ગુજરાત રાજ્યના શિક્ષણવિભાગના પત્ર-ક્રમાંક
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PHYSICS

Standard 11

(Semester II)

PLEDGE

India is my country.
All Indians are my brothers and sisters.
I love my country, and I am proud of its rich and varied heritage.
I shall always strive to be worthy of it.
I shall respect my parents, teachers and all my elders and treat everyone with courtesy.
I pledge my devotion to my country and its people.
My happiness lies in their well-being and prosperity.

રાજ્ય સરકારની વિનામૂલ્યે યોજના હેઠળનું પુસ્તક

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PREFACE

The Gujarat State Secondary and Higher Secondary Education Board has prepared new syllabi in accordance with the new national syllabi prepared by N.C.E.R.T. based on N.C.F. 2005 and core-curriculum. These syllabi are sanctioned by the Government of Gujarat.

It is a pleasure for the Gujarat State Board of School Textbooks, to place before the students this textbook of **Physics, Standard 11, (Semester II)** prepared according to the new syllabus.

Before publishing the textbook, its manuscript has been fully reviewed by experts and teachers teaching at this level. Following suggestions given by teachers and experts, we have made necessary changes in the manuscript before publishing the textbook.

The Board has taken special care to ensure that this textbook is interesting, useful and free from errors. However, we welcome any suggestion from people interested in education, to improve the quality of the textbook.

Dr. Bharat Pandit
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Date : 05-08-2015

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FUNDAMENTAL DUTIES

It shall be the duty of every citizen of India : *

(a) to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;

(b) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(c) to uphold and protect the sovereignty, unity and integrity of India;

(d) to defend the country and render national service when called upon to do so;

(e) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practice heritages derogatory to the dignity of women;

(f) to value and preserve the rich heritage of our composite culture;

(g) to protect and improve the natural environment including forests, lakes, rivers and wild-life, and to have compassion for living creatures;

(h) to develop the scientific temper, humanism and the spirit of inquiry and reform;

(I) to safeguard public property and to abjure violence;

(j) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievement;

(k) to provide opportunities for education by the parent or the gurdian, to his child or a ward between the age of 6 and 14 years as the case may be.

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*Constitution of India : Section 51 A

INDEX

About This Textbook...

We have pleasure in presenting this textbook of physics of Standard 11 to you. This book is on the syllabi based on the courses of National Curriculum Framework (NCF), Core-Curriculum and National Council of Educational Research and Training (NCERT) and has been sanctioned by the State Government keeping in view the National Education Policy.

The State Government has implemented the semester system in Standard 11. The semester system will reduce the educational load of the students and increase the interest towards study.

In this Textbook of Physics for Standard-11, Eight chapters are included in Semester I and Semester II each, looking into the depth of the topics, time which will be available for classroom teaching, etc...

The real understanding of the theories of physics is obtained only through solving related problems. Hence, for the new concept, solved problems are given. One of the positive sides of the book is that at the end of each chapter extended summary is given. On the basis of this one can see the whole contents of the chapter at a glance.

Keeping in view the formats of various entrance test conducted on all India basis, we have included MCQs, Short questions, objective questions and problems in this book. At the end of the book, Hints for solving the problems are also included so that students themselves can solve the problems. The Appendices given at the end of the book will also be very useful.

This book is published in quite a new look in four-colour printing so that the figures included in the book are much clear. It has been observed, generally, that students do not preserve old text-books, once they go to the higher standard. In the semester system, each semester has its own importance and the look of the book is also very nice so the students would like to preserve this book and it will become a reference book in future.

The previous textbook got excellent support from students, teachers and experts. So a substantial portion from that book is taken in this book either in its original form or with some changes. We are thankful to that team of authors. We are also thankful to the teachers who remained present in the Review workshop and gave their inputs to make this textbook error-free.

Proper care has been taken by authors, subject advisors and reviewers while preparing this book to see that it becomes error-free and concepts are properly developed. We welcome suggestions and comments for the importance of the textbook in future.

Authors/Editors

CHAPTER 1

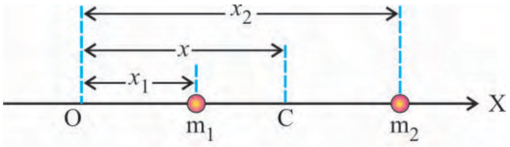
DYNAMICS OF A SYSTEM OF PARTICLES

1.1 Introduction

In Semester-I, we have studied the linear motion of a particle. In this chapter, we will study about how to find out the centre of mass of a system of two particles, the centre of mass of a system of *n*-particles and the centre of mass of a rigid body. Further, we will study the kinetic theory of a system of particles in which the conservation of linear momentum is explained.

1.2 Centre of Mass of a System of Particles in One Dimension

As shown in Figure 1.1 consider two particles having mass m_1 and m_2 lying on X-axis at distances of x_1 and x_2 respectively from the origin (O).



Centre of mass of a system of two particles of masses

Figure 1.1

The centre of mass of this system is that point whose distance from origin O is given by

$$\therefore \ x = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \qquad (1.2.1)$$

Here, x is the mass-weighted average position of x_1 and x_2. If both particles are of the same mass, then $m_1 = m_2 = m$.

$$\therefore \quad x = \frac{mx_1 + mx_2}{m + m}$$

$$\therefore \quad x = \frac{x_1 + x_2}{2} \qquad (1.2.2)$$

Thus, **the centre of mass of the two particles of equal mass lies at the centre (on the line joining the two particles) between the two particles.**

Similarly, if n particles of mass m_1, m_2,...., m_n are lying on X-axis at distances x_1, x_2,...., x_n respectively from the origin 'O', then the centre of mass of the system of n-particles is

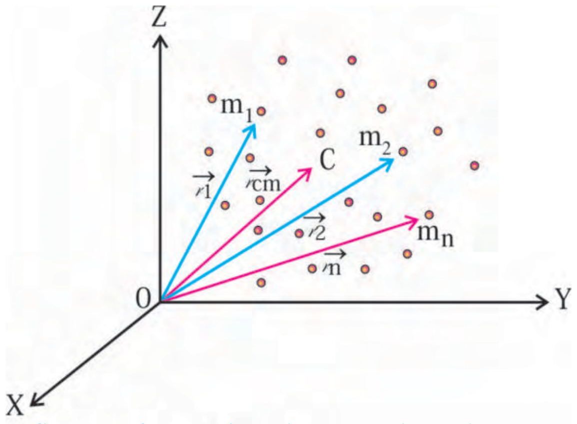
$$x = \frac{m_1x_1 + m_2x_2 + \ldots. + m_nx_n}{m_1 + m_2 + \ldots. + m_n}$$

$$\therefore \quad x = \frac{\Sigma m_i x_i}{\Sigma m_i} \qquad (1.2.3)$$

$$\therefore \quad x = \frac{\Sigma m_i x_i}{\text{M}} \qquad (1.2.4)$$

Where M $= \Sigma m_i =$ total mass of the system of n-particles.

1.3 Centre of Mass of a System of *n*-Particles in Three Dimensions



System of *n*-particles in three dimensions

Figure 1.2

Figure 1.2 shows a system of n-particles in three dimensions. Let the position vectors of the particles of mass m_1, m_2, ... , m_n, with respect to the origin 'O' of the co-ordinate system are $\vec{r}_1$, $\vec{r}_2$, ..., $\vec{r}_n$, respectively. The position vector of centre of mass of the system is given by following equation.

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots. + m_n\vec{r}_n}{m_1 + m_2 + \ldots. + m_n} \qquad (1.3.1)$$

$$\therefore \quad \vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots. + m_n\vec{r}_n}{\text{M}}$$

or

$$\text{M}\,\vec{r}_{cm} = m_1\,\vec{r}_1 + m_2\vec{r}_2 + \ldots.. + m_n\,\vec{r}_n \qquad (1.3.2)$$

where,

$$\text{M} = m_1 + m_2 + \ldots. + m_n$$

= total mass of system of n-particles

1.3.1 Motion of Centre of Mass and Newton's Second Law of Motion :

If the mass of each particle of the system of n-particles does not change with time, then differentiating equation (1.3.2) with respect to time.

$$\text{M}\frac{d\vec{r}_{cm}}{dt} = m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + \ldots. + m_n\frac{d\vec{r}_n}{dt}$$

$$\therefore \ \text{M}\vec{v}_{cm} = m_1\,\vec{v}_1 + m_2\,\vec{v}_2 + \ldots. + m_n\,\vec{v}_n \qquad (1.3.3)$$

Here, $\vec{v}_{cm} = \frac{d\vec{r}_{cm}}{dt}$ is the velocity of centre of mass, and $\vec{v}_1, \vec{v}_2$...... are the velocities of respective particles.

$$\therefore \ \text{M}\vec{v}_{cm} = \vec{\text{P}}_1 + \vec{\text{P}}_2 + \ldots. + \vec{\text{P}}_n \qquad (1.3.4)$$

$$\therefore \ \text{M}\vec{v}_{cm} = \vec{\text{P}} \qquad (1.3.5)$$

Where $\vec{\text{P}}_1$, $\vec{\text{P}}_2$, ... are the momenta of respective particles, and

$\vec{\text{P}} = \vec{\text{P}}_1 + \vec{\text{P}}_2 + \ldots. + \vec{\text{P}}_n$ is the total momentum of the system of n particles.

Equation (1.3.5) shows that **the total linear momentum of system of particles is equal to the product of total mass of the system and velocity of the centre of mass of the system.**

Differentiating equation (1.3.4) with respect to time.

$$M\frac{d\vec{v}_{cm}}{dt} = \frac{d\vec{P}_1}{dt} + \frac{d\vec{P}_2}{dt} + ... + \frac{d\vec{P}_n}{dt}$$

$$\therefore M\frac{d\vec{v}_{cm}}{dt} = \vec{F}_1 + \vec{F}_2 + ... + \vec{F}_n = \vec{F} \quad (1.3.6)$$

$$= m_1\vec{a}_1 + m_2\vec{a}_2 + ... + m_n\vec{a}_n \quad (1.3.7)$$

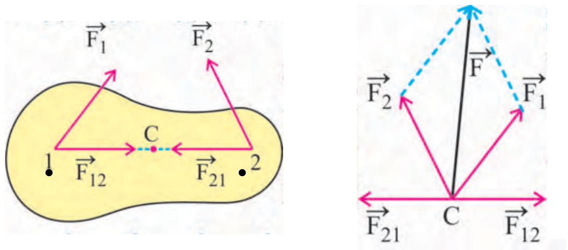
In equation (1.3.6) $\vec{F}_1$, $\vec{F}_2$, ... , $\vec{F}_n$ are the forces acting on the respective particles of the system and $\vec{F}$ is the resultant force. In equation (1.3.7), $\vec{a}_1$, $\vec{a}_2$, ..., $\vec{a}_n$ are the accelerations of the respective particles produced due to these forces.

From equation (1.3.5)

$$M\frac{d\vec{v}_{cm}}{dt} = \frac{d\vec{P}}{dt} = M\,\vec{a}_{cm} \quad (1.3.8)$$

The forces acting on the particles of a system are of two kinds :

(1) Internal forces prevailing among the particles of the system, and (2) External forces.



(*a*) (*b*)

Different types of forces acting on a system of two particles

Figure 1.3

For a system of two particles as shown in Figure 1.3, let the external forces acting on particles 1 and 2 are respectively $\vec{F}_1$ and $\vec{F}_2$, and the mutual forces of interaction acting between them are $\vec{F}_{12}$ and $\vec{F}_{21}$.

While discussing the overall motion of the system, all these forces may be considered to be acting on the centre of mass 'C' [See Figure 1.3(*b*)]. According to the Newton's Third Law of Motion, $\vec{F}_{12} = -\vec{F}_{21}$, and hence the resultant internal force becomes zero. Thus, in equation (1.3.6) the resultant force $\vec{F}$ acting on the system of particles is only the resultant external force. From equation (1.3.6) and (1.3.8).

$$M\frac{\vec{v}_{cm}}{dt} = M\vec{a}_{cm} = \vec{F} = \frac{d\vec{P}}{dt} \quad (1.3.9)$$

Equation (1.3.9) shows that **the resultant external force acting on a system is equal to the rate of change of total linear momentum of the system.** This is the Newton's Second Law of Motion for a system of particles. Further, equation (1.3.9) shows that **the centre of mass of the system moves under the influence of the resultant external force $\vec{F}$ as if the whole mass of the system is concentrated at its centre of mass.**

Newton's Second Law of Motion, for a particle, can be written without the help of the Third Law. But for a system of particles, the help of Newton's Third Law is required to obtain the Second Law. This fact is known as interdependence of Newton's Laws of Motion.

Illustration 1 : The particles of mass m_1, m_2 and m_3 are placed on the vertices of an equilateral triangle of sides '*a*'. Find the centre of mass of this system with respect to the position of particle of mass m_1.

Solution :

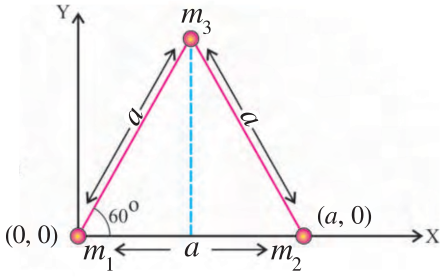


Figure 1.4

The angles of the three corners of an equilateral triangle are equal (60°). Hence, as shown in Figure (1.4) if we place the particle of mass m_1 at the origin (0, 0), and particle of mass m_2 along the X-axis at distance of 'a' from the origin at (a, 0) position, then the co-ordinates of particle of mass m_3 are

$$(a \cos 60°, a \sin60°) = \left(\frac{a}{2}, \frac{\sqrt{3}a}{2}\right)$$

Thus, the position vectors of the particles of mass m_1, m_2 and m_3 are respectively

$$\vec{r}_1 = (0, 0), \ \vec{r}_2 = (a, 0) \text{ and}$$

$$\vec{r}_3 = \left(\frac{a}{2}, \frac{\sqrt{3}a}{2}\right)$$

Hence, according to definition, the position vector of centre of mass of the system of three particles is

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1 + m_2 + m_3}$$

$$= \frac{m_1(0,0) + m_2(a,0) + m_3\left(\frac{a}{2}, \frac{\sqrt{3}a}{2}\right)}{m_1 + m_2 + m_3}$$

$$= \frac{\left(m_2 a + \frac{m_3 a}{2}, \ \frac{\sqrt{3}}{2} m_3 a\right)}{m_1 + m_2 + m_3}$$

$$\therefore \vec{r}_{cm} = \left[\frac{\left(m_2 + \frac{m_3}{2}\right)a}{m_1 + m_2 + m_3}, \frac{\sqrt{3}\ m_3 \frac{a}{2}}{m_1 + m_2 + m_3}\right]$$

Illustration 2 : In a system of three particles, the linear momenta of the three particles are (1, 2, 3), (4, 5, 6) and (5, 6, 7). These components are in kg m s^{-1}. If the velocity of centre of mass of the system is (30, 39, 48) m s^{-1}, then find the total mass of the system.

Solution : Here $\vec{P}_1 = (1, 2, 3)$ kg m s^{-1}

$\vec{P}_2 = (4, 5, 6)$ kg m s^{-1}

$\vec{P}_3 = (5, 6, 7)$ kg m s^{-1}

$\vec{v}_{cm} = (30, 39, 48)$ m s^{-1}

Now, $M\vec{v}_{cm} = \vec{P} = \vec{P}_1 + \vec{P}_2 + \vec{P}_3$

$\therefore$ M(30, 39, 48) = (1, 2, 3) + (4, 5, 6) + (5, 6, 7)

$\therefore$ (30 M, 39 M, 48 M) = (10, 13, 16)

Comparing respective co-efficients on both sides

$$30 \text{ M} = 10 \Rightarrow \text{M} = \frac{1}{3} \text{ kg}$$

$$39 \text{ M} = 13 \Rightarrow \text{M} = \frac{1}{3} \text{ kg}$$

$$48 \text{ M} = 16 \Rightarrow \text{M} = \frac{1}{3} \text{ kg}$$

Thus, the total mass of the system is $\frac{1}{3}$ kg.

Illustration 3 : At time t = 0, a stone of 0.1 kg is released freely from a high rise building. Another stone of 0.2 kg is released from the same position after 0.1 s.

(1) At t = 0.3 s time, what will be the distance of centre of mass of the two stones from original position ? (Neither stone has yet reached the ground).

(2) How fast is the centre of mass of the system of two stones moving at that time ?

(3) What will be the momentum of the system of two stones at this time ?

Solution : Mass of stone 1 is m_1 = 0.1 kg

Mass of stone 2 is m_2 = 0.2 kg

Initial speed of stone 1 is v_{01} = 0 m s^{-1}

Initial speed of stone 2 is v_{02} = 0 m s^{-1}

(1) Here both the stones are moving in one direction so their velocity and momentum vectors can be regarded as scalers. At t = 0.3 s time, the distance travelled by stone 1 is

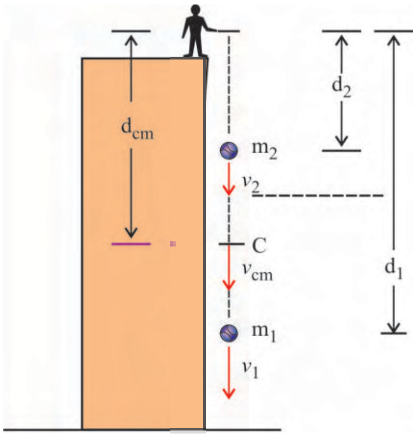


Figure 1.5

$$d_1 = v_{01}\ t + \frac{1}{2}\ g\ t^2$$

$$= 0 + \frac{1}{2}\ (9.8)\ (0.3)^2$$

$$d_1 = 0.441 \text{ m} \qquad (1)$$

Stone 2 is released after 0.1 s. Hence, at time $t = 0.3$ s, the time taken by stone 2 to fall down is $t' = 0.3 \text{ s} - 0.1 \text{ s} = 0.2$ s.

Hence, in time $t' = 0.2$ s (i.e. at $t = 0.3$ s), the distance travelled by stone 2 is

$$d_2 = v_{02}t' + \frac{1}{2}\ g\ t'^2$$

$$= 0 + \frac{1}{2}\ (9.8)\ (0.2)^2$$

$$d_2 = 0.196 \text{ m} \qquad (2)$$

Hence, at time $t = 0.3$ s, the distance of centre of mass of the system of the two stones is

$$d_{cm} = \frac{m_1 d_1 + m_2 d_2}{m_1 + m_2}$$

$$= \frac{(0.1)(0.441) + (0.2)(0.196)}{0.1+0.2}$$

$$\therefore\ d_{cm} = 0.277 \text{ m} \qquad (3)$$

(2) At time $t = 0.3$ s, the speed of stone 1 is

$$v_1 = v_{01} + gt = 0 + (9.8)(0.3)$$

$$\therefore\ v_1 = 2.94 \text{ m s}^{-1} \qquad (4)$$

At time $t = 0.3$ s, the time interval for fall of the stone 2 is $t' = 0.2$ s. Hence, after $t' = 0.2$ s time, the speed of stone 2 is

$$v_2 = v_{02} + gt' = 0 + (9.8)(0.2)$$

$$\therefore\ v_2 = 1.96 \text{ m s}^{-1} \qquad (5)$$

Hence at $t = 0.3$ s, the velocity of centre of mass of the system of two stones is

$$v_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

$$\therefore\ v_{cm} = \frac{(0.1)(2.94) + (0.2)(1.96)}{0.1 + 0.2}$$

$$\therefore\ v_{cm} = 2.29 \text{ m s}^{-1} \qquad (6)$$

(3) At time $t = 0.3$ s, the total momentum of the system of two stones

$$P = P_1 + P_2 = m_1 v_1 + m_2 v_2$$

$$\therefore\ P = (0.1)\ (2.94) + (0.2)\ (1.96)$$

$$\therefore\ P = 0.686 \text{ kg m s}^{-1}$$

$$= 0.69 \text{ kg m s}^{-1} \qquad (7)$$

Illustration 4 : Different forces acting on a lamina body (two dimensional) of mass 2 kg are shown in Figure 1.6. Calculate the linear acceleration of the centre of mass of the body.

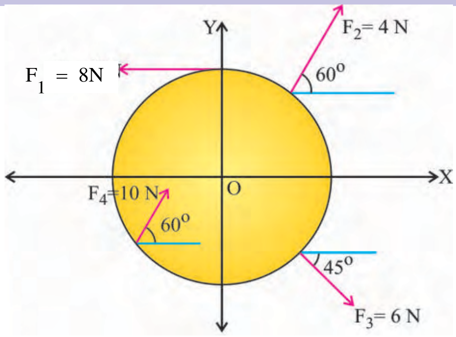


Figure 1.6

Solution : Writing all forces in the form of their components,

$$\vec{F_1} = (-8,\ 0) \text{ N}$$

$$\vec{F_2} = (4 \cos 60°,\ 4 \sin 60°) = (2,\ 2\sqrt{3}) \text{N}$$

$$\vec{F_3} = [6 \cos (-45°),\ 6 \sin (-45°)]$$

$$= (6 \cos 45°,\ - 6 \sin 45°)$$

$$\therefore\ \vec{F_3} = \left(\frac{6}{\sqrt{2}},\ \frac{-6}{\sqrt{2}}\right) \text{N}$$

$$\vec{F_4} = (10 \cos 60°,\ 10 \sin 60°) = (5,\ 5\sqrt{3}) \text{N}$$

Now,

$$\mathrm{M}\vec{a}_{cm} = \vec{\mathrm{F}}_1 + \vec{\mathrm{F}}_2 + \vec{\mathrm{F}}_3 + \vec{\mathrm{F}}_4$$

Where M = 2 kg

$$\therefore \vec{a}_{cm} = \frac{1}{2}(\vec{\mathrm{F}}_1 + \vec{\mathrm{F}}_2 + \vec{\mathrm{F}}_3 + \vec{\mathrm{F}}_4)$$

$$= \frac{1}{2}[(-8 + 2 + \frac{6}{\sqrt{2}} + 5),$$

$$(2\sqrt{3} - \frac{6}{\sqrt{2}} + 5\sqrt{3})]$$

$$\therefore \vec{a}_{cm} = \frac{1}{2}[(-1 + \frac{6}{\sqrt{2}}), (7\sqrt{3} - \frac{6}{\sqrt{2}})]\mathrm{m\ s}^{-2}$$

1.4 Law of Conservation of Linear Momentum

If the resultant force acting on a system is zero, then from equation (1.3.9)

$$\vec{\mathrm{F}} = \frac{d\vec{\mathrm{P}}}{dt} = 0 \qquad (1.4.1)$$

$$\therefore \vec{\mathrm{P}} = \vec{\mathrm{P}}_1 + \vec{\mathrm{P}}_2 + ... + \vec{\mathrm{P}}_n = \text{constant} \qquad (1.4.2)$$

This shows that, **"if the resultant external force acting on a system is zero, then the total linear momentum of the system remains constant."** This statement is known as the **law of conservation of linear momentum.** In absence of resultant external force, the momenta of individual particles $\vec{\mathrm{P}}_1$, $\vec{\mathrm{P}}_2$, ... may change, but these changes occur in such a way that the vector sum of changes in momenta always becomes zero. As the total change in the linear momentum is zero, the total momentum remains constant.

e.g. The gas molecules in a closed container move randomly in the container. During the inter-atomic collisions or the collision of the molecules with the wall of the container, their momentum changes individually. But the vector sum of the changes in momenta of all the particles is zero.

It means that their total momentum remains constant. (If the vector sum of the changes in momenta of the gas molecules were in a particular direction, then what would have happened ? Think).

The law of conservation of linear momentum is fundamental and universal. This law is equally true for the systems as big as that of planets and as small as that of tiny particles like electrons, protons, etc.

From equation (1.3.9)

$$\vec{\mathrm{F}} = \mathrm{M}\vec{a}_{cm} = \mathrm{M}\frac{d\vec{v}_{cm}}{dt} = 0$$

$$\therefore \vec{a}_{cm} = 0 \text{ and } \vec{v}_{cm} = \text{constant}$$

Which shows that, **if the resultant external force is zero, then the acceleration of the centre of mass is zero. It means that the velocity of centre of mass remains constant.** Thus, in absence of external force the centre of mass of a system remains stationary if it was stationary or moves with constant velocity if it was in motion.

Let us see the following illustration :

Suppose, a chemical bomb is stationary. The initial momentum and kinetic energy of the bomb are zero. When the bomb explodes, its fragments are thrown in air. Though these fragments have different momenta in different directions, but the magnitudes and directions of these fragments would be such that

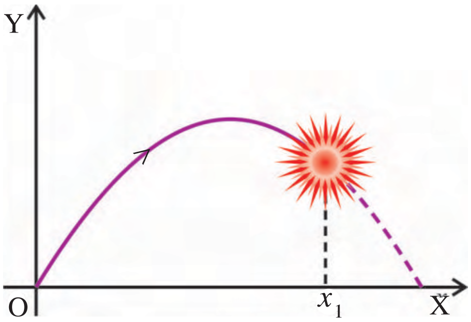
$$\vec{\mathrm{P}}_1 + \vec{\mathrm{P}}_2 + ... + \vec{\mathrm{P}}_n = 0$$

Here $\vec{\mathrm{P}}_1$, $\vec{\mathrm{P}}_2$, ... are the momenta of the fragments.

Here, the centre of mass of the system of fragments remains at the same point, where it was located before explosion of the bomb.

The kinetic energy of the bomb before explosion was zero, but the sum of kinetic energy of the fragments is not zero. Thus the kinetic energy of the system got changed. In the chapter of work, energy and power you came to know that the change in kinetic energy of the system is equal to the work done by the resultant external force. Here, resultant external force is zero. Then how does the kinetic energy of bomb change ? The fact is that chemical bomb possesses internal energy due to the chemical bonds between its complex molecules (and due to some additional reasons). When bomb explodes, the chemical bonds are broken and a part of the internal energy associated with them is converted into heat energy, and the remaining part in the form of kinetic energy of the fragments. Thus, in this case, the work is done at the cost of internal energy **which leads to the more general form of the work energy theorem.**

Here, the bomb was stationary. But, if the bomb was moving with constant velocity and explode during motion, then according to the law of conservation of linear momentum, its fragments would move in such directions, that the vector sum of their momenta become equal to the momentum of the bomb before explosion. The centre of mass would move such that its original velocity ($\vec{v}_{cm}$) is maintained (See Figure 1.7).



Motion of Centre of Mass of Fragments of Bomb after Explision

Figure 1.7

Illustration 5 : A bomb of mass 50 kg moving uniformly with a velocity of 10 m/s explodes spontaneously into two fragments of 40 kg and 10 kg. If the velocity of the smaller fragment is zero, then calculate the velocity of the smaller fragment.

Solution : Bomb is moving with uniform (constant) velocity. Hence, the external force acting on it is zero.

Initial linear momentum = Final linear momentum

$$\therefore \ M\vec{v} = m_1\vec{v}_1 + m_2\vec{v}_2$$

Where, M = total mass of the bomb = 50 kg

m_1 = mass of the larger fragment = 40 kg

m_2 = mass of the smaller fragment = 10 kg

$\vec{v}$ = velocity of the bomb = 10 m/s

$\vec{v}_1$ = velocity of larger fragment = 0

$\vec{v}_2$ = velocity of smaller fragment = ?

Hence,

$$Mv = m_2\vec{v}_2$$

$$\therefore \ \vec{v}_2 = \frac{M}{m_2}\vec{v} = \frac{50}{10} \times 10 = 50 \text{ m/s},$$

in the direction of $\vec{v}$

Illustration 6 : A sphere of mass 4 kg collides with a wall, at an angle of 30° with the wall and rebounds in the direction making an angle of 60° with its original direction of motion. Find the force on the wall if the ball remains in contact with the wall for 0.1 s. The initial and final velocities are the same, equal to 1 m s^{-1}.

Solution : The given situation is shown in the Figure 1.8.

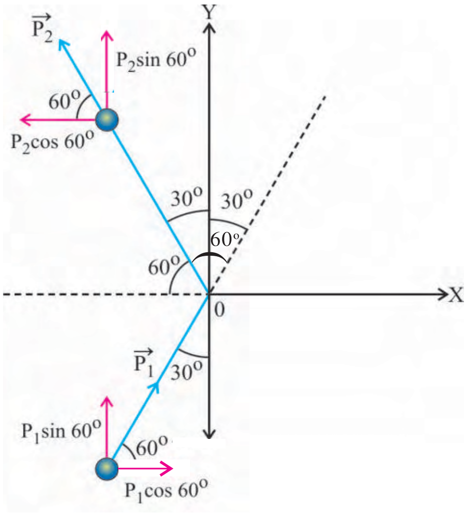


Figure 1.8

Here $\vec{P}_1$ = initial momentum of sphere

$$= mv \cos 60\hat{i} + mv \sin 60\hat{j}$$

$\vec{P}_2$ = final momentum of sphere

$$= -mv \cos 60\hat{i} + mv \sin 60\hat{j}$$

Hence, change in momentum of the sphere

$$\Delta\vec{P} = \vec{P}_2 - \vec{P}_1$$

$$= -mv \cos 60\hat{i} + mv \sin 60\hat{j}$$

$$-mv \cos 60\hat{i} - mv \sin 60\hat{j}$$

$$= -2mv \cos 60\hat{i}$$

$$\therefore \ \vec{r} = -2 \times 4 \times 1 \times \frac{1}{2}\hat{i}$$

$$= -4\hat{i} \text{ kg m s}^{-1}$$

Hence, momentum gained by the wall

$$= 4\hat{i} \text{ kg m s}^{-1}$$

∴ Force exerted on the wall

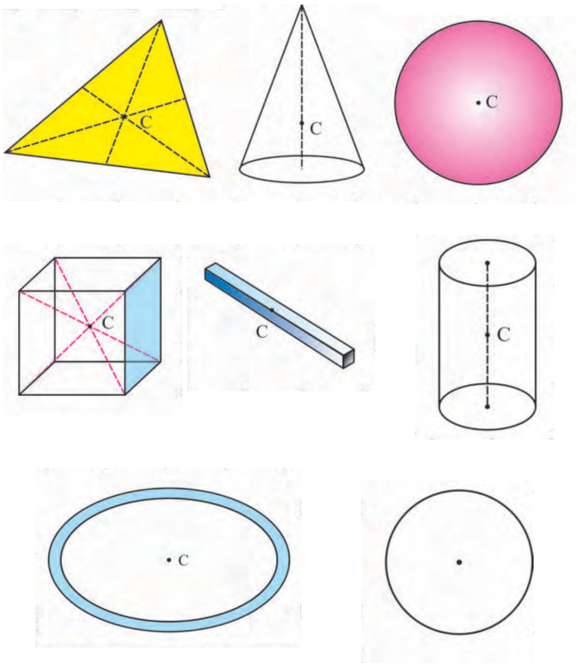
$$= \frac{\text{momentum gained by the wall}}{\text{time of contact}}$$

$$= \frac{4\hat{i}}{0.1} = 40\,\hat{i}\ \text{N}$$

Thus 40 N force acts on the wall in positive X-direction.

1.5 Centre of Mass of a Rigid Body

A system of particles in which the relative positions of particles remain invariant is called a rigid body. The location of centre of mass of a rigid body depends on the distribution of mass in the body and the shape of the body. The centre of mass of a rigid body can be anywhere inside or outside the body. For example, the centre of mass of a disc of uniform distribution of mass is at its geometric centre within the matter, whereas the centre of mass of a ring of uniform mass distribution lies at its geometrical centre which is outside of its matter. The centre of mass of a rod of uniform cross-section lies at its geometrical centre. The position of the centre of mass of symmetric bodies with uniform mass distribution can be easily obtained theoretically. The centre of mass ‘C’ of certain symmetric bodies are shown in Figure 1.9.



Centre of mass of some symmetric bodies

Figure 1.9

1.5.1 Theoretical Method for Estimation of the Centre of Mass of a Solid Body :

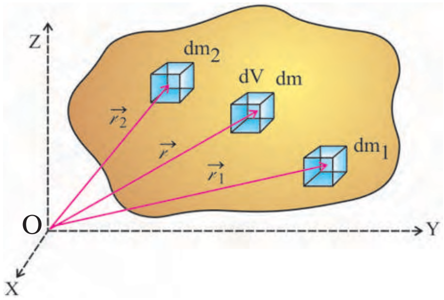


Figure 1.10

We know now that a solid body is made up of microscopic particles (molecules, ions or atoms) distributed continuously inside the body.

As shown in Figure 1.10, consider a small volume element dV, containing mass dm. Here dm is called the mass element, whose position vector is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

This way the whole solid body can be considered to be made up of such small mass elements. Let the solid body is made up of small mass elements dm_1, dm_2, ..., dm_n having position vectors $\vec{r}_1$, $\vec{r}_2$, ..., $\vec{r}_n$, respectively. Hence, according to definition, the position vector of centre of mass of the solid body is

$$\vec{r}_{cm} = \frac{dm_1\vec{r}_1 + dm_2\vec{r}_2 + \ldots.. + dm_n\vec{r}_n}{dm_1 + dm_2 + \ldots.. + dm_n} \qquad (1.5.1)$$

As the mass distribution is continuous, the summation in equation (1.5.1) can be represented as an integration.

$$\therefore\ \vec{r}_{cm} = \frac{\int \vec{r}\,dm}{\int dm}$$

$$\therefore\ \vec{r}_{cm} = \frac{\int \vec{r}\,dm}{\text{M}} \qquad (1.5.2)$$

Where, $\text{M} = \int dm$

= total mass of the solid body

Representing equation (1.5.2) in terms of its vector components,

$$(x_{cm}\hat{i} + y_{cm}\hat{j} + z_{cm}\hat{k}) \qquad (1.5.3)$$

$$= \frac{1}{\text{M}}\int (x\hat{i} + y\hat{j} + z\hat{k})dm$$

$$\left.\begin{aligned} \therefore x_{cm} &= \frac{1}{M}\int xdm \\ y_{cm} &= \frac{1}{M}\int ydm \\ z_{cm} &= \frac{1}{M}\int zdm \end{aligned}\right\} \qquad (1.5.4)$$

1.5.2 Centre of mass of a solid body of uniform density and specific geometrical shape :

The centre of mass of a solid body of uniform mass density and specific geometrical shape can be calculated using the symmetry of the body. Using the law of symmetry we can easily prove that the centre of mass of such bodies lies at their geometric centre.

Let us see the following Illustration :

Suppose we have to find out the centre of mass of given triangular plate :

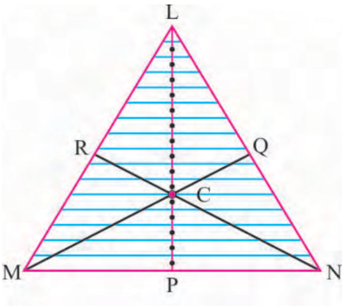
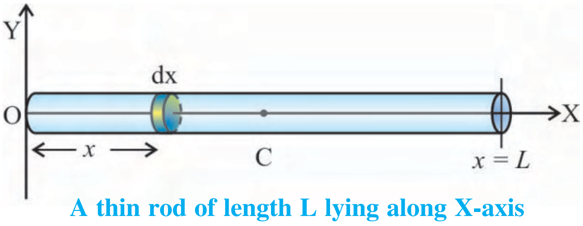


Figure 1.11

As shown in Figure 1.11 imagine the triangular plate to be divided into parallel thin stripes parallel to side MN of the triangle. According to the law of symmetry, the centre of mass of each stripe will be lying at its geometric centre. Thus the centre of mass of the triangular plate will be lying along the bisector LP. Similarly, considering the triangular plate to be divided into thin stripes parallel to the sides ML and LN of the triangle, we get bisectors NR and MQ, respectively. Then, the centre of mass of the triangular sheet will be lying at the intersection point 'C' of the three bisectors.

1.6 Centre of Mass of a Thin Rod of Uniform Density :

Figure 1.12 shows a thin rod of mass 'M' and length 'L' having uniform area of cross section and uniform linear mass density 'λ'. Put the rod along X-axis such that its one end lies at the origin of the co-ordinate system.



A thin rod of length L lying along X-axis

Figure 1.12

Consider a line element 'dx' on the rod at a distance 'x' from the origin.

The mass per unit length of the rod,

$$\lambda = \frac{M}{L}$$

Hence, the mass of the line element dx is,

$$dm = \lambda dx = \frac{M}{L}dx$$

According to definition, the centre of mass of the rod is

$$x_{cm} = \frac{1}{M}\int xdm$$

$$= \frac{1}{M}\int_0^L x \,.\, \frac{M}{L}dx$$

$$= \frac{1}{L}\int_0^L xdx$$

$$= \frac{1}{L}\left[\frac{x^2}{2}\right]_0^L$$

$$= \frac{1}{L}\left[\frac{L^2}{2} - 0\right]$$

$$\therefore x_{cm} = \frac{L}{2}$$

Thus, the centre of mass of the thin rod of uniform mass density lies at mid point of its length, i.e. at its geometric centre.

Illustration 7 :

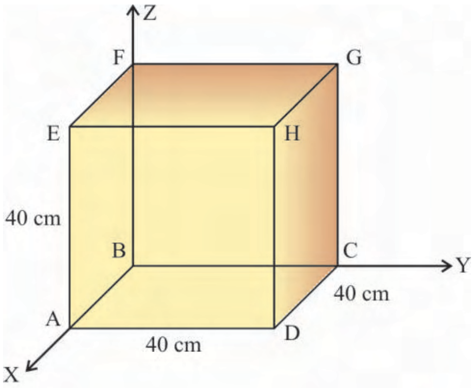


Figure 1.13

Figure 1.13 shows a cubical box made up of negligible thin metal sheet of uniform mass density. If the length of each side of the box is 40 cm, then

(*a*) Find out the co-ordinates (x_{cm}, y_{cm}, z_{cm}) of centre of mass of the box.

(*b*) If the box is open from upper side, (EFGH sheet is absent) then find out the co-ordinates $(x'_{cm}, y'_{cm}, z'_{cm})$ of the centre of mass of the box.

Solution : Each sheet of the box is negligibly thin and have uniform density. Hence, according to the law of symmetry, the centre of mass of each plate will be lying at its geometrical centre. Hence, calculating the centre of mass of each plate :

Name of Plate	Co-ordinates of Centre
ABCD	(20, 20, 0) cm
EFGH	(20, 20, 40) cm
ABFE	(20, 0, 20) cm
DCGH	(20, 40, 20) cm
BCGF	(0, 20, 20) cm
ADHE	(40, 20, 20) cm

(*a*) Considering that the whole mass (say M) of each sheet (plate) is concentrated at its centre of mass (The area and surface density of each plate is same, so the mass of each plate is also same, M = ρ × A), the position of centre of mass of such a system is

$$r_{cm} = (x_{cm}, y_{cm}, z_{cm})$$

$$= \frac{\left\{\begin{matrix} M(20, 20, 0) + M(20, 20, 40) \\ + M(20, 0, 20) + M(20, 40, 20) \\ + M(0, 20, 20) + M(40, 20, 20) \end{matrix}\right\}}{6M}$$

$$= \frac{M(120, 120, 120)}{6M}$$

$$\therefore\ r_{cm} = (20, 20, 20) \text{ cm}$$

(*b*) If the box is open from upper surface, then EFGH plate is absent, and hence the centre of mass of the remaining system is

$$r'_{cm} = (x'_{cm}, y'_{cm}, z'_{cm})$$

$$= \frac{\left\{\begin{matrix} M(20, 20, 0) + M(20, 0, 20) \\ + M(20, 40, 20) + M(0, 20, 20) \\ + M(40, 20, 20) \end{matrix}\right\}}{5M}$$

$$= \frac{M(100, 100, 80)}{5M}$$

$$= (20, 20, 16) \text{ cm}$$

SUMMARY

1. **Centre of mass of a system of two particles :** The centre of mass of a system of two particles of masses m_1 and m_2 lying on X-axis at x_1 and x_2 distances respectively from origin is *a* point at a distance

   $$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \text{ from the origin.}$$

2. **Centre of mass of a system of *n*–particles :** If there are *n*–particles in a system, and C is representing the centre of mass of the system, then 'C' is the point where the whole mass of system of *n*–particles can be considered to be concentrated. For a system of *n*–particles in a three dimensional space, if $m_1, m_2, ..., m_n$ are the masses of the particles and $\vec{r_1}, \vec{r_2}, ..., \vec{r_n}$ are respective position vectors of the particles, then the position vector of centre of mass of the system is

   $$\vec{r}_{cm} = \frac{m_1 \vec{r_1} + m_2 \vec{r_2} + ... + m_n \vec{r_n}}{m_1 + m_2 + ... + m_n}$$

3. The velocity of centre of mass of system of n–particles

$$\vec{v}_{cm} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2 + \ldots + m_n \vec{v}_n}{\mathrm{M}}$$

where, $\mathrm{M} = m_1 + m_2 + \ldots\ldots + m_n$

4. Acceleration of centre of mass of a system of n–particles

$$\vec{a}_{cm} = \frac{m_1 \vec{a}_1 + m_2 \vec{a}_2 + \ldots + m_n \vec{a}_n}{\mathrm{M}}$$

5. Newton's Second Law of Motion for a system of particles :

$$\vec{\mathrm{F}} = \frac{d\vec{\mathrm{P}}}{dt} = \mathrm{M}\frac{d\vec{v}_{cm}}{dt} = \mathrm{M}\vec{a}_{cm}$$

6. **Conservation of linear momentum :** If the resultant external force acting on a system is zero, then the total linear momentum of the system remains constant. In the absence of resultant external force, the centre of mass of a system remains stationary if it was stationary and moves with constant velocity if it was in motion.

