{1}{4}} + 2 \times \left[\left(\frac{2}{3}\right)^3\right]^{\frac{-1}{3}}$

$$= 3 \times 5^{3 \times \frac{1}{3}} - 5 \times 2^{6 \times \frac{1}{6}} + 4^{4 \times \frac{1}{4}} + 2 \times \left(\frac{2}{3}\right)^{3 \times \left(\frac{-1}{3}\right)}$$

$$= 3 \times 5 - 5 \times 2 + 4 + 2 \times \left(\frac{2}{3}\right)^{-1} = 15 - 10 + 4 + 2 \times \frac{3}{2} = 12.$$

EXAMPLE 9 **Prove that** $\left(\frac{x^a}{x^b}\right)^c \cdot \left(\frac{x^b}{x^c}\right)^a \cdot \left(\frac{x^c}{x^a}\right)^b = 1.$

Solution $\left(\frac{x^a}{x^b}\right)^c \cdot \left(\frac{x^b}{x^c}\right)^a \cdot \left(\frac{x^c}{x^a}\right)^b = (x^{a-b})^c \cdot (x^{b-c})^a \cdot (x^{c-a})^b$

$$= x^{c(a-b)} \cdot x^{a(b-c)} \cdot x^{b(c-a)} = x^{ca-bc} \cdot x^{ab-ac} \cdot x^{bc-ab}$$

$$= x^{ca-bc+ab-ac+bc-ab} = x^0 = 1.$$

EXAMPLE 10 **Evaluate m when $7^{27} \div 7^2 = 7^{-2} \times 7^{5m+2}$.**

Solution Given, $\frac{7^{27}}{7^2} = 7^{-2} \times 7^{5m+2}$ or $7^{27-2} = 7^{-2+5m+2}$ or $7^{25} = 7^{5m}$.

$\therefore$ $25 = 5m$ or $m = 5$.

Remember These

1. $x^0 = 1,\ x^1 = x,\ x^{-1} = \frac{1}{x},\ x^{-m} = \frac{1}{x^m}$
2. $x^m \times x^n = x^{m+n},\ x^m \times x^n \times x^p = x^{m+n+p},\ \frac{x^m}{x^n} = x^{m-n},$

   $(x^m)^n = x^{mn},\ (xy)^m = x^m y^m,\ (xyz)^n = x^n \times y^n \times z^n,\ \left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$
3. $(x^{1/n})^n = x,\ \ x^{1/n} = \sqrt[n]{x},\ \ x^{m/n} = (x^m)^{1/n} = (x^{1/n})^m$

EXERCISE 3

1. Express in the form a^n with positive index.

   (i) $(5^2)^3$ (ii) $(2^{-3})^4$ (iii) $(3^{-2})^{-4}$ (iv) $[(-2)^{-2}]^{-3}$

   (v) $[(3)^{-a}]^{-2/a}$ (vi) $[(-7^2)^0]^3$ (vii) $[(3^2)^{-3}]^0$

2. Express in the form a^n.

   (i) $2^9 \div 2^4$ (ii) $7^4 \times 7^5 \div 7^7$ (iii) $(3^5 \div 3^3)^4$ (iv) $(5^3)^3 \div 5^4$

3. Find the value of each of the following.

   (i) $\left(\frac{1}{3}\right)^0 + \left(-\frac{1}{3}\right)^0 + \left[\left(\frac{1}{3}\right)^0\right]^2$ (ii) $\frac{(-3)^4 \times (-2^2)^3}{24 \times 2^2}$ (iii) $\frac{(7^{-2})^3 \times 7^{-4}}{7^{-8}}$

4. Simplify the following.

(i) $[(8^2)^{-3}]^{-2} \div [(4^{-3})^{-2}]^2 \times (2^{-3})^4$ (ii) $[(25)^{-3}]^{-2} \div [\{(-5)^2\}^3]^2$

(iii) $(2^{-4} + 2^{-3}) \times (2^{-4} - 2^{-3})$

5. Simplify:

(i) $\left(\frac{2a^3}{6x^{-2}}\right)^3$ (ii) $\left(\frac{8a^3b^{-4}}{12a^{-2}b^2}\right)^{-2}$

6. Evaluate:

(i) $(27)^{1/3}$ (ii) $\sqrt[3]{-125}$ (iii) $\sqrt[4]{256}$ (iv) $(16)^{-3/4}$

(v) $(32)^{-3/5}$ (vi) $\left(\frac{1}{81}\right)^{3/4}$ (vii) $\sqrt[3]{\left(\frac{8}{27}\right)^2}$

7. Simplify:

(i) $\sqrt[3]{27x^6}$ (ii) $(-32x^5)^{1/5}$ (iii) $\left(\frac{8a^3}{27b^6}\right)^{2/3}$ (iv) $\left(\frac{x^{12}}{64y^{18}}\right)^{1/6}$

8. Find the value of each expression.

(i) $8^{-2/3} + (-32)^{3/5} - \left(\frac{49}{4}\right)^{1/2}$ (ii) $(27)^{4/3} - 2 \times (16)^{5/4} + \frac{1}{3} \times 9^{3/2} + (9^{-1/2})^{-1}$

9. Use $x^2 - y^2 = (x+y)(x-y)$ to prove that $\left(\frac{x^{a^2}}{x^{b^2}}\right)^{\frac{1}{a+b}} \cdot \left(\frac{x^{b^2}}{x^{c^2}}\right)^{\frac{1}{b+c}} \cdot \left(\frac{x^{c^2}}{x^{a^2}}\right)^{\frac{1}{c+a}} = 1.$

10. Prove that $\left(\frac{x^{b+c}}{x^a}\right)^{b-c} \cdot \left(\frac{x^{c+a}}{x^b}\right)^{c-a} \cdot \left(\frac{x^{a+b}}{x^c}\right)^{a-b} = 1.$

11. Find the value of n from the given equation.

(i) $\frac{3^{10}}{3^{-2}} = (81)^{2n-1}$ (ii) $5^{12} \div \sqrt[4]{625} = 5^{3n-1}$

ANSWERS

1. (i) 5^6 (ii) $\left(\frac{1}{2}\right)^{12}$ (iii) 3^8 (iv) 2^6 (v) 3^2 (vi) 1 (vii) 1

2. (i) 2^5 (ii) 7^2 (iii) 3^8 (iv) 5^5 **3.** (i) 3 (ii) –54 (iii) $\frac{1}{49}$

4. (i) 1 (ii) 1 (iii) $\frac{-3}{256}$ **5.** (i) $\frac{a^9x^6}{27}$ (ii) $\frac{9b^{12}}{4a^{10}}$

6. (i) 3 (ii) –5 (iii) 4 (iv) $\frac{1}{8}$ (v) $\frac{1}{8}$ (vi) $\frac{1}{27}$ (vii) $\frac{4}{9}$

7. (i) $3x^2$ (ii) $-2x$ (iii) $\frac{4a^2}{9b^4}$ (iv) $\frac{x^2}{2y^3}$ **8.** (i) $-11\frac{1}{4}$ (ii) 29

11. (i) 2 (ii) 4

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