7. **Rigid body :** A system of particles in which the relative positions of particles remain invariant is called a rigid body.

8. **The centre of mass of a rigid body :** The location of centre of mass of a rigid body depends on the distribution of mass in the body and the shape of the body. The centre of mass of symmetric bodies lies at their geometric centre.

9. In general form, the co-ordinates of centre of mass of a rigid body are

$$x_{cm} = \frac{1}{\mathrm{M}}\int x\,dm,\ y_{cm} = \frac{1}{\mathrm{M}}\int y\,dm,\ z_{cm} = \frac{1}{\mathrm{M}}\int z\,dm$$

EXERCISES

Choose the correct option from the given options :

1. Suppose your mass is 50 kg. How fast should you run so that your linear momentum becomes equal to that of a bicycle rider of 100 kg moving along a straight road with a speed of 20 km/h ?

   (A) 40 m/s (B) 11.11 m/s (C) 20 km/h (D) 10 km/h

2. A bus of 2400 kg is moving on a straight road with a speed of 60 km/h. A car of 1600 kg is following the bus with a speed of 80 km/h. How fast is the centre of mass of the system of two vehicles moving ?

   (A) 70 km/h (B) 75 km/h (C) 72 km/h (D) 68 km/h

3. The momentum of a stone at time 't' is

   $[(0.5 \text{ kg m/s}^3)t^2 + (3.0 \text{ kg m/s})]\hat{i} + [1.5 \text{ kg m/s}^2]t\hat{j}]$. How much force is acting on it ?

   (A) $(t\hat{i} + 1.5\hat{j})$ N (B) $(0.5t\hat{i} + 1.5\hat{j})$ N

   (C) $[(0.5t + 3)\hat{i} + 1.5\hat{j}]$ N (D) $(0.5\hat{i} + 1.5\hat{j})$ N

4. A bird of 2 kg is flying with a constant velocity of $(2\hat{i} - 4\hat{j})$ m/s, and another bird of 3 kg with $(2\hat{i} + 6\hat{j})$ m/s. Then the velocity of centre of mass of the system of two birds is .... m/s.

(A) $2\hat{i} + 5.2\hat{j}$ (B) $2\hat{i} + 2\hat{j}$ (C) $2\hat{i} - 2\hat{j}$ (D) $10\hat{i} + 10\hat{j}$

5. A quill of 0.100 g is falling with a velocity of $(-0.05\hat{j})$ m/s. When blown from lower side, its velocity changes to $(0.20\hat{i} + 0.15\hat{j})$ m/s. The change in its momentum will be .... kg m/s.

(A) $2 \times 10^{-2}\hat{i} + 2 \times 10^{-2}\hat{j}$ (B) $2 \times 10^{-5}\hat{i} + 2 \times 10^{-5}\hat{j}$

(C) $2 \times 10^{-2}\hat{i} + 1 \times 10^{-2}\hat{j}$ (D) $2 \times 10^{-2}\hat{i} - 2 \times 10^{-2}\hat{j}$

6. A monkey sitting on a tree drops a 10 g seed of rose-apple on a crocodile, at rest below the tree. If the seed falls in the mouth of the crocodile in 2 s time and becomes stationary, then the momentum gained by the crocodile (in addition to the seed) is .... kg m/s. ($g = 9.8$ m s^{-2})

(A) 0.196 (B) −0.196 (C) 19.6 (D) −19.6

7. As shown in Figure (1.14), the stones of 30, 60, 90 and 120 g are placed at 3, 6, 9 and 12 hour symbols respectively of a weightless dial of clock having radius of 10 cm. Find the co-ordinates of centre of mass of this system.

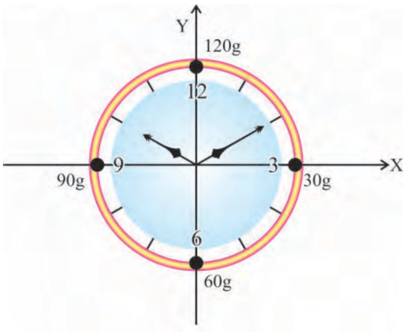


Figure 1.14

(A) (2, −2) cm (B) (0, 0) cm (C) (−2, 2) cm (D) (−4, 4) cm

8. In cricket match, a baller throws a ball of 0.5 kg with a speed of 20 m/s. When a batsman swings the bat, the ball strikes with the bat normal to it, and returns in opposite direction with speed of 30 m/s. If the time of contact of the ball with the bat is 0.1 s, then the force acting on the bat is .... N.

(A) 250 (B) 25 (C) 50 (D) 125

9. A boy standing on the terrace of 10 storeyed building, drops four stones of different mass. At one moment, if the stone of 500 g is at 8th floor, stone of 400 g is at 6th floor, stone of 1 kg is at 3rd floor and a stone of 600 g is reached at 1st floor, then at that time, the centre of mass of the system of four stones is at .... floor.

(A) 7th (B) 5th (C) 3rd (D) 4th

10. As shown in Figure 1.15 the centre of mass of a thin metal sheet of uniform density is .... cm.

(A) (10.00, 14.28)

(B) (11.67, 16.67)

(C) (8.75, 12.50)

(D) (7.78, 11.11)

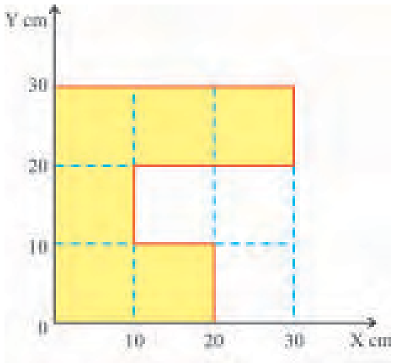


Figure 1.15

ANSWERS

1. (B)	**2.** (D)	**3.** (A)	**4.** (B)	**5.** (B)
6. (A)	**7.** (C)	**8.** (A)	**9.** (D)	**10.** (B)

Answer the following questions in short :

1. What is the meaning of inter-dependence of Newton's Laws of Motion ?
2. Give definition of a rigid body.
3. Give two illustrations of rigid bodies in which the centre of mass lies in the matter of the body.
4. Where does the centre of mass of a thin rod of uniform mass density lie ?
5. What do you mean by the mass element *dm* of a solid body ?
6. When a stationary bomb explodes, then from where does its fragments get kinetic energy ?

Answer the following questions :

1. Write down the expression for the centre of mass of a system of n-particles in three dimensions and obtain the expression for its velocity.
2. State the law of conservation of linear momentum and explain.
3. How does the illustration of chemical bomb lead to the more general form of work energy theorem ? Explain.
4. Write down the equation for the velocity of centre of mass of a system of *n*-particles, and derive the Newton's Second Law of Motion for it.
5. Explain the theoretical method for estimating the centre of mass of a solid body.
6. Obtain the position of centre of mass of a thin rod of uniform density with respect to the one end of the rod.

Solve the following problems :

1. The distance between the centres of carbon and oxygen atoms in a carbon monoxide (CO) molecule is 1.130×10^{-10} m. Find the position of centre of mass of CO molecule with respect to carbon atom.

   (Atomic mass of carbon = 12 g mol^{-1}, and atomic mass of oxygen = 16 g mol^{-1})

   [**Ans. :** 0.64 $\overset{o}{A}$]

2. The velocity vectors of three "particles" of masses 1 kg, 2 kg and 3 kg

are respectively (1, 2, 3), (3, 4, 5) and (6, 7, 8). The velocity vectors are in m s^{-1}. Find the velocity vector of centre of mass of this system of particles. [**Ans. :** $\frac{1}{6}(25, 31, 37)$ m s^{-1}]

3. A car of 1000 kg is at rest at a traffic signal. At the instant, the light turns green, the car starts to move with a constant acceleration of 4.0 m s^{-2} on a straight road. At the same instant, a truck of 2000 kg travelling at a constant speed of 8.0 m s^{-1} overtakes and passes the car.

(*a*) How far will be the centre of mass of the car-truck system from the traffic light after 3 sec. ?

(*b*) What will be the speed of the centre of mass of the car-truck system then ?

[**Ans. :** (*a*) 22.0 m, (*b*) 9.33 m s^{-1}]

4. A dog having mass of 40 kg and a cat of 20 kg mass are standing on both sides of a roti at distance of 15–15 *m* each (See Figure 1.16). Both start to run at the same instant to eat the roti in such a way that the centre of mass of the system made up of the dog and the cat remains stationary. In the table, the position of the dog at different instants is represented, with respect to the origin lying at the roti. Calculate the position of the cat, the velocities, momenta and total momentum of both of them.

Which will reach to the roti first ? Dog or Cat ? Is the momentum conserved in this case ? Why ?

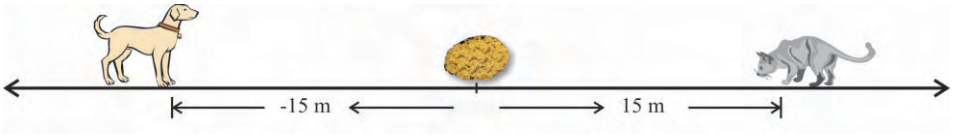


Figure 1.16

Time	Distance from Roti		Centre of mass	Velocity ms^{-1}		Momentum kg ms^{-1}		Total momentum
t			of dog-cat	Dog	Cat	Dog	Cat	$P = P_1 + P_2$
sec	Dog $x_1(m)$	Cat $x_2(m)$	$x_{cm}(m)$	v_1	v_2	P_1	P_2	kg m s^{-1}
0	−15.0	15	.....(constant)					
2	−12.5	.....	„					
4	−10.0	.....	„					
6	−7.5	.....	„					

Ans. :

Time	Dog	Cat	Centre of mass	Dog	Cat	Dog	Cat	Total
t	$x_1(m)$	$x_2(m)$	$x_{cm}(m)$	v_1	v_2	P_1	P_2	$P = P_1 + P_2$
sec			$x_{cm}(m)$			kg ms^{-1}	kg ms^{-1}	kg ms^{-1}
0	−15.0	15.0	−5.0 *m* (const.)	0	0	0	0	0
2	−12.5	10.0	−5.0 *m*	1.25	−2.5	50	−50	0
4	−10.0	5.0	−5.0 *m*	1.25	−2.5	50	−50	0
6	−7.5	0	−5.0 *m*	1.25	−2.5	50	−50	0

At $t = 6$ sec, $x_1 = -7.5\ m$, $x_2 = 0\ m$ and Roti is at origin $x = 0$.
Hence, cat will reach first.
Total momentum remains constant. Hence, momentum is conserved. It is due to the fact that centre of mass remains stationary (in this example).

5. The distance between two particles of masses m_1 and m_2 is r. If the distances of these particles from the centre of mass of the system are r_1 and r_2, respectively, they show that

$$r_1 = r\left[\frac{m_2}{m_1 + m_2}\right] \text{ and } r_2 = r\left[\frac{m_1}{m_1 + m_2}\right]$$

6. As shown in Figure 1.17 three indentical spheres 1, 2 and 3, each of radius R, are arranged on a horizontal surface so as to touch one another. The mass of each sphere is m. Determine the position of centre of mass of this system, taking the centre of sphere 1 as origin. Z-axis is in the direction perpendicular to the plane of the figure.

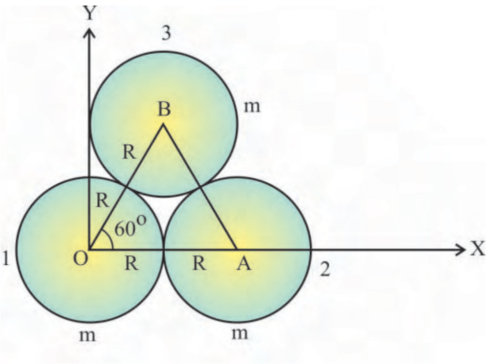
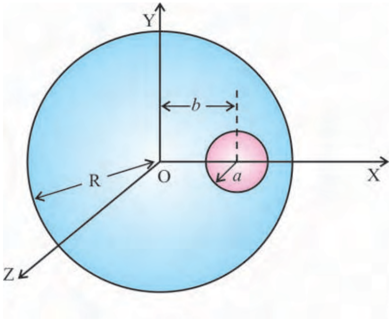


Figure 1.17

[**Ans. :** $(\mathrm{R}, \frac{\mathrm{R}}{\sqrt{3}}, 0)\ m$]

7. A small sphere of radius a is cut from a homogeneous sphere of radius R as shown in Figure 1.18. Find the position of centre of mass of the remaining part with respect to the centre of mass of the original sphere.



[**Ans. :** $\left(\frac{-a^3 b}{\mathrm{R}^3 - a^3}, 0, 0\right)$]

Figure 1.18

8. Figure shows the stationary positions of three "particles". Find out the co-ordinates of centre of mass for the system of particles. As shown in Figure 1.19, if the external forces $\mathrm{F}_1 = 6.0$ N, $\mathrm{F}_2 = 12.0$ N and $\mathrm{F}_3 = 14.0$ N are acting on the particle then find out the acceleration and the direction of acceleration of the centre of mass.

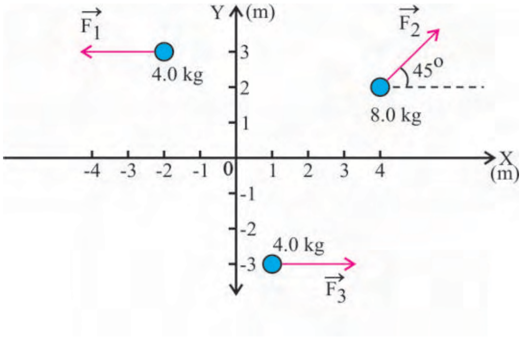


Figure 1.19

[**Ans. :** $\vec{r}_{cm} = (1.75, 1.00)$ m, $\vec{a}_{cm} = (1.03, 0.53)$ m s^{-2},
$|\vec{a}| = a = 1.16$ m s^{-2}. The direction making an angle of $\theta = 27°$ with X–axis.]

9. Figure 1.20 shows an extremely thin disc of radius R, having uniform mass density of ρ. A small disc of radius $\frac{R}{2}$ is cut from it. Find the centre of mass of the remaining part of the disc with respect to the centre of mass of the original disc.

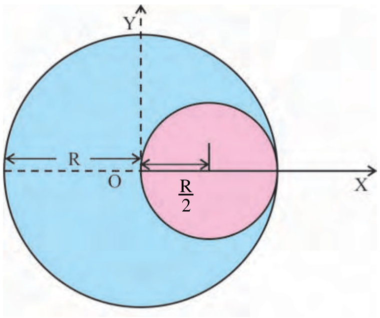


Figure 1.20

[**Ans. :** $(-\frac{R}{6}, 0)$]

●

Prof. Satyendranath Bose (1894-1974)

Satyendranath Bose was born on the 1st of January 1894 in Calcutta. He studied at the University of Calcutta, then taught there in 1916, taught at the University of Dacca (1921-45), and then returned to Calcutta (1945-56). He did important works in quantum theory, in particular on Planck's black body radiation law. Bose sent his work in Planck's Law and the Hypothesis of Light Quanta (1924) to Einstein. It was enthusiastically endorsed by Einstein. The paper was translated into German by Einstein. Bose also worked on statistical mechanics leading to the Bose-Einstein statistics.

Dirac gave the name boson to the particles obeying this statistics. Satyendranath Bose and Albert Einstein together published a series of papers on the physics of particles with integer spins (bosons). Satyendranath Bose passed away on February 4, 1974 at the age of 80.

CHAPTER 2

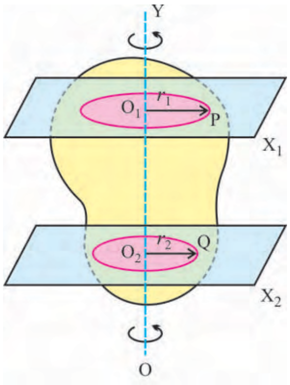
ROTATIONAL MOTION

2.1 Introduction

Student Friends, you must have seen the motion of fan, top and also the motion of merry-go-round. You also know that. **the Earth is revolving round about its own axis.** In the present chapter we shall study such type of motion. This type of motion is called rotational motion. In the beginning we shall discuss the rotational motion of the rigid body about a fixed axis. At the last we shall discuss the motion of the rigid body, rolling without slipping.

The system of particles in which the relative distance between the particles remain invariant is called the rigid body. Rigid body is an ideal concept. From physics point of view a solid body and a rigid body do not mean the same. A solid body can be deformed while a rigid body cannot. But for many practical purposes a solid body can be treated as a rigid body.

2.2 Rotational Kinematics and Dynamics



Rotational motion of rigid body

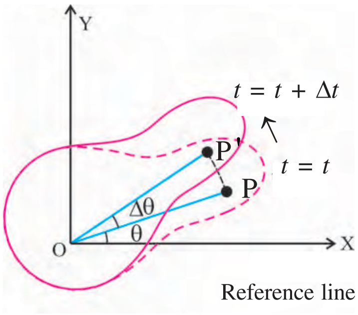
Figure 2.1

If all the particles of a rigid body perform circular motion and the centres of these circles are steady on a definite straight line called axis of rotation it is a geometrical line and the motion of the rigid body is called the rotational motion. In figure 2.1, two particles P and Q of a rigid body are shown. The rigid body rotates about the axis OY. O_1 and r_1 are the centre and the radius respectively of the circle on which particle P moves. Similarly O_2 and r_2 are the centre and the radius respectively of the circle on which the particle Q moves. The circular paths of particles P and Q are in the planes normal to the axis of rotation OY.

First, we shall describe the rotational motion without mentioning its causes. This branch of Physics is called the **rotational kinematics**, and the branch in which the rotational motion is described along with the causes of the rotational motion and the properties of the body is called **rotational dynamics.**

2.3 Relation between Variables of Rotational Motion And the Variables of Linear Motion

(*a*) Angular Displacement :



Angular Displacement

Figure 2.2

Suppose a rigid body performs rotational motion about a fixed rotational axis OZ which is perpendicular to the plane of paper as shown in Figure 2.2.

The positions of the cross-sections of the rigid body with the plane of paper at time t and $t + \Delta t$ are shown by dotted line and the continuous line respectively.

Consider a particle P of the rigid body. At any time the angle made by the line joining it to the centre of its circular path (O) (which is also the radius of its circular path) with a definite reference line (as shown in the figure) is called the angular position of that particle at that time. As shown in the figure, the particle P subtends an angle θ with the reference line OX, at time t. It is the angular position of that particle P at time t. The particle performs circular motion in the XY–plane and reaches from P to P', at time $t + \Delta t$, and its angular position is $\theta + \Delta\theta$ this time.

The change in the angular position of the particle is called its angular displacement. Thus the angular displacement of the particle P in time interval Δt is $\Delta\theta$.

(Any line can be taken as a reference line. Generally, the positive X-axis is taken as the reference line). Since the relative distances between the particles of the rigid body remain invariant (unchanged), all particles experience equal angular displacement in the same time-interval. Hence the rotational motion of the rigid body can be described by the motion of some one representative particle out of its innumerable particles. Thus, in the above discussion the angular displacement $\Delta\theta$ is the angular displacement of the rigid body. Its SI unit is radian.

(*b*) Angular speed and angular velocity :

According to the definition, the average angular speed during the time-interval Δt is

$$<\omega> = \frac{\text{Angular displacement}}{\text{Time-interval}}$$

Here the angular displacement $\Delta\theta$ occurs in the time-interval Δt, hence

$$<\omega> = \frac{\Delta\theta}{\Delta t} \tag{2.3.1}$$

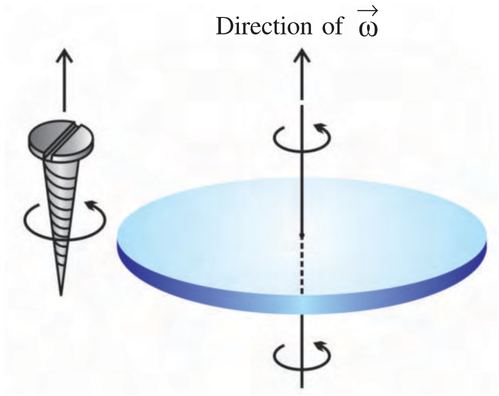
In the limit $\Delta t \rightarrow 0$, this ratio will become the instantaneous angular speed of the particle P at time t.

$$\therefore \ \omega = \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t}$$

$$\therefore \ \omega = \frac{d\theta}{dt} \tag{2.3.2}$$

This is also the angular speed of the entire rigid body at time t (From now onwards we will understand angular speed as instantaneous angular speed except specifically mentioned). The unit of ω is rad s^{-1} or rotation s^{-1}. When a proper direction is linked with angular speed, it is called angular velocity. Conventionally the direction of angular velocity is determined from the right hand screw rule.

A right hand screw is adjusted parallel to the rotational axis as shown in the Figure 2.3, and is rotated in the same sense as the rotation of the body, the direction of shifting of the screw is taken as the direction of angular velocity $\vec{\omega}$.



Right hand screw rule

Figure 2.3

(c) Scalar relation between angular velocity and linear velocity :

As shown in the Figure 2.2 the particle P, travels a linear distance equal to arc PP' in time-interval Δt. Hence, by definition average linear

$$\text{speed } <v> = \frac{\text{arc PP'}}{\text{time-interval } \Delta t}$$

If the radius of the circular path of the particle P (the perpendicular distance of the particle P from the rotational axis) is r, then arc PP' $= r\ \Delta\theta$

$$\therefore\ <v> = \frac{r\Delta\theta}{\Delta t}$$

$$= r<\omega> \qquad (2.3.3)$$

In the limit $\Delta t \rightarrow 0$ the value of the above ratio gives the value of instantaneous linear velocity.

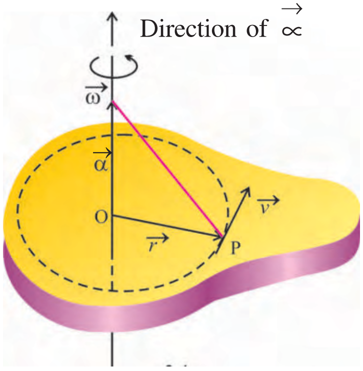
$$\therefore\ v = \lim_{\Delta t \rightarrow 0} \frac{r\Delta\theta}{\Delta t}$$

$$= r\frac{d\theta}{dt}$$

$$\therefore\ v = r\omega \qquad (2.3.4)$$

This shows scalar relation between linear velocity and angular velocity of a rigid body.

(d) Vector relation between angular velocity and linear velocity :



Vector relation between the linear velocity and angular velocity

Figure 2.4(*a*)

The situation of the position vector $\vec{r}$ and the linear velocity $\vec{v}$ for a particle P of the rigid body with respect to the centre of its circular path in a plane perpendicular to the rotational axis, are as shown in the Figure 2.4(*a*). And the angular velocity $\vec{\omega}$ is according to right hand screw rule along the rotational axis (as shown in the figure).

Linear velocity is a vector. In circular motion the direction of linear velocity at any point is along the tangent drawn to the circle at that point. In the equation $v = r\omega$ the left hand side is the value of the linear velocity while on the right hand side r and ω are the values of vector quantities $\vec{r}$ and $\vec{\omega}$. This fact suggests that we should take such a product of $\vec{r}$ and $\vec{\omega}$ that its product is also a vector, which is known as the vector product (cross product) of two vector products. Here direction of $\vec{\omega} \times \vec{r}$ according to right hand screw rule is in the direction of $\vec{v}$ and $\vec{\omega} \perp \vec{r}$. Hence, $\vec{\omega} \times \vec{r}$ = $\omega r \sin 90 = \omega r$ = magnitude of $\vec{v}$.

Hence, we can write the vector relation between $\vec{v}$ and $\vec{\omega}$ as

$$\vec{v} = \vec{\omega} \times \vec{r} \qquad (2.3.5)$$

(e) Angular acceleration :

Suppose instantaneous angular velocities of the particle P at time t and $t + \Delta t$ are $\vec{\omega}$ and $\vec{\omega} + \Delta\vec{\omega}$ respectively.

Hence by definition,

Average angular acceleration

$$<\vec{\alpha}> = \frac{\Delta\vec{\omega}}{\Delta t} \qquad (2.3.6)$$

In the limit $\Delta t \to 0$, the value of the above ratio gives the instantaneous angular acceleration $\vec{\alpha}$ of the particle P at time t.

$$\therefore\ \vec{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\omega}}{\Delta t}$$

$$\therefore\ \vec{\alpha} = \frac{d\vec{\omega}}{dt} \qquad (2.3.7)$$

The direction of $\vec{\alpha}$ is in the direction of $\Delta\vec{\omega}$ (change in angular velocity). In case of the fixed rotational axis the direction of $\Delta\vec{\omega}$ is along the rotational axis, hence the direction of $\vec{\alpha}$ is also along the rotational axis. See Figure 2.4(*a*).

The unit of $\vec{\alpha}$ is rad s^{-2} or rotation s^{-2}.

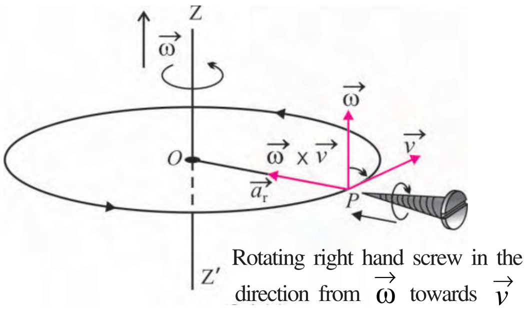
(*f*) Relation between Linear Acceleration and Angular Acceleration :

The derivative of linear velocity with respect to time gives linear acceleration ($\vec{a}$). Differentiating equation (2.3.5) with respect to time, we get

$$\frac{d\vec{v}}{dt} = \vec{a} = \vec{\omega} \times \frac{d\vec{r}}{dt} + \frac{d\vec{\omega}}{dt} \times \vec{r}$$

$$\text{Since } \frac{d\vec{r}}{dt} = \vec{v} \text{ and } \frac{d\vec{\omega}}{dt} = \vec{\alpha}$$

$$\vec{a} = \vec{\omega} \times \vec{v} + \vec{\alpha} \times \vec{r} \qquad (2.3.8)$$



Radial component of linear acceleration

Figure 2.4(*b*)

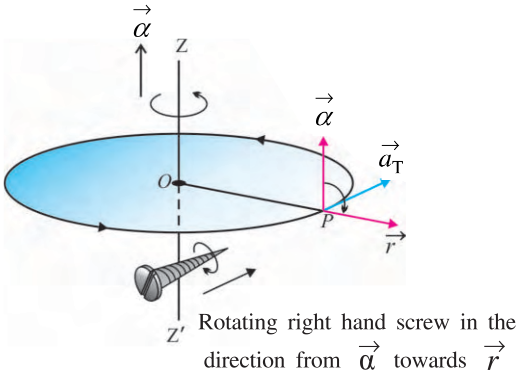
The two vector components of linear acceleration $\vec{a}$ are $\vec{\omega} \times \vec{v}$ and $\vec{\alpha} \times \vec{r}$.

According to Figure 2.4(*b*), using right hand screw rule, the direction of $\vec{\omega} \times \vec{v}$ is found to be the radial direction towards the centre. Hence, $\vec{\omega} \times \vec{v}$ is called the **radial component** of linear acceleration $\vec{a}$. It is denoted by $\vec{a}_r$. Its magnitude is $\omega v \sin\frac{\pi}{2} = \omega v = \frac{v^2}{r} = r\omega^2$

$$\ldots(\because\ v = r\omega)$$

Similarly the direction of $\vec{\alpha} \times \vec{r}$ is found to be along the tangent to the circular path. Hence it is called the **tangential** component of the linear acceleration. (See Figure 2.4 (*b*)). It is denoted as $\vec{a}_T$. Its magnitude is $\alpha r \sin\frac{\pi}{2} = \alpha r$

$$\therefore\ \vec{a} = \vec{a}_r + \vec{a}_T$$



Tangential component of linear acceleration

Figure 2.4 (*c*)

The radial component $\vec{a}_r$ and the tangential component $\vec{a}_T$ are mutually perpendicular. Hence the magnitude of $\vec{a}$ is

$$a = \sqrt{a_r^2 + a_T^2} = \sqrt{\omega^2 v^2 + \alpha^2 r^2} \qquad (2.3.9)$$

If the rigid body is rotating with constant angular velocity, that is, its angular acceleration $\alpha = 0$, then the tangential component of its linear acceleration becomes zero but the radial component remains non-zero. This condition is found in the uniform circular motion. You know very well that in uniform circular motion the centripetal acceleration is $\frac{v^2}{r}$.

In the above discussion we have seen that angular displacement (θ), angular velocity ($\vec{\omega}$) and angular acceleration $\vec{\alpha}$ are equal for all particles of the rigid body. Thus θ, $\vec{\omega}$ and $\vec{\alpha}$ are the characteristics of the rigid body and they are called the variables of the rotational kinematics.

Here, note that the description of motion of a particle of the rigid body rotating about a fixed axis can be made in respect of linear variables ($\vec{r}$, $\vec{v}$ and $\vec{a}$) and rotational variables (θ, $\vec{\omega}$, $\vec{\alpha}$). But when all particles of the rigid body are to be considered, the rotational variables may be used so that the motion of the entire body is easily described.

Illustration 1 : The length of the second-hand of a clock is 20 cm. Find the values of (1) angular velocity (2) linear velocity (3) angular acceleration (4) radial acceleration (5) tangential acceleration (6) linear acceleration, for the particle at the tip of the second-hand.

Solution :

$r = 20$ cm

(1) The second-hand makes angular displacement of 2π radian in one minute (60 seconds.) $\therefore\ \omega = \frac{2\pi}{60} = \frac{\pi}{30}$ rad s^{-1}

(2) Linear velocity $v = \omega r = \frac{\pi}{30} \times 20 = \frac{2}{3}\pi$ cm s^{-1}

(3) The second-hand of a clock moves with constant angular speed. $\therefore\ \alpha = 0$ rad s^{-1}

(4) Radial acceleration $= a_r = \frac{v^2}{r}$

$$= \left(\frac{2\pi}{3}\right)^2 \left(\frac{1}{20}\right) = \frac{\pi}{45}^2 \text{ cm s}^{-2}$$

(5) Tangential acceleration $= a_T = \alpha r = 0$

(6) Linear acceleration

$$a = \sqrt{a_r^2 + a_T^2} = a_r = \frac{\pi}{45}^2 \text{ cm s}^{-2}$$

(Calculate these quantities for minute hand of length 15 cm and hour hand of length 10 cm by yourself.)

2.4 Equations of Rotational Motion with Constant Angular Acceleration

Suppose at $t = 0$ time the angular position of a particle of a rigid body is $\theta = 0$ and its angular velocity is ω_0.

At $t = t$ time its angular position is $\theta = \theta$ and angular velocity $= \omega$.

If the rigid body is rotating about a fixed axis, then the directions of $\vec{\omega}_0$, $\vec{\omega}$ and its constant angular acceleration $\vec{\alpha}$ are all along the fixed axis. Hence relations of θ, $\vec{\omega}$ and $\vec{\alpha}$ can be written in the scalar form : Since α is constant, according to definition

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{\omega - \omega_0}{t} \qquad (2.4.1)$$

OR $$\omega = \omega_0 + \alpha t \qquad (2.4.2)$$

This equation is similar to the equation $v = v_0 + at$ in linear motion.

Here, the angular acceleration is constant, hence using average angular velocity we can find the angular displacement.

$\therefore$ Angular displacement

θ = (average angular velocity) (t)

$$\therefore\ \theta = \left(\frac{\omega + \omega_0}{2}\right)t \qquad (2.4.3)$$

This equation is similar to the equation $x = \left(\frac{v + v_0}{2}\right)t$ in linear motion.

Substituting the value of ω from equation (2.4.2) in equation (2.3.3) we get

$$\theta = \left(\frac{\omega_0 + \alpha t + \omega_0}{2}\right) t$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 \qquad (2.4.4)$$

This equation is similar to the equation $x = v_0 t + \frac{1}{2}at^2$ in linear motion.

Substituting the value of t from equation (2.4.1) in equation (2.4.3), we get

$$\theta = \left(\frac{\omega + \omega_0}{2}\right)\left(\frac{\omega - \omega_0}{\alpha}\right)$$

$$\therefore\ 2\alpha\theta = \omega^2 - \omega_0^{\,2} \qquad (2.4.5)$$

This equation is similar to the equation $2ax = v^2 - v_0^{\,2}$ in linear motion.

> **Illustration 2 :** A mini–train in a children's park moving at a linear velocity of 18 km/h stops in 10 s due to constant angular deceleration produced in it. If the radius of the wheels of the mini-train is 30 cm, find the angular deceleration of the wheel.

Solution :

v_0 = 18 km/h = 5 m/s; r = 30 cm = 0.3 m

$$\omega_1 = \frac{v_1}{r} = \frac{5}{0.3} = \frac{50}{3} \text{ rad/s}$$

$\omega_2 = 0, \qquad t = 10$ s

$$\therefore\ \alpha = \frac{\omega_2 - \omega_1}{t} = \frac{0 - \frac{50}{3}}{10}$$

$$= \frac{-5}{3} = -1.666 \text{ rad/s}^2$$

> **Illustration 3 :** A truck is moving at a speed of 54 km/h. The radius of its wheels is 50 cm. On applying the brakes the wheels stop after 20 rotation. What will be the linear distance travelled by the truck during this ? Also find the angular acceleration of the wheels.

Solution : Here, v_1 = 54 km/h = 15 m/s; r = 50 cm = 0.5 m, θ = 20 rotations = 20 × 2π rad = 40π rad; d = ?, α = ?

$$v_1 = r\omega_1\ \therefore\ \omega_1 = \frac{v_1}{r} = \frac{15}{0.5} = 30 \text{ rad/s}$$

$$\omega_2 = 0;\ \alpha = \frac{\omega_2^{\,2} - \omega_1^{\,2}}{2\theta} = \frac{0 - 900}{2 \times 40\pi}$$

$$= -3.58 \text{ rad/s}^2$$

Now, 1 rotation = $2\pi r$ linear distance

$\therefore$ 20 rotations = 20 × $2\pi r$ distance.

$\therefore$ linear distance travelled by the truck

d = 20 × 2 × 3.14 × 0.5

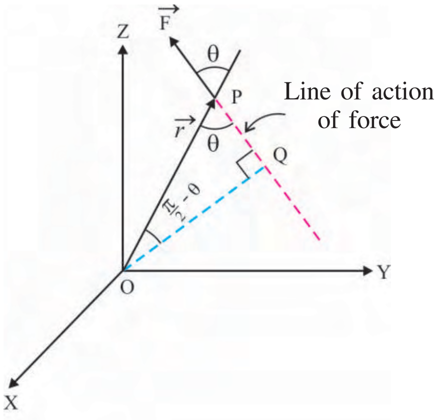
= 62.8 m

2.5 Torque : Up till now we have discussed rotational motion of rigid body without bothering the causes for it. Now we will think about the cause for it.

Torque is an important physical quantity of the rotational dynamics. Torque plays a similar role in relational motion as the force plays in the linear motion.

We will first discuss the torque acting on a particle and then will discuss the torque acting on the system of particles.

(*a*) Torque acting on a particle :



Torque acting on a particle

Figure 2.5

As shown in the Figure 2.5, suppose a force $\vec{F}$ acts on a particle P. The position vector of P with respect to origin O is $\vec{r}$. The angle between $\vec{r}$ and $\vec{F}$ is θ. Here, the particle P is not necessarily be a particle of a rigid body.

The vector product of $\vec{r}$ and $\vec{F}$ is called the torque ($\vec{\tau}$) acting on the particle P, with respect to the point O.

$$\therefore\ \vec{\tau} = \vec{r} \times \vec{F} \qquad (2.5.1)$$

$$\therefore\ \tau = rF\sin\theta$$

From Figure 2.5, $r\sin\theta$ = OQ = the perpendicular distance of the line of action of force from O.

$\therefore\ \tau$ = (F) (perpendicular distance of line of action of force from O)

= Moment of force with respect to point O (by definition)

Thus, torque is the moment of force with respect to a given reference point. Its dimensional formula is $M^1 L^2 T^{-2}$ and its unit is N m.

Remember that,

(i) According to the right hand screw rule the direction of torque ($\vec{\tau}$) is perpendicular to the plane formed by $\vec{r}$ and $\vec{F}$.

(ii) Since the value of torque ($\vec{\tau}$) depends on the reference point, in defining the torque, the reference point must be mentioned.

(b) Torque Acting on the System of Particles :

The mutual internal forces between the particles of a system are equal and opposite, the resultant force and hence the torque produced due to them becomes zero. Hence we will not consider the internal forces in our discussion.

Suppose for a system of particles the position vector of different particles are $\vec{r_1}, \vec{r_2}, \dots, \vec{r_n}$ and the respective forces acting on them are $\vec{F_1}, \vec{F_2}, \dots, \vec{F_n}$. The resultant torque on the system means the vector sum of the torque acting on every particle of the system.

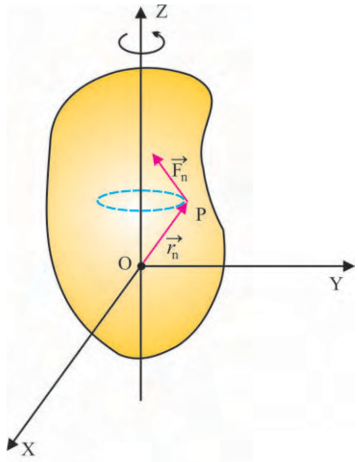
$$\vec{\tau} = \vec{\tau_1} + \vec{\tau_2} + \dots\dots + \vec{\tau_n} \qquad (2.5.2)$$

∴ Resultant torque

$$\vec{\tau} = (\vec{r_1} \times \vec{F_1}) + (\vec{r_2} \times \vec{F_2}) + \dots\dots + (\vec{r_n} \times \vec{F_n})$$

$$= \sum_{i=1}^{n} (\vec{r_i} \times \vec{F_i}) \qquad (2.5.3)$$

(c) Torque acting on the rigid body :



Torque acting on the rigid body

Figure 2.6

Suppose a rigid body rotates about a fixed axis OZ, as shown in the Figure 2.6. The forces acting on the particles with position vectors $\vec{r_1}, \vec{r_2}, \dots, \vec{r_n}$ are $\vec{F_1}, \vec{F_2}, \dots, \vec{F_n}$ respectively.

Considering the force $\vec{F_n}$ acting on the particle with position vector $\vec{r_n}$ the torque $\vec{\tau_n}$ acting on it is

$$\vec{\tau_n} = \vec{r_n} \times \vec{F_n}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_n & y_n & z_n \\ F_{nx} & F_{ny} & F_{nz} \end{vmatrix}$$

$$\therefore \vec{\tau_n} = (y_n F_{nz} - z_n F_{ny})\hat{i} + (z_n F_{nx} - x_n F_{nz})\hat{j} + (x_n F_{ny} - y_n F_{nx})\hat{k} \qquad (2.5.4)$$

From equation (2.5.4) the torque acting on the entire body can be written as a vector sum of the torques acting on all particles, as follows :

$$\vec{\tau_n} = \sum_n (y_n F_{nz} - z_n F_{ny})\hat{i} + (z_n F_{nx} - x_n F_{nz})\hat{j} + (x_n F_{ny} - y_n F_{nx})\hat{k} \qquad (2.5.5)$$

For the rotational motion of the rigid body about Z-aixs, only the Z–component of the above mentioned torque is responsible. For the rotational motion about X–axis the X–component and about Y–axis the Y–component of the torque is responsible. As a general case if the unit vector on the rotational axis is $\hat{n}$; the $\vec{\tau} \cdot \hat{n}$ component of the torque is responsible for the rotational motion, about that axis.

To produce the rotational motion of the rigid body external forces are required to be applied, but not on all the particles of it. As for example, we do not apply forces on all the particles of a door to open it or shut.

Since the relative distances between all the particles of a rigid body remain invariant, the torque produced by applying a force on any one particle becomes the torque on the entire rigid body. If the force acting on any one particle with position vector $\vec{r}$ is $\vec{F}$, then the torque on the rigid body can be taken as $\vec{\tau} = \vec{r} \times \vec{F}$.

Illustration 4 : The force acting on a particle $\vec{r} = (4, 6, 12)$ m of a rigid body is $\vec{F} = (6, 8, 10)$ N. Find the magnitude of the torque producing the rotational motion about an axis along which the unit vector is $\frac{1}{\sqrt{3}}(1, 1, 1)$.

Solution : $\vec{\tau} = \vec{r} \times \vec{F}$

The magnitude of the torque with respect to the axis on which the unit vector is $\hat{n}$, is

$\tau_n = (\vec{r} \times \vec{F}) \cdot \hat{n}$

$$\text{Now, } \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 6 & 12 \\ 6 & 8 & 10 \end{vmatrix}$$

$$= (-36)\hat{i} - (-32)\hat{j} + (-4)\hat{k}$$

$$\therefore \vec{\tau} = (-36, 32, -4) \text{ Nm}$$

Magnitude of the torque responsible for rotational motion

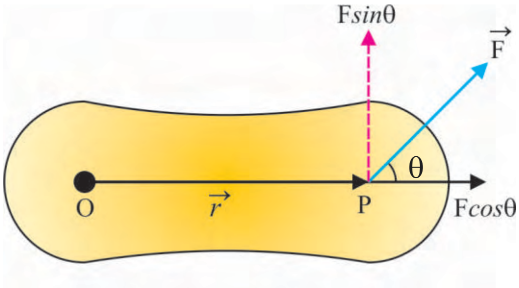
Now, $(\vec{r} \times \vec{F}) \cdot \hat{n}$

$$= (-36, 32, -4) \cdot \frac{1}{\sqrt{3}} (1, 1, 1)$$

$$= \frac{1}{\sqrt{3}}(-36 + 32 - 4)$$

$$= -\frac{8}{\sqrt{3}} \text{ N m} \therefore \text{ The magnitude is } \frac{8}{\sqrt{3}} \text{ N m}$$

(*d*) Physical interpretation of the definition of torque :



Effective component of the torque

Figure 2.7

Suppose a force $\vec{F}$ acts on the particle P of the rigid body as shown in the Figure 2.7. Here the force is taken in a plane perpendicular to the rotational axis, which is coming out from the plane of paper, from point O.

The position vector of point P with respect to the centre of its circular path is $\vec{r}$. The angle between $\vec{F}$ and $\vec{r}$ is θ. To understand the effectiveness of $\vec{F}$ in producing the rotational motion, consider two components of $\vec{F}$.

(i) The component of $\vec{F}$ parallel to $\vec{r}$ is say $F_1 = F \cos\theta$. Hence $\vec{r} \times \vec{F_1} = 0$. This does not produce torque. Thus it does not produce the rotational motion.

(ii) The component of $\vec{F}$ perpendicular to $\vec{r}$ is $F_2 = F \sin\theta$. This component produces the rotational motion. If the magnitude of F and/or θ is more, then $\vec{F}$ becomes more effective. Moreover, our common experience tells us that if the position vector $\vec{r}$ of the point of action of $\vec{F}$ is more, then also $\vec{F}$ becomes more effective in producing rotation. Thus the quantity responsible for producing rotation is not only $\vec{F}$ but is r F sinθ. This quantity is called the torque. Writing the above formula in the vector form

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{2.5.6}$$

Remember that torque is the measure of the effectiveness of the force in producing the rotational motion.

(*e*) Couple : Two forces of equal magnitude and opposite directions which are not collinear form a couple. As shown in the Figure 2.8, forces $\vec{F_1}$ and $\vec{F_2}$ act on two particles P and Q of the rigid body having position vectors $\vec{r_1}$ and $\vec{r_2}$ respectively. Here, $|\vec{F_1}| = |\vec{F_2}|$ and the directions of $\vec{F_1}$ and $\vec{F_2}$ are mutually opposite. The resultant torque of the torques $\vec{\tau_1}$ and $\vec{\tau_2}$ produced due to the forces $\vec{F_1}$ and $\vec{F_2}$ is called the moment of the couple $(\vec{\tau})$.

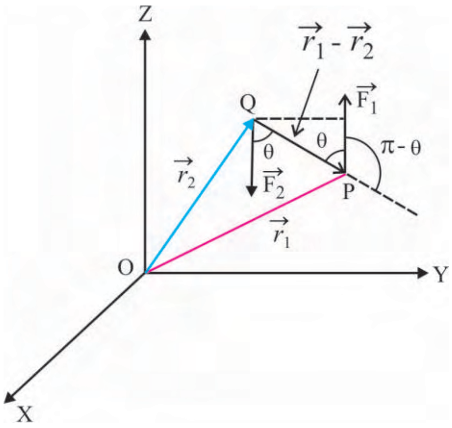


Figure 2.8

$$\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2$$

$$\therefore \vec{\tau} = (\vec{r}_1 \times \vec{F}_1) + (\vec{r}_2 \times \vec{F}_2)$$

$$= (\vec{r}_1 \times \vec{F}_1) - (\vec{r}_2 \times \vec{F}_1)$$

$$(\because \vec{F}_2 = -\vec{F}_1)$$

$$\therefore \vec{\tau} = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_1$$

$$= |\vec{r}_1 - \vec{r}_2|(F_1) \sin(\pi - \theta)$$

$$= |\vec{r}_1 - \vec{r}_2|(F_1) \sin\theta$$

Where $(\pi - \theta)$ is the angle between $(\vec{r}_1 - \vec{r}_2)$ and $\vec{F}_1$

From the figure $|\vec{r}_1 - \vec{r}_2| \sin\theta$ = perpendicular distance between the two forces.

$\therefore$ Moment of couple = (F_1) (perpendicular distance between the two forces),

= (magnitude of any one of the two forces) (perpendicular distance between the two forces) (2.5.8)

Student friends, do you know that you are also using couple in practice ? When you are driving scooter or car, to turn the vehicle, the forces you apply on the steering, produce couple.

(*f*) Equilibrium of a rigid body :

Now we shall discuss the equilibrium of the rigid body under the influence of many forces acting on it. If the external forces acting on the rigid body are $\vec{F}_1, \vec{F}_2, \ldots\ldots, \vec{F}_n$ and if resultant force $\vec{F} = \vec{F}_1 + \vec{F}_2 + \ldots.. + \vec{F}_n = 0$ (2.5.9) then the rigid body remains in translational equilibrium. Writing the above equation in the form of the components of forces, $\sum_i F_{xi} = 0;$ $\sum_i F_{yi} = 0;$ and $\sum_i F_{zi} = 0$ (2.5.9 *a*)

If the torques produced by the above mentioned forces are $\vec{\tau}_1, \vec{\tau}_2, \ldots.., \vec{\tau}_n$ then the rigid body remains in rotational equilibrium when

$$\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 + \ldots.. + \vec{\tau}_n = 0. \qquad (2.5.10)$$

That is, if rigid body is stationary, it will remain stationary and if it is performing rotational motion, it will continue rotational motion with constant angular velocity.

Writing this equation in the form of components of torques.

$$\sum_i \tau_{xi} = 0; \ \sum_i \tau_{yi} = 0; \text{ and } \sum_i \tau_{zi} = 0 \qquad (2.5.10\ a)$$

Illustration 5 : As shown in the figure a block of mass m moves with constant velocity under the influence of a force F acting in the direction making an angle θ with the horizontal. If the frictional force between the surface of the block and the horizontal surface is f_k, find the distance of line of action of normal reaction N from O. Length of the block is L and height is h.

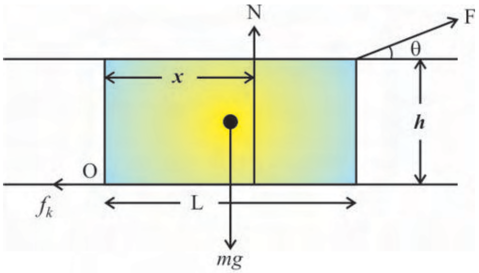


Figure 2.9

Solution : The block does not perform rotational motion in spite of being under the influence of various forces. Hence, it is in rotational equilibrium. In this condition the vector sum of the torques produced due to different forces should be zero. Taking all the torques with reference to the point O, we get, $\tau = f_k(0) - (mg)\left(\frac{L}{2}\right) + N(x) - (F\cos\theta)(h) + F\sin\theta(L) = 0.$

(Here the torque in the clockwise direction is taken negative and the torque in the anticlockwise direction is taken positive).

$$\therefore \quad N(x) = (mg)\left(\frac{L}{2}\right) + (F\cos\theta)(h) - F\sin\theta\,(L) \qquad (1)$$

Now, for translation equilibrium,

$mg = N + F\sin\theta$ and $F\cos\theta = f_k$

$\therefore$ $N = mg - F\sin\theta$

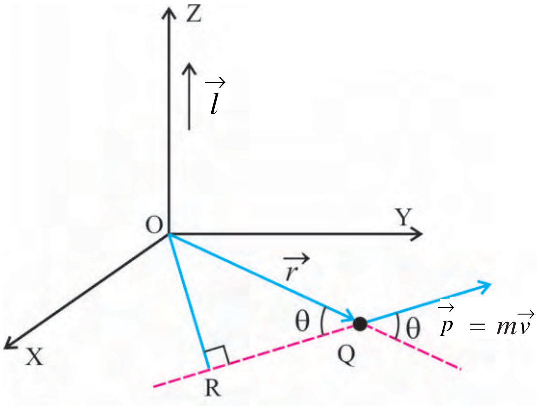
Substituting this value in equation (1), and making x the subject of the formula we get,

$$x = \frac{(mg)\left(\frac{L}{2}\right) + (F\cos\theta)(h) - (F\sin\theta)(L)}{mg - F\sin\theta}$$

2.6 Angular Momentum

(*a*) Angular Momentum of a Particle :

Suppose the position vector of a particle of mass m is $\overrightarrow{OQ} = r$ in a Cartesian co-ordinate system as shown in the Figure 2.10. The linear velocity of this particle is $\vec{v}$ and its linear momentum is $\vec{p} = m\vec{v}$. Here, the particle Q is not necessarily a particle of a rigid body. Suppose the angle between $\vec{p}$ and $\vec{r}$ is θ. We have taken the particle and its motion in the (X – Y) plane only for simplicity. The vector product of $\vec{r}$ and $\vec{p}$ is called the angular momentum $\vec{l}$ of the particle with reference to the point O.



Angular momentum

Figure 2.10

$$\vec{l} = \vec{r} \times \vec{p} \qquad (2.6.1)$$

The SI unit of $\vec{l}$ is kg m^2s^{-1} or Js.

(i) The magnitude of $\vec{l}$ depends on the selection of the reference point, hence the reference point must be mentioned in its definition.

(ii) The direction of $\vec{l}$ is given by the right hand screw rule for the vector product. In the present case the direction of $\vec{l}$ is in OZ direction.

(iii) Now $|\vec{l}| = |\vec{r} \times \vec{p}| = r\,p\,\sin\theta$

But from Figure 2.10,

$r\sin\theta = OR$

$\therefore \quad l = (p)$ (distance OR)

Thus the angular momentum of the particle = (linear momentum) (perpendicular distance of the vector of linear momentum from reference point)

= moment of linear momentum with reference to point O.

Note : Cartesian components of angular momentum of a particle :

By definition, the angular momentum is

$$\vec{l} = \vec{r} \times \vec{p}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$$= (yp_z - zp_y)\hat{i} + (zp_x - xp_z)\hat{j} + (xp_y - yp_x)\hat{k}$$

$$\vec{l} = l_x\hat{i} + l_y\hat{j} + l_z\hat{k}$$

Here, l_x, l_y and l_z are the components of angular momentum with reference to X, Y and Z axes respectively.

(*b*) The relation between angular momentum of a particle and torque acting on it :

Differentiating equation (2.6.1) with respect to time, we get

$$\frac{d\vec{l}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p}$$

But $\frac{d\vec{p}}{dt}$ = rate of change of linear momentum = $\vec{F}$ (force) and $\frac{d\vec{r}}{dt} = \vec{v}$ (velocity)

$$\therefore \frac{d\vec{l}}{dt} = \vec{r} \times \vec{F} + \vec{v} \times \vec{p}$$

But $\vec{v}$ and $\vec{p}$ being in the same direction the vector product $\vec{v} \times \vec{p} = 0$

$$\therefore \frac{d\vec{l}}{dt} = \vec{r} \times \vec{F} = \vec{\tau} \qquad (2.6.2)$$

Thus the time rate of change of angular momentum is equal to torque. This result is similar to Newton's second law of motion "the time rate of change of linear momentum is equal to force."

(*c*) Angular momentum of system of particles :

Suppose the angular momentum of particles of a system made up of n particles are $\vec{l}_1, \vec{l}_2, \ldots, \vec{l}_n$.

Hence the total angular momentum $\vec{L}$ is,

$$\vec{L} = \vec{l}_1 + \vec{l}_2 + \ldots + \vec{l}_n \qquad (2.6.3)$$

$$\therefore \frac{d\vec{L}}{dt} = \frac{d\vec{l}_1}{dt} + \frac{d\vec{l}_2}{dt} + \ldots + \frac{d\vec{l}_n}{dt} \qquad (2.6.4)$$

Using equation (2.6.2)

$$\frac{d\vec{L}}{dt} = \vec{\tau}_1 + \vec{\tau}_2 + \ldots. + \vec{\tau}_n$$

$$\therefore \frac{d\vec{L}}{dt} = \vec{\tau} \qquad (2.6.5)$$

Thus the rate of change of angular momentum of a system of particles is equal to the resultant external torque.

(*d*) Angular momentum of a rigid body :

The relative distance between the particles of a rigid body remain invariant, hence it is a special case of the system of particles. We know that every particle of the rigid body performs circular motion in a plane perpendicular to the rotational axis. If we take the centre of the circular path of every particle as reference point, the angular momentum of respective particle is found to be along the rotational axis. Moreover, for every particle $\vec{r}$ and $\vec{p}$ are mutually perpendicular.

We know that,

$$\vec{L} = \vec{l}_1 + \vec{l}_2 + \ldots + \vec{l}_n$$

Using equation (2.6.1)

$$\vec{L} = \vec{r}_1 \times \vec{P}_1 + \vec{r}_2 \times \vec{P}_2 + \ldots + \vec{r}_n \times \vec{P}_n$$

Here vector $\vec{r}$ and $\vec{p}$ being mutually perpendicular, the magnitude of $\vec{L}$ is,

$$|\vec{L}| = r_1\, p_1 + r_2 p_2 + \ldots + r_n p_n$$

($\because$ $\vec{r} \perp \vec{p}$, hence $|\vec{r} \times \vec{p}| = rp \sin 90° = rp$)

$$\therefore |\vec{L}| = r_1 m_1 v_1 + r_2 m_2 v_2 + \ldots + r_n m_n v_n$$

($\because$ $P = mv$)

Here, angular speed of each particle is same.

$$|\vec{L}| = m_1 r_1^2 \omega + m_2 r_2^2 \omega + \ldots + m_n r_n^2 \omega$$

($\because$ $v = r\omega$)

$$= (m_1 r_1^2 + m_2 r_2^2 + \ldots + m_n r_n^2)\omega$$

$$\therefore |\vec{L}| = I\ |\vec{\omega}| \qquad (2.6.6)$$

Here $I = m_1 r_1^2 + m_2 r_2^2 + \ldots + m_n r_n^2$

$$= \sum_{i=1}^{n} m_i r_i^2$$

I is called the moment of inertia of the rigid body about the given axis of rotation. More details about it are given in the article 2.9. In the present case $\vec{\omega}$ and $\vec{L}$ both being parallel to the rotational axis, I can be taken as a scalar. In this condition the equation (2.6.6) can be written in the vector form as under.

$$\vec{L} = I\vec{\omega} \qquad (2.6.7)$$

$$\therefore \frac{d\vec{L}}{dt} = I\frac{d\vec{\omega}}{dt} \qquad (2.6.8)$$

Combining equations (2.6.5) and (2.6.8),

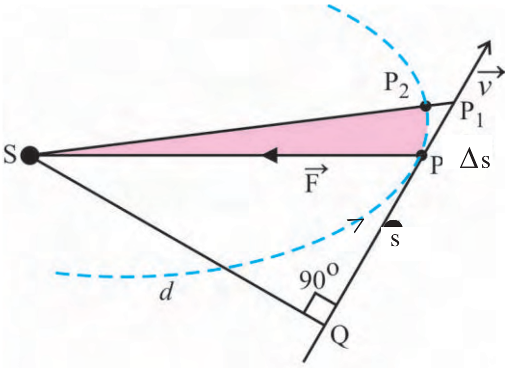
$$\frac{d\vec{L}}{dt} = \mathrm{I}\frac{d\vec{\omega}}{dt} = \mathrm{I}\vec{\alpha} = \vec{\tau} \qquad (2.6.9)$$

Law of conservation of angular momentum :

From equation (2.6.9), if $\vec{\tau} = 0$, $\vec{\mathrm{L}} =$ constant.

Thus, **"if the resultant torque acting on the rigid body is zero, the angular momentum of the rigid body remains constant."** This statement is called law of conservation of angular momentum.

2.7 Geometrical Representation of the Law of Conservation of Angular Momentum



Geometrical representation of the law of conservation of angular momentum

Figure 2.11

As shown in the Figure 2.11, a planet P revolves around the sun in an elliptical orbit (which is shown as dotted line). Suppose the linear velocity of the planet at P is $\vec{v}$.

∴ The angular momentum of the planet with respect to sun is

$$\mathrm{L} = mvd \qquad (2.7.1)$$

Now, the area of triangle SQP, is

$$\mathrm{A} = \frac{1}{2}(\mathrm{SQ})\ (\mathrm{PQ})$$

$$= \frac{1}{2}(d)\ (s) \qquad (\because\ \mathrm{PQ} = s)$$

In time Δt, the planet moves from P to P_2. During this, if the increase in the area of the triangle SQP is ΔA, then

$$\Delta\mathrm{A} = \frac{1}{2}(d)(\Delta s)$$

Now in the $lim\ \Delta t \rightarrow 0$, the areas of the triangles SPP_2 and SPP_1 become equal.

∴ The time rate of change of area swept by the line joining the planet with the sun is

$$\frac{d\mathrm{A}}{dt} = \frac{1}{2}(d)\left(\frac{ds}{dt}\right) = \frac{1}{2}(d)(v)$$

Multiplying both the sides of the equation by m, we get $m\frac{d\mathrm{A}}{dt} = \frac{1}{2}mvd$ (2.7.2)

Substituting the value of mvd from equation (2.7.1),

$$m\frac{d\mathrm{A}}{dt} = \frac{1}{2}\mathrm{L} \qquad (2.7.3)$$

Now the line of action of the gravitational force on planet due to the sun, passes through S, the torque due to this force with respect to sun becomes zero.

Hence the angular momentum of the planet remains constant.

$$\therefore\ \frac{d\mathrm{A}}{dt} = \text{constant} \qquad (2.7.4)$$

Equation (2.7.4), represents Kepler's second law of planetary motion. **"The area swept by the line joining the sun and the planet in unit time (which is called the areal velocity) is constant."**

Thus, the areal velocity being constant is the geometrical representation of the law of conservation of angular momentum.

2.8 Moment of Inertia

Suppose the masses of different particles of the rigid body are $m_1, m_2, \ldots, m_n$ and their perpendicular distances from the given axis are respectively $r_1, r_2, \ldots, r_n$, then $m_1r_1^2 + m_2r_2^2 + \ldots + m_nr_n^2$ is called the moment of inertia (I) of the rigid body about that axis.

That is, $\mathrm{I} = m_1r_1^2 + m_2r_2^2 + \ldots + m_nr_n^2$

$$= \sum_i m_ir_i^2$$

The magnitude of the moment of inertia depends on the selection of axis and the

distribution of mass about it. The SI unit of moment of inertia is kg m^2. Its dimensional formula is $M^1L^2T^0$

The equation $\vec{L} = I\vec{\omega}$ resembles with the equation $\vec{p} = m\vec{v}$ of the linear motion, and the equation $\vec{\tau} = I\vec{\alpha}$ resembles with the equation $\vec{F} = m\vec{a}$ of the linear motion. In reference to this resemblance we can say that the moment of inertia plays the same role in rotational motion as the mass plays in the linear motion.

Illustration 6 : If we accept the earth as a solid sphere of uniform density and imagine that it contracts so that the radius becomes half without change in mass, then what will be the length of day which at present is of 24 hours ?

Solution : If we accept that no external torque acts on the Earth we can take its angular momentum as constant. Using equation (2.6.6) and comparing the angular moments of the Earth, in the two cases,

$$I_1 \omega_1 = I_2\omega_2 \qquad (2.6.10)$$

Now for the solid sphere $I = \frac{2}{5}MR^2$ about its diameter, where M = mass of the sphere and R = radius of the sphere. (See Table-2.1).

$$\therefore I_1 = \frac{2}{5}MR_1^2 \text{ and } I_2 = \frac{2}{5}MR_2^2$$

But $R_1 = 2R_2$ Substituting these values in question (2.6.10), $\omega_2 = 4\omega_1$

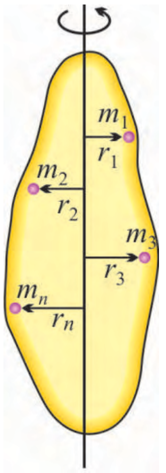
Thus the new rate of rotation ω_2 becomes four times the present rate ω_1, and hence the present day of 24 hours becomes day of 6 hours.

2.9 Radius of Gyration

Suppose a rigid body has mass M. It is made up of n particles each having mass *m*.

$$\therefore m_1 = m_2 = \dots = m_n = m \;\therefore\; M = nm$$

As shown in the Figure 2.12, the moment of inertia of inertia of the body about the given axis is $I = mr_1^2 + mr_2^2 + \dots + mr_n^2$



Radius of Gyration

Figure 2.12

Here, r_1, r_2, … r_n are the perpendicular distances of the respective particles of the body from the axis.

$$\therefore I = \frac{mn(r_1^2 + r_2^2 + \dots + r_n^2)}{n}$$

$$= M\frac{(r_1^2 + r_2^2 + \dots + r_n^2)}{n}$$

$$= MK^2 \qquad (2.9.1)$$

$$\text{Where } K^2 = \frac{r_1^2 + r_2^2 + \dots + r_n^2}{n}$$

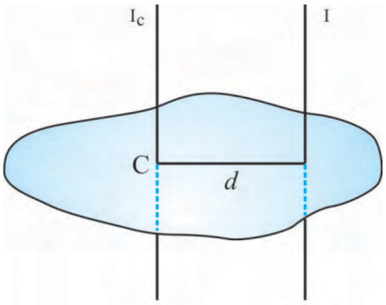
$$= \langle r^2 \rangle \qquad (2.9.2)$$

$$\therefore K = \sqrt{\frac{r_1^2 + r_2^2 + \dots + r_n^2}{n}} \qquad (2.9.3)$$

Thus, K^2 shows the mean (average) value of the squares of the perpendicular distances of the particles of the body from the axis. K is called the **radius of gyration** of the body about the given axis. Its SI unit is m.

2.10 Two Theorems Regarding Moment of Inertia

(i) Theorem of Parallel axis : The statement of this theorem is "The moment of



Theorem of parallel axis

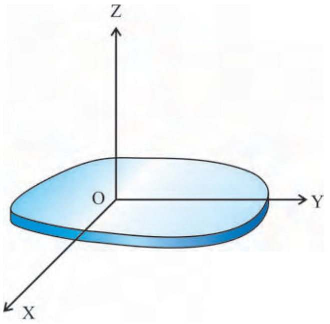
Figure 2.13

inertia (I) of the body about any axis is equal to the sum of the moment of inertia about a parallel axis passing through its centre of mass and the product of its mass with the square of the perpendicular distance between the two axes.” See Figure 2.13.

$$I = I_c + Md^2 \qquad (2.10.1)$$

(ii) Theorem of perpendicular axis :

This theorem is applicable only to the planar bodies. If we take X and Y axes in the plane of a planar body (See Figure 2.14), then the moment of inertia of the body about the Z axis which is perpendicular to the plane of the body is equal to the sum of the moments of inertia of the body about X and Y axes.



Theorem of perpendicular axis

Figure 2.14

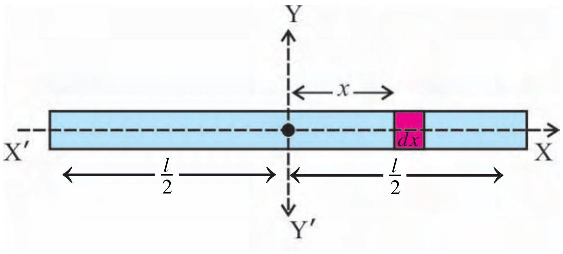
$$I_Z = I_X + I_Y \qquad (2.10.2)$$

Where I_X and I_Y are the moments of inertia of the body about X and Y axes respectively. When planar body is in YZ plane, $I_x = I_Y + I_Z$ and if it is in XZ plane then $I_Y = I_X + I_Z$

Mass is the inertia for linear motion, the moment of inertia is the inertia for rotational motion.

2.11 Calculation of Moment of Inertia And Radius of Gyration

(*a*) Moment of inertia of a thin rod about an axis passing through its centre and perpendicular to its length :



Moment of inertia of thin rod

Figure 2.15

Consider a thin rod of mass M and length *l*, with uniform cross section and uniform mass distribution as shown in Figure 2.15. Consider the axis YY' passing through its centre O and perpendicular to its length. The origin of the co-ordinate system is coinsiding with centre of the rod and the *x*–axis is along the length of the rod. Consider a small element of rod with length *dx* at distance *x* from the origin.

Mass per unit length of rod $\lambda = \frac{M}{l}$

mass of element of length *dx* is

$$\lambda dx = \frac{M}{l} dx.$$

The moment of inertia of this element about YY' axis is $dI = \frac{M}{l} dx \cdot x^2$ (2.11.1)

To find the moment of inertia of the entire rod about YY' axis, we integrate equation (2.11.1) from $x = -l/2$ to $x = l/2$

$$\therefore \ I = \int_{-l/2}^{l/2} \frac{M}{l} dx \ x^2 = \frac{M}{l}\left[\frac{x^3}{3}\right]_{-l/2}^{+l/2}$$

$$= \frac{M}{3l}\left[\frac{l^3}{8} + \frac{l^3}{8}\right]$$

$$I = \frac{Ml^2}{12} \qquad (2.11.2)$$

For uniform thin rod its centre of a mass is at its geometrical centre. Thus, this moment of inertia is the moment of inertia I_c about an axis passing through the centre of mass.

Radius of gyration : When equation 2.11.2 is compared with $I = MK^2$

$$K^2 = \frac{l^2}{12}$$

$$\therefore \ K = \frac{l}{\sqrt{12}}$$

Illustration 7 : Find the moment of inertia I about an axis passing through the end and perpendicular to the length of a rod of uniform cross section having mass M and length *l* and also find radius of gyration.

Solution : Suppose the mass of the rod is M. The distance of the end from the centre of the rod is $d = l/2$. According to equation (2.11.2) the moment of inertia of this rod about an axis passing through its centre and perpendicular to its length is $I_c = \frac{Ml^2}{12}$

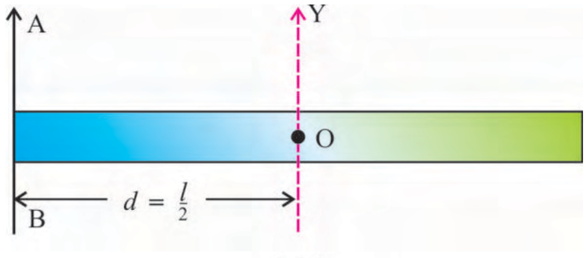


Figure 2.16

Using theorem of parallel axis, moment of inertia of the rod about an axis passing through its end and perpendicular to the length is

$$I = I_c + Md^2$$

$$= \frac{Ml^2}{12} + \frac{Ml^2}{4} \qquad (d = l/2)$$

$$\therefore I = \frac{Ml^2}{3}$$

Now comparing with $I = MK^2$

$$K^2 = \frac{l^2}{3} \quad \therefore K = \frac{l}{\sqrt{3}}$$

Illustration 8 : Find the moment of inertia of a uniform circular disc about an axis passing through its geometrical centre and perpendicular to its plane and radius of gyration :

Solution :

Consider a uniform circular disc with mass M and radius R as shown in the Figure 2.17. We will find moment of inertia of this disc about the axis zz' passing through its geometrical centre and perpendicular to its plane.

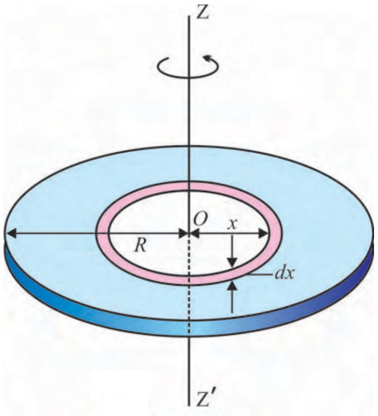


Figure 2.17

Here, area of the disc $A = \pi R^2$ and mass per unit area of the disc

$$\sigma = \frac{\text{Mass of the disc}}{\text{Area of the disc}} = \frac{M}{\pi R^2}$$

Let us imagine this disc consisting of several concentric rings with different radii and their centre is O as shown in the Figure 2.17.

Let us consider one of such rings with radius x and width dx as shown in the figure.

Area of this ring $a = 2\pi x \cdot dx$ and

mass $m = \sigma \cdot a = \frac{M}{\pi R^2}(2\pi x \cdot dx) = \frac{2Mx}{R^2} dx.$

If dI is the moment of inertia of this ring about the axis zz' then

$$dI = (\text{mass of the ring})(\text{radius of the ring})^2$$

$$= \frac{2Mx}{R^2} dx \cdot x^2 \qquad (1)$$

The total of the moment of inertia of such concentric rings with different radii gives the moment of inertia of the disc as a whole about the zz' axis.

For this integrating equation (1) in the interval $x = 0$ to $x = R$

$$I = \int dI = \int_0^R \frac{2Mx^3}{R^2} \cdot dx$$

$$I = \frac{2M}{R^2}\int_0^R x^3 \cdot dx$$

$$= \frac{2M}{R^2}\left[\frac{x^4}{4}\right]_0^R$$

$$= \frac{2M}{R^2}\left[\frac{R^4}{4} - 0\right]$$

$$\therefore I = \frac{1}{2}MR^2 \qquad (2)$$

Now comparing the equation (2) with $I = MK^2$

$$K^2 = \frac{1}{2}R^2$$

Radius of gyration $K = \frac{R}{\sqrt{2}}$

Illustration 9 : Find the moment of inertia of a disc of uniform density about an axis coinciding with its diameter.

Solution :

Suppose the mass of disc is M and radius is *r*. Z axis is perpendicular to the plane of the disc and passing through its centre. From table 1, the moment of inertia about this axis is

$$I_z = \frac{MR^2}{2}.$$

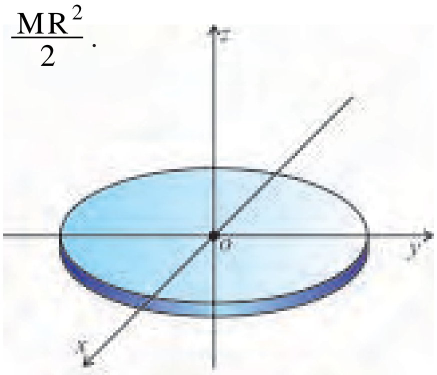


Figure 2.18

If the moment of inertia of the disc about X and Y axis are I_x and I_y respectively, according to the theorem of perpendicular axis (See Figure 2.18).

$$I_z = I_x + I_y$$

The disc is symmetric about *x* and *y* axes

$$\therefore\ I_x = I_y \qquad \therefore\ I_z = 2I_x$$

Moreover $I_z = \dfrac{MR^2}{2}$

$$\therefore\ \frac{MR^2}{2} = 2I_x$$

$$\therefore\ I_x = \frac{MR^2}{4}$$

Illustration 10 : The mass of a hollow cylinder is 4 kg and its radius is 0.1 m. It is capable of rotating about its geometrical axis. By winding a thin string around the cylinder a force of 50 N is applied at the free end of the string, tangentially to the cylinder surface. So it starts rotating. Find the answers to the following questions :

(1) torque acting on the cylinder (2) angular acceleration of the cylinder (3) angular velocity at the end of 4 s. (4) angular momentum at the end of 4 s. (5) rotational kinetic energy at the end of 4 s. (6) angular displacement during 4 s. (7) work done on the cylinder during 4 s. (8) power at the end of 4 s.

Solution :

(1) The torque on the cylinder :

$$\vec{\tau} = \vec{r} \times \vec{F} = rF \sin\theta\ \hat{n}$$

$$\therefore |\vec{\tau}| = rF \qquad (\because\ \theta = \frac{\pi}{2})$$

$$= (0.1)\ (50) = 5\ \text{N m}$$

(2) Angular acceleration of the cylinder (α) :

Here $\tau = I\alpha = mr^2\alpha$

$$\therefore\ 5 = (4)\ (0{\cdot}1)^2\ (\alpha) = 0.04\ \alpha$$

$$\therefore\ \alpha = 125\ \text{rad s}^{-2}$$

(3) Angular velocity (ω) :

$$\omega = \omega_0 + \alpha t = 0 + (125)\ (4)$$

$$= 5000\ \text{rad s}^{-1}$$

(4) Angular momentum (L) :

$$L = I\omega = mr^2\omega$$

$$\therefore\ L = (4)\ (0{\cdot}1)^2\ (500) = (0{\cdot}04)\ (500)$$

$$= 20\ \text{kg}\ m^2\ \text{s}^{-1}$$

(5) Rotational kinetic energy (E) :

$$E = \frac{1}{2}I\omega^2 = \frac{1}{2}mr^2\ \omega^2$$

$$= \frac{1}{2}(4)\ (0{\cdot}1)^2\ (500)^2 = 5000\ \text{J}$$

(6) Angular displacement in 4 s :

$$\theta = \left[\frac{\omega_0+\omega}{2}\right]t = \left[\frac{0+500}{2}\right]4$$

$$= 1000\ \text{rad}$$

(7) Work done in 4 s W = the kinetic energy gained by the cylinder in this time = 5000 J

or work $W = \tau\theta = 5 \times 1000 = 5000$ J

(8) Power at the end of 4 s is

$$P = \tau\omega = 5 \times 500 = 2500\ \text{watt}$$

Table 2.1 : Moment of inertia and radius of gyration for some symmetric bodies

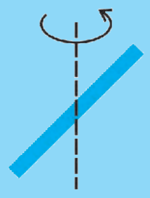
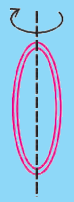
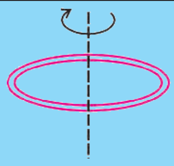
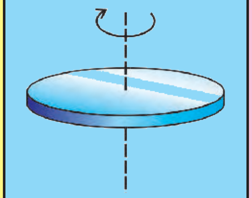
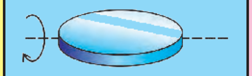
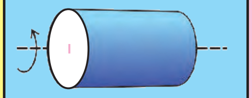
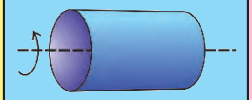
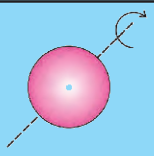
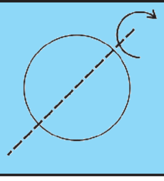
Body	Axis	Figure	I	K
Thin rod of Length L	Passing through its centre and perpendicular to its length		$\frac{1}{12}ML^2$	$\frac{L}{2\sqrt{3}}$
Ring of radius R	Any diameter		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
Ring of radius R	Passing through its centre and perpendicular to its plane		MR^2	R
Circular disc of radius R	Passing through its centre and perpendicular to its plane		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
Circular disc of radius R	Any diameter		$\frac{1}{4}MR^2$	$\frac{R}{2}$
Hollow cylinder of radius R	Geometrical axis of the cylinder		MR^2	R
Solid cylinder of radius R	Geometrical axis of the cylinder		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
Solid sphere of radius R	Any diameter		$\frac{2}{5}MR^2$	$\sqrt{\frac{2}{5}}R$
Hollow sphere of radius R	Any diameter		$\frac{2}{3}MR^2$	$\sqrt{\frac{2}{3}}R$

Table 2.2 : Comparison between physical quantities of linear motion and rotational motion

Translational motion	Rotational motion
Linear displacement, $\vec{d}$	Angular displacement, θ
Linear velocity, $\vec{v}$	Angular velocity, $\vec{\omega}$
Linear acceleration, $\vec{a} = \frac{d\vec{v}}{dt}$	Angular acceleration, $\vec{\alpha} = \frac{d\vec{\omega}}{dt}$
Mass, m	Moment of inertia, I
Linear Momentum, $\vec{p} = m\vec{v}$	Angular momentum, $\vec{L} = I\vec{\omega}$
Force, $\vec{F} = ma$	Torque, $\vec{\tau} = I\vec{\alpha}$
Newton's Second Law of Motion, $\vec{F} = \frac{d\vec{p}}{dt}$	A result similar to Newton's Second Law, $\vec{\tau} = \frac{d\vec{L}}{dt}$
Translational kinetic energy $K = \frac{1}{2}mv^2$	Rotational kinetic energy $K = \frac{1}{2}I\omega^2$
Work, $W = \vec{F}\cdot\vec{d}$	Work, $W = \tau\theta$
Power, $P = Fv$	Power, $P = \tau\omega$
Equations of linear motion taking place with constant linear acceleration $v = v_0 + at$ $d = v_0t + \frac{1}{2}at^2$ $2ad = v^2 - v_0^2$	Equations of rotational motion taking place with constant angular acceleration : $\omega = \omega_0 + \alpha t$ $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ $2\alpha\theta = \omega^2 - \omega_0^2$

Illustration 11 : A circular turn-tabale rotates in the horizontal plane with an angular speed of 20 rpm about a vertical axis passing through its centre. A man of 60 kg mass is standing on the edge of this table. If the man goes from the edge to the centre, what would be the angular speed of the turn-table now ? Consider the mass as a point particle and the turn-table as a uniform disc. The mass of the turn-table is 200 kg.

Solution : mass of man m = 60 kg, mass of turntable m=200 kg, W= 20rpm Here, the external torque on the system is zero. hence its angular momentum remains constant.

∴ initial angular momentum of (turn-table + man) = their final angular momentum

$$\left(\frac{MR^2}{2} + mR^2\right)\omega_1 = \frac{MR^2}{2}\omega_2$$

$$\therefore \left(\frac{M}{2} + m\right)\omega_1 = \frac{M}{2}\omega_2$$

$$\therefore (100 + 60)(20) = 100\omega_2$$

$$\therefore \omega_2 = 32 \ rpm$$

Note : In this illustration the final kinetic energy will be found to be more than the initial kinetic energy. The increase in the kinetic energy is the work done by the man in going from the edge to the centre. To calculate this take the radius of the turn-table R = 1·5 m.

Illustration 12 : A child of mass m is sitting on the board of the merry-go-round rotating about an axis passing through its centre and perpendicular to its plane, at 1 m away from the axis. With what angular velocity the merry-go-round be rotated so that the child is on the verge of sliding on the board of it ? The coefficient of friction between the child and the surface of the board is 0.25. Take g = 10 m s^{-2}.

Solution : Different forces acting on the child at point P are shown in the Figure 2.19.

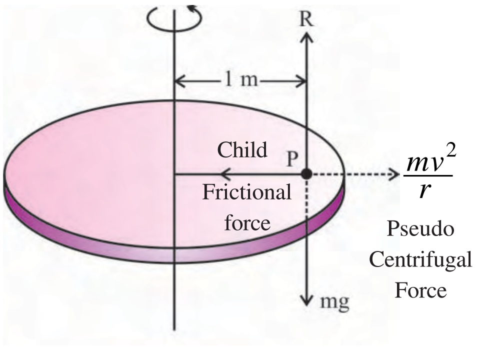


Figure 2.19

Here, R = normal reaction and $\frac{mv^2}{r}$ = centrifugal (fictitious) force. When the frictional force μR becomes $\frac{mv^2}{r}$, the child is on the verge of sliding on the board of the merry-go-round.

$$\frac{mv^2}{r} = \mu R = \mu mg$$

$$\therefore \; r^2\omega^2 = r\mu g \qquad (\because \; v = r\omega)$$

$$\therefore \; \omega = \sqrt{\frac{\mu g}{r}}$$

$$= \sqrt{\frac{0{\cdot}25\times10}{1}}$$

$$= 1{\cdot}58 \text{ rad s}^{-1}$$

Illustration 13 : A string is wound around a disc of rdius r and mass M and at the free end of the string a body of mass m is suspended. The body is then allowed to descend. Show that the angular acceleration of the disc is $\alpha = \frac{mg}{R\left(m+\frac{M}{2}\right)}$.

Solution :

The suspended body and the forces acting on the disc are shown in Figure 2.20.

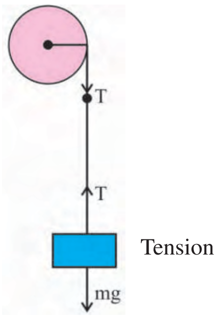


Figure 2.20

The equation of the linear motion of the suspended body is $ma = mg - T$ (Where T = tension in the string)

$$\therefore \; T = m\,(g - a)$$

Now the torque on the disc τ = RT

$$(\because \; \vec{\tau} = \vec{r} \times \vec{F})$$

$$\therefore \; I\,\alpha = R\,T \;\therefore\; \alpha = \frac{RT}{I} = \frac{Rm(g-a)}{I}$$

$$\therefore \; = \frac{Rm(g-a)}{MR^2/2} \;\therefore\; \alpha = \frac{2m}{RM}\,(g - a)$$

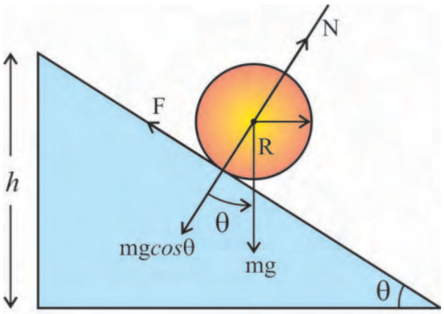
But $a = R\alpha$

$$\therefore \; \alpha = \frac{2mg}{RM} - \frac{2mR\alpha}{RM} = \frac{mg}{R\left(m+\frac{M}{2}\right)}$$

2.12 Rigid Bodies Rolling without Slipping

When a rigid body is rolling without slipping its motion is combination of linear (translational) motion and the rotational motion. The centre of mass of the rigid body performs translational motion and the body itself rotates about its own axis.

In the description of such combined motion, both of the above mentioned motions can be described independently.



Rigid body rolling without slipping

Figure 2.21

As shown in Figure (2.21) suppose a rigid body rolls down without slipping along an inclined plane of height h and angle θ. Here, the mass of the body is *m*, moment of inertia is I, geometrical radius is R and the radius of gyration is K. When the body reaches the bottom of the inclined plane, its potential energy decreases by mgh. According to the law of conservation of energy mechanical energy, this decrease in potential energy is converted as increase in the kinetic energy. Here, the kinetic energy of the body is $= \begin{pmatrix}\text{Translation}\\ \text{kinetic energy}\end{pmatrix} + \begin{pmatrix}\text{Rotational}\\ \text{kinetic energy}\end{pmatrix}$

$$= \frac{1}{2}mv^2 + \frac{1}{2}\mathrm{I}\omega^2$$

According to the law of conservation of mechanical energy,

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}\mathrm{I}\omega^2 \qquad (2.12.1)$$

Now using $\omega = v/\mathrm{R}$ and $\mathrm{I} = \mathrm{MK}^2$ in equation (2.12.1)

(**Note :** $\omega = v/\mathrm{R}$ equation is applicable only when the body is rolling without slipping. For the body rolling with slipping this equation cannot be applied.)

$$v^2 = \frac{2gh}{\left[1+\frac{\mathrm{K}^2}{\mathrm{R}^2}\right]} \qquad (2.12.2)$$

If the length of the slope is d, and the body starting from rest, moves with linear acceleration a to reach the bottom.

$\therefore\ v^2 = 2ad$

From the geometry of the figure,

$$d = \frac{h}{\sin\theta}$$

$$\therefore\ v^2 = \frac{2ah}{\sin\theta} \qquad (2.12.3)$$

Combining equations (2.11.2) and (2.12.3) we get,

$$a = \frac{g\sin\theta}{\left[1+\frac{\mathrm{K}^2}{\mathrm{R}^2}\right]} \qquad (2.12.4)$$

Here, the linear acceleration a is along the inclined plane, its value should be equal to the component of g along the inclined plane g sinθ. But according to the equation (2.12.4) its value is found as $\dfrac{g\sin\theta}{\left[1+\frac{\mathrm{K}^2}{\mathrm{R}^2}\right]}$

$\therefore$ decrease in linear acceleration,

$$= g\sin\theta - \frac{g\sin\theta}{\left[1+\frac{\mathrm{K}^2}{\mathrm{R}^2}\right]}$$

$$= g\sin\theta\left[\frac{\mathrm{K}^2}{\mathrm{K}^2+\mathrm{R}^2}\right]$$

This decrease in the linear acceleration is due to the frictional force F acting on the rolling body.

The work done against the frictional force, results in the rotational kinetic energy and hence only even in the presence of frictional force we have been able to use the law of conservation of mechanical energy.

Thus, the frictional force

$$\mathrm{F} = mg\sin\theta\left[\frac{\mathrm{K}^2}{\mathrm{K}^2+\mathrm{R}^2}\right] \qquad (2.12.5)$$

Now, as shown in the Figure 2.21, the normal reaction N and $mg\cos\theta$ balance each other, hence $\mathrm{N} = mg\cos\theta$ (2.22.6)

Dividing equation (2.11.5) by equation (2.11.6)

$$\frac{\mathrm{F}}{\mathrm{N}} = \left[\frac{\mathrm{K}^2}{\mathrm{K}^2+\mathrm{R}^2}\right]\tan\theta$$

But $\frac{F}{N} = \mu_s$ (coefficient of static friction)

$$\therefore \ \mu_s = \left[\frac{K^2}{K^2+R^2}\right] tan\theta$$

$$= \frac{1}{\left[1+\frac{R^2}{K^2}\right]} tan\theta \qquad (2.12.7)$$

Here, the line on the surface of the rolling body, which touches the inclined plane at a given instant, is stationary instantaneously and hence in the above equation (2.11.7), the static coefficient of friction is used.

Hence, from equation (2.12.7) we can say that if

$$\mu_s \geq \frac{1}{\left[1+\frac{R^2}{K^2}\right]} tan\theta \qquad (2.12.8)$$

condition is satisfied, then only the body can roll down the slope without slipping. **Special cases :**

(1) Thin ring :

From Table 2.1, for thin ring K = R

Substituting this value in equation (2.12.8)

$$\mu_s \geq \frac{1}{2} tan\theta \qquad (2.12.9)$$

(2) Circular disc : $K = \frac{R}{\sqrt{2}}$ (from Table 2.1)

Substituting this value in equation (2.12.8)

$$\mu_s \geq \frac{1}{3} tan\theta \qquad (2.12.10)$$

(3) Solid sphere : $K = \sqrt{\frac{2}{5}} R$ (from Table 2.1)

Substituting this value in equation (2.12.8)

$$\mu_s \geq \frac{2}{7} tan\theta \qquad (2.12.11)$$

SUMMARY

1. **Rigid Body :** The system of particles in which the relative distance between the paricles remain invariant, is called rigid body.

   Rotational kinematics : A branch of physics in which the rotational motion is described without mentioning its causes, is called the rotational kinematics.

   Rotational Dynamics : A branch of physics in which the rotational motion is described along with its causes and properties of the body, is called rotational dynamics.

2. **Angular Speed :** $\omega = \frac{d\theta}{dt}$. Its SI unit is rad s^{-1} or rotational s^{-1} scalar relation between angular velocity and linear velocity.

   $v = r\omega$

   Vector relation between angular velocity and linear velocity

   $\vec{v} = \vec{\omega} \times \vec{r}$

   Right Hand Screw Rule :

   A right hand screw is adjusted (kept) parallel to the rotational axis and is rotated in the same sense as the rotation of the body, the direction in which the screw shifts is taken as direction of angular velocity $\vec{\omega}$.

   Angular acceleration

   $\vec{\alpha} = \frac{d\vec{\omega}}{dt}$. Its SI unit is rad s^{-2} or rotation s^{-2}.

   Vector relation between linear acceleration $\vec{a}$ and angular acceleration.

   $\vec{a} = \vec{\omega} \times \vec{v} + \vec{\alpha} \times \vec{r} = \vec{a}_r + \vec{a}_T$

$\vec{\omega} \times \vec{v}$ is called radial component a_r of the linear acceleration.

$$a_r = \omega v = \frac{v^2}{r} = r\omega^2$$

$\vec{\alpha} \times \vec{r}$ is called tangential component a_T of the linear acceleration

$$a_T = \alpha r$$

Magnitude of linear acceleration

$$a = \sqrt{a^2{}_r + a^2{}_T} = \sqrt{\omega^2 v^2 + \alpha^2 r^2}$$

3. Resemblance between the equations of motions of rotational motion with constant angular acceleration and linear motion with constant linear acceleration.

Linear motion	**Rotational motion**
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x = v_0 t + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$x = \left(\frac{v + v_0}{2}\right)t$	$\theta = \left(\frac{\omega + \omega_0}{2}\right) t$
$x = \left(\frac{v^2 - v^2{}_0}{2a}\right)$	$\theta = \left(\frac{\omega^2 - \omega_0^2}{2\alpha}\right)$

4. Torque plays a similar role in rotational motion as the force plays in the linear motion.

Torque $\vec{\tau} = \vec{r} \times \vec{F}$ = Moment of force.

Its direction can be obtained by right hand screw rule.

Component of the torque $\vec{\tau} \cdot \hat{n}$ is responsible for the rotational motion. Where $\hat{n}$ is the unit vector along the stationary axis of rotation.

Torque is the measure of the effectiveness of the force in producing rotational motion.

Moment of couple = (magnitude of any one of the forces) (perpendicular distance between the two forces)

If the forces acting on the rigid body are

$\vec{F}_1, \vec{F}_2$ ..... $\vec{F}_n$ and if $\vec{F}_1 + \vec{F}_2 + \dots\dots\dots + \vec{F}_n = 0$,

then rigid body remains in translational equilibrium.

If the torques produced by the above mentioned force are $\vec{\tau}_1, \vec{\tau}_2, \dots\dots \vec{\tau}_n$ and $\vec{\tau}_1 + \vec{\tau}_2 + \dots\dots + \vec{\tau}_n = 0$, then the rigid body remains in rotational equilibrium.

5. Moment of linear momentum is called angular momentum.

angular momentum $\vec{l} = \vec{r} \times \vec{p}$

Time rate of change of angular momentum gives torque $\frac{d\vec{l}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$

Torque acting on the system of particles $\frac{d\vec{l}}{dt} = \vec{\tau}$

For rigid body $\vec{L} = I\vec{\omega}$

Where I is the moment of inertia. $I = m_1r^2_1 + m_2r^2_2 + \ldots\ldots\ldots + m_nr^2_n$

$$\frac{d\vec{l}}{dt} = I\vec{\alpha} = \vec{\tau}$$

6. Law of Conservation of Angular Momentum.

"If the resultant torque acting on the rigid body is zero, the angular momentum of the rigid body remains constant."

$$\frac{d\vec{l}}{dt} = 0 \Rightarrow \vec{L} = \text{constant}$$

7. Kepler's Second Law for planetary motion can be obtained from the physical representation of the law of conservation of angular momentum which is as under.

"The area swept by the line joining the sun and the planet in unit time (which is called areal velocity) is constant."

i.e. $\frac{dA}{dt} = \text{constant}$. here $\frac{dA}{dt}$ is called areal velocity.

8. In general for rigid body $I = MK^2$

where $K = \sqrt{\frac{r_1^2 + r_2^2 + \ldots\ldots + r_n^2}{n}}$. Here, K is called radius of gyration.

9. Theorem of parallel axis for moment of inertia

$I = I_C + Md^2$ where I_C is the moment of inertia about the axis passing through the centre of mass and M is the mass of body and I is moment of inertia about the axis parallel to the axis passing through the centre of mass and located at a perpendicular distance (*d*) between the two axis.

Theorem of perpendicular axis for moment of inertia

If I_x, I_y and I_z are the moment of inertia about X, Y and Z axis respectively, then $I_z = I_x + I_y$

10. Condition for the body to roll without slipping.

$$\mu_s \geq \left[\frac{1}{1+\frac{R^2}{K^2}}\right] \tan\theta$$

Expressions for the linear velocity and linear acceleration for the body rolling on the slope without slipping are

$$v = \left[\frac{2gh}{1+\frac{K^2}{R^2}}\right]^{\frac{1}{2}} \text{ and } a = \left[\frac{g\sin\theta}{\left(1+\frac{K^2}{R^2}\right)}\right] \text{ respectively.}$$

EXERCISES

Choose the correct option from the given options :

1. If the angular speed of a particle 10 cm away from the axis of rotation of a rigid body is 12 rad s^{-1}, what will be the angular speed of a particle 20 cm away from the axis of rotation ?

(A) 2 rad s^{-1} (B) 15 rad s^{-1} (C) 12 rad s^{-1} (D) 10 rad s^{-1}

2. The angular speed of a particle 10 cm away from the axis of rotation is 20 rad s^{-1}. What will be its linear speed ?

(A) 1 cm s^{-1} (B) 20 cm s^{-1} (C) 200 cm s^{-1} (D) 400 cm s^{-1}

3. What is the angular speed of the minute hand of a clock ?

(A) $\frac{\pi}{43200}$ rad s^{-1} (B) $\frac{\pi}{1800}$ rad s^{-1}

(C) $\frac{\pi}{6}$ rad s^{-1} (D) $\frac{\pi}{12}$ rad s^{-1}

4. A wheel initially at rest acquires an angular velocity of 64 rad s^{-1} in 4 s. Hence its constant angular acceleration is ....

(A) 64 rad s^{-2} (B) 128 rad s^{-2} (C) 16 rad s^{-2} (D) 4 rad s^{-2}

5. An artificial satellite orbiting round the Earth has mass of 500 kg. What will be its areal velocity if its angular momentum is 4×10^7 J s ?

(A) 2×10^4 m^2 s^{-1} (B) 0

(C) 2×10^7 m^2 s^{-1} (D) 4×10^4 m^2 s^{-1}

6. Suppose the Earth suddenly contracts and its radius becomes $\frac{R}{4}$ (R = present radius of the Earth) keeping its mass same. Then what will be the length of the day ?

(A) 1.5 *h* (B) 6 *h* (C) 48 *h* (D) 36 *h*

7. There are two identical eggs. One is raw and other is boiled and both are rotated with same angular speed. Which one will come to rest earlier ?

(A) Can't say anything.

(B) both will come to rest simultaneously

(C) boiled

(D) raw

8. A hollow cylinder and a solid sphere have same mass and the same radius. Both are rotated by applying equal torques for the same time interval. The cylinder rotates about its diameter. Which one of them will have greater angular speed ?

(A) nothing can be said

(B) both will have same angular speed

(C) Cylinder

(D) Sphere

9. A plane is inclined at an angle 30°. A solid cylinder is kept at the top of the plane. If the co-efficient of static friction between surfaces of inclined plane and cylinder is 0.35, can cylinder roll without slipping on the inclined plan ?

(A) Cylinder will remain stationary on the inclined plane

(B) nothing can be said

(C) Yes

(D) No

10. A cylinder is rolling down the inclined plane without slipping. On which factor does the velocity of a cylinder at the bottom of an inclined plane depend ?

(A) mass of the cylinder (B) length of the cylinder

(C) height of the inclined plane (D) radius of the cylinder

11. The mass and radius of a circular disc are 4 kg and 2 m respectively. The moment of inertia of the disc about axis passing through its centre and perpendicular to its plane is....

(A) 24 kg m^2 (B) 8 kg m^2 (C) 16 kg m^2 (D) 11 kg m^2

12. What will be the effect on the length (24 hours) of a day, if snow on the poles of the Earth melts and water comes at the equator ?

(A) day becomes shorter

(B) day becomes longer

(C) no change in the length of the day

(D) length of the day and night will become same.

13. If torque acting on rigid body is zero, then which of the following will remain constant.

(A) Linear momentum (B) Angular momentum

(C) Force (D) Impulse of force

14. A fly wheel starts rotating from rest and acquires rotational speed of 240 revolution s^{-1} in 4 minutes. The average angular acceleration is ....

(A) 1 revolution s^{-2} (B) 3 revolution s^{-2}

(C) 4 revolution s^{-2} (D) 2 revolution s^{-2}

15. Two identical spheres are rolling down the slope. One is solid and other is hollow, the ratio of moment of inertia of solid (axis of rotation is diameter) to that of the hollow is ....

(A) $\frac{1}{3}$ (B) $\frac{3}{5}$ (C) $\frac{2}{3}$ (D) $\frac{2}{5}$

16. If axis of rotation of two identical cylinders, one solid and other hollow, is taken as their geometrical axis. Then the ratio of radius of gyration of solid one to that of hollow is ....

(A) $\frac{1}{2}$ (B) $\frac{1}{\sqrt{2}}$ (C) 2 (D) $\sqrt{2}$

17. A thin ring of mass M and radius R is rotating about axis passing through its centre and perpendicular to its plane with angular velocity ω. Two point objects of mass m are attached gently to the opposite ends of its any diameter, then angular velocity of the ring will be

(A) $\left(\frac{M}{M+2m}\right)\omega$ (B) $\left(\frac{M}{M+m}\right)\omega$

(C) $\left(\frac{M+2m}{M}\right)\omega$ (D) $\left(\frac{M-2m}{M+2m}\right)\omega$

18. A ring of radius r and mass m rotates about the axis passing through the centre and perpendicular to its plane. The kinetic energy is ....

(A) $\frac{1}{2}mr^2\omega^2$ (B) $\frac{1}{2}mr\omega^2$ (C) $mr^2\omega^2$ (D) $mr\omega^2$

19. The ratio of magnitude of the orbital angular velocity of geostationary satellite to that of earth's rotation about its own axis is

(A) 3 : 1 (B) 4 : 3 (C) 1 : 1 (D) 1 : 2

20. Areal velocity of a planet rotating round the sun

(A) keeps on increasing (B) remains constant

(C) keeps on decreasing (D) nothing can be said

ANSWERS

1.	(C)	**2.**	(C)	**3.**	(B)	**4.**	(C)	**5.**	(D)	**6.**	(A)
7.	(C)	**8.**	(D)	**9.**	(C)	**10.**	(C)	**11.**	(B)	**12.**	(B)
13.	(B)	**14.**	(A)	**15.**	(B)	**16.**	(B)	**17.**	(A)	**18.**	(A)
19.	(C)	**20.**	(B)								

Answer the following questions in short :

1. Give SI units of angular velocity and angular acceleration.

2. What is the value of the tangential component of linear acceleration of the representative particle of the rigid body rotating with constant angular velocity ?

3. In the rotational motion of the rigid body the angular variables of all particles are equal ?

4. Which physical quantity does play the same role in rotational motion as the force plays in the linear motion ?

5. How is the direction of torque is determined ?

6. Which component of the torque is responsible for rotational motion about Z axis ?

7. What is the effectiveness of force producing the rotational motion known as ?

8. Give the formula for the moment of couple.

9. What is the moment of linear momentum known as ?

10. What does the time-rate of change of angular momentum indicate ?

11. State the law of conservation of angular momentum.

12. What is the time rate of the area swept out by the line joining the planet with the sun known as ?

13. State theorem of parallel axes for the moment of inertia.

14. State the theorem of perpendicular axes for the moment of inertia.

15. Write the condition in the form of the formula for a body to roll down along slope without slipping.

Answer the following questions :

1. Define angular displacement of rigid body and obtain formula for instantaneous angular speed.
2. Obtain the relation between the linear speed and the angular speed for a representative particle of a rigid body.
3. Explain the right hand screw rule for the direction of angular velocity and establish the vector relation between the linear velocity and the angular velocity.
4. Obtain the relation between the linear acceleration and the angular acceleration for a representative particle of a rigid body.
5. Derive the equations for the rotational motion with constant angular acceleration.
6. State the conditions for the equilibrium of a rigid body.
7. Give the physical explanation of the definition of torque.
8. What is couple ? Obtain the formula for the moment of couple.
9. Obtain the relation between angular momentum and torque.
10. Obtain formula $\vec{L} = I\vec{\omega}$ for the angular momentum of the rigid body.
11. Obtain the formula $v^2 = \left[\frac{2gh}{1 + \frac{K^2}{R^2}}\right]$ for the rigid body rolling without slipping on an inclined plane of angle θ.
12. Assuming the velocity of the body (rolling without slipping) reaching the bottom of the slope of length d is $v = \sqrt{\frac{2gh}{1 + \frac{K^2}{R^2}}}$, obtain formula for its linear acceleration and frictional force.
13. Assuming the linear acceleration of the body rolling on the slope without slipping is $a = \frac{g \sin\theta}{1 + \frac{K^2}{R^2}}$, obtain the formula for static coefficient of friction.

Solve the following problems :

1. A rigid body acquires angular speed of 100 rad s^{-1} after undergoing angular displacement of 600 rad in 12 s. Find its constant angular acceleration and initial angular speed.

[**Ans. :** 8.33 rad s^{-1}; 0 rad s^{-1}]

2. Initial angular speed of a wheel is 20 rad s^{-1}. Its angular displacement in 10 s is 100 rad. How many rotations will it make from the beginning to the time till it stops ? Find its angular acceleration also.

[**Ans. :** $\theta = \frac{50}{\pi}$ rotations; $\alpha = -2$ rad s^{-2}]

3. A ring of mass 20 kg and radius 1 *m*, is rotating about the axis passing throught its centre and perpendicular to its plane. The angular speed of the ring changes from 5 rad s^{-1} to 25 rad s^{-1} in 4 s. Find (1) the magnitude of the torque acting on it (2) work done by the torque during 4 s.

[**Ans. :** τ = 100 N *m*; W = 6000 J]

4. The velocity vector of a particle is (2, 3, 6) unit when its position vector is (4, 6, 12) unit. Find the angular momentum of the particle. Mass of the particle is 50 unit.

[**Ans. :** zero]

5. A hollow cylinder rolls (about its geometrical axis) without slipping on an inclined plane of angle θ. Find its linear acceleration in the direction parallel to the surface of the inclined plane.

[**Ans. :** 0.5 $g\sin\theta$)]

6. The position vector of two point-like objects of masses 100 kg and 200 kg are (2, 4, 6) m and (3, 5, 7) m respectively. Find the moment of inertia of this system about z–axis.

[**Ans. :** 8800 kg m^2]

7. Mass of a solid sphere is 8 kg. Find its linear velocity at the bottom of an inclined plane of height 70 m (after rolling down without slipping from the top of the inclined plane). Also find its rotational kinetic energy at the bottom of the plane. (Take g = 10 m s^{-2})

[**Ans. :** $v = 10\sqrt{10}$ m s^{-1}; rotational K.E. = 16 × 10^2 J]

8. Find the angular momentum of earth due to earth's rotational motion about its own axis. Mass of the Earth = 6 × 10^{24} kg and radius of the Earth = 6400 km [**Ans. :** 7.15 × 10^{33} kg m^2 s^{-1}]

9. Moment of inertia of a body about an axis 3 m from its centre of mass is 8200 kg m^2. Find the moment of inertia of this body about the axis 5 m away from its centre of mass and parallel to the above mentioned axis. Mass of the body is 200 kg.

[**Ans. :** 11400 kg m^2]

10. Four point objects of equal mass m are placed at the corners of a square with side '*a*'. Find the moment of inertia of the system about the axis passing through the centre of the square and perpendicular to its plane.

[**Ans. :** 2 ma^2]

11. Four spheres each of mass of M and radius R are placed at the corners of a square having side '*a*'. Find the moment of inertia of the system about an axis along one of the sides of square. [**Ans.** :$2\left(\frac{4}{5}MR^2+Ma^2\right)$]

12. Four point like masses of 1 kg, 2 kg, 3 kg and 4 kg are attached to a rod of negligible mass as shown in the figure. Calculate the moment of inertia of the system about axis AB.

[**Ans. :** 200 kg m^2]

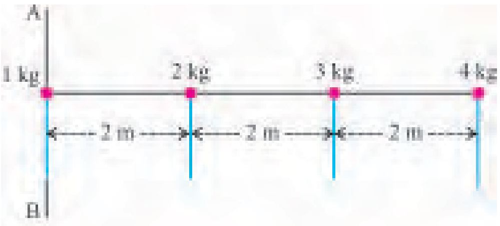


Figure 2.22

13. A disc rolls without slipping with constant velocity. What fraction of total kinetic energy of the disc is in the form of rotational kinetic energy ?

[**Ans.** : $\frac{1}{3}$]

14. Prove that the moment of inertia of a uniform circular ring of mass M and radius R about its geometrical axis, is MR^2.

15. Figure 2.25 shows the forces acting on a light rod. Write the formula for the resultant force. At what distance from A, will this resultant force act ?

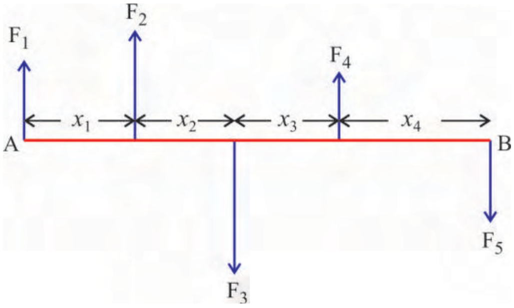


Figure 2.23

[**Ans.** : $\vec{F} = F_1(\hat{j}) + F_2(\hat{j}) + F_3(-\hat{j}) + F_4(\hat{j}) + F_5(-\hat{j})$

$$x = \frac{x_1F_2 - (x_1+x_2)F_3 + (x_1+x_2+x_3)F_4 - (x_1+x_2+x_3+x_4)F_5}{F_1+F_2+F_4-F_3-F_5}$$]

●

Sir Jagdish Chandra Bose (1858-1937)

Jagdish Chandra Bose was born in Bengal in November 30, 1858. He got his B.A degree from Cambridge University and a B.Sc. from the London University. He did experiments involving refraction, diffraction and polarization. He did his original scientific work in the area of Microwaves. He produced extremely short waves and did considerable improvement upon Hertz's detector of electric waves. He produced a compact apparatus for generating electromagnetic waves of wavelengths 25 to 5 mm. Bose turned his attention on response of electromagnetic waves on plants by the end of the 19th century. He was appointed Professor Emeritus after he retired from the Presidency College in 1915. He was also elected Fellow of the Royal Society in 1920. On November 23, 1937 Jagdish Chandra Bose passed away at Giridih in Bihar.

CHAPTER 3

GRAVITATION

3.1 Introduction

The stars in the sky and the planets revolving around the sun have been attracting the attention of the scientists since ancient time.

First scientific study of the solar system was carried out by the Greeks. The principle of Greek astronomy proposed by Ptolemy, nearly 2000 years ago, is known as **geo-centric theory**.

According to this theory the Earth is stationary at the centre of the universe and all celestial bodies – stars, sun, planets all of them-are revolving around the Earth. Ptolemy proposed their motions to be circular. According to him the planets move on circular paths and the centres of those circles move on larger circles. But Aryabhatt in the fifth century, proposed a theory that all planets revolve on the circles with the sun at the centre.

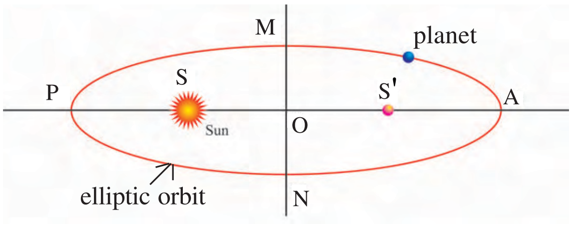
Then, almost one thousand years later Nicolaus Copernicus (1473–1543) of Poland proposed a **definitive model** about the planets revolving on perfect circles with the sun at the centre. This is known as **helio-centric theory.** Thus it was a support to the theory of Arybhatt. Copernicus model was not accepted by the recognised institutions of that time. But Galileo supported his theory.

Tyco Brahe (1546–1601) of Denmark had accumulated many observations, about planetary motion by direct eye, during his life-time. These observations were studied by Johannes Kepler (1571–1640) who gave three laws of planetary motion. They are known as Kepler's laws. In this chapter we will study these laws, Newton's Law of Gravitation and the satellites.

3.2 Kelpler's Laws

From the study of the observations recorded by Tycho Brahe, Johannes Kelper gave three laws of planetary motion. They are called Kepler's laws. They are as follows.

First Law (Law of Orbits) : "All the planets move in the elliptical orbits with the sun situated at one of the foci."



Elliptic orbit of the planet

Figure 3.1

PA = 2a, MN = 2b

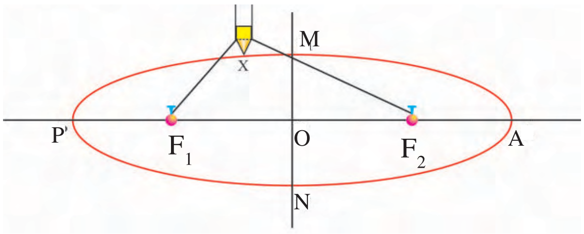
OP = OA = a = semi-major axis

In the Figure 3.1 the ellipse PNAM showing the path of a planet has two foci S and S'.

This law of orbit suggests different shapes from the circular orbits suggested by Copernicus.

[**Only for information :** An ellipse can be drawn as under :

Keep the ends of a string of length l fixed at points F_1 and F_2, where $F_1F_2 < l$. Now keep the tip of a pencil with the string and move it such that the string remains tight. The curve PNAM obtained in this way is an ellipse as in Figure 3.2.



Ellipse can thus be drawn

Figure 3.2

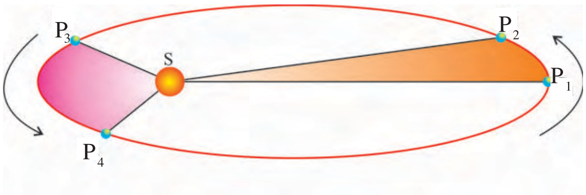
OP = a = OA

OM = b = ON

Here, $F_1X + F_2X$ = constant. It shows the characteristic of an ellipse.

Moreover, if $a = b$, the ellipse becomes a circle.]

Second Law (Law of Areas) : "The line joining the Sun and the planet sweeps equal areas in equal intervals of time." (See Figure 3.3).



Areal velocity is constant

Figure 3.3

When the planet is away from the sun, it goes from P_1 to P_2 in certain time-interval Δt and when it is near the sun it goes from P_3 to P_4 in the same time-interval.

Hence, according to this law, area of SP_1P_2 = area of SP_3P_4.

This law has been obtained from the obervation that a planet moves slower in the orbit when it is far away from the sun and it moves faster when it is near to the sun.

We can call the area swept in unit time as the areal velocity (= area / time) and this law indicates that the areal velocity is constant. You have already seen this aspect in Chapter 2.

Third Law (Law of Periods) : "The square of the time-period (T) of the revolution of a planet is proportional to the cube of the semi-major axis (a) of its elliptical orbit." That is, $T^2 \propto a^3$.

The **time-period (T)** means the time required to complete one revolution. It is also called the **period** or the **periodic time.**

From the examples of a few planets given in the following table you can see that T^2/a^3 = constant and hence $T^2 \propto a^3$.

Table 3.1 : (Values of T^2/a^3 for a few planets) (This table is only for information.)

Planet	a m	T year	T^2/a^3 year^2/m^3
Mercury	5.79×10^{10}	0.24	2.95×10^{-34}
Earth	15×10^{10}	1.0	2.96×10^{-34}
Mars	22.8×10^{10}	1.88	2.98×10^{-34}
Saturn	143×10^{10}	29.5	2.98×10^{-34}

[Discovery of Gravitation-Only for Information :

Newton saw an apple falling down

Figure 3.4

According to a legend, Newton sitting under a tree, saw an apple falling down. He (instead of eating it !) got involved in deep thinking as to "Why it fell downward only." As a result of such thinking Newton discoverd the law of gravitation. His line of thinking was somewhat like the following : (i) The gravitational acceleration near the Earth's surface was already known to be 9.8 m/s^2. So the acceleration of apple is $a_{\text{apple}} = 9.8$ m/s^2. (ii) The acceleration of the moon revolving circularly around the Earth is $a_{\text{moon}} = v^2/r_m$ towards the centre of the Earth, where r_m = radius of moon's orbit = 3.84×10^5 km. The time period of revolution of moon around the Earth is T_m = 27.3 day. From this $v = 2\pi\, r_m/T_m$ can be found and by putting it in the above equation it is found that $a_{\text{moon}} = 0.0027$ m/s^2.

$$\therefore \frac{a_{apple}}{a_{moon}} = \frac{9.8}{0.0027} = 3600 \qquad (1)$$

Moreover the ratio of their distances from the centre of the Earth is,

$$\frac{r_{apple}}{r_{moon}} = \frac{6400\ km}{3.84 \times 10^5\ km} = \frac{1}{60} \qquad (2)$$

Where r_{apple} = distance equal to Earth's radius. From results (1) and (2) Newton found that the acceleration of a body is inversely proportional to the square of the distance from the centre of the Earth, ($a \propto \frac{1}{r^2}$). Hence the force by the Earth on the body of mass m is $\propto \frac{m}{r^2}$.

But according to Newton's third law of motion this body also exerts the same force on the Earth in opposite direction. Hence the value of force would also be proportional to the mass of the Earth (M).

Thus we get $F \propto \frac{Mm}{r^2}$ or $F = \frac{GMm}{r^2}$.

where G = constant.

This great scientific discovery is based on Newton's revolutionary idea. Newton believed that **the laws of nature are the same for the terrestrial bodies and for the celestial bodies.**

Hence, the force between the Earth and the apple and the force between the Earth and the moon must be governed by the same law. Today we may feel this statement to be quite obvious but in those days it was believed that the laws for terrestrial bodies are different from the laws for celestial bodies. Hence Newton's idea was indeed revolutionary.]

3.3 Newton's Universal Law of Gravitation

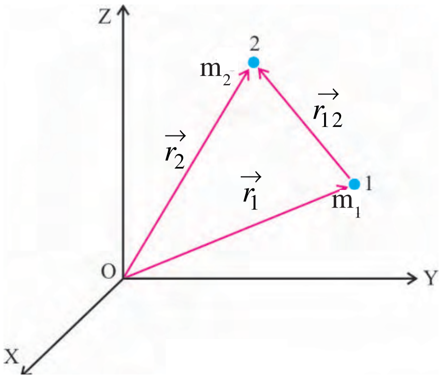
Newton's universal law of gravitation is as follows : **"Every particle in the universe attracts every other particle with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them."** The direction of this force is along the line joining them. This force is called the gravitational force.

According to this law, the magnitude of the force on the **particle 1** of mass m_1, by the other particle 2 of mass m_2, lying at distance r from it is

$$|\vec{F}_{12}| = \frac{G m_1 m_2}{r^2} \qquad (3.3.1)$$

The direction of this force is from particle 1 to the particle 2 (in the direction of $\vec{r}_{12}$), (See Figure 3.5).

Here, G is a constant and it is called the universal constant of gravitation, because its value is the same at all places at all times in the whole of the universe. The value of G was first determined by Cavendish experimentally. Thereafter many other scientists also have determined its value more precisely. At present the accepted value of G is 6.67×10^{-11} N m^2/kg^2. The dimensional formula for G is $M^{-1} L^3 T^{-2}$. In order to write the equation (3.3.1) in the vector form consider Figure 3.5.



To obtain formula for gravitational force in the vector form

Figure 3.5

From the figure,

$$\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$$

$$\hat{r}_{12} = \frac{\vec{r}_{12}}{|\vec{r}_{12}|} = \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_{12}|}$$

$$= \frac{\vec{r}_2 - \vec{r}_1}{r} \qquad (3.3.2)$$

Here, $r = |\vec{r}_{12}|$

It is clear from the figure, that

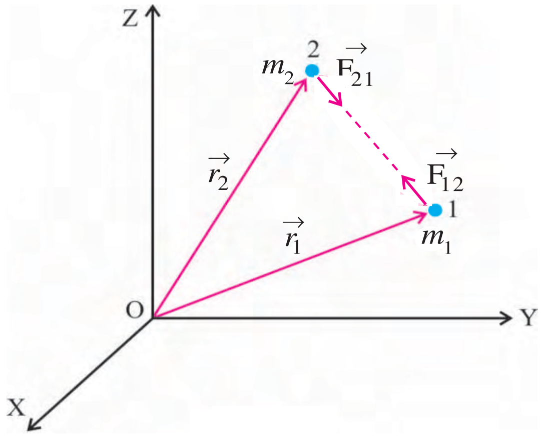
$$\overset{\vec{F}_{12}}{\begin{bmatrix}\text{Force on}\\ 1 \text{ by } 2\end{bmatrix}} = \frac{G m_1 m_2}{r^2}\hat{r}_{12} \qquad (3.3.3)$$

Since the gravitational forces are mutually interactive forces, the force exerted on particle 1 by particle 2, $\left(\vec{F}_{12}\right)$ is the **same in magnitude** and in opposite direction to the force exerted on particle 2 by particle 1, $\left(\vec{F}_{21}\right)$.

$$\therefore \overset{\vec{F}_{21}}{\begin{bmatrix}\text{Force on}\\ 2 \text{ by } 1\end{bmatrix}} = \frac{-G m_1 m_2}{r^2}\hat{r}_{12}$$

$$= \frac{G m_1 m_2}{r^2}\hat{r}_{21} \qquad (3.3.4)$$

Both these forces $\vec{F}_{12}$ and $\vec{F}_{21}$ are shown in the Figure 3.6.



Mutual forces on two particles

Figure 3.6

Force due to an extended object : An extended object can be considered as a collection of point masses. (i.e. particles)

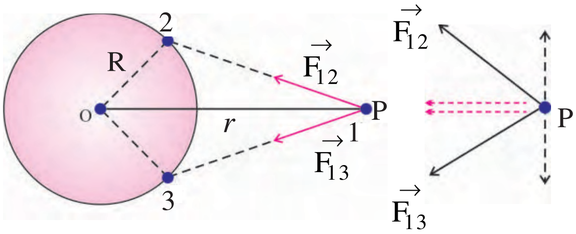
The force due to such an extended object **on a point mass** is equal to the vector sum of the forces exerted on it by all the point masses in the extended object. Thus the force on particle 1 by an extended object is,

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \ldots \qquad (3.3.5)$$

$$= \frac{G m_1 m_2}{r_{12}^2}\hat{r}_{12} + \frac{G m_1 m_3}{r_{13}^2}\hat{r}_{13} + \frac{G m_1 m_4}{r_{14}^2}\hat{r}_{14} + \ldots .(3.3.6)$$

In the same way we can find the total force **on an extended object** by **another extended object** by the vector sum of the forces on every point mass of one object by every point mass of the other object. This can be done easily with the help of calculus. We take note of two aspects : (1) The gravitational force by a hollow spherical shell of uniform density on a particle outside the shell is equal to the force which can be obtained by considering the entire mass of the shell as concentrated on its centre.

[Qualitative explanation – only for information :



[For r > R, the force due to the shell is towards the centre of the shell

Figure 3.7

The forces on the particle 1 by the particles 2 and 3 on the shell are $\vec{F}_{12}$ and $\vec{F}_{13}$.

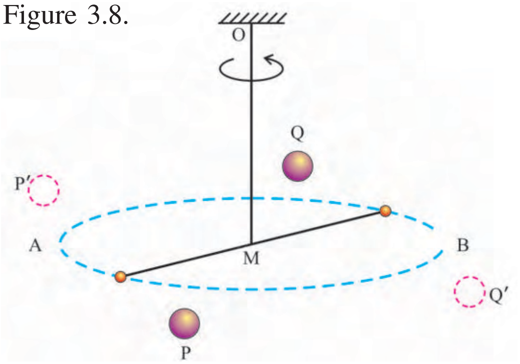
Consider their components (i) parallel to OP and (ii) perpendicular to OP. The components perpendicular to OP are cancelled and the components parallel to OP are added. Such a process can be thought for the particles on symmetric positions with respect to line OP on the shell. Thus, it can be seen that the resultant force is towards the centre. We shall accept without giving proof that its magnitude is obtained as mentioned above.]

(2) The force on a particle at any point inside a hollow spherical shell of uniform density is zero.

[Qualitative explanation – only for information : Different particles of the shell attract the given particle in different directions and the resultant of those forces becomes zero. This also we will accept without giving proof.]

3.4 Universal Constant of Gravitation

The value of the constant G appearing in the formula (3.3.1) showing Newton's universal law of gravitation, was first dertermined by Cavendish an English Scientist experimentally in 1798. The experimental arrangement is schematically shown in Figure 3.8.



Arrangement of Cavendish's experiment

Figure 3.8

From a rigid support a long rod is suspended using a thin metallic wire. Two small equal lead spheres A and B are attached at the ends of the rod. Two other equal large lead spheres are brought near the small spheres on opposite sides at equal distances. The forces on the small spheres due to the large spheres are equal in magnitude and opposite in directions. These forces produce torque. Hence the rod rotates about wire OM. Thus wire OM is twisted and the restoring torque (due to elasticity) is produced in the wire.

When the torque due to the gravitational forces equals the restoring torque, this system becomes steady (i.e. it comes in equilibrium). In this condition the positions of large spheres P and Q (or P' and Q') are on lines perpendicular to AB.

Suppose, mass of each large sphere = M

mass of each small sphere = m

Distance between their centres in equilibrium condition = AP = BQ = r.

Angle of twist in the wire in equilibrium condition = θ

The restoring torque per unit twist = k

Length of rod, AB = l

∴ The gravitational force on the small sphere due to the large sphere $= \dfrac{GMm}{r^2}$ (3.4.1)

The total torque due to both such forces

$$= \left(\frac{GMm}{r^2}\right)(l) \qquad (3.4.2)$$

and the restoring torque $\tau = k\theta$ (3.4.3)

In equilibrium condition, $\left(\dfrac{GMm}{r^2}\right)(l) = k\theta$

(3.4.4)

$$\therefore \ G = \frac{k\theta r^2}{Mml} \qquad (3.4.5)$$

[Here the value of θ is obtained with the help of a small mirror attached to the wire, using lamp and scale method. These are not shown in the figure. Moreover the value of k is obtained from some separate experiment of other kind in which known torque τ is applied and the twist in the wire θ is measured which gives $k = \frac{\tau}{\theta}$.]

Thus by measuring θ, G can be evaluated.

Illustration 1 : The position vector of the objects of masses 25 kg and 10 kg are (4, 7, 5) m and (1, 3, 5)m respectively. Obtain the vector representing the gravitational force on 25 kg object by 10 kg object. (Take G = 6.67×10^{-11} Nm^2/kg^2).

Solution : Here, $m_1 = 25$ kg, $m_2 = 10$ kg,

$\vec{r}_1 = (4, 7, 5)\ m$, $\vec{r}_2 = (1, 3, 5)\ m$, $\vec{F}_{12} = ?$

$$\vec{F}_{12}\left[\begin{array}{c}\text{Force on}\\ 1\text{ by }2\end{array}\right] = \frac{G m_1\ m_2}{r^2}\hat{r}_{12} \qquad (1)$$

$$\vec{r}_{12} = \vec{r}_2 - \vec{r}_1 = (1, 3, 5) - (4, 7, 5) = (-3, -4, 0)\ m$$

$$\therefore\ r = |\vec{r}_{12}| = \sqrt{(-3)^2 + (-4)^2 + (0)^2} = 5\ m$$

$$\text{and } \hat{r}_{12} = \frac{\vec{r}_{12}}{|\vec{r}_{12}|} = \frac{(-3, -4, 0)}{5}$$

$$= (-0.6, -0.8, 0)\ m$$

Substituting these values in equation (1)

$$\vec{F}_{12} = (6.67 \times 10^{-11})\frac{(25 \times 10)}{5^2}(-0.6, -0.8, 0)$$

$$= (6.67 \times 10^{-10})\ (-0.6\hat{i} - 0.8\hat{j})\text{N}$$

Illustration 2 : At each vertex of an equilateral triangle a particle of mass *m* kg is kept. What is the gravitational force acting on a mass M kg placed at the centroid of the triangle ? The distance of centroid from the vertex is 1 *m*.

Solution : If we choose the axes (as shown in Figure 3.9) whose origin is at the centroid of the triangle, then $\angle XGC = \angle X'GB = 30^\circ$.

The force on the particle at G, due to particle

$$\text{at A is, } \vec{F}_{GA} = \frac{G\,m\ (M)}{1^2}\hat{j} \qquad (1)$$

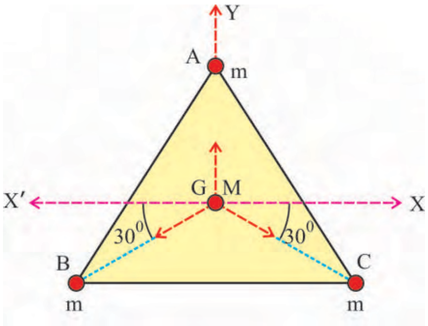


Figure 3.9

Similarly, forces due to particles at B and C are respectively

$$\vec{F}_{GB} = \frac{G\,(m)\ (M)}{1^2}[-\hat{i}\ \cos 30^\circ - \hat{j}\sin 30^\circ] \quad (2)$$

$$\text{And } \vec{F}_{GC} = \frac{G\,(m)\ (M)}{(1^2)}[\hat{i}\ \cos 30^\circ - \hat{j}\sin 30^\circ] \quad (3)$$

∴ The resultant force on the particle at point G is $\vec{F} = \vec{F}_{GA} + \vec{F}_{GB} + \vec{F}_{GC}$

$$= \frac{G\,m\ (M)}{1^2}\hat{j}$$

$$+ \frac{G\,m\ (M)}{1^2}[-\hat{i}\ \cos 30^\circ - \hat{j}\sin 30^\circ]$$

$$+ \frac{G\,m\ (M)}{1^2}[\hat{i}\cos 30^\circ - \hat{j}\sin 30^\circ]$$

$$= 0$$

Note : Using the law of triangle of vectors also you can obtain the above result. Moreover, here it can be seen that the vectors showing the forces form a closed loop and hence also we can say that the resultant force is zero.

3.5 Gravitational Acceleration And Variations In It

(a) Acceleration due to gravity :

The acceleration produced in the body due to the gravitational force is called the gravitational acceleration or the acceleration due to gravity (g).

Considering Earth as a perfect sphere of uniform density we shall consider the acceleration due to Earth's gravity at different points. We can imagine Earth to be made up of innumerable concentric hollow spherical shells. Now a particle outside the Earth is **also outside all** these shells. Hence, to find the gravitational force on that particle, we can consider the mass of every shell as concentrated at the centre of Earth (as explained in the article 3.3). Thus to find the force on that particle due to entire Earth, we can consider the entire mass of Earth to be concentrated at its centre.

Let the mass of Earth be M_e and radius be R_e. The gravitational force of Earth on the particle of mass m, at distance r from the centre (here $r > R_e$); is $F = \frac{GM_e m}{r^2}$.

Hence, from Newton's second law of motion we can write, the acceleration due to gravity

$$g = \frac{F}{m} = \frac{GM_e}{r^2} \qquad (3.5.1)$$

Now, for the particle on the surface of Earth, $r = R_e$.

∴ Acceleration due to gravity for the particle on the Earth's surface is $g_e = \frac{GM_e}{R_e^{\ 2}}$ (3.5.2)

As we have considered the Earth as a perfect sphere the value of g_e at all points on the Earth's surface would be the same. In fact, Earth is not completely spherical but is slightly bulged out at the equator and flattened at the poles. The radius of Earth at equator is nearly 21 km more than the radius at the poles. Hence the value of g_e at the poles is slightly more than that at the equator. But the variation in the value of g_e at different places on Earth's surface is extremely small and hence for practical purposes the value of g_e at every point on the Earth's surface is taken the same. The empirical value of g_e is found to be equal to 9.8 m/s^2. You may calculate the value of g_e by taking $M_e = 6 \times 10^{24}$ kg and $R_e = 6400$ km in the above equation.

Illustration 3 : If the radius of Earth suddenly decreases to 60% of the present value (with mass of the Earth remaining the same) what would be the percentage change in the magnitude of the gravitational acceleration g_e, on the surface of the Earth ?

Solution : Original value of gravitational acceleration $g_e = \frac{GM_e}{R^2_{\ e}}$

New radius of Earth $R' = \frac{60}{100} R_e$

$= 0.6\ R_e$

New value of gravitational acceleration

$$g' = \frac{GM_e}{R'^2} = \frac{GM_e}{(0.6\ R_e)^2} = \frac{g_e}{0.36}$$

$$= \frac{25}{9} g_e$$

Increase in the gravitational acceleration is

$$= g' - g_e = \frac{25}{9} g_e - g_e = \frac{16}{9} g_e$$

∴ Percentage increase in the magnitude of gravitational acceleration $= \frac{\text{increase}}{\text{original value}} \times 100$

$$= \frac{16}{9} \times \frac{g_e}{g_e} \times 100$$

$= 177.8\ \%$

Illustration 4 : If the mass and the radius of the Earth both decrease by 1 %, what will be the percentage change in the gravitational acceleration at the surface ?

Solution : The original value of gravitational acceleration $g = \frac{GM_e}{R_e^{\ 2}}$

If $M_e' = 0.99$ Me and $R_e' = 0.99\ R_e$, then new value of gravitational acceleration

$$g' = \frac{GM'_e}{R'_e{}^2} = \frac{G \times 0.99\ M_e}{(0.99\ R_e)^2}$$

$$= 1.01 \left(\frac{GM_e}{R_e^{\ 2}} \right)$$

$= 1.01\ g$

∴ Change in the gravitational acceleration

$= g' - g = 1.01\ g - g = 0.01\ g$

∴ Percentage change in the gravitational acceleration $= \frac{\text{change}}{\text{original value}} \times 100$

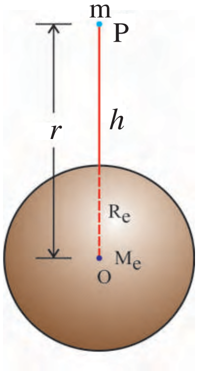
$$= \frac{0.01\ g}{g} \times 100$$

$= 1\ \%$

Thus the magnitude of g increases by 1 %

3.5(*b*) Variation in gravitational acceleration *g* with altitude :

The acceleration due to gravity at the Earth's surface is given by $g_e = \frac{GM_e}{R_e^{\ 2}}$



Gravitational acceleration at height *h* from the Earth's surface

Figure 3.10

The point P at height h from the Earth's surface is at distance $r = R_e + h$ from the centre of the Earth.

∴ The gravitational force of the Earth on a body of mass *m* at this point is

$$F(h) = \frac{GM_e m}{(R_e + h)^2} \qquad (3.5.3)$$

∴ at P gravitational acceleration is

$$g(h) = \frac{GM_e}{(R_e + h)^2} \qquad (3.5.4)$$

$$\therefore \frac{g(h)}{g_e} = \frac{R_e^{\ 2}}{(R_e + h)^2}$$

$$= \frac{R_e^{\ 2}}{R_e^{\ 2}\left[1 + \frac{h}{R_e}\right]^2} \qquad (3.5.5)$$

$$\therefore g(h) = \frac{g_e}{\left[1 + \frac{h}{R_e}\right]^2} \qquad (3.5.6)$$

It is clear from this that $g(h) < g_e$

From equation (3.5.6), $g(h) = g_e\left[1 + \frac{h}{R_e}\right]^{-2}$

(3.5.7)

$$= g_e\left[1 - \frac{2h}{R_e} + \begin{matrix}\text{terms with powers greater} \\ \text{than 1 of } \frac{h}{R_e}\end{matrix}\right]$$

(3.5.8)

..... (using binomial theorem).

If $h << R_e$, we can neglect the terms having powers greater than 1 of $\frac{h}{R_e}$. In such a condition $g(h) = g_e\left[1 - \frac{2h}{R_e}\right]$ (3.5.9)

Equation (3.5.6) can be used for any height (*h*) but equation (3.5.9) can be used only when $h << R_e$.

We can take the value of g almost equal to g_e for small heights from the Earth's surface. Let us understand this by an example : To find *g* for $h = 10$ km height from Earth's surface, we put $R_e = 6400$ km and $g_e = 9.8$ m/s^2 in equation (3.5.9).

$$\therefore g(h = 10 \text{ km}) = 9.8\left[1 - \frac{(2)(10)}{6400}\right]$$
$$= 9.8 - 0.028$$
$$= 9.772$$
$$\simeq 9.8 \text{ m/s}^2$$

Thus on the Earth's entire surface and for small heights from surface we can take $g = g_e = 9.8$ m/s^2 for practical purposes.

Illustration 5 : Prove that the ratio of the rate of change of *g* at a height equal to the Earth's radius from the surface of the Earth to the value of *g* at the surface of the Earth is equal to $\frac{-1}{4R_e}$.

Solution : The gravitational acceleration at distance $r \geq R_e$ from the centre of the Earth is $g(r) = GM/r^2$.

Differentiating with respect to *r*,

$$\left[\frac{dg(r)}{dr}\right] = \frac{-2GM_e}{r^3}$$

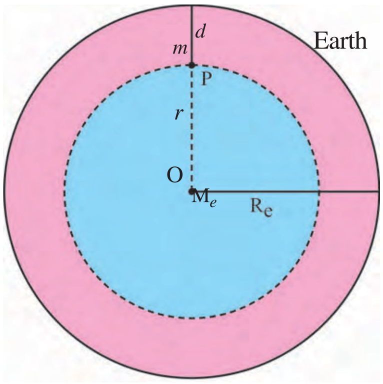
and $r = R_e + h = R_e + R_e = 2R_e$

$$\therefore \left[\frac{dg(r)}{dr}\right]_{2R_e} = \frac{-2GM_e}{(2R_e)^3} = \frac{-2GM_e}{8R_e^{\ 3}}$$

But at the surface of the Earth $g_e = \dfrac{GM_e}{R_e^{\ 2}}$

$$\therefore \frac{\left[\frac{dg(r)}{dr}\right]_{2R_e}}{g_e} = \frac{-2GM_e}{8R_e^{\ 3}} \times \frac{R_e^{\ 2}}{GM_e} = \frac{-1}{4R_e}$$

3.5(*c*) Variation in the gravitational acceleration *g* with depth from the surface of the Earth :



Variation in *g* with depth from Earth's surface

Figure 3.11

Consider a particle of mass *m* at point P at a depth *d* from the surface of the Earth. It is at distance $r = R_e - d$ from the centre of the Earth.

To find the Earth's gravitational force on this particle, we can imagine the Earth as made up of a small solid sphere of radius $r = R_e - d$ and a spherical shell of thickness d **over it.** This particle at point P is situated **inside this hollow spherical shell.** Hence, the gravitational force on this particle due to the shell is zero (as explained in the article 3.3). Moreover, this particle is also on the **outer surface** of the small sphere (shaded) of radius *r*. Hence the gravitational force on this particle can be obtained by considering the entire mass (M') of the **small sphere** at its centre O.

If the density of the Earth is ρ, then,

$$\rho = \frac{\text{total mass}}{\text{total volume}} = \frac{M_e}{\frac{4}{3}\pi R_e^{\ 3}} \qquad (3.5.10)$$

∴ The mass of the small sphere of radius *r* is,

$$M' = \text{(volume) (density)}$$

$$= \left(\frac{4}{3}\pi r^3\right)(\rho) \qquad (3.5.11)$$

∴ Gravitational acceleration at P,

$$g(r) = \frac{GM'}{r^2}$$

$$= \frac{G}{r^2}\left(\frac{4}{3}\pi r^3\right)(\rho)$$

$$= \frac{4}{3}\pi G\rho r \qquad (3.5.12)$$

From this equation, the gravitational acceleration at the surface of the Earth (putting $r = R_e$) is

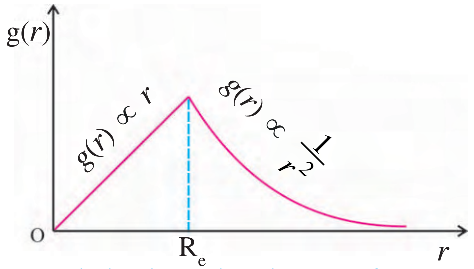
$$g_e = \frac{4}{3}\pi G\rho R_e \qquad (3.5.13)$$

From equations (3.5.12) and (3.5.13)

$$\frac{g(r)}{g_e} = \frac{r}{R_e} \qquad (3.5.14)$$

$$\therefore\ g(r) = g_e\left(\frac{r}{R_e}\right) \qquad (3.5.15)$$

From equation (3.5.12) and (3.5.15) it is clear that $g(r)$ is proportional to distance r from the centre of Earth **upto the surface.** Thus, the gravitational acceleration at a point inside the Earth is directly proportional to the distance of that point from the centre of the Earth. Moreover, for region outside the Earth, $g(r) = GM_e/r^2$, shows that $g(r) \propto \dfrac{1}{r^2}$. Hence starting from the centre of the Earth, $g(r)$ increases in direct proportion as r increases and then outside the surface $g(r)$ decreases as inverse square of distance. Such variations in g are shown in Figure 3.12.



Variation in *g* with distance *r* from centre of Earth

Figure 3.12

By substituting $r = R_e - d$ in equation (3.5.15) the gravitational acceleration is obtained in terms of depth d from the Earth's surface. We denote it as $g(d)$.

$$\therefore \quad g(d) \quad = \frac{g_e}{R_e}(R_e - d)$$

$$= g_e\left[1 - \frac{d}{R_e}\right] \qquad (3.5.16)$$

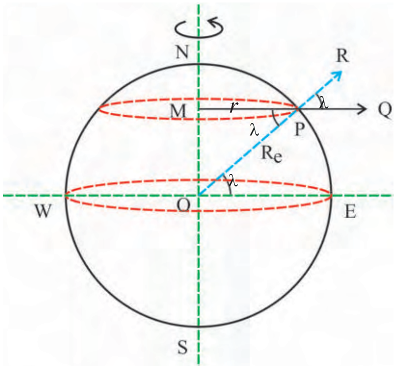
This shows that the gravitational acceleration at depth d has a smaller value than that at the Earth's surface.

Thus the acceleration due to Earth's gravity is maximum on its surface and from there on going above or below it decreases. It becomes zero at the Earth's centre. This is a notable fact.

3.5(*d*) Variation in effective gravitational acceleration g' with latitude due to earth's rotation :

The angle made by the line joining a given place on the Earth's surface to the centre of the Earth with the equatorial line is called the latitude (λ) of that place. Hence, for the equator lattitude $\lambda = 0^o$ and for the poles latitude $\lambda = 90^o$.

As shown in the figure the latitude of the place P on the Earth's surface is $\lambda = \angle POE$. At this position consider a particle of mass m. We have to think of two forces acting on it.



Variation in effective g' with latitude due to Earth's rotation

Figure 3.13

(1) Earth's gravitational force = mg (in $\overrightarrow{PO}$ direction) (3.5.17)

(2) To understand the other force consider the rotation of the Earth. The Earth has an acceleration due to its rotational motion. So this particle is in the accelerated frame of reference. At this point the acceleration of the frame of reference is = $\frac{v^2}{r}$ in $\overrightarrow{PM}$ direction (that is, towards the centre of the circular path). Hence we have to consider a fictitious acceleration $\frac{v^2}{r}$ and the fictitious force $\frac{mv^2}{r}$ in the opposite direction, that is in $\overrightarrow{MPQ}$ direction. The component of this force in $\overrightarrow{PR}$ direction =

$$\frac{mv^2}{r}\cos\lambda. \qquad (3.5.18)$$

This is the second force to be considered.

Thus cosidering two forces given by equations (3.5.17) and (3.5.18), the effective force on the particle at P, towards the centre of the Earth is $mg' = mg - \frac{mv^2}{r}\cos\lambda$. (3.5.19)

Where g' = effective gravitational acceleration at this place obtained by considering rotation of Earth.

g = gravitational acceleration at this place without considering rotation of Earth.

$$\therefore \; g' = g - \frac{v^2}{r}\cos\lambda \qquad (3.5.20)$$

But $v = r\omega$ where ω = angular speed of the Earth.

$$\therefore \; g' = g - \frac{(r\omega)^2}{r}\cos\lambda \qquad (3.5.21)$$

$$= g - r\omega^2\cos\lambda \qquad (3.5.22)$$

From the figure, $r = MP = R_e\cos\lambda$ (3.5.23)

$$\therefore \; g' = g - R_e\omega^2\cos^2\lambda \qquad (3.5.24)$$

$$\text{or } g' = g\left[1 - \frac{R_e\omega^2\cos^2\lambda}{g}\right] \qquad (3.5.25)$$

From this equation (3.5.24) or (3.5.25), we get information about the variation in g with latitude, due to the Earth's rotation. We note two special cases :

(i) At equator, $\lambda = 0^o$, $\therefore \cos\lambda = 1$, $\therefore g' = g - R_e\omega^2$, which shows the minimum value of the effective gravitational acceleration.

(ii) At poles, $\lambda = 90^o$, $\cos\lambda = 0$, $\therefore g' = g$, which shows the maximum value of the effective gravitational acceleration.

Illustration 6 : Find the period of rotation of the Earth about its own axis in terms of R_e and g for which the effective acceleration due to gravity becomes zero at the equator ?

Solution : At the equator the latitude $\lambda = 0^o$. The effective acceleration due to gravity at a place having latitude λ on the Earth's surface is given by $g' = g - R_e\omega^2\cos^2\lambda$... (from equation 3.5.24). R_e = radius of the Earth.

g = gravitational acceleration at the Earth's surface without considering rotation.

ω = angular speed of Earth's rotation = $\frac{2\pi}{T}$

We want to find the time-period T for $g' = 0$ at the equator.

$$\therefore\ 0 = g - R_e\omega^2\cos^2(0^o)$$

$$\therefore\ g = R_e\omega^2 \quad ...(\cos 0^o = 1)$$

$$= R_e\left(\frac{4\pi^2}{T^2}\right)$$

$$\therefore\ T^2 = 4\pi^2\frac{R_e}{g} \qquad \therefore\ T = 2\pi\sqrt{\frac{R_e}{g}}$$

3.6 Gravitational Intensity

The gravitational force on a body by the other one is given by Newton's law of gravitation (equation 3.3.1). This process of **action at a distance** in which force is exerted mutually on two bodies separated by some distance is explained to occur through the **field** as under :

(1) Every object produces a gravitational field around it, due to its mass. (2) This field exerts a force on another body brought (or lying) in this field. Hence it is important to know about the strength of such a gravitational field.

"The gravitational force exerted by the given body on a body of unit mass at a given point is called the intensity of gravitational field ($\vec{I}$) at that point." It is also known as the **gravitational field or gravitational intensity.**

Using Newton's law of gravitation we can write the formula for the gravitational intensity. Consider a body of mass M at the origin of co-ordinate system O. The gravitational intensity ($\vec{I}$) due to it at some point P is,

$$\vec{I} = \frac{-GM(1)}{r^2}\hat{r} = \frac{-GM}{r^2}\hat{r} \quad ...\ (3.6.1)$$

where $\hat{r} = \vec{OP}$ and $\hat{r}$ = unit vector in the direction of $\hat{r}$ (i.e. $\vec{OP}$).

In magnitude we can write $I = \frac{GM}{r^2}$ ... (3.6.2)

Its unit is N/kg and the dimensional formula is $M^0L^1T^{-2}$.

Now if a body of mass m is put (or lying) at this point P, the gravitational force exerted by the field on it is

$$\vec{F} = \vec{I}\,m = \frac{-GMm}{r^2}\hat{r} \qquad (3.6.3)$$

Equation (3.6.2) shows that the gravitational intensity due to Earth at a point has the **same value** as the gravitational acceleration at that point. But these two quantities are different and their units are different but equivalent. [∵ N/kg = m/s^2]. It is obvious that I – r graph for the Earth's gravitational field would be the same as g – r graph (like Figure 3.12).

[In future you will learn the formula, electric force = (electric intensity $\vec{E}$) × (charge q) in case of electricity.]

Illustration 7 : The gravitational intensity at a point is $\vec{I} = 10^{-9}(\hat{i} + \hat{j})$ N/kg. If a body of 10 kg mass is placed at this point, find the magnitude of force on it and the magnitude of its acceleration.

Solution :

$$\vec{F} = (\vec{I})(m)$$

$$= (10^{-9})(\hat{i} + \hat{j})(10)$$

$$= 10^{-8}\hat{i} + 10^{-8}\hat{j}\ \text{N}$$

$$\therefore\ |\vec{F}| = \sqrt{(10^{-8})^2 + (10^{-8})^2}$$

$$= 10^{-8}\sqrt{2}$$

$$= 1.414 \times 10^{-8}\ \text{N}$$

$$g = \frac{|\vec{F}|}{m} = \frac{1.414\times10^{-8}}{10}$$

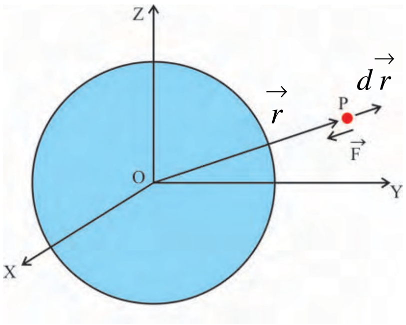
$$= 1.414 \times 10^{-9}\ \text{m/s}^2$$

3.7 Gravitational Potential and Gravitational Potential Energy in the Earth's Gravitational Field

(*a*) Gravitational potential :

Every object produces a gravitational field around it. A characteristic of such a field is defined as a quantity called the **gravitational potential** as under :

"The negative of the work done by the gravitational force in bringing a body of unit mass, from infinite distance to the given point in the gravitational field is called the gravitational potential (ϕ) at that point." The unit of gravitational potential is J kg^{-1} and dimensional formula is $M^0L^2T^{-2}$.



Work done by gravitational force in small displacement

Figure 3.14

Consider Figure 3.14 to obtain the formula for the gravitational potential in the Earth's gravitational field.

We put the origin of the co-ordinate system at the centre of the Earth. Mass of the Earth is M_e and radius is R_e. The position vector of point P at distance r from the centre of the Earth is $\overrightarrow{OP} = \vec{r}$. Here, $r \geq R_e$. At this point the Earth's gravitational force on a body of **unit mass** is

$$\vec{F} = \frac{-GM_e(1)}{r^2}\hat{r}$$

$$= \frac{-GM_e}{r^2}\hat{r} \qquad (3.7.1)$$

This force is not constant but changes with distance. But during an infinitely small displacement $d\vec{r}$ the force can be taken as constant. Hence, during such a small displacement, the work done by the gravitational force is

$$dW = \vec{F} \cdot d\vec{r} = \left(\frac{-GM_e}{r^2}\hat{r}\right) \cdot (dr\hat{r}) \qquad (3.7.2)$$

$$= \frac{-GM_e}{r^2}dr \qquad (3.7.3)$$

The entire path from point P to infinite distance can be divided in large number of infinitely small intervals. Taking the force as constant during every such interval, we can calculate the work done during that interval, and by adding all such works we get the total work W. As this process is a continuous one, the summation can be written as integration.

Hence, in this case, work done by the gravitational force in moving this body from point P at distance *r* to infinite distance is

$$W_{r \to \infty} = \int dW = \int_r^\infty \left(-\frac{GM_e}{r^2}\right) dr \qquad (3.7.4)$$

$$= -GM_e \int_r^\infty \frac{1}{r^2} dr \qquad (3.7.5)$$

$$= -GM_e \left[-\frac{1}{r}\right]_r^\infty \qquad (3.7.6)$$

$$= \frac{-GM_e}{r} \qquad (3.7.7)$$

Now if we bring this body from infinite distance to the point P at distance *r*; the work done $(W_{\infty \to r})$ **by the gravitational force** will be the same as that given by equation (3.7.7) **but with opposite sign,** $[W_{\infty \to r} = -W_{r \to \infty}]$, because gravitational force is a conservative force.

$$\therefore \; W_{\infty \to r} = \frac{GM_e}{r} \qquad (3.7.8)$$

The negative of this work $(W_{\infty \to r})$ is by definition, called the gravitational potential ϕ at point P.

∴ Gravitational potential at P is,

$$\phi = \frac{-GM_e}{r} \qquad (3.7.9)$$

From this the gravitational potential at the Earth's surface (putting $r = R_e$) is,

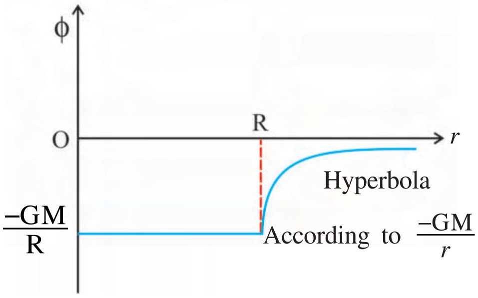
$$\phi_e = \frac{-GM_e}{R_e} \qquad (3.7.10)$$

We note a few points about the gravitational potential :

(1) The gravitational potential at infinite distance from the centre of the Earth = 0.

(2) **The gravitational potential at all points inside** a uniform spherical shell **is the same,** and is equal to the value at the surface that is, equal to $\frac{-GM}{R}$, where M = mass of the shell and R = radius of the shell. The reason for this is that the gravitational force at all points inside the shell is zero, hence **no work is done in the motion of the body inside the shell.** The work during the motion from infinite distance to surface only comes in the calculation.

(3) The variation in the gravitational potential with distance *r* from the centre of the shell, having mass M and radius R is shown in the Figure 3.15(*b*)



Variation in ϕ with distance r

Figure 3.15

(*b*) Gravitational Potential Energy : "The negative of the work done by the gravitational force in bringing a given body (of mass *m*) in the gravitational field of the Earth from infinite distance at the given point is called the gravitational potential energy U of that body at that point." It is actually the gravitational potential energy of the system of the Earth + that body.

Considering definitions of gravitational potential, gravitational potential energy and using equation (3.7.9), the gravitational potential energy of a body of mass *m* at a distance *r* from the Earth's centre ($r \geq R_e$) is,

$$U = \phi m = \frac{-GM_e m}{r} \qquad (3.7.10)$$

Hence the gravitational potential energy of the body of mass m, lying on the surface of the Earth ($r = R_e$) is,

$$U_e = \frac{-GM_e m}{R_e} \qquad (3.7.11)$$

We can also say that the gravitational potential is the gravitational potential energy of unit mass.

At infinite distance from the centre of the Earth the gravitational force of the Earth on that body is zero and according to the above definition we can say that its gravitational potential energy is also zero.

The **absolute value** of the potential energy (or potential) has **no importance** at all, only the **change** in its value **is important.** Hence the reference point for zero potential energy (or zero potential) can be taken anywhere. (You may recall that in the chapter of 'Work Energy and Power' we had taken zero potential energy at the surface of the Earth, while here we have taken zero potential energy at infinite distance. But in both the cases only the changes are important, hence no contradiction is produced.)

Here the potential energy U is of the system consisting of the Earth and the body. But in this process the position of Earth or its velocity is not appreciably changed, hence it is also conventionally mentioned as the potential energy **of the body.** Whenever such a mention is made we have to understand that this potential energy is actually **of that system** but the entire change in that potential energy appears to be experienced by the **body alone.**

In future, we are also going to consider a satellite. In that case the potential energy is of the system consisting of the Earth and the satellite. But we shall mention it as potential energy of the satellite.

Illustration 8 : A particle of mass m is placed on each vertex of a square of side *l* as shown in Figure 3.16. Calculate the gravitational potential energy of this system of four particles. Also calculate the gravitational potential at the centre of the square.

Solution : Here we can write the potential energy due to every pair of particles as

$U_{ij} = \frac{-Gm_i m_j}{r_{ij}}$, where m_i and m_j are the masses of particles i and j respectively, and r_{ij} is the distance between them $m_i = m_j = m$.

∴ Total potential energy

$$U = -Gm^2\left[\sum_{i<j}\frac{1}{r_{ij}}\right]$$

$$= -Gm^2\left[\frac{1}{r_{12}}+\frac{1}{r_{13}}+\frac{1}{r_{14}}+\frac{1}{r_{23}}+\frac{1}{r_{24}}+\frac{1}{r_{34}}\right]$$

$$= -Gm^2\left[\frac{1}{l}+\frac{1}{\sqrt{2}l}+\frac{1}{l}+\frac{1}{l}+\frac{1}{\sqrt{2}l}+\frac{1}{l}\right]$$

$$= -Gm^2\left[\frac{4+\sqrt{2}}{l}\right]$$

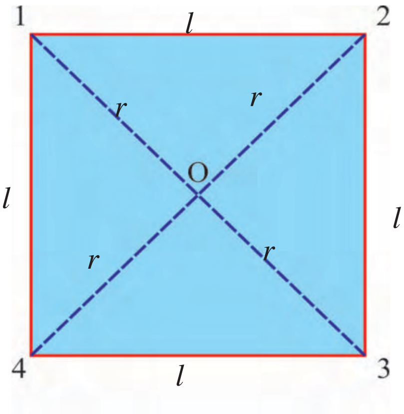


Figure 3.16

$$r_{13} = r_{24} = \sqrt{2}\,l$$

$$r_{01} = r_{02} = r_{03} = r_{04} = r$$

The gravitational potential at the centre, due to each particle is same.

∴ The total gravitational potential at the centre of the square is

ϕ = 4 (potential due to every particle)

$$= 4\left(\frac{-Gm}{r}\right);\ \text{where } r = \frac{\sqrt{2}l}{2}$$

$$\therefore\ \phi = \frac{-4\sqrt{2}Gm}{l}$$

3.8 Escape Energy and Escape Speed

If we throw a stone upwards with our hand, it goes to a certain height and then falls back towards the Earth. If we throw it with larger and larger initial speed we can send it to greater and greater heights. From this a natural question may arise : can we throw the stone with such an initial speed that it does not return back to Earth ? It means, it goes to infinite distance from Earth forever and then there is no attraction on it by the Earth. To get the answer let us consider its energy.

The gravitational potential energy of a body of mass m lying on the Earth's surface is $= \frac{-GM_e m}{R_e}$ and its kinetic energy is zero. So its total energy is $= \frac{-GM_e m}{R_e}$

If we supply energy $\frac{+GM_e m}{R_e}$ to this body in the form of kinetic energy, then it can go upto a point where its total energy becomes $\frac{+GM_e m}{R_e} + \left(\frac{-GM_e m}{R_e}\right) = 0$.

It means, it will go to infinite distance from the Earth and there its potential energy is zero and kinetic energy is also zero. In this condition the body escapes from the binding with the Earth forever and does not return back. (If we give kinetic energy more than $GM_e m/R_e$ to the body then at infinite distance its potential energy becomes zero but it has still certain kinetic energy remaining with it.)

"The minimum energy to be supplied to the body to make it free from Earth's gravitational field (in other words from binding with the Earth) is called the escape energy of that body." It is often called the binding energy of the body.

Thus, the escape energy of the body of mass m lying on the surface of the Earth $= \frac{GM_e m}{R_e}$

(3.8.1)

The speed to be given to the body to give the kinetic energy equal to its escape energy is called the escape speed (v_e) which is often called the escape velocity also.

$$\therefore\ \frac{1}{2}mv_e^2 = \frac{GM_e m}{R_e} \qquad (3.8.2)$$

$$\therefore\ \text{Escape speed } v_e = \sqrt{\frac{2GM_e}{R_e}} \qquad (3.8.3)$$

$$= \sqrt{2gR_e} \qquad (3.8.3a)$$

where g = gravitational acceleration at Earth's surface = $\frac{GM_e}{R_e^{\,2}}$. This shows that the escape speed of the body does not depend on its own mass (but depends on the mass and radius of the other body from the binding of which it has to escape).

By putting the values of G, M_e and R_e in equation (3.8.3), we get v_e = 11.2 km/s. If the initial speed of the body is equal to or greater than its escape speed (v_e), it will escape from the gravitational field of Earth forever.

If the speed required for the body lying on the surface of moon, to make it free from the moon's gravitation is v_e', then $v_e' = \sqrt{\frac{2GM_m}{R_m}}$ where M_m = mass of the moon and R_m = radius of the moon. In that case v_e' = 2.3 km/s which is nearly $\left(\frac{1}{6}\right)$ times the escape speed at the Earth's surface. Moon has no atmosphere because of this reason. If the gas molecules are formed on its surface then at the temperature prevailing there, those molecules have speeds greater than the above mentioned value. Hence, they escape the gravitational field of the moon forever.

If the density of a body is so high that the escape speed (v_e) at its surface is > velocity of light C, then nothing will be able to escape from its surface forever (not even light !). Such a body is called **black hole.** We have to remember that no meterial particle can have velocity greater than or equal to the velocity of light C.

$(C = 3 \times 10^8 \text{ m s}^{-1})$

Illustration 9 : For an object lying on the surface of the Earth the escape speed is 11.2 km/s. If an object on the Earth is thrown away with a speed three times this value, find its speed after it has escaped from the gravitational field of the Earth.

Solution : The initial speed of the object = $v = 3v_e$, where v_e = escape speed = 11.2 km/s.

Suppose the speed of this object after it escaped from the Earth's gravitational field (that is at infinite distance) = v'.

According to the law of conservation of mechanical energy,

$$\begin{Bmatrix}\text{Kinetic energy +}\\ \text{potential energy}\\ \text{at the Earth}\\ \text{surface}\end{Bmatrix} = \begin{Bmatrix}\text{Kinetic energy +}\\ \text{potential energy}\\ \text{at infinite}\\ \text{distance}\end{Bmatrix}$$

$$\therefore \ \frac{1}{2}mv^2 + \left(\frac{-GM_em}{R_e}\right) = \left[\frac{1}{2}mv'^2 + 0\right] \qquad ....(1)$$

($\because$ at infinite distance potential en. = 0)

$$\text{But } v_e = \sqrt{\frac{2GM_e}{R_e}} \quad \therefore \frac{GM_e}{R_e} = \frac{v_e^{\,2}}{2}$$

Putting this value in equation (1) and writing $v = 3v_e$ (given), we get,

$$\frac{1}{2}m(9v_e^{\,2}) + \left(\frac{-v_e^{\,2}m}{2}\right) = \frac{1}{2}mv'^2$$

$$\therefore \ 9v_e^{\,2} - v_e^{\,2} = v'^2$$

$$\therefore \ v' = \sqrt{8}\,v_e = \left(\sqrt{8}\right)(11.2)$$

$$= 31.63 \text{ km/s}$$

Illustration 10 : An object is allowed to fall freely towards the Earth from a distance r (>R_e) from the centre of the Earth. Find the speed of the object when it strikes the surface of the Earth.

Solution : Allowing the body to fall freely from distance r, from the centre of Earth, its initial velocity is zero. ∴ its kinetic energy = 0. Its potential energy = $\frac{-GM_em}{r}$, where m = mass of body.

When it strikes the surface of the Earth, if its velocity is v and the kinetic energy = $\frac{1}{2}mv^2$, its potential energy here = $\frac{-GM_em}{R_e}$

Neglecting the air resistance, the conservation of mechanical energy gives,

$$\begin{Bmatrix}\text{Kinetic energy +}\\ \text{potential energy}\\ \text{at distance } r \text{ from}\\ \text{Earth's surface}\end{Bmatrix} = \begin{Bmatrix}\text{Kinetic energy +}\\ \text{potential energy}\\ \textit{at}\\ \text{Earth's surface}\end{Bmatrix}$$

$$\therefore \left\{0+\left(\frac{-GM_e m}{r}\right)\right\} = \left\{\frac{1}{2}mv^2+\left(\frac{-GM_e m}{R_e}\right)\right\}$$

$$\therefore v^2 = 2GM_e\left[\frac{1}{R_e} - \frac{1}{r}\right] \qquad (1)$$

This gives the required speed v.

To obtain the answer in terms of g, we write

$$g = \frac{GM_e}{R_e^{\ 2}} \quad \therefore GM_e = gR_e^{\ 2}$$

$\therefore$ From equation (1)

$$v^2 = 2g\ R_e^{\ 2}\left[\frac{1}{R_e} - \frac{1}{r}\right] \qquad (2)$$

$$\therefore v = \left[2gR_e^{\ 2}\left(\frac{1}{R_e} - \frac{1}{r}\right)\right]^{\frac{1}{2}} \qquad (3)$$

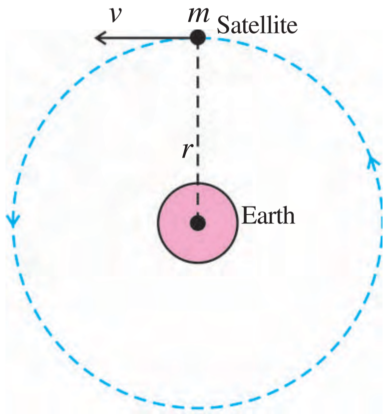
Note : If it falls freely from a very large distance ($r \to \infty$) from the Earth's surface then equations (1) and (2) will give $v = \sqrt{\frac{2GM_e}{R_e}} = \sqrt{2R_e g}$. This is the same as the formula for the escape speed.

3.9 Satellites

A body revolving around a planet is called its satellite. The orbital motion of the satellite depends on the gravitational force by the planet and the initial conditions. Satellites can be classified into two categories : (1) natural satellite (2) artificial satellite.

Moon is the natural satellite of the Earth. Moreover, Jupiter and other planets also have their moons (means satellites). The periodic time of our moon's revolution around the Earth is 27.3 day and the periodic time of rotation of moon about its own axis is also nearly the same.

The first artificial satellite made by the mankind was **"Sputnic"** put into orbit around the Earth by Russian scientists in 1957. Our Indian scientists have also successfully launched 'Aryabhatta' and INSAT series of satellites. Presently hundreds of satellites launched by many countries of the world orbit around the Earth. They are used for scientific, engineering, communication, whether forecast, spying and military purposes. In the present article we shall study the dynamics of the satellite and geo-stationary (or geo-synchronous) as well as polar satellites.



Orbital motion of a satellite

Figure 3.17

Suppose a satellite of mass m is launched at distance r from the centre of the Earth and its speed in the circular orbit is v_0. It is also called the orbital speed or the orbital velocity. Here $r = R_e + h$ where R_e = radius of the Earth, h = height of the satellite from the Earth's surface. The necessary centripetal force ($mv_0^{\ 2}/r$) for this circular motion of the satellite is provided by the Earth's gravitational force on it.

$$\therefore \frac{mv_0^{\ 2}}{r} = \frac{GM_e m}{r^2} \qquad (3.9.1)$$

$\therefore$ The orbital speed of the satellite is

$$v_0 = \sqrt{\frac{GM_e}{r}} \qquad (3.9.2)$$

From equation (3.9.1), the kinetic energy of the satellite is, $K = \frac{1}{2}mv_0^{\ 2} = \frac{GM_e m}{2r}$ (3.9.3)

The potential energy of this satellite (actually of the system of Earth + satellite) is (from equation 3.7.10)

$$U = \frac{-GM_e m}{r} \qquad (3.9.4)$$

$\therefore$ Total energy of the satellite, is

E = kinetic energy K + potential energy U

$$= \frac{GM_e m}{2r} - \frac{GM_e m}{r} \qquad (3.9.5)$$

$$= \frac{-GM_e m}{2r} \qquad (3.9.6)$$

This total energy is negative, which indicates that this satellite is in the bound state. You will be able to see from equations (3.9.3), (3.9.4) and (3.9.6) that if the kinetic energy of the satellite is x, its potential energy is $-2x$ and the total energy is $-x$. Hence its binding energy also equal to its escape energy is x.

Time period (T) of the satellite : The time taken by the satellite to complete one revolution around Earth is called its time-period or the periodic time or the period (T) of revolution. During this time the distance travelled by it is equal to the circumference ($= 2\pi r$) of the circular path.

$$\therefore \text{ The orbital speed } v_0 = \frac{2\pi r}{T} \qquad (3.9.7)$$

$\therefore$ From equation (3.9.1),

$$\frac{m}{r}\left(\frac{4\pi^2 r^2}{T^2}\right) = \frac{GM_e m}{r^2} \qquad (3.9.8)$$

$$\therefore\ T^2 = \left(\frac{4\pi^2}{GM_e}\right) r^3 \qquad (3.9.9)$$

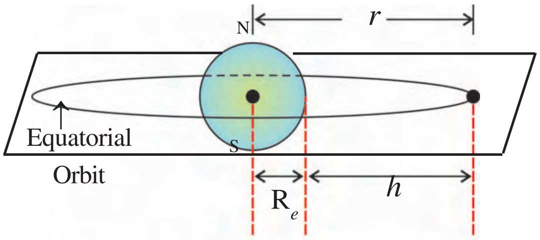
Since all quantities in the bracket are constant we can say that $T^2 \propto r^3$ (3.9.10)

Thus, "the square of the orbital time-period of the satellite is directly proportional to the cube of the orbital radius." This is Kepler's third law with reference to the circular orbit of the satellite.

From equation (3.9.9),

$$T = \left(\frac{4\pi^2 r^3}{GM_e}\right)^{\frac{1}{2}} \qquad (3.9.11)$$

Geo-stationary satellite : The Earth's satellite having orbital periodic time of 24 hours (equal to the periodic time of rotation of the Earth about its own axis), is called the geo-stationary satellite (or geo-synchronous satellite), because it appears always stationary as viewed from the Earth. Such a geo-stationary satellite revolve around the Earth in the equatorial plane in east-west direction. See Figure 3.18(*a*)



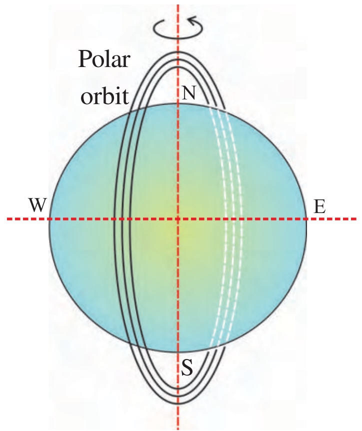
Geo-stationary satellite

Figure 3.18(*a*)

For geo–stationary satellite by putting $G = 6.67 \times 10^{-11}$ Nm^2 kg^{-2}, $M_e = 5.98 \times 10^{24}$ kg and T = 24 × 3600 s, in equation (3.9.11), we get r = 42260 km. Hence the height of this geo-stationary satellite from the Earth's surface is $h = r - R_e = 42260 - 6400 = 35860$ km. A satellite cannot remain geo-stationary for any other height except this one.

These satellites are used in tele-communication. Moreover they are also used in **Global Positioning System (GPS)** in which a person gets information about various ways and the shortest route to go from his present position to his destination, alongwith the map displayed on the screen of the monitor.

Polar satellite : These satellites revolve around the Earth in north-south direction. Their heights from the surface of the Earth is nearly 800 km. Since the Earth rotates in the east-west direction, these satellites (Their time-period is almost 100 min.) can view every section of the Earth many times in a day. With the help of a camera kept inside this satellite it can see a thin strip of the Earth in every rotation. In the next rotation it will see the region of the next strip and thus can see the entire Earth many times in a day. They are useful in remote sensing, meteorology, environmental study, spying etc.



Polar orbit

Figure 3.18(*b*)

Illustration 11 : Imagine a simple pendulum suspended from a support which is at infinite height from the surface of the Earth. The bob of the pendulum is close to the surface of the Earth. Show that the period of such a pendulum (of infinite length) is $T = 2\pi\sqrt{\frac{R_e}{g}}$.

Solution : Since the point of suspension is at infinite height, the small path of the motion of the bob can be considered almost a linear one. Mass of the bob = m.

When the bob is released from A the $mg\cos\theta$ component of the gravitational force F_G ($= mg$), provides the necessary restoring force towards B.

i.e. restoring force on the bob, $F = -mg\cos\theta$ (since the force is restoring negative sign is put).

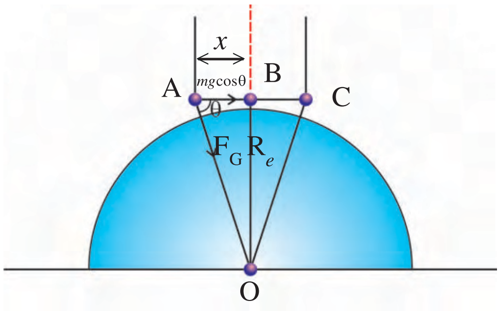


Figure 3.19

Figure 3.19 shows $\cos\theta = \frac{x}{R_e}$. (Since the bob is close to the surface of the Earth, we can take AO = BO = R_e.

$$\therefore\ F = -mg\left(\frac{x}{R_e}\right)$$

$$\therefore\ F = -kx \qquad (1)$$

where k = force constant = $\frac{mg}{R_e}$

∴ Equation (1) indicates that the pendulum performs simple harmonic motion.

$$\therefore\ T = 2\pi\sqrt{\frac{m}{k}} \text{ gives,}$$

$$T = 2\pi\sqrt{\frac{m}{mg / R_e}}$$

$$= 2\pi\sqrt{\frac{R_e}{g}}$$

Illustration 12 : Show that the binding energy of a satellite revolving around the Earth and remaining close to the surface of the Earth is $\frac{1}{2}mgR_e$.

Solution : Here the necessary centripetal force for the circular motion of the satellite (mass = m), is provided by Earth's gravitational force on it,

$$\therefore\ \frac{mv^2}{R_e} = \frac{GM_em}{R_e^{\ 2}} = gm\ (\because\ g = \frac{GM_e}{R_e^{\ 2}})$$

This gives Kinetic energy of

satellite $= \frac{1}{2}mv^2 = \frac{1}{2}mgR_e$

The potential energy $= \frac{-GM_em}{R_e}$

$$= \frac{-GM_em}{R_e^{\ 2}}R_e$$

$$= -gmR_e$$

∴ Total energy = Kinetic energy + Potential energy $= \frac{1}{2}mgR_e - gmR_e$

$$= -\frac{1}{2}mgR_e$$

∴ The binding energy $= \frac{1}{2}mgR_e$

Illustration 13 : Two objects of masses 1 kg and 2 kg respectively are released from rest when their separation is 10*m*. Assuming that only mutual gravitational forces act on them, find the velocity of each of them when separation becomes 5*m*.

(Take $G = 6.66 \times 10^{-11}$ Nm^2/kg^2)

Solution : Initially velocities of both the particles are zero and hence their kinetic energies are zero (i.e. $v_1 = v_2 = 0$; $K_1 = K_2 = 0$)

When the separation is $5m$, their velocities are v_1' and v_2' and kinetic energies are K_1' and K_2' respectively.

For this system initial potential energy

$$U_1 = \frac{-Gm_1m_2}{r_1}$$

$$= \frac{-(6.67\times10^{-11})(1\times2)}{10}$$

$$= -13.32 \times 10^{-12} \text{ J}$$

Final potential energy

$$U_2 = \frac{-Gm_1m_2}{r_2}$$

$$= \frac{-(6.66\times10^{-11})(1\times2)}{5}$$

$$= -26.64 \times 10^{-12} \text{ J}$$

$\therefore$ Change in potential energy $\Delta U = U_2 - U_1$

$$= -26.64 \times 10^{-12} - (-13.32 \times 10^{-12})$$

$$= -13.32 \times 10^{-12} \text{ J}$$

According to the law of conservation of mechanical energy

$K + U = \text{constant} \;\therefore\; \Delta K + \Delta U = 0$

$\therefore\; \Delta K = -\Delta U$

$\therefore\; (K_1' + K_2') - 0 = -(U_2 - U_1)$

$$\therefore (\tfrac{1}{2}m_1v_1'^2 + \tfrac{1}{2}m_2v_2'^2) - (0) = 13.32 \times 10^{-12} \text{ J}$$

$$\therefore \frac{v_1'^2}{2} + v_2'^2 = 13.32 \times 10^{-12} \qquad (1)$$

According to the law of conservation of momentum, final total momentum = initial total momentum.

$$\therefore m_1\vec{v_1}' + m_2\vec{v_2}' = 0$$

$$\therefore m_1\vec{v_1}' = -m_2\vec{v_2}'$$

$$\therefore \vec{v_1}' = -\frac{m_2}{m_1}\vec{v_2}'$$

$$\therefore |\vec{v_1}'| = \left(\frac{m_2}{m_1}\right)(|\vec{v_2}'|)$$

$$\therefore v_1' = 2v_2' \qquad (2)$$

From equations (1) and (2)

$$\frac{4v_2'^2}{2} + v_2'^2 = 13.32 \times 10^{-12}$$

$$\therefore 3v_2'^2 = 13.32 \times 10^{-12}$$

$$\therefore v_2'^2 = 4.44 \times 10^{-12} = 444 \times 10^{-14}$$

$$\therefore v_2'. = 21.07 \times 10^{-7} \text{ m/s}$$

$$\therefore v_1' = 42.14 \times 10^{-7} \text{ m/s}$$

Illustration 14 : Two satellites S_1 and S_2 revolve around a planet in two different but coplanar circular orbits in the same direction. If their periods are 31.4 *h* and 62.8 *h* and the radius of orbit of S_1 is 4000 km, find (i) the radius of the orbit of S_2 (ii) the magnitudes of the velocities of the two satellites.

Solution :

(i) $T^2 \propto r^3$

$$\therefore \frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}$$

$$\therefore r_2^3 = r_1^3\left(\frac{T_2^2}{T_1^2}\right)$$

$$= (4000)^3\left(\frac{62.8^2}{31.4^2}\right)$$

$$\therefore r_2 = (4000)(4)^{\frac{1}{3}} = (4000)(1.588)$$

$$= 6352 \text{ km}$$

(ii) $$v_1 = \frac{2\pi r_1}{T_1} = \frac{(2)(3.14)(4000)}{31.4}$$

$$= 800 \text{ km/h}$$

$$v_2 = \frac{2\pi r_2}{T_2} = \frac{(2)(3.14)(6352)}{62.8}$$

$$= 635.2 \text{ km/h}$$

Tides in the Ocean

(Only for Information)

Friends, you may have the idea that the reason for the tides in the ocean is the gravitation. In this phenomenon sun and moon both play a part. Actually the gravitational force by the sun on the Earth is nearly 175 times that exerted by the moon on the Earth. However, in the phenomenon of tides the contribution by the moon is more than that by the sun which is nearly 2.17 times that by the sun. This is a fact. What could be the reason for this ?

The reason for this is that the calculations reveal that the tide-generating force (tidal force) depends on the **rate of change of the gravitational force with distance and not on the magnitude of the gravitational force itself.** Hence, in spite of $F_{\text{by sun}} > F_{\text{by moon}}$, since $\frac{d}{dr}(F_{\text{by moon}}) > \frac{d}{dr}(F_{\text{by sun}})$, the contribution by the moon is more in the phenomenon of tides.

$F = \frac{GMm}{r^2}$ gives $\frac{d(F)}{dr} = \frac{-2GMm}{r^3}$. You will be able to verify for yourself the above, by taking m = unit mass of water in these formulae. (Take $M_s = 2 \times 10^{30}$ kg, $r_s = 1.5 \times 10^{11}$ m, $M_m = 7.36 \times 10^{22}$ kg, $r_m = 3.84 \times 10^8$m).

(This is only a simple explanation. The phenomenon of tide is a complex one. To some extent local parameters-like distance of sea-bottom from the sea-shore, structure of Earth below and close to the sea bottom, rotation of Earth etc. do also play part in it.)

We shall only take note that the tidal force depends on $\frac{1}{r^3}$ and hence contribution by the moon is found to be more than that by the sun according to the above formulae.

SUMMARY

1. Out of Ptolemy's geo-centric theory and Copernicus' helio centric theory at present helio-centric theory is accepted.
2. Kepler's Laws : (1) "All planets move in elliptical orbits with the sun at one of the foci." (2) "The line joining the sun and the planet sweeps equal areas in equal intervals of time." (3) "The square of the time period (T) of the revolution of a planet is proportional to the cube of the semi-major axis (a) of its elliptical orbit" ($T^2 \propto a^3$)
3. Newton's universal law of gravitation : "Every particle in the universe attracts every other particle with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them." i.e. $F = \frac{Gm_1m_2}{r^2}$

   In vector form $\left[\begin{matrix}\vec{F}_{12} \\ force\ on \\ 1\ by\ 2\end{matrix}\right] = \frac{Gm_1m_2}{r^2}\hat{r}_{12}$

   where $\hat{r}_{12} = \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_{12}|}$; $\vec{r}_{12} = (\vec{r}_2 - \vec{r}_1)$; $|\vec{r}_{12}| = r$

Moreoever $\vec{F}_{12} = -\vec{F}_{21}$

Notable points : (i) The gravitational force due to a hollow spherical shell on a particle at a point outside the shell is equal to that obtained by considering the entire mass of the shell as concentrated on the centre of the shell. (ii) The gravitational force on a particle at any point inside the hollow spherical shell is zero.

4. The value of G was first determined by Cavendish experimentally. At present the accepted value of G is 6.67×10^{-11} N m^2/kg^2.

5. The acceleration produced in the body due to the gravitational force is called the acceleration due to gravity (or gravitational acceleration) g. For a point on the surface of the Earth the formula for g is $g_e = \dfrac{GM_e}{R_e{}^2}$, and its value is 9.8 m/s^2.

The value of g at poles is slightly more than that at the equator, but the difference is very small.

The gravitational acceleration at a height h from the surface of the Earth is given by $g(h) = \dfrac{g_e}{\left(1 + \dfrac{h}{R_e}\right)^2}$. For very small heights from the Earth's surface we can take $g(h) \approx g_e$.

The gravitational acceleration at a depth d inside the Earth from its surface is given by $g(d) = g_e\left[1 - \dfrac{d}{R_e}\right]$. The gravitational acceleration at the Earth's centre is zero.

Because of the rotation of the Earth the effective gravitational acceleration on the surface of the Earth at a place with latitude λ, is given by

$$g' = g\left[1 - \frac{R_e\omega^2\cos^2\lambda}{g}\right]$$

6. "At a given point the gravitational force on a body of unit mass is known as the intensity of the gravitational field (I) at that point." $I = \dfrac{GM}{r^2}$

From this the gravitational force on the body of mass m at that point is F = (I)(m). The gravitational intensity at the Earth's centre is zero. The values of I and g are equal.

7. "The negative of the work done by the gravitational force in bringing a body of **unit mass** from infinite distance to the given point in the gravitational field is called the gravitational potential (ϕ) at that point." The gravitational potential at distance r (>R_e) from the centre of the Earth is $\phi = \frac{-GM_e}{r}$, and the gravitational potential at the surface of the Earth is $\phi_e = \frac{-GM_e}{R_e}$. The unit of gravitational potential is J kg^{-1} and its dimentional formula is $M^0L^2T^{-2}$.

"The negative of the work done by the gravitational force in bringing a body of given mass (m) from infinite distance to the given point in the gravitational field is called the gravitational potential energy (U) of the system of the Earth and that body, which is usually mentioned as the gravitational potential energy of that body at that point. The gravitational potential energy of the body of mass m at distance r from the centre of the Earth is $U = \frac{-GM_em}{r} = \phi m$ and the gravitational potential energy at the surface of the Earth is $U_e = \frac{-GM_em}{R_e} = \phi_e m$

The value of gravitational potential is not important but only the difference in the values is important.

For gravitational potential energy also only the difference in values is important.

8. For a body lying (stationary) on the surface of the Earth the total energy = its potential energy $= \frac{-GM_em}{R_e}$

$\therefore$ Its escape energy = binding energy $= \frac{GM_em}{R_e}$ and escape speed $v_e = \sqrt{\frac{2GM_e}{R_e}} = 11.2$ km/s

For the body on the surface of the moon the escape speed is 2.3 km/s.

9. For a satellite revolving around the Earth the orbital velocity $v_0 = \sqrt{\frac{GM_e}{r}}$ and the total energy of the satellite $= \frac{-GM_em}{2r}$

$\therefore$ Its escape energy = binding energy $= \frac{GM_em}{2r}$. For a geostationary satellite the time-period is 24 hours = 24 × 3600 s. It's height from the surface of the Earth is h = 35800 km (approx). They revolve in east-west direction in the equatorial plane.

Polar satellites revolve in north-south direction.

EXERCISES

Choose the correct option from the given options :

1. Which of the following has the unit N m^2/kg^2 ?

(A) linear momentum (B) gravitational force

(C) universal constant of gravitation (D) gravitational acceleration.

2. Using orbital radius r and the corresponding periodic time T of different satellites revolving around a planet, what would be the slope of the graph of log r – log T ?

(A) $\frac{3}{2}$ (B) 3 (C) $\frac{2}{3}$ (D) 2

3. If the gravitational acceleration at the Earth's surface is 9.81 m/s^2, what is its value at a height equal to the diameter of the Earth from its surface ?

(A) 4.905 m/s^2 (B) 2.452 m/s^2 (C) 3.27 m/s^2 (D) 1.09 m/s^2

4. If the gravitational potential at the Earth's surface is Φ_e, what is the gravitational potential at a height from Earth's surface equal to its radius ?

(A) $\frac{\Phi_e}{2}$ (B) $\frac{\Phi_e}{4}$ (C) Φ_e (D) $\frac{\Phi_e}{3}$

5. If we take the gravitational acceleration at the Earth's surface as 10 m/s^2 and radius of the Earth as 6400 km, the decrease in the value of the gravitational acceleration g at a depth of 64 km from its surface would be .... m/s^2.

(A) 0.1 (B) 0.2 (C) 0.05 (D) 0.3

6. What would be the fictitious (pseudo) acceleration of the body lying on the equator of Earth in the radial direction away from the Earth's centre due to its rotation ?

(A) ωR_e (B) $\omega^2 R_e$ (C) $\omega R_e^{\ 2}$ (D) $\omega^2 R_e^{\ 2}$

Where ω = angular speed of the Earth,

R_e = radius of the Earth.

7. For different satellites revolving around a planet in different circular orbits, which of the following shows the relation between the angular momentum L and the orbital radius r ?

(A) $L \propto \frac{1}{\sqrt{r}}$ (B) $L \propto r^2$ (C) $L \propto \sqrt{r}$ (D) $L \propto \frac{1}{r^2}$

8. At all points inside a uniform spherical shell......

(A) gravitational intensity and gravitational potential both are zero.

(B) gravitational intensity and gravitational potential both are non-zero.

(C) gravitational intensity is non-zero and gravitational potential is zero.

(D) gravitational intensity is zero and gravitational potential is non-zero but equal.

9. Which of the following alternatives represents the dimensional formulae of the gravitational potential and gravitational potential energy respectively ?

(A) $M^1L^1T^{-1}$, $M^1L^2T^{-2}$ (B) $M^0L^2T^{-2}$, $M^1L^2T^{-2}$

(C) $M^0L^2T^{-2}$, $M^1L^2T^2$ (D) $M^1L^2T^{-1}$, $M^2L^1T^{-1}$

10. For a planet revolving around the sun

(A) linear speed and angular speed are constant

(B) areal velocity and angular momentum are constant

(C) linear speed and areal velocity are constant

(D) areal velocity is constant but angular momentum changes.

11. Two satellites revolving around a planet in the same orbit have the ratio of their masses $\frac{m_1}{m_2} = \frac{1}{2}$. The ratio of their orbital velocities $\frac{v_1}{v_2}$ = ...

(A) 1 (B) $\frac{1}{2}$ (C) 2 (D) 4

12. If the time period of a satellite in the orbit of radius r around a planet is T, then the time period of a satellite in the orbit of radius $4r$ is T' = ...

(A) 4T (B) 2T (C) 8T (D) 16T

13. Radii of two planets are r_1 and r_2 respectively and their densities are ρ_1 and ρ_2 respectively. The gravitational accelerations on their surfaces are g_1 and g_2 respectively. $\therefore \frac{g_1}{g_2}$ = ...

(A) $\frac{r_1\rho_1}{r_2\rho_2}$ (B) $\frac{r_2\rho_2}{r_1\rho_1}$ (C) $\frac{r_1}{r_2}\cdot\frac{\rho_2}{\rho_1}$ (D) $\frac{r_2}{r_1}\cdot\frac{\rho_1}{\rho_2}$

14. What kind of relation exists between the kinetic energy (E_k) and the orbital radius (r) of the satellites revolving around the Earth ?

(A) $E_k \propto r$ (B) $E_k \propto \frac{1}{r}$ (C) $E_k \propto r^2$ (D) $E_k \propto \frac{1}{r^2}$

15. If the Earth shrinks (but not cut !) in such a way that its radius becomes $\frac{R_e}{2}$ from R_e, what can we say about the values of gravitational acceleration g and the gravitational potential ϕ, at a point at distance R_e from its centre in the two cases ?

(A) the values of g and ϕ both become half.

(B) the value of g becomes half and the value of ϕ remains the same as before.

(C) the value of g remains the same as before and the value of ϕ becomes half.

(D) the values of g and ϕ both remain the same as before.

ANSWERS

1. (C) **2.** (C) **3.** (D) **4.** (A) **5.** (A) **6.** (B)
7. (C) **8.** (D) **9.** (B) **10.** (B) **11.** (A) **12.** (C)
13. (A) **14.** (B) **15.** (D)

Answer the following questions in short :

1. Out of the equator and the pole of the Earth where does the gravitational acceleration g have a larger value ? Why ?

2. Give the values of gravitational acceleration and the gravitational intensity at the centre of the Earth.

3. The magnitude of gravitational intensity at a point is 0.7 N/kg. What would be the magnitude of the gravitational force on a body of 5 kg mass at this point ? [**Ans. :** 3.5 N]

4. "The value of the escape velocity v_e for a stationary body on the surface of a planet is directly proportional to the mass and the radius of the planet". Is this statement true ? If not, correct it and write.

5. Give two uses of a polar statellite.

6. If the kinetic energy of a statellite is 6×10^9 J, what is its potential energy ? What is its total energy ?

7. The potential energy of a statellite is -8×10^9 J. What is its binding energy (or escape energy) ?

8. For different planets the masses are M_1, M_2, M_3, the radii are R_1, R_2, R_3 and the gravitational accelerations at their surfaces are g_1, g_2, g_3 respectively. From the following graphs for them, arrange their masses in the descending order.

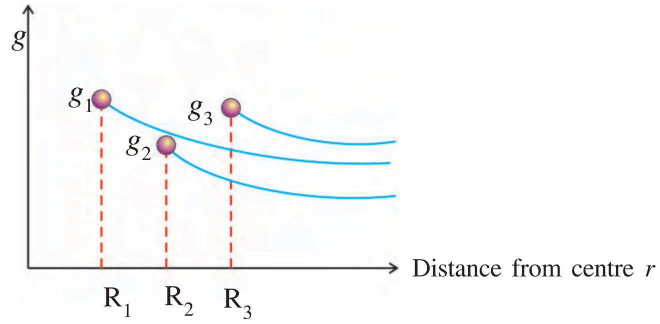


Figure 3.20

(**Hint :** Think from $g = \frac{GM}{r^2}$ for some definite distance $r > R_3$]

[**Ans. :** $M_3 > M_1 > M_2$]

Answer the following questions :

1. State Newton's universal law of gravitation. Write and explain its formula in the vector form.

2. Obtain the formula for the orbital velocity of a statellite of the Earth.

3. Obtain the formula for the time-period of a statellite of the Earth.

4. For an object lying on the surface of the Earth, obtain the formula for its escape speed.

5. Obtain the formula for the gravitational acceleration at a depth d from the Earth's surface.

6. Obtain the formula for the total energy of a satellite.

7. Define gravitational intensity. Write its formula. Give its unit and the dimensional formula.

8. Define gravitational potential. Give its unit and dimensional formula.

9. Define gravitational potential energy. Give its unit and dimentional formula.

10. Obtain the formula for the Earth's gravitational potential at distance r ($> R_e$) from its centre.

11. Obtain the formula for the variation in effective gravitational acceleration with latitude due to Earth's rotation.

12. The semi-major axis of a planet revolving around sun is a and at this distance the mechanical energy of the planet is $\frac{-GMm}{2a}$ where M = mass of sun and m = mass of the planet. Find its velocity when its distance from the sun is r.

[**Ans. :** $v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}$]

[**Hint :** Use the law of conservation of mechanical energy].

13. Give the distinguishing points between g and G.

Solve the following problems :

1. A space craft goes from the Earth directly to the sun. How far from the centre of the Earth the gravitational forces exerted on it by the Earth and by the sun would be of equal magnitude ? The distance between the Earth and the sun is 1.49×10^8 km. The masses of the sun and the Earth are 2×10^{30} kg and 6×10^{24} kg respectively. [**Ans. :** 25.7×10^4 km]

2. If the Earth were a sphere made completely of gold (!), what would have been the magnitude of gravitational acceleration on its surface ? The radius of the Earth = 6400 km., density of gold = 19.3×10^3 kg/*m*^{3}, G = 6.67×10^{-11} Nm^2/kg^2.

[**Ans. :** 34.49 m s^{-2}]

3. The radius of the circular orbit of the Earth revolving around the sun is 1.5×10^8 km. The orbital speed of the Earth is 30 km/s. Calculate the mass of the sun from this data. G = 6.67×10^{-11} Nm^2/kg^2.

[**Ans. :** 2.02×10^{30}kg]

4. A satellite revolves around the Earth at a height from surface equal to the radius of the Earth. Calculate its (i) orbital speed (ii) time period. Take G = 6.67×10^{-11} Nm^2/kg^2, radius of the Earth = 6400 km and mass of the Earth = 6×10^{24} kg.

[**Ans. :** $v_0 = 5.59 \times 10^3$ m/s, T = 14370 s]

5. A satellite of 200 kg revolves around the Earth at a height of 1000 km from the surface of the Earth. Calculate (i) escape energy (ii) escape speed of this satellite. Take G = 6.67×10^{-11} Nm^2/kg^2; The radius of the Earth = 6400 km and the mass of the Earth = 6×10^{24} kg.

[**Ans. :** 5.4×10^9 J; $v_e = 7.35 \times 10^3$ m/s.]

6. An artificial satellite revolves around the Earth, remaining close to the surface of the Earth. Show that its time-period is $T = 2\pi\sqrt{\frac{R_e}{g}}$.

7. Show that the ratio of the linear (orbital) speed of a satellite revolving round the Earth and remaining close to the surface of the Earth to the escape speed of an object lying on the Earth is equal to $\frac{1}{\sqrt{2}}$.

8. The mass and radius of the Earth are M_1, R_1 and those for the moon are M_2, R_2 respectively. The distance between their centres is *d*. With what velocity should an object of mass *m* be thrown away from the mid-point of the line joining them so that it escapes to infinity ?

[**Ans. :** $v_e = \sqrt{\frac{4G}{d}(M_1 + M_2)}$]

9. Consider different planets revolving in different circular orbits around a star of very large mass. If the gravitational force between the planet and the star varies as $r^{-5/2}$, r = distance between them how does the square of the orbital period T depend on the distance r ? [**Ans. :** $T^2 \propto r^{7/2}$]

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CHAPTER 4

MECHANICAL PROPERTIES OF SOLIDS

4.1 Introduction

In this chapter, we shall study the structure of solid substances and their mechanical properties. We shall study elasticity, which is one of the important mechanical properties of solids. In the last two decades of the twentieth century many advances have taken place in solid state physics and liquid state physics. It has become possible to determine many physical quantities related to elasticity with the help of computers along with experimental techniques. In this chapter our discussion shall be confined to primary information about elasticity and practical applications of elasticity of solids.

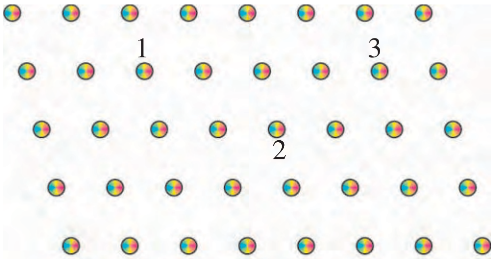
4.2 Solids

One of the characteristics of solids is that in certain physical condition the average distance between constituents remains constant. They may oscillate about their mean positions with the amplitude depending on their temperature. But average distance between any two constituent particles remains constant. If the distance between particles in equilibrium position is changed, internal forces acting between them also change in such a way that the average distance between them remains the same. Thus, when the particles are displaced from their mean positions, a force comes into existence, trying to bring the particles back to their equilibrium position. Such a force is called restoring force.

Solids can be broadly classified in three categories, according to arrangement of constituent particles. (Such classification can be done on the basis of some other criteria also.) They are (i) Crystalline Solids (ii) Non-crystalline Solids (iii) Semi Crystalline Solids.

(i) Crystalline Solids : Arrangement of constituent particles in this type of solids, is a regular geometrical array in three dimensions. To understand this arrangement of points in two dimensions is shown in figure.

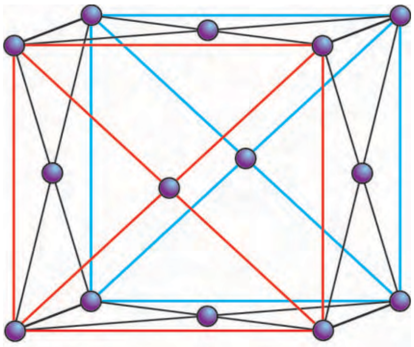
This is a very small part of an infinite arrangement of points. Here, if you observe from keeping yourself at any point 1, 2, or 3, the arrangement of points appears identical. This type of arrangement of points in three dimensions is called a lattice. Lattice is a mathematical concept. If groups of atoms, molecules or ions identical in all respects (which are known as basis) are placed on the lattice points, crystal is formed. Depending on the type of interaction among the constituent particles various types of crystals are formed. But under given condition a solid assumes the constitution for which its internal energy is the minimum.



Lattice

Figure 4.1

A crystal can also be thought of as made up of many identical blocks. One such building block of constituent particles (ions) of copper ions is shown in Figure 4.2. Here, in this arrangement one ion is on every vertex of a cube and one ion is on the centre of each face of the cube. Putting such units side by side in three dimensions, we get the crystal of copper.



Unit cell of crystal of copper

Figure 4.2

A branch of physics in which crystal structures are studied, is called crystallography. Study of crystal structure can be carried out using x–rays, electron beams or neutron beams.

Crystalline solids have definite melting point due to **long range order** existing in them. Crystalline solids are classified into four groups, depending on the type of constituent particles and the bonding existing between them. They are as follows :

Molecular Solids : Constituent particles of such solids are molecules. Molecules are formed due to covalent bonds (sharing of electrons). Molecule may be a polar molecule or a non-polar molecule. If in a molecule, centre of positive charges and centre of negative charges coincide, the molecule is said to be non-polar molecule, else it is a polar molecule. Molecules of iodine (I_2), phosphorus (P_4) and sulphur (S_8) are non-polar and molecules of H_2O, CO_2 are polar molecules. If a solid consists of polar molecules, dipole-dipole attraction force is responsible for formation of such a solid. Vander-Walls force is responsible for the other types of molecular solid. Since these intermolecular forces are weak, melting point and boiling point of such solids are low compared to other solids. They are poor conductors of heat and electricity. Examples of such solids are S_8, P_4, and I_2.

Ionic Solids : In such solids the constituent particles are ions. Under the resultant effect of electrostatic attraction and quantum mechanical repulsion among these ions, bonds are formed. These attractive forces are quite strong, so this solid material is usually hard and has fairly high melting point. Ionic solids are poor conductors of electricity e.g. NaCl.

Covalent Solids : The constituent particles in such solids are atoms. Atoms in such solids are connected to the nearest neighbours by covalent bonds. If any atom is imagined at the centre of a tetrahedron, its four nearest neighbours are at the vertices of this tetrahedron. Due to extension of such an arrangement in three dimensions, covalent solid is formed. Diamond, silicon, germenium etc. are such solids. Such solids are quite hard and have high melting points. Such solids behave as semiconductors. They are also known as **network solids.**

Metallic Solids : If on lattice points of solids, positive ions of metals are placed, metallic solids are formed. During formation of metallic solid, atoms lose their valence electrons and they become positive ions. Such free electrons perform random motion in space between ions. Hence, such solids are good conductors of heat and electricity.

(ii) Non-crystalline substances : In some solids the constituent particles are not arranged in a regular array. Such solids are called non-crystalline solids. e.g. wood, glass.

There are certain substances which are capable of forming a crystal structure. But in the molten state of such a substance, at a temperature higher than its solidification temperature, is cooled rapidly, its constituent particles do not get enough time to adjust themselves in a regular array. Hence, they form a solid substance having **short range order.** Such substances are called **glassy** or **amorphous solids.** Here, meaning of short range is that few constituent particles (say four to five) are bonded together to form a structure and exhibit a **local ordering.** Such independent units are randomly arranged to form the extended solid.

It should be clearly noted that given an opportunity (i.e. allowing sufficient time) such glassy substances could have formed a crystalline structure. But there are certain substances which always remain non-crystalline, no matter whatever opportunity is given to them.

Here, question arises as to if glassy substances do not have long range order just like liquids, why they can't flow like liquid ? The intermolecular forces in glassy solids are much stronger than those in liquids. Due to this reason glassy substance cannot flow like a liquid. Now it would have been clear that the intermolecular (or interatomic) forces play an important role in deciding the phase of matter.

The amorphous solids do not have a definite melting point. Different bonds have different strengths and as the material is heated the weaker bonds break earlier starting the melting process and it becomes soft. The stronger bonds break at higher temperatures to complete the melting process.

(iii) Semi-crystalline substances : The molecule of polyethylene used extensively in our daily life, is represented by $(-CH_2-)_n$ where n is the number of repeated units. These molecules are long chain molecules and are called macromolecules. Protein molecules fall under this category. When a substance, made from such molecules, is cooled from the liquid phase or the molten phase, the molecules acquire a configuration such that in some regions the molecular chain is arranged in a regular manner and in other regions in an irregular manner. Such substances are called semi-crystalline or polymers and they are very important in modern material science.

4.3 Elasticity

In mechanics, we have seen that force can change the state of motion of a body and is also capable of changing its shape. But the second effect of force is not studied so far. In fact an ideal rigid body is an imagination only. In reality, all the solids can be deformed under the effect of an external force. The extent of their deformation depends on their ability to resist the change. All the bodies cannot resist such a change equally. Some of the bodies which are deformed on application of an external force are able to restore their original shape when the deforming force is removed. The extent to which the shape of a body is restored, when the deforming force is removed depends on the type of material. **The inherent property of a body due to which body tries to restore the normal (natural) shape or to oppose the change in shape is known as elasticity.**

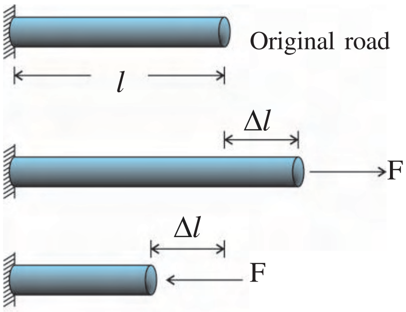
If a body can completely regain its original shape after removal of the deforming force, it is called a **perfect elastic body.** If a body remains in the defromed state and does not even partially regain its original shape after the removal of deforming force, it is called a perfect **non-elastic body** or a **plastic body.** If a body partially regains its original shape it is called **partially elastic.** Most of the bodies are found to be partially-elastic.

In order to study elasticity, we have to define two useful quantities viz. stress and strain. Let us begin with strain.

4.3.1 Strain :

When an external force is applied on a body its length, volume or shape changes and corresponding to each of these we define strain (ε). Strains are of three types. Longitudinal strain, Volume strain and shearing Strain.

(i) Longitudinal Strain : The ratio of change in length of a body (Δl), when deforming force is applied to the original length (l) (i.e. fractional change in length), is defined as longitudinal strain.



Longitudinal strain

Figure 4.3

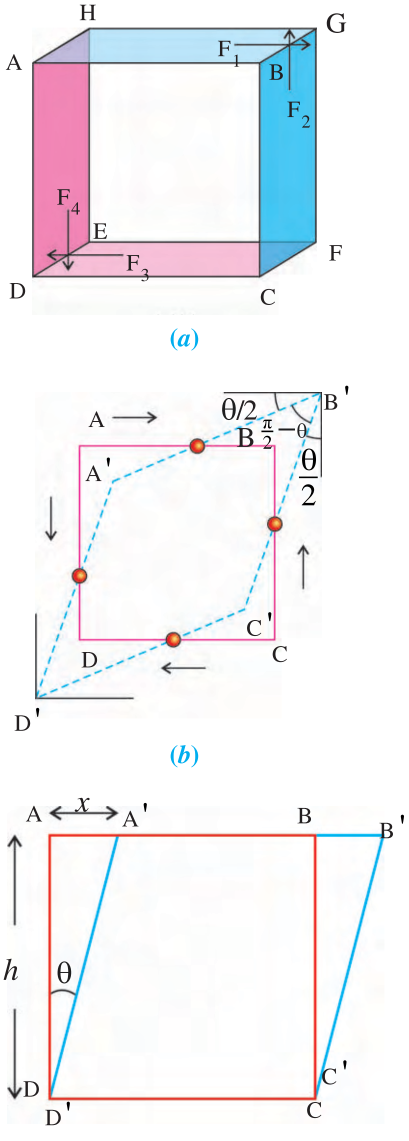
Thus longitudinal strain $\varepsilon_l = \frac{\Delta l}{l}$ (4.3.1)

If the length of a rod increases, the corresponding longitudinal strain is called **tensile strain** and if due to application of external force, length of the rod reduces the strain is called **compressive strain.**

(ii) Volume Strain : When a body is acted upon by the forces everywhere on its surface in the direction perpendicular to the surface, the volume of the body changes. The fractional change in the volume of a body is defined as volume strain. If V is the volume of undeformed body and ΔV is the change in volume, then

volume strain $\varepsilon_V = \frac{\Delta V}{V}$ (4.3.2)

(iii) Shearing Strain : A force tangential to a cross-section of a body produces the change in its shape. Now, the change in shape can't be measured quantitatively unlike length or volume. Hence, in order to understand shearing strain consider Figure 4.4(*a*). In this figure, a body with square cross-section is shown. Suppose the forces of the same magnitude parallel to (i.e. tangentially) the surfaces AHGB, BGFC, DEFC and DAHE are applied on it. Note that the resultant of these forces is zero as well as the resultant torque produced by these forces is also zero. Hence the object is in translational and rotational equilibrium. Because of the couples of forces acting in mutually opposite directions, these forces are displaced and deformation in its shape takes place. Figure 4.4(*b*) shows the shape that plane ABCD assumes due to such deformation. For the sake of clarity the deformation has been magnified.



(*a*)

(*b*)

(*c*)

Shearing strain

Figure 4.4

Due to such deformation, the angle between AB and BC is no more $\frac{\pi}{2}$ but it becomes $\frac{\pi}{2} - \theta$. To measure this deformation, we rotate A'B'C'D' (about an axis which is perpendicular to its plane) such that its edge D'C' coincides with its undeformed position DC. The drawing is shown in Figure 4.4(*c*). Here, tanθ is called shearing strain. If the value of θ (in radian) is small, *tan*$\theta \approx \theta$ and then θ is called the shearing strain (ε_S).

Shearing strain $\varepsilon_S = \theta = \frac{x}{h}$ (4.3.3)

All types of stain are dimensionless physical quantities.

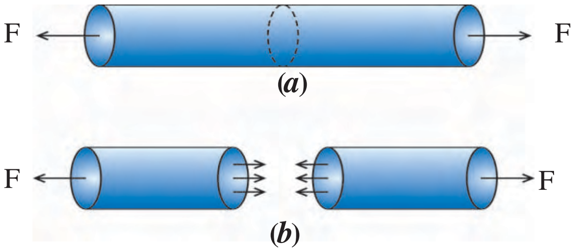
4.3.2 Stress : Elastic bodies regain their original shape after removal of deforming force. This is possible only if a restoring force, opposing the strain, arises in it. The restoring force arising per unit cross sectional area of a deformed body is defined as stress. If the body is in static equilibrium, external force is equal to restoring force. If the restoring force is F and cross sectional area is A, stress (σ) is given by,

$$\text{stress } \sigma = \frac{\text{Force}}{\text{Area}} = \frac{F}{A} \qquad (4.3.4)$$

SI unit of stress is Nm^{-2} or pascal (Pa) . Its dimensional formula is $M^1L^{-1}T^{-2}$.

(i) Longitudinal stress

Consider a bar and its section (shown by a dotted line) as shown in the Figure 4.5.



Tensile stress

Figure 4.5

The rod is in equilibrium under the effect of two external forces which are equal and in mutually opposite directions. In this condition, the portion of the rad situated on the left and right sides of the above mentioned section, pull this section in mutually opposite directions with forces having the same magnitude.

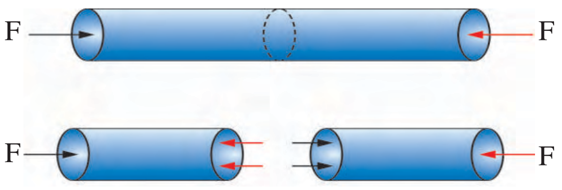
If the section is not near the ends of the rod, these pulls are uniformly distributed over the entire cross-section. Such forces are shown in Figure 4.5(*b*). Here, for better understanding, the portion separated by this section are shown separately.

When external force is applied on the rod the inter-molecular distance in it changes. So, the forces are so produced that they try to bring the molecules back to their equilibrium position. These forces are restoring forces. In Figure 4.5(*b*) restoring forces are shown everywhere on the cross section by small arrows. Since the restoring force arises between each pair of molecules, it is distributed uniformly over the entire section.

The restoring forces, produced due to deformation of a body under the effect of an external force, at different cross-sections are the same but as they are distributed over different cross sectional area it becomes necessary to mention the area of cross-section also.

Here, in the discussion so far, we have considered external force which causes increase in the length of the rod. Resulting stress due to such a force is called tensile stress.

If, due to application of external forces length of the rod decreases, the resulting stress is called compressive stress.



Compressive stress

Figure 4.6

(ii) Volume or Hydraulic Stress : Suppose a body is subjected to forces acting over the entire surface of it.

The forces are perpendicular to the surface locally and also the magnitude of force acting on an element of the surface is proportional to magnitude of the area element. Application of such forces cause change in the volume of the body, and as a result volume strain is produced in the body. When a solid body is immersed in a fluid such a situation arises.

If the pressure at the location of immersed solid is P, the force on any area A is PA, directly perpendicular to area. In equilibrium condition the force per unit area is volume stress.

$$\text{Volume stress } \sigma_v = \frac{F}{A} = \frac{PA}{A} = P \qquad (4.3.5)$$

Thus volume stress is same as the pressure. Hence, we can say that pressure is a specific type of stress due to which only the volume of a body changes.

(iii) Shearing Stress (Tangential Stress) : If the force acting on a body is tangential to a surface of the body as shown in Figure 4.4, it causes shearing strain in the body and the corresponding stress is called shearing stress. Thus,

$$\text{Shearing Stress} = \frac{\text{Tangential force } (F_t)}{\text{Area } (A)} \tag{4.3.6}$$

It may also happen that the force acting on a body is neither perpendicular nor tangential to the surface. In this case, components of force perpendicular to the surface and tangential to the surface can be considered as shown in Figure 4.7.

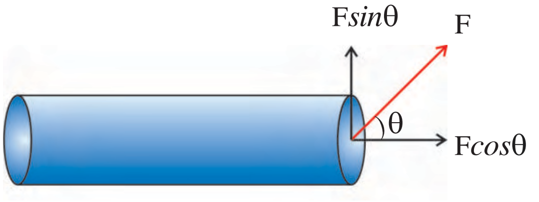


Figure 4.7

Here, force acting on a body is shown. This force makes an angle θ with the area vector (a vector having magnitude equal to area of the surface outward perpendicular to the surface). As shown in the diagram $F\cos\theta$ is the component perpendicular to the surface and $F\sin\theta$ is the component tangential to the surface. So, $F\cos\theta$ has tensile effect whereas $F\sin\theta$ has shearing effect. In this case both tensile stress and shearing stress (also tensile strain and shearing strain) are produced in the body.

Difference between pressure and stress : Pressure is the force per unit area. Though the dimensions of pressure and stress are same, they aren't two names of the same quantity.

When the whole body is acted upon by forces, acting perpendicularly everywhere on it, the force per unit area is called the pressure, (Figure 4.8).

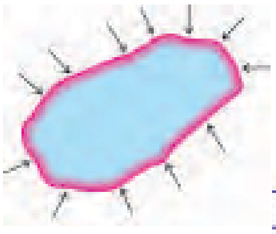


Figure 4.8

Figure 4.9

Stress is also a force per unit area but it can be different on different surfaces. Also it is not necessary that the force should be perpendicular to the surface. It is also possible that there is stress on one surface and there is no stress on the other surface (Figure 4.9)

Illustration 1 : As shown in the Figure 4.9, 10 N force is applied at two ends of a rod. Calculate tensile stress and shearing stress for section PR Area of cross-section PQ is 10 cm^2.

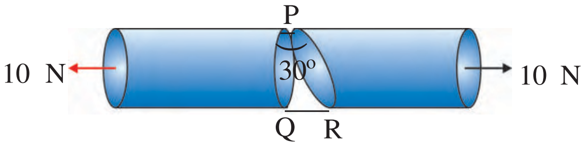


Figure 4.10

Solution :

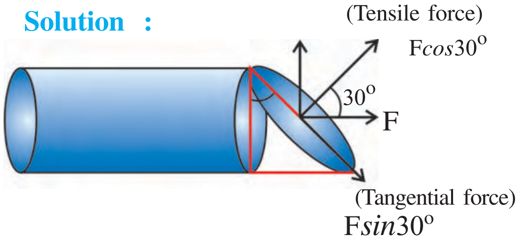


Figure 4.11

Here the angle between cross-section PQ and section PR is 30°, So,

$$\frac{\text{Magnitude of area of cross-section PQ}}{\text{Magnitude of area of section PR}} = \cos 30$$

$$= \frac{\sqrt{3}}{2}$$

Magnitude of Area of Section PR

$$= \left(\frac{\text{Magnitude of area of cross-section PQ}}{\frac{\sqrt{3}}{2}} \right)$$

$$= \frac{2 \times 10 \times 10^{-4}}{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}} \times 10^{-3}\, m^2$$

Also angle between force F and area vector of section PR is 30°. (How ? Think over.)

So, for section PR tensile force is

$$F_l = F\cos 30 = 10 \times \frac{\sqrt{3}}{2} = 5\sqrt{3}\,\text{N}$$

and tangential force

$$F_t = F\sin 30 = 10 \times \frac{1}{2} = 5 \text{ N}$$

∴ For section PR,

$$\text{Tensile stress}(\sigma) = \frac{\text{Tensile force}}{\text{Area of section PR}}$$

$$= \frac{5\sqrt{3}}{\frac{2}{\sqrt{3}} \times 10^{-3}}$$

$$= 7.5 \times 10^3 \text{ N/m}^2$$

$$\text{Shearing stress } \sigma_t = \frac{\text{Tangential force}}{\text{Area of section PR}}$$

$$= \frac{5}{\frac{2}{\sqrt{3}} \times 10^{-3}}$$

$$= \frac{5\sqrt{3}}{2} \times 10^3 = 4.33 \text{ N/}m^2$$

4.4 Relation Between Longitudinal Stress and Longitudinal Strain

To study the relation between longitudinal stress and longitudinal strain, a wire is stretched using external force. Corresponding to various values of stress, fractional values of strain (or percentage values) are found out. The relation between the stress and strain can be studied by plotting stress-strain (%) graph. Such a graph is shown in Figure 4.12.

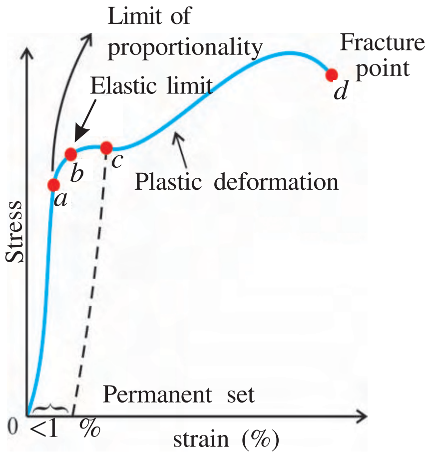


Figure 4.12

During the initial portion of the curve strain is less than 1% (i.e. from 0 to *a*) stress and strain are proportional to each other. Here point *a* is called limit of proportionality.

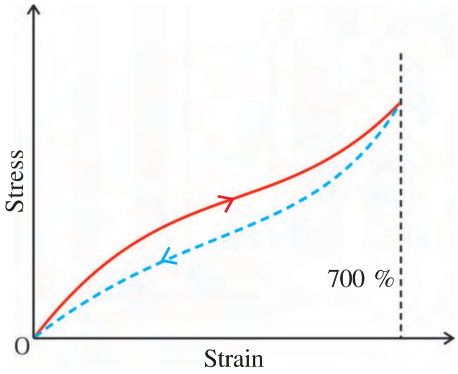
From points *a* to *b* on the graph, stress and strain are not directly proportional, but if the load is removed at any point between 0 and *b* the curve will be retraced and the material would return to its original length. In this sense the material is said to be elastic or to exhibit elastic behaviour upto point *b*. The point *b* is called the **elastic limit** or the **yield point.**

The strain increases rapidly between points *b* and *c* when the load is removed at some point between *b* and *c*, the material traces the path shown by dotted line and acquires the state in which a permanent deformation (defect) is produced. The material is said to have **permanent set** in this state.

Further increase of deforming force beyond point C results in larger increase in strain. In this condition the planes of the material having maximum shearing strain slide over each other. This phenomenon is called **plastic deformation.**

At point *d*, the body gets fractured and hence it is called the **fracture point.** The stress corresponding to *d* is called the **breaking stress.** If in a metal large plastic deformation takes place between elastic limit *b* and fracture point *d* the metal is said to be **ductile.** If fracture occurs soon after the elastic limit is passed, the metal is said to be **brittle.**

However, some materials (like rubber) behave differently from what have been described above. We know that some types of rubber can be pulled to several times its length and still returns to its original shape. Figure 4.13 shows a graph of stress versus strain for a typical sample of vulcanized rubber. It is amusing to see that up to 700% strain can be produced in it. Substances in which very large strain can be produced are called **elastomers.** In our body tissues of aorta (artery carying blood from heart to various parts of body) is an example of elastomer.



Hysteresis for a vulcanized rubber

Figure 4.13

The noteworthy points of this graph are (i) During no portion of the curve, stress is directly proportional to strain. (ii) When deforming force is removed the object returns to the original shape, but not along the original path. The work done by the material in returning to its original shape is less than the work required to deform it. This means that certain amount of energy is absorbed by the material. This energy gets dissipated in the form of heat. This phenomenon is called **elastic hysteresis.**

Elastic hysteresis has an important application in shockobsorbers. If a padding of vulcanized rubber is placed between a vibrating system and the support on which it is placed, the rubber is compressed and released in every cycle of vibration. As energy is absorbed in the rubber in each cycle, only a part of the energy of vibration is transmitted to the support. Hence, effect of vibration on the support is reduced.

4.5 Hooke's Law and Elastic Modulii

In 1678 Robert Hooke showed, experimentally that "For small deformations the stress and strain are directly proportional to each other". This statement is known as Hooke's Law so,

stress $\propto$ strain.

$\therefore$ stress = constant $\times$ strain

$$\therefore \sigma = k\varepsilon \qquad (4.5.1)$$

Here constant k appearing in the equation 4.5.1 is known as modulus of elasticity. Its unit is Nm^{-2} or Pa.

Hooke's Law is an empirical law, also found to be valid for very small strain (about 1%) as shown in Figure 4.12 for most of the materials. However, for substances like rubber this linear relationship is not exhibited.

4.5.1 Young's Modulus :

We have seen that for small strain, stress and strain are directly proportional to each other tensile stress and tensile strain are considered, equation 4.5.1 can be written as,

$$\sigma_l = Y\varepsilon_l \qquad (4.5.2)$$

Here the elastic modulus is known as Young's Modulus (Y)

Experimental arrangement to determine Y-Young's modulus is shown in the Figure 4.14.

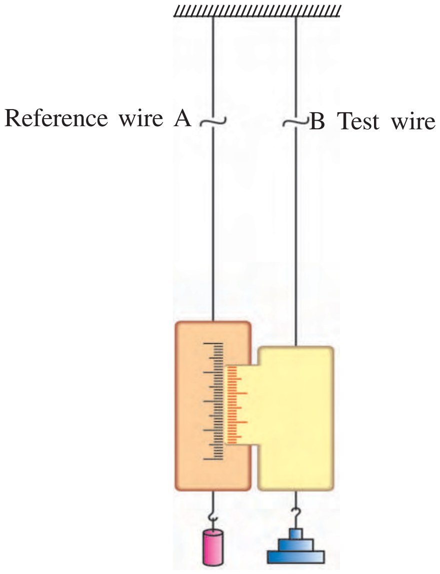


Figure 4.14

Wire A is called the reference wire and wire B is the test wire. In the hook, attached to wire A some fixed mass is suspended at the end of wire A.

Different masses(m) are suspended at the end of test wire B and corresponding to the resulting tensile force (mg), increase in length (Δl) is measured using the vernier's scale, attached to reference wire.

$$\text{Here, } \sigma_l = \frac{\text{tensile force}(F_l)}{\text{Area (A)}} = \frac{mg}{\pi r^2} \qquad (4.5.3)$$

where r is radius of test wire.

And tensile strain $\varepsilon_l = \frac{\Delta l}{l}$ (4.5.4)

where l is the original length of test wire.

From equations (4.5.2), (4.5.3) and (4.5.4) we get

$$\frac{mg}{\pi r^2} = Y\frac{\Delta l}{l}$$

$$\therefore\ Y = \frac{mgl}{\pi r^2 \Delta l} \quad (4.5.5)$$

Young's Modulus is the characteristic of material of the substance.

In most of the substance the Young modulus is found to be of the same value for tensile stress and for the compressive stress but for bones and concrete it is not so.

> **Illustration 2 :** An object of mass 5 kg is suspended by a copper wire of length 2 m and diameter 5 mm. Calculate the increase in the length of the wire. In order not to exceed the elastic limit, what should be the minimum diameter of the wire ? For copper, elastic limit = 1.5×10^9 dyne/cm². Young's modulus. (Y) = 1.1×10^{12} dyne/cm²

Solution :

$Y = 1.1 \times 10^{12}$ dyne/cm

$L = 2$ m $= 200$ cm

$d = 5$ mm $= 0.5$ cm

$\therefore\ r = 0.25$ cm

$F = mg = 5 \times 10^3 \times 980$ dyne

l = increase in length

$$Y = \frac{FL}{\pi r^2 l}$$

$$\therefore\ l = \frac{FL}{\pi r^2 Y}$$

$$= \frac{5.0 \times 10^3 \times 980 \times 200}{3.14 \times (0.25)^2 \times 1.1 \times 10^{12}}$$

$$= 4.99 \times 10^{-3} \text{ cm}$$

For copper, elastic limit = 1.5×10^9 dyne/cm² (given)

If the minimum diameter required is d', then

The maximum stress produced in the wire

$$= \frac{F}{\pi\left(\frac{d'}{2}\right)^2} = \frac{4F}{\pi d'^2} = 1.5 \times 10^9$$

$$\therefore\ d'^2 = \frac{4 \times 5 \times 10^3 \times 980}{3.14 \times 1.5 \times 10^9}$$

$$= 41.6 \times 10^{-4}$$

$$\therefore\ d' = 6.45 \times 10^{-2} \text{ cm}$$

4.5.2 Bulk Modulus :

In equation 4.5.1, if the stress and strain considered are volume stress and volume strain, the elastic modulus is known as **bulk modulus (B).**

Thus for small deformation, ratio of volume stress to volume strain is called bulk modulus.

So, Bulk modulus $= \frac{\text{Volume stress}}{\text{Volume strain}}$.

$$\therefore\ \text{Bulk modulus B} = -\frac{P}{\left(\frac{\Delta V}{V}\right)} \quad (4.5.6)$$

Here, the negative sign makes B positive as volume of the body decreases on application of pressure.

The reciprocal of bulk modulus is called **compressibility (K).**

4.5.3 Modulus of Rigidity (Shear modulus) :

Ratio of shearing stress to shearing strain for small deformation of a body is called modulus of rigidity (η). Thus, from equation 4.3.3.

Modulus of rigidity (η)

$$= \frac{\text{shearing (Tangential) stress}}{\text{shearing strain}}$$

$$= \frac{F_t/A}{\theta} \quad \text{But, } \theta = \frac{x}{h}$$

$$\therefore\ \eta = \frac{F_t/A}{x/h}$$

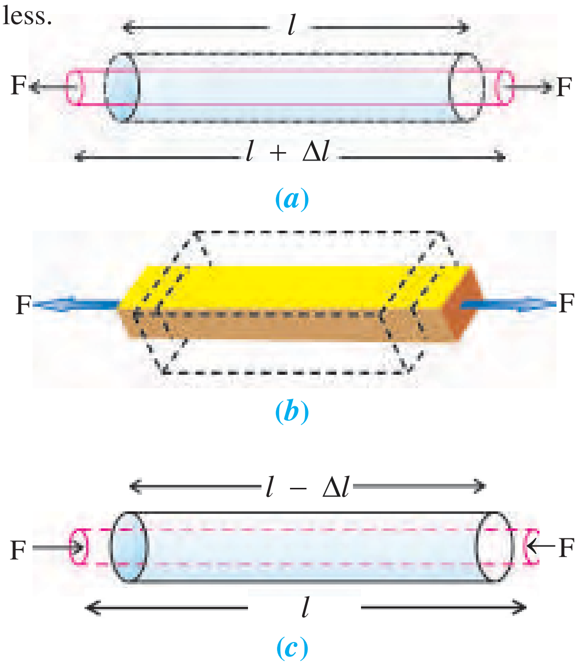
$$\therefore\ \eta = \frac{F_t h}{Ax} \quad (4.5.7)$$

4.6 Poisson's Ratio

When a tensile force is applied on an object, its length increases but the dimensions perpendicular to direction of tensile force reduce.

Similarly, if the body is subjected to compressive force, the length decreases and its dimensions perpendicular to compressive force increase. Such a change in lateral dimensions perpendicular to the applied force) per unit original dimension is called lateral strain.

The ratio of lateral strain to longitudinal strain is known as Poisson's ratio. It is denoted by μ. Since it is ratio of strains, it is dimensionless.



Changes in dimension due to stress

Figure 4.15

As shown in the Figure 4.15 in case of a cylindrical rod under tensile force

$$\text{longitudinal strain} = \frac{\Delta l}{l}$$

$$\text{and lateral strain} = \frac{\Delta d}{d}$$

where d is the diameter of the rod.

$$\mu = \frac{\text{Lateral strain}\left(\frac{\Delta d}{d}\right)}{\text{Longitudinal strain}\left(\frac{\Delta l}{l}\right)}$$

$$\therefore \frac{\Delta d}{d} = -\mu\frac{\Delta l}{l} \therefore \frac{\Delta r}{r} = -\mu\frac{\Delta l}{l} \quad (4.6.1)$$

Here, negative sign appears due to opposite nature of changes in lateral and tensile dimensions. If a bar having rectangular cross section is subjected to tensile force, there will be increase in its length and decrease in breadth and thickness. So, lateral strains would be $\frac{\Delta b}{b}$ and $\frac{\Delta h}{h}$ where b is the breadth and h is thickness of the bar and Δb and Δh are corresponding changes in them.

$$\text{So, } \frac{\Delta b}{b} = -\mu\frac{\Delta l}{l}$$

$$\text{and } \frac{\Delta h}{h} = -\mu\frac{\Delta l}{l} \quad (4.6.2)$$

Change in volume due to longitudinal forces :

Due to application of tensile force, lateral dimension decreases and length increases. As a result there is a change in its volume (usually volume increase). Let us consider the case of a cylindrical rod of length l and radius r.

$$\text{Since } V = \pi r^2 l$$

$\therefore \frac{\Delta V}{V} = 2\frac{\Delta r}{r} + \frac{\Delta l}{l}$ (for very small change)

From equation 4.6.1

$$\therefore \frac{\Delta V}{V} = -2\mu\frac{\Delta l}{l} + \frac{\Delta l}{l} \quad (4.6.3)$$

$$\therefore \frac{\Delta V}{V} = \frac{\Delta l}{l}(1 - 2\mu)$$

$$\therefore \frac{\Delta V}{V} = \varepsilon_l\ (1 - 2\mu) \quad (4.6.4)$$

Equation 4.6.4 suggests that since ΔV > 0, value of μ cannot exceed 0.5.

Here, we have discussed the case of a cylindrical rod. However, the results obtained here are also valid for body having other types of cross-sections.

Illustration 3 : A bar is subjected to tensile force. Show that the rate of change in volume of the bar with respect to length is $\frac{\Delta V}{\Delta l}$ = A(1 − 2μ) where A is cross sectional area of the bar. Here changes are small.

Solution : From equation 4.6.4 we have,

$$\frac{\Delta V}{V} = \varepsilon_l\ (1 - 2\mu)$$

Since volume = cross sectional Area (A) × length (l)

$$\therefore \frac{\Delta V}{A\,l} = \frac{\Delta l}{l}(1 - 2\mu)$$

$$\therefore \frac{\Delta V}{\Delta L} = A\,(1 - 2\mu)$$

Elastic constants for some materials are given in the Table 4.1.

Table 4.1

Elastic Constants (For Information Only)

Material	Young modulus $\times 10^{11}$Pa	Shear modulus $\times 10^{11}$Pa	Bulk modulus $\times 10^{11}$Pa	Poisson's Ratio
Aluminium	0.7	0.3	0.7	0.16
Brass	0.91	0.36	0.61	0.26
Copper	1.1	0.42	1.4	0.32
Iron	1.9	0.70	1.0	0.27
Steel	2.0	0.84	1.6	0.19
Tungston	3.6	1.5	2.0	0.20

4.7 Elastic Potential Energy

When external force is applied to an object, it gets deformed. Also restoring force is developed in it. So deformation takes place against this restoring force. Thus, work is done against this restoring force. This work is stored in the object in the form of elastic potential energy. Remember, this potential energy acquired by the body is due to the new configuration of the body.

Let us obtain the expression for the potential energy gained by the object due to a tensile force.

Consider a bar of length L and cross sectional area A. Suppose due to tensile forces length of the bar increases by x. If young's modulus of the material is Y.

$$Y = \frac{F/A}{x/L}$$

So, the restoring force is given by

$$F = \frac{YA}{L}x$$

Now the work done against the restoring force for further small increase in length dx is

$$dw = F dx$$

∴ Total work done in increasing the length of the bar or wire by ΔL is,

$$w = \int_0^{\Delta L} \left(\frac{YA}{L}\right) x dx$$

$$= \frac{AY}{L}\int_0^{\Delta L} x dx$$

$$= \frac{AY}{L}\left[\frac{x^2}{2}\right]_0^{\Delta L}$$

This work done is the elastic potential energy, stored in the bar or wire.

$$\therefore U = \frac{AY}{L}(\Delta L)^2 \qquad (4.7.1)$$

Let us think little more.

Equation 4.7.1 can also be written as

$$\frac{U}{\text{Volume of the object}} = \frac{U}{LA}$$

$$= \frac{1}{2}Y\left(\frac{\Delta L}{L}\right)^2 = \frac{1}{2}Y\left(\frac{\Delta L}{L}\right)^2$$

$$= \frac{1}{2}\frac{\text{Stress}}{\text{Strain}} \times (\text{Strain})^2$$

∴ Elastic potential energy per unit volume

$$= \frac{1}{2} \text{ stress} \times \text{strain} \qquad (4.7.2)$$

Energy per unit volume is also called energy density.

4.8 Applications of Elastic Behaviour of Materials

(i) When a material is used for practical purposes, it will be in the condition of certain stress. For example, cranes are used for lifting and moving heavy loads from one place to another. Cranes have a thick metal rope (cable) to which the load is attached and so the cable is under stress. The maximum load that a cable can carry or the maximum acceleration that can be produced in the attached load should be such that the material of the cable does not exceed its elastic limit. For example, suppose the magnitude of stress at elastic limit (for the material of the cable) is 30×10^7 N m^{-2}. If the cross-sectional area of the cable is A and the load to be carried by it is M, then

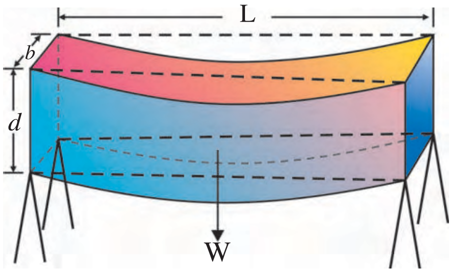


Figure 4.16

Longitudinal stress $\sigma_n = \frac{F_n}{A} = \frac{Mg}{A}$

$$\therefore \; A = \frac{Mg}{\sigma_n} \tag{4.8.1}$$

Here, the cross-section (A) of the cable should be such that its value is much greater than $\frac{Mg}{\sigma_n}$. If M = 10^4 kg, taking $g = 3.1\pi$ m s^{-2}, we get

$$A = \pi r^2 = \frac{(10^4)(3.1\pi)}{(30 \times 10^7)}$$

$\therefore$ Radius of the cable, $r \approx 10^{-2}$ m = 1 cm

Hence, the radius of the cable should be quite more than 1 cm. But a cable of this radius becomes practically a rigid rod. So the cables are always made of a number of thin wires braided together.

(ii) A bridge has to be designed such that it can withstand the load of the flowing traffic, the force of winds and its own weight. Similarly, in the design of buildings use of beams and columns is very common. In both these problems, the bending of beams under a load is of prime importance. The beam should not bend too much or break.

Let us, therefore, consider the case of a beam, having rectangular cross-sections, loaded at the centre and supported near its ends as shown in Figure 4.16. A bar of length L, breadth *b*, and thickness *d* when loaded at the centre by a load (W) sags by an amount δ, given by

$$\delta = \frac{WL^3}{4bd^3Y} \tag{4.8.2}$$

Here, δ is called the bending of a beam.

We accept this formula without proof.

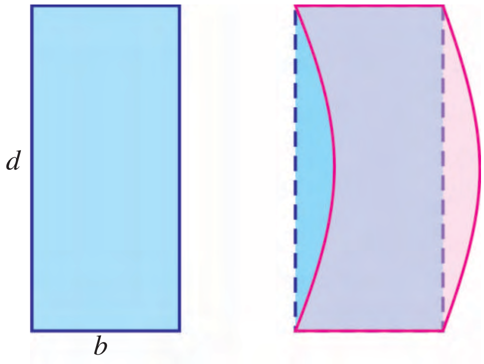


Figure 4.17

From this equation, we see that to reduce the bending of a beam for a given load, one should use a material with a large Young's modulus Y. Moreover, for a given material, increasing the thickness *d* rather than the breadth *b* is more effective in reducing the bending. This is so because δ is proportional to d^{-3} and only to b^{-1}. However, a deep bar may have a tendency to buckle as shown in Figure 4.17. To avoid this, a common compromise is the cross-sectional shape (I) as shown in Figure 4.18. Such a section provides a large load bearing surface and enough depth to prevent bending.

Figure 4.18

(ii) In the end of the chapter let us consider an interesting example of nature.

Consider a mountain of height h and density ρ (assumed uniform). Then the force per unit area at the bottom because of its weight is $h\rho g$ and it acts in a vertically downward direction. Since the sides of the mountain are free, it suffers a shearing stress approximately of the order of $h\rho g$. If the elastic limit for the rocks of mountain is taken to be 3 × 10^8 N m^{-2} and density ρ equal to 3 × 10^3 kg m^{-3} then

$$h_{max}\rho g = 3 \times 10^8 \text{ N m}^{-2}$$

$$\therefore\ h_{max} = \frac{3\times10^8}{3\times10^3\times9.8} \simeq 10^4\ \text{m}$$

$$= 10\ \text{km}$$

Thus elastic limit for the rocks imposes a limit on the maximum height of mountains. Height of Mount Everest is 8848 m i.e. 8.848 km. which is within this limit.

Illustration 4 : Length of a wire under the effect of tensile force F_1 is l_1 and its length under the effect of tensile force F_2 is l_2. Prove that its normal length $l = \dfrac{F_2 l_1 - F_1 l_2}{F_2 - F_1}$.

Solution :

Since $\Delta l = \dfrac{Fl}{AY}$,

$$l_1 = l + \frac{F_1 l}{AY} \qquad (1)$$

$$\text{and } l_2 = l + \frac{F_2 l}{AY} \qquad (2)$$

Multiplying equation 1 by F_2 and equation 2 by F_1 and then subtracting equation 2 from 1 we get,

$$F_2 l_1 - F_1 l_2 = F_2 l + \frac{F_1 F_2 l}{AY} - F_1 l - \frac{F_1 F_2 l}{AY}$$

$$\therefore\ F_2 l_1 - F_1 l_2 = (F_2 - F_1) l$$

$$\therefore\ l = \frac{F_2 l_1 - F_1 l_2}{F_2 - F_1}$$

Illustration 5 : The pressure at certain depth in a sea is 80 atm. If the density of water at the surface of the sea is 1.03×10^3 kg/m^3 and the compressibility of water is 45.8×10^{-11} Pa^{-1}, calculate the density of water at the mentioned depth.

1 atm = 1.013×10^5 Pa

Solution : Suppose the density of water at the mentioned depth is ρ' and the density on the surface is ρ. For a given mass M of water, let the volumes of water at the surface and at the mentioned depth are V and V$'$ respectively.

$$\therefore\ V = \frac{M}{\rho} \text{ and } V' = \frac{M}{\rho'}$$

$$\therefore\ \text{Decrease in volume} = \Delta V$$

$$= V - V'$$

$$= M\left[\frac{1}{\rho} - \frac{1}{\rho'}\right]$$

$$\therefore\ \text{volume strain} = \frac{\Delta V}{V} = M\left[\frac{1}{\rho} - \frac{1}{\rho'}\right] \times \frac{\rho}{M}$$

$$= 1 - \frac{\rho}{\rho'}$$

$$\text{But, compressibility } K = \frac{\Delta V}{PV} = \frac{1}{P}\left[1 - \frac{\rho}{\rho'}\right]$$

$$\therefore\ 45.8 \times 10^{-11} = \frac{1}{80\times1.013\times10^5}$$

$$\times \left[1 - \frac{1.03\times10^3}{\rho'}\right]$$

$$\therefore\ \rho' = 1.034 \times 10^3\ \text{kg/m}^3$$

Illustration 6 : A steel wire of length 5 m and diameter 10^{-3} m is hanged vertically from a ceiling of 5.22 m height. A sphere of radius 0.1 m and mass 8 π kg is tied to the free end of the wire. When this sphere is oscillated like a simple pendulum, it touches the floor of the room in its lower most position. Calculate the velocity of the sphere in its lower most position. Young's modulus for steel = 1.994×10^{11} Nm^{-2}

Solution :

Radius of sphere r = 0.1 m

Initial length L = 5 m

Increase in the length of the wire

$$\Delta L = 5.22 - (L + 2r)$$

$$= 5.22 - (5 + 2 \times 0.1)$$

$$= 0.02\ \text{m}$$

Radius of the wire $r_o = 5 \times 10^{-4}$ m

If the tension produced in the wire at the lowest point is T, then

$$Y = \frac{T/A}{\Delta L/L} \text{ gives}$$

$$T = \frac{YA\Delta L}{L} = \frac{Y(\pi r^2_0)\Delta L}{L}$$

$$= \frac{1.994\times10^{11}\times\pi\times(5\times10^{-4})^2\times0.02}{5}$$

$$= 199.4\pi\ \text{N}$$

But, net force T − Mg = $\frac{Mv^2}{R}$ where R = radius of the path of oscillations of the sphere = 5.22 − 0.1 = 5.12 m

$$\therefore\ 199.4\pi - 8\pi \times 9.8 = \frac{8\pi \times v^2}{5.12}$$

$$\therefore\ 199.4 - 78.4 = \frac{8v^2}{5.12}$$

$$\therefore\ 121 = \frac{8v^2}{5.12}$$

$$\therefore\ v = 8.8\ \text{ms}^{-1}$$

Illustration 7 : A mass of 15 kg is tied at the end of a steel wire of length 1m. It is whirled in a vertical plane with angular velocity 1 rad/s. Cross sectional area of the wire is 0.06 cm^2. Calculate the elongation of the wire when the mass is at its lowest position.

Y_{steel} = 2 × 10^{11} N m^{-2}

Solution :

m = 15 kg, l = 1m, ω = 1 rad/s A = 0.06 cm^2 = 6 × 10^{-6}m^2

Y_{steel} = 2 × 10^{11} N m^{-2}

At the lowest position total force acting on the body is, the sum of gravitational force and centrifugal force, (mv^2/r)

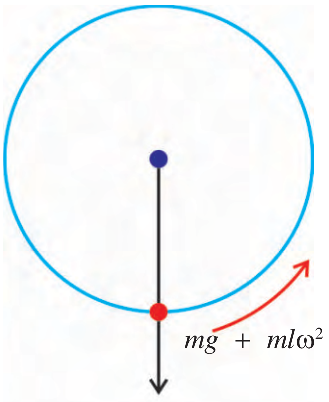


Figure 4.19

F = $mg + mv^2/r$ putting $v = l\omega$ and $r = l$,

F = $mg + ml\omega^2$

= 15(9.8 + 1 × (1)^2)

= 15 (10.8) = 162 N

$$\therefore\ \text{Stress}\ \sigma = \frac{F}{A} = \frac{162}{6 \times 10^{-6}}$$

$$= 27 \times 10^6\ \text{N m}^{-2}$$

Now, since $Y = \frac{\sigma}{\varepsilon_l}$

$$\therefore\ \frac{\Delta l}{l} Y = \sigma$$

$$\therefore\ \Delta l = \frac{\sigma l}{Y}$$

$$= \frac{27 \times 10^6 \times 1}{2 \times 10^{11}} = 13.5 \times 10^{-5}\ \text{m}$$

$$= 0.135 \times 10^{-3}\ \text{m}$$

$$= 0.135\ \text{mm}$$

Illustration 8 : Length and cross sectional area of a wire are 5 m and 2.5 mm^2. Calculate work required to be done to increase its length by 1 mm. Young's modulus of material of wire = 2 × 10^{11} N m^{-2}.

Solution : l = 5m, Δl = 1mm = 10^{-3}m

A = 2.5 mm^2 = 2.5 × 10^{-6} m^2, γ = 2 × 10^{11} N m^{-2}

Here, work done W is given by

$$W = \frac{1}{2}\text{stress} \times \text{strain} \times \text{volume}$$

$$= \frac{1}{2}(\gamma \times \varepsilon_l) \times \varepsilon_l \times V$$

$$= \frac{1}{2}\gamma \times \left(\frac{\Delta l}{l}\right)^2 \times V$$

$$= \frac{1}{2} \times 2 \times 10^{11} \times \left(\frac{10^{-3}}{5}\right)^2 \times 2.5 \times 10^{-6} \times 5 \qquad (V = Al)$$

$$= 5 \times 10^{-2}\ \text{J}$$

SUMMARY

1. Solids can be classified into three groups : (i) Crystalline solids (ii) Amorphous solids (iii) Semi-crystalline solids.
2. In Crystalline solids atoms, molecules or ions are arranged in geometrical array in space. Such a geometrical array of points in space is called lattice.
3. Crystalline solids are made of identical blocks (units).
4. Crystalline solids have long range order and so they have definite melting points.
5. In amorphous solids there is no order in arrangement of constituent particles. During formation of such solids constituent particles do not get enough time for regular arrangement of particles.
6. In semi-crystalline solids in some regions there is regular arrangement of particles whereas in some regions arrangement of particles is not regular.
7. When an external force acts on a body, the body is deformed. Property of an object to resist such a deformation is called elasticity.
8. If a deformed body regians its original shape after removal of deforming force, the body is said to be perfect elastic body.
9. If a deformed body cannot regain its original shape partially, even after removal of deforming force, the body is said to be perfect plastic body.
10. When an external force acts on a body its dimensions change. Ratio of change in dimension to original dimension is called strain. Strain can be classified into three types. Strain is dimensionless.
11. Tensile (compressive) strain (ε_l) is ratio of change in the length to original length of a body.
12. Ratio of change in volume of a body to its original volume is called volume strain.
13. Shearing strain is produced due to tangential force acting on any surface.
14. When deforming force is acting on a body, restoring force produced per unit area is called stress. Its unit is N m^{-2}.
15. Stress produced corresponding to longitudinal strain, volume strain and shearing stain, are longitudinal stress, volume stress and shearing stress respectively.
16. When an external force acts on a body for the restoring force, inter-molecular forces are responsible.
17. If a force acting on a body is not perpendicular to any surface, component of a force perpendicular to suface produced tensile (compressive) strain and a component of the force parallel to surface causes shearing strain.
18. Stress and pressure both are forces per unit area and yet they are in different physical quantities.
19. If tensile strain is less than 1 % tensile stress is directly proportional to tensile strain. The maximum stress for which after removal of the external force the body regains original dimension on the original path is called proportionality limit. The maximum stress for which body can regain its original shape is called limit of elasticity.
20. If in a body large plastic deformation can be produced which is said to be ductile. If a body breacks when limit of elasticity is crossed, it is called brittle.

21. In a substance like rubber, 700 % strain can be produced. Such bodies are called elastomers.

22. When external force is applied to rubber, large deformation is produced in it. When deforming force is removed, it regains original state but not on original path. Here, energy spent in deforming the body is more than energy released by the body while regaining the shape. This phenomenon is known as elastic hysteresis. It is used in shockabsorbers.

23. Hooke's Law : For small deformations stress is directly proportional to strain.

24. For small deformations ratio of stress to strain is called modulus of elasticity. Young modulus (Y), Bulk modulus (B) and modulus of rigidity (η) are the moduli corresponding to longitudinal strain, volume strain and shearing strain respectively. Unit of modulus of elasticity is N m^{-2}.

25. When an axial force (tensile force or compressive force) is applied on a body, its length and lateral dimensions change. Ratio of fractional change in lateral dimension to fractional change in axial dimension is called Poisson's ratio. Its symbol is μ. It is dimensionless. Value of μ is less than 0.5.

26. When external force is applied, body achieves a new configuration due to deformation. And hence it gains potential energy. This potential energy is called elastic potential energy. It is given by

$$U = \frac{1}{2} \text{ stress} \times \text{strain} \times \text{volume}$$

EXERCISES

Choose the correct option from the given options :

1. A wire is stretched to double the length. Which of the following statements is false in this context ?

(A) Its volume increases.
(B) Its longitudinal strain is I.
(C) Stress = Young's modulus
(D) Stress = 2 Young's modulus.

2. Which is the dimensional formula for modulus of rigidity ?

(A) $M^1L^1T^{-2}$
(B) $M^1L^{-1}T^{-2}$
(C) $M^1L^{-2}T^{-1}$
(D) $M^1L^{-2}T^{-2}$

3. When more then 20 kg mass is tied to the end of wire it breaks. What is maximum mass that can be tied to the end of a wire of same material with half the radius ?

(A) 20 kg
(B) 5 kg
(C) 80 kg
(D) 160 kg

4. Length of a metallic rod of mass *m*, and cross-sectional area A is L. If mass M is suspended at the lower end of this rod suspended vertically stress at the cross-section situated at $\frac{3l}{4}$ distance from its lower end is ....

(A) M*g*/A
(B) (M + *m*/4) *g*/A
(C) (M + $\frac{3}{4}m$)g/A
(D) M + *m*) *g*/A

5. Here, values of lengths and diameters of wires of same material are given. If same mass is suspended at the end which wire will have the maximum extension ?

(A) $l = 0.5$ m, $d = 0.05$ mm (B) $l = 1$m, $d = 1$mm

(C) $l = 2$m, $d = 2$mm (D) $l = 3$m, $d = 3$mm

6. When 100 N tensile force is applied to a rod of 10^{-6} m^2 cross sectional area, its length increases by 1 %. So, Young's modulus of material is ....

(A) 10^{12} *Pa* (B) 10^{11} *Pa* (C) 10^{10} *Pa* (D) 10^{2} *Pa*

7. A composite wire is made by joining ends of two wires of equal dimensions, one of copper and the other of steel. When a weight is attached to its end the ratio of increase in their lengths is .... $Y_{steel} = \frac{20}{7} Y_{copper}$

(A) 20:7 (B) 10:7 (C) 7:20 (D) 1:7

8. A rubber ball when taken to the bottom of a 100 m deep lake suffer decrease in volume by 1 %. Hence, the bulk modulus of rubber is .... ($g = 10$ m s^{-2})

(A) 10^6 *Pa* (B) 10^8 *Pa* (C) 10^7 *Pa* (D) 10^9 *Pa*

9. Young's modulus of a rigid body is ....

(A) 0 (B) 1 (C) $\propto$ (D) 0.5

10. Pressure on an object increases from 1.01×10^5 *Pa* to 1.165×10^5 *Pa*. Its volume decreases by 10% at constant temperature. Bulk modulus of material is ....

(A) 1.55×10^5 *Pa* (B) 51.2×10^5 *Pa*

(C) 102.4×10^5 *Pa* (D) 204.8×10^5 *Pa*

11. When 200 N force is applied on an object, its length increases by 1 mm. So potential energy stored in it due to this change is ....

(A) 0.2 J (B) 10 J (C) 20 J (D) 0.1 J

12. A wire is tied to a rigid support. Its length increases by l when force F acts at its free end. So work done is ....

(A) $\frac{1}{2}$ (B) Fl (C) 2Fl (D) $\frac{1}{2}$Fl

13. For perfect plastic body Young's modulus is ....

(A) l (B) zero (C) ∞ (D) 2

14. Dimensionally modulus of elasticity is equivalent to ....

(A) Force (B) Stress (C) Strain (D) none of these

15. Cross-sactional area of a wire of length L is A. Young's modulus of material is Y. If this wire acts as a spring what is the value of force constant ?

(A) $\frac{YA}{L}$ (B) $\frac{YA}{2L}$ (C) $\frac{2YA}{L}$ (D) $\frac{YL}{A}$

16. In a metal wire when 10 N tensile force is applied, its length becomes 5.001 m and when 20 N tensile force is applied, its length became 5.002 m so its original length is ....

(A) 5.001 m (B) 4.009 m (C) 5.0 m (D) 4.008 m

ANSWERS

1. (D) **2.** (B) **3.** (B) **4.** (B) **5.** (A) **6.** (C)
7. (A) **8.** (B) **9.** (C) **10.** (A) **11.** (D) **12.** (D)
13. (B) **14.** (B) **15.** (A) **16.** (C)

Answer the following questions in short :

1. Which forces are responsible for the formation of molecular crystals ?

2. Define perfect elastic body.

3. Give dimensional formula of strain.

4. Give the reason of restoring force produced in a body when an external force acts on it.

5. Define compressibility. Also give its dimensional formula.

6. Which is more elastic, rubber or steel ? Why ?

7. Give reason : Springs are made from steel and not from copper.

8. What happens to the energy spent in changing the dimensions of an elastic body ?

9. When a rod is stretched to increase its length by Δl, increase in its potential energy is U. What will be the change in its potential energy if it is compressed to decrease its length by Δl ?

10. For a wire breaking force is F. If the thickness of wire is doubled what will be the value of breaking force ?

Answer the following questions :

1. Write a short note on ionic crystals.

2. What is meant by strain ? Explain shearing strain with the help of an example.

3. Discuss the effect of force acting on a body making angle θ with a normal drawn to its surface.

4. Explain experimental method to determine Young's modulus.

5. Explain the difference between stress and pressure.

6. Define Poisson's ratio and show that its value is less then 0.5.

7. Derive an expression for elastic potential energy.

Solve the following problems :

1. A steel wire is hanged vertically. What should be its maximum length so that it does not break by its own weight. Density of steel = 7.8×10^3 kg m^{-3}, for steel breaking stress = 7.8×10^9 dyn/cm^2. (Ans : 1.02×10^4 m)

2. Figure shows a composite rod of cross-sectional area 10^{-4} m^2 made by joining three rods AB, BC and CD of different materials end to end. The composite rod is suspended vertically and an object of 10 kg is hung by it. $L_{AB} = 0.1$ m, $L_{BC} = 0.2$ m $L_{CD} = 0.15$ m. Calculate displacement of B, C and D. $Y_{AB} = 2.5 \times 10^{10}$ Pa, $Y_{BC} = 4 \times 10^{10}$ Pa $Y_{CD} = 1 \times 10^{10}$ Pa.

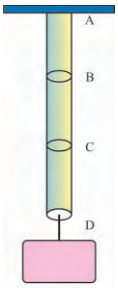


Figure 4.20

[**Ans. :** Displacement of B = 3.9×10^{-6} m, Displacement of C = 8.8×10^{-6} m, Displacement of D = 2.3×10^{-5} m]

3. A wire of length L and cross section area A is kept on a horizontal surface and one of its end is fixed at point 0. A ball of mass m is tied to its other end and the system is rotated with angular velocity ω. Show that increase in its length $\Delta l = \frac{m\omega^2 L^2}{AY}$. Y is Young's modulus.

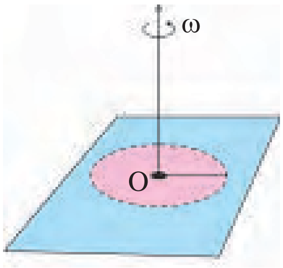


Figure 4.21

4. As shown in figure, masses of 2 kg and 4 kg are tied to two ends of a wire passed over a pulley. Cross-sectional area of wire is 2 cm^2. Calculate longitudinal strain produced in wire. g = 10 m s^{-2} Y = 2 × 10^{11} Pa.

[**Ans** : 6.6 × 10^{-7}]

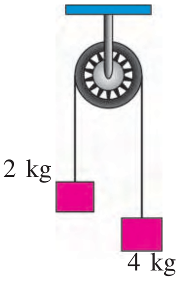


Figure 4.22

5. A wire of length 5 m and diameter 2 mm is hanging from a ceiling. A mass of 5 kg is suspended at its lower end. Calculate increase in its volume. Poisson's ratio of material = 0.2, Y = 2 × 10^{11} Pa. g = 10 m s^{-2}. Also calculate change in potential energy of wire.

[**Ans.** : ΔV = 7.5 × 10^{-10} m^3, 10^{-2} J]

6. A steel wire of cross section 1 mm^2 is heated at 60°C and tied between two ends firmly. Calculate change in tension when temperature becomes 30°C co-efficient of linear expansion for steel is $\propto = 1.1 \times 10^{-5}$ °C^{-1}, Y = 2 × 10^{11} P$_a$. (change in length of wire due to change in temperature (Δt) is $\Delta l = \propto l \Delta t$) [**Ans.** : 66 N]

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CHAPTER 5

FLUID MECHANICS

5.1 Introduction

A substance which can flow is known as fluid. As liquids and gases can flow, they are called fluids. Molten glass and tar can flow although slowly, they are also included in fluids.

Fluid mechanics comprises of fluid statics and fluid dynamics. In fluid statics the forces and pressures acting on a stationary fluid are studied while fluid dynamics includes motion of fluid and properties of its motion. Fluid dynamics is studied in two sections : Hydrodynamics and Aerodynamics.

We will discuss pressure of fluids and Pascal's Law in fluid statics. In fluid dynamics, characteristics of fluid flow, Bernoullie's theorem and its applications and viscosity will be studied. Finally, we will also discuss the surface tension of stationary liquids. Let us begin with fluid statics.

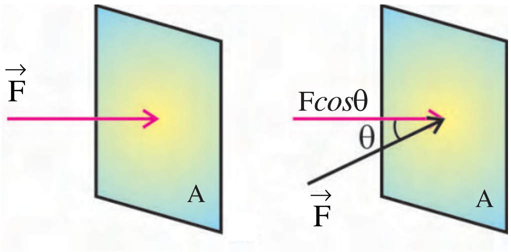
5.2 Pressure and Density

'Magnitude of force acting on a surface per unit area in a direction perpendicular to it, is called the pressure on the surface.'

If F is the magnitude of force acting perpendicular to area A, then the pressure acting on this surface is given by,

$$\text{Pressure (P)} = \frac{\text{Force(F)}}{\text{Area(A)}} \qquad (5.2.1)$$

If force is not perpendicular to the surface, then the component of the force perpendicular to the surface is taken into account for the pressure on the surface (See Figure 5.1).



Pressure

Figure 5.1

If the force ($\vec{F}$) makes an angle θ with the normal drawn to the surface, F*cos*θ is the force perpendicular to the surface. So, as per the definition, pressure is given by,

$$P = \frac{\text{Force(F)}}{\text{Area(A)}} \qquad (5.2.2)$$

Unit of pressure is Newton/(metre)^2, (N/m^2) which is also known as Pascal (P*a*), in memory of French scientist Blaise Pascal (1623–1662). Pressure is a scalar quantity.

Apart from pascal, other units of pressure are bar, atmosphere (atm) and torr.

1 P_a = 1 N m^{-2}

1 bar = 10^5 P_a

and 1 atm = 1.013 × 10^5 P_a

1 torr = 133.28 P_a

One atm pressure is the pressure equal to the pressure exterted by the atmosphere at sea level. It is also expressed in terms of height of mercury column, as cm–Hg or mm–Hg

1 atm = 76 cm Hg = 760 mm–Hg

Density : The ratio of mass to the volume of an object is known as density of the object. If the volume of a body of mass m is V, its density (ρ) is given by

$$\rho = \frac{m}{V} \qquad (5.2.3)$$

It is clear that unit of density is kg m^{-3}. Normally liquids are incompressible. (Percentage change in the volume of most of the liquids is of the order of 0.005 %) So, their densities are constant at a given temperature. Density of a gas depends on its pressure. In Table 5.1 densities of some fluids are given.

Table 5.1 : Densities of fluids at NTP

(Only For Information)

Liquid	Density (kg m^{-3})	Gas	Density (kg m^{-3})
Water	1 × 10^3	Air	1.29
Sea water	1.03 × 10^3	Oxygen	1.43
Mercury	13.6 × 10^3	Hydrogen	9.0 × 10^{-2}
Ethyl Alcohol	0.806 × 10^3	Inter stellar space	10^{-18}–10^{-21}
Blood	1.06 × 10^3		

Water is taken as a standard substance. By comparing the density of a given body to that of water we get specific density. **'Specific density of an object is the ratio of density of an object to density of pure water at 277 K.** Thus,

Specific density

$$= \frac{\text{Density of an object}}{\text{Density of pure water at 277 K}}$$

Specific density is dimensionless. It is also known as relative density or specific gravity. Reciprocal of density is called specific volume.

If we take water having the same volume as that of the given object, the specific density can be obtained as,

$$\text{specific density} = \frac{\text{Mass of the object}}{\text{Mass of water of the same volume at 277 K}}$$

This equation is very useful in determining the specific density of a substance because, it is not necessary to know the density of given object to detemine the specific density.

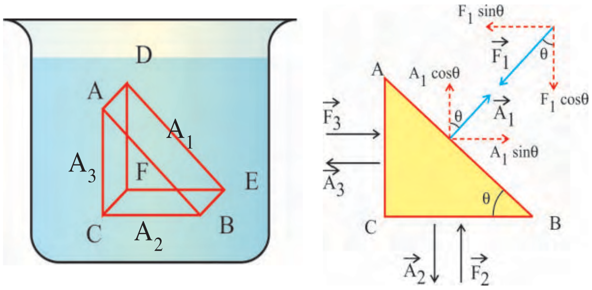
5.3 Pascal's Law and its Applications

Pascal's Law : 'Pressure in an incompressible fluid in equilibrium is the same everywhere, if the effect of gravity is neglected.'

This statement can easily be verified as follows :

Consider a small element in the interior of a liquid at rest. The liquid element is in the shape of a prism consisting of two right angled triangle surfaces.

Let the areas of surfaces ADEB, CFEB and ADFC be A_1, A_2 and A_3.



Verification of Pascal's Law

Figure 5.2

It is clear from Figure 5.2 that

$$A_2 = A_1 \cos\theta \text{ and } A_3 = A_1 \sin\theta$$

Also, since liquid element is in equilibrium,

$F_2 = F_1 cos\theta$ and $F_3 = F_1 sin\theta$

Now pressure on the surface ADEB is $P_1 = \frac{F_1}{A_1}$

Pressure on the surface CFEB is

$$P_2 = \frac{F_2}{A_2} = \frac{F_1 cos\theta}{A_1 cos\theta} = \frac{F_1}{A_1}$$

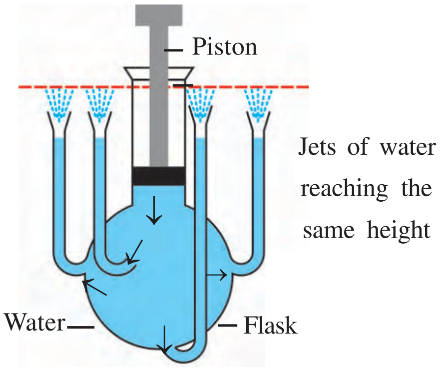
and pressure on the surface ADFC is

$$P_3 = \frac{F_3}{A_3} = \frac{F_1 cos\theta}{A_1 cos\theta} = \frac{F_1}{A_1}$$

So, $P_1 = P_2 = P_3$

Since θ is arbitrary this result holds for any surface. Thus Pascal's law is varified.

An obvious consequence of Pascal's law is that "A change in pressure applied to an enclosed is transmitted undiminished to every portion of the fluid and the walls of the container". It is perpendicular to the walls of the container. This statement is known as Pascal's law of transmission of fluid pressure.

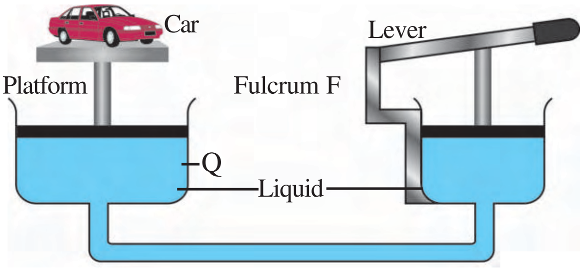


Transmission of pressure in the fluid

Figure 5.3

This can be demonstrated using a flask of glass, having small tubes jetting out from every part. (Figure 5.3) Fill some coloured water in it. Push the piston attached to it downwards. Water will rise in all tubes to the same height. This shows that change in pressure at any part of the enclosed liquid is transmitted equally in all directions.

Hydraulic Lift : Hydraulic lift works on Pascal's law. It is a device which consists of two cylinders of cross section A_1 and A_2, ($A_1 << A_2$) connected by a horizontal pipe. (Figure 5.4) These two cylinders are fitted with smooth, air tight pistons.



Hydraulic Jack

Figure 5.4

It contains liquid in it, as shown in the figure.

Suppose a force F_1 is applied on the pistion with cross sectional area A_1.

So, pressure produced due to it is,

$$P = \frac{F_1}{A_1}$$

This pressure exerted on liquid in a closed vessel is transmitted unchanged on to the piston with larger cross sectional area through liquid. Hence pressure on the second piston is

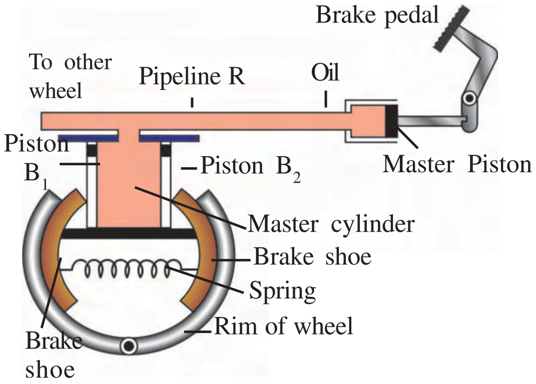
$$P = \frac{F_2}{A_2}$$

$$\therefore \frac{F_2}{A_2} = \frac{F_1}{A_1}$$

$$\therefore F_2 = F_1\left(\frac{A_2}{A_1}\right)$$

Here, $A_1 << A_2$, $\therefore F_1 << F_2$. Thus with less effort (F_1), heavy load can be lifted at the other end.

Hydraulic Brakes : Brake system used in automobiles are hydraulic brakes, which are based on Pascal's law. When driver applies a small force on the brake-pedal, the master piston moves in the master cylinder and the pressure caused by this is transmitted undiminished through the brake-oil, gets applied on the piston of larger area. Thus a greater force is applied on this piston, which pushes the brake-shoes to come in contact with the brake-liner. Thus, with a small force applied on the pedal, a greater retarding force is applied on the wheel.



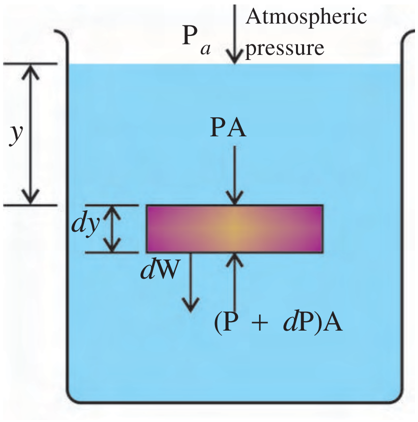
Hydraulic Brake

Figure 5.5

Door closers and shockabsorbers of automobiles work on Pascal's Law.

5.4 Pressure Due to Fluid Column

Suppose a liquid of density ρ is in static equilibrium in a container. Consider an imaginary cylindrical fluid element of height dy and cross-sectional area A at the depth y from the surface of liquid. As shown in Figure 5.6 the volume of this cylindrical element is Ady, its mass is ρ.A.dy and its weight is $dw = \rho.g.$A.dy.



Pressure due to liquid column

Figure 5.6

Suppose the pressure on upper and lower faces of the cylindrical element are P and P + dP respectively, as shown in Figure 5.6. Hence the force on upper surface in downward direction will be PA, while the force on lower face, in upward direction is (P + dp)A.

$$\text{PA} + dW = (\text{P} + dp)\text{A}$$

$$\therefore \ \text{PA} + \rho g \text{A} dy = \text{PA} + \text{A}dp$$

$$\therefore \ \rho g \text{A} dy = \text{A}dp$$

$$\therefore \ \frac{dp}{dy} = \rho g \qquad (5.4.1)$$

This equation shows that the rate of change in pressure with depth (or height) depends on physical quantity ρg, known as weight density (weight of a body per unit volume). Since most of the liquids are fairly incompressible, ρg is constant for small heights of liquid column. For fluids like air value of density ρ_1 depends on height from earth's suface, temperature etc. and hence value of weight density cannot be treated to be taken as constant for it.

As shown in Figure 5.6 container being open the pressure on upper free surface is equal to the atmospheric pressure. Thus for $y = 0$ P = P$_a$. Pressure P at depth $y = h$ can be determined by integrating equation 5.4.1,

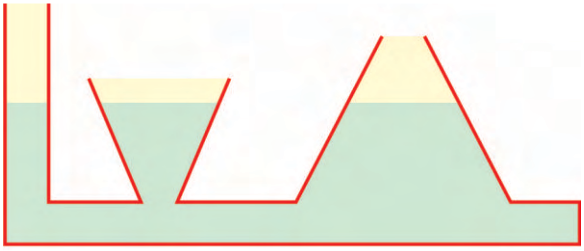
$$\int_{Pa}^{P} dp = \int_{0}^{h} \rho g dy$$

$$\therefore \ \text{P} - \text{P}_a = \rho g h$$

$$\therefore \ \text{P} = \text{P}_a + \rho g h \qquad (5.4.2)$$

Here, $\text{P} = \text{P}_a + \rho g h$ is known as absolute pressure, while the difference $\text{P} - \text{P}_a$ is known as the gauge pressure or hydrostatic pressure at that point.

The pressure at any point in a liquid neither depends on the shape of container in which it is filled nor on its area. This fact is known as **hydrostatic paradox.** (See Figure 5.7). When liquid is filled in containers of different shapes and sizes but interconnected, height of liquid column is found to be same everywhere.



Hydrostatic paradox

Figure 5.7

If two points are in the same horizontal level in the liquid equation 5.4.2 shows that pressure at these two points will be the same in stationary liquid.

5.5 Archimedes Principle : "When a body is partially or fully immersed in a liquid the buoyant force acting on it is equal to the weight of the liquid displaced by it and it acts in the upward direction at the centre of mass of the displaced liquid."

Thus if density of fluid is ρ_f and volume of the body immersed is V,

buoyant force is $F_b = \rho_f Vg$

This is equal to decrease in weight of the body immersed.

Law of Floatation : When the weight W of a body is equal to the weight of the liquid displaced by the part of body immersed in it, the body floats on the surface of the liquid.

(i) If $W > F_b$, the body sinks in the liquid

(ii) If $W = F_b$, the body can remain in equilibrium at any depth in liquid.

(iii) If $W < F_b$, the body floats on the liquid surface, and remain partially immersed.

Illustration 1 : As shown in the figure 5.8 two cylindrical vessels A and B are interconnected. Vessel A contains water up to 2 m height and vessel B contains kerosene. Liquids are separated by movable, airtight disc C. If height of kerosene is to be maintained at 2 m, calculate the mass to be placed on the piston kept in vessel B. Also calculate the force acting on disc C due to this mass. Area of piston = 100 cm^2. Area of disc C = 10 cm^2. Density of water = 10^3 kg m^{-3}, specific density of kerosene = 0.8.

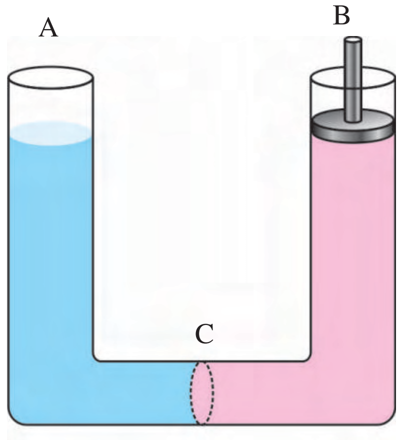


Figure 5.8

Solution : Area of piston A_1 = 100 cm^2 = 10^{-2} m^2

Area of disc A_2 = 10 cm^2 = 10^{-3} m^2

Density of water = 10^3 kg m^{-3} = ρ_ω

$$\text{Now, } \frac{\text{Density of kerosene}}{\text{Density of water}} = 0.8$$

$\therefore$ Density of kerosene ρ_k = 0.8 × density of water = $0.8 \times 10^3 = 800$ kg m^{-3}

If height of kerosene is maintained at 2 m,

Pressure of water column = $\frac{mg}{A_1}$ + Pressure of kerosene column

$$\therefore h\rho_w g = h\rho_k g + \frac{mg}{A_1}$$

$$\therefore 2 \times 10^3 = 2 \times 800 + \frac{m}{10^{-2}}$$

$$\therefore 2000 - 1600 = \frac{m}{10^{-2}}$$

$$\therefore 400 \times 10^{-2} = m$$

$$\therefore m = 4 \text{ kg}$$

Now pressure due to mass m is transmitted undiminished to disc C. So, pressure due to 4 kg mass

$$= \frac{\text{Force on disc C}}{\text{Area of disc C}}$$

$$\therefore \frac{mg}{A_1} = \frac{F_C}{A_2}$$

$$\therefore F_C = mg\frac{A_2}{A_1}$$

$$= \frac{4 \times 9.8 \times 10^{-3}}{10^{-2}}$$

$$= 3.92 \text{ N}$$

Illustration 2 : As shown in Figure 5.9. lower portion of the manometer tube contains fluid of density ρ_2 and the upper part contains fluid of density ρ_1 ($\rho_1 > \rho_2$). If pressures on the top of these two arms are P_1 and P_2, calculate pressure difference ($P_1 - P_2$).

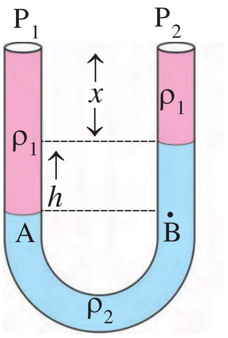


Figure 5.9

Solution : Consider points A and B, as shown in the diagram, at equal height from bottom.

For these points

$P_A = P_B$

$\therefore P_1 + (h + x)\rho_1 g = x\rho_1 g + h\rho_2 g + P_2$

$\therefore P_1 - P_2 = x\rho_1 g + h\rho_2 g - h\rho_1 g - x\rho_1 g$

$\therefore P_1 - P_2 = (\rho_2 - \rho_1)gh$

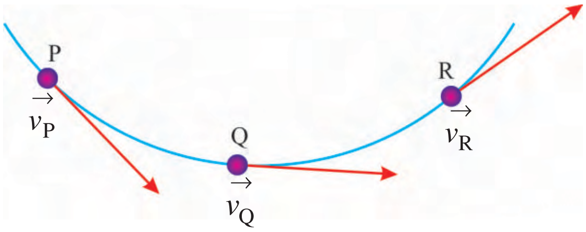
5.6 Fluid Dynamics

While studying the motion of a particle, we had to concentrate on the motion of the particle only and hence we didn't find it much difficult. But in the motion of a fluid when a very large number of particles are in motion, it becomes a formidable task to follow the motion of each of these particles. **J. L. Lagrange** developed a procedure in which he generalised the concepts of particle mechanics; but we shall not discuss it here. There is a treatment, developed by **Euler** which is more convenient for most purposes. In it we give up the attempt to specify the history of each fluid particle and instead specify the density, pressure and velocity of the fluid at each point in space at each instant of time. Of course, we can't afford to forget the particles of the fluid completely because, finally the motion of the fluid is attributed to the motion of its particles.

In the study of the motion of a fluid, we will consider ideal and simple situations. Let us first be familiar with some of the characteristics of the fluid flow.

Characteristics of Fluid Flow :

(1) Steady flow : If in a fluid flow, velocity of the fluid at each point remains constant with time, the flow is known as steady flow. This means that the velocity of any particle of the fluid remains the same while passing through a given point. To understand this, consider three representative points P, Q and R shown in the Figure 5.10. Let the velocities of each particle passing through these points be $\vec{v}_P$, $\vec{v}_Q$ and $\vec{v}_R$ respectively. These velocities remain constant with time. It is not necessary that the velocities of a particle at different points be the same, but velocity of the particles passing through the same point does not change with time, i.e. it is **not necessary** that $\vec{v}_P = \vec{v}_P = \vec{v}_R$, but $\vec{v}_P$, $\vec{v}_Q$ and $\vec{v}_R$ should remain constant with time. These conditions can be achieved at low flow speeds e.g. a gently flowing stream.



Characteristics of steady flow

Figure 5.10

(2) Unsteady flow : If in a fluid flow, velocity of the fluid at a given point keeps on changing with time, the flow is known as unsteady flow. For example, the motion of water during ebb and tide.

(3) Turbulent flow : In a fluid flow, if the velocity of the fluid changes erratically from point to point as well as from time to time, the flow is known as a turbulent flow. Waterfalls, breaking of the sea waves are the examples of turbulent flow.

(4) Irrotational flow : If the element of a fluid at each point has no net angular velocity about that point, the fluid flow is called irrotational.

If the flow is irrotational, a small paddle wheel placed in the flow (as shown in Figure 5.11) will move without rotating.

Motion of a small paddle-wheel

Figure 5.11

(5) Rotational flow : If the element of a fluid at each point has net angular velocity about that point, the fluid flow is called rotational. A paddle wheel placed in such a flow rotates while moving. The rotational motion is turbulent. Rotational flow includes vortex motion such as whirlpools, the air thrown out by exhaust fans etc.

(6) Incompressible flow : If the density of a fluid remains constant with time everywhere (in a given flow), the flow is said to be incompressible. Generally, liquids can usually be considered as flowing incompressibly. But even a highly compressible gas may sometimes undergo insignificant changes in density, its flow is then practically incompressible. For example, the flow of air relative to the wings of an aeroplane flying with velocity quite less than that of sound waves can be considered almost incompressible.

(7) Compressible flow : If in a fluid flow, the density changes with position and time, the flow is known as compressible flow.

(8) Non-viscous flow : The flow of a fluid having small co-efficient of viscosity is known as non-viscous flow. In other words, a flow of a readily flowing fluid is called non-viscous flow. The flow of water in normal conditions is an example of non-viscous flow.

(9) Viscous flow : The flow of a fluid which has large co-efficient of viscosity is called a viscous flow. Thus, a flow of a fluid which cannot flow readily is called a viscous flow. The flow of castor oil is an example of a viscous flow.

In the beginning, we shall consider steady, irrotational, incompressible and non-viscous flow only. But our assumptions are too ideal for the real situations. It is not possible to have such an ideal fluid.

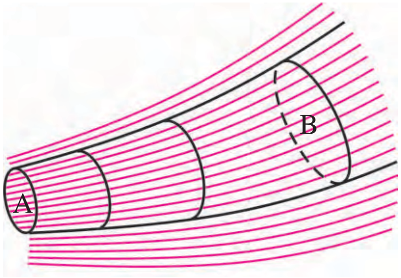
5.6.1 Streamlines, Tube of flow :

The path of motion of a fluid particle is called a **line of flow.** Normally, the direction and the magnitude of the velocity of a fluid particle keeps on changing on its path of motion and hence all the particles passing through a point in a flow may not move on the same path. But the situation in a steady flow is interesting.

In a steady flow, velocity of each particle arriving at a point remains constant with time. In Figure 5.10 let the velocity of a particle arriving at point P be $\vec{v}_P$. It does not change with time. Thus, each particle arriving at P has velocity $\vec{v}_P$ and at that point each particle proceeds in the same direction. When each particle passing through P goes to Q, its velocity $\vec{v}_Q$ (at point Q) also remains constant with time and it proceeds further to R, where its velocity $\vec{v}_R$ also remains constant. Thus, the path of motion of a particle passing through P is PQR. This path of motion does not change with time. Such a steady path of motion is called a **streamline.** The flow for which such streamlines can be defined is called streamline flow. In unsteady flow, flow lines can be defined but not the streamlines.

In a steady flow, streamlines can never intersect each other. If they do, at the point of intersection a particle may move in any direction out of two tangents drawn at that point, which is not possible.

Tube of flow : In principle, we can draw a streamline through every point in the fluid flow. As shown in Figure 5.12, if we imagine a bundle of streamlines passing through the boundary of any surface, this tubular region is called a tube of flow.



Flow tube

Figure 5.12

The wall of the tube of flow is made of streamlines. As streamlines can never intersect each other, a particle of a fluid cannot pass through the wall of a tube of flow. Hence, the tube behaves somewhat like a pipe of the same shape.

5.7 Equation of Continuity

Consider a tube of flow as shown in Figure 5.13. Let the velocity of fluid at cross section P, of area A_1, be v_1 and at cross section Q, of area A_2, be v_2.

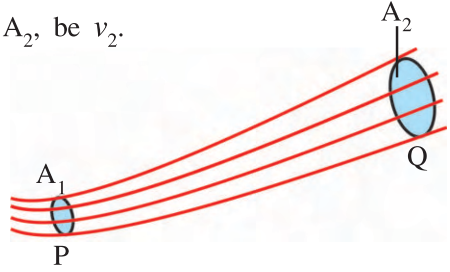


Figure 5.13

So, through the cross section P in unit time fluid can travel distance equal to v_1. Thus the volume of the fluid entering at P in unit time is A_1v_1. If the density of incompressible fluid is ρ, mass of fluid entering at P in unit time is $\rho A_1 v_1$.

Since, **mass of fluid passing through cross section per unit time is called mass flux,**

mass flux at P = $\rho A_1 v_1$ (5.7.1)

Similarly mass flux at Q = $\rho A_2 v_2$ (5.7.2)

Since the liquid cannot pass through the wall and the fluid can neither be destroyed nor be created, mass flux at P and Q should be equal. So, from 5.7.1 and 5.7.2.

$$\rho A_1 v_1 = \rho A_2 v_2$$

$$\therefore\ A_1 v_1 = A_2 v_2 \qquad (5.7.3)$$

or for any cross-section of tube of flow

$$Av = \text{constant} \qquad (5.7.4)$$

Equation 5.7.3 or 5.7.4 is known as the equation of continuity. The product of area of a cross-section and velocity of fluid at this cross section is called volume-flux. Equation 5.7.4 shows that speed of fluid is larger in narrower section of tube and vice versa. In a narrow part of the tube the density of streamlines is larger, fluid speed is more. So it can be concluded that closely packed stremlines indicate larger speed of fluid and vice versa.

5.8 Bernoulli's Equations and its Applications

Bernoulli's equation is a fundamental relation in fluid dynamics. This equation does not represent a new principle of fluid mechanics. It can be obtained using work-energy theorm.

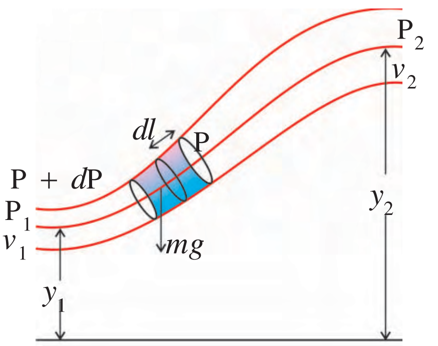


Figure 5.14

Let us consider a streamline flow which is steady, irrotational, incompressible and non-viscous, through flow-tube as shown in figure. Consider a small fluid-element, having area A and length dl. Central streamline passing through this fluid element passes through heights y_1 and y_2 from reference level. At height y_1, pressure and speed of fluid are P_1 and v_1 and at height y_2 they are P_2 and v_2 respectively. This fluid element of mass m, is acted upon by two forces (1) force due to pressure difference (Adp) and (2) gravitational force mg. Suppose this fluid element is displaced by distance dl.

So, here work done due to the first force is $Adl\ dp$ and work done against gravitational force is $-\ mgdy$ (change in potential energy) where dy is change in height. If initially its kinetic energy is $\frac{1}{2}mv^2$, change in kinetic energy during displacement dy, is $d(\frac{1}{2}mv^2) = mvdv$

As per work-energy theorem

$$mvdv = Adldp - mg\ dy \qquad (5.8.1)$$

Since Adl is the volume of fluid element, equation 5.8.1 becomes

$$\frac{m}{Adl}vdv = -dp - \frac{m}{Adl}gdy \qquad (5.8.2)$$

Here m/Adl is the density (ρ) of the fluid and since fluid is incompressible it is constant. So, equation 5.8.2 can be written as

$$\rho vdv = -dp - \rho gdy$$

$$\therefore\ \rho\int_{v_1}^{v_2} vdv = -\int_{P_2}^{P_2} dp - \rho g\int_{y_1}^{y_2} dy$$

$$\therefore\ \rho\left[\frac{v^2}{2}\right]_{v_1}^{v_2} = -[\ P\ ]_{P_1}^{P_2} - \rho g[\ y\ ]_{y_1}^{y_2}$$

$$\therefore\ \frac{1}{2}\rho(v_2{}^2 - v_1{}^2) = -(P_2 - P_1) - \rho g\ (y_2 - y_1)$$

$$\therefore\ P_1 + \rho gy_1 + \frac{1}{2}\rho v_1{}^2 = P_2 + \rho gy_2 + \frac{1}{2}\rho v_2{}^2 \qquad (5.8.3)$$

$$\therefore\ P + \rho gy + \frac{1}{2}\rho v^2 = \text{constant} \qquad (5.8.4)$$

Equation 5.8.3 or 5.8.4 is known as Bernoulli's equation. It should be noted that all the terms of their equation are to be calculated on the same streamline. If the flow is irrotational it can be proved that the constant appearing in equation 5.8.4 is same for all streamlines.

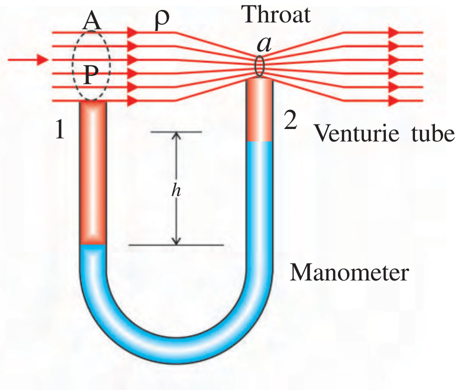
If equation 5.8.4 is divided by ρg we get

$$\frac{P}{\rho g} + \frac{v^2}{2g} + y = \text{constant} \qquad (5.8.5)$$

This is another form of Bernoulli's equation. The first term in this equation is known as pressure head, second term is known as velocity head and third terms as elevation head.

Applications of Bernoulli's Equation

(1) Venturie meter : This apparatus is used to measure speed of fluid. The construction of venturie meter is shown in Figure 5.15. A manometer is connected with a venturie tube having specific design. The narrow part of the tube is called throat. Broader end of the apparatus has cross-sectional area 'A' and the cross-sectional area of throat is 'a'. Speeds near the broader end and near throat are v_1 and v_2. While, pressures are P_1 and P_2 respectively. Density of manometer – fluid is ρ_2 and that of fluid, where velocity is to be measured is ρ_1.



Venturie Meter

Figure 5.15

Using Bernoulli's equation at points '1' and '2'.

$$P_1 + \frac{1}{2}\rho_1 v_1^2 + \rho_1 g y_1 = P_2 + \frac{1}{2}\rho_1 g v_2^2 + \rho_1 g y_2$$

Points '1' and '2' are horizontal. $\therefore\ y_1 = y_2$.

$$\therefore\ P_1 + \frac{1}{2}\rho_1 v_1^2 = P_2 + \frac{1}{2}\rho_1 v_2^2$$

$$\therefore\ P_1 - P_2 = \frac{1}{2}\rho_1(v_2^2 - v_1^2) \qquad (5.8.6)$$

Here, for manometer $P_1 - P_2 = (\rho_2 - \rho_1)gh$ (see illustration 2)

Inserting value of $P_1 - P_2$ in equation 5.8.6 we get

$$(\rho_2 - \rho_1)\ gh = \frac{1}{2}\rho_1(v_2^2 - v_1^2) \qquad (5.8.7)$$

From equation of continuity $Av_1 = av_2$

$$\therefore\ v_2 = \frac{A v_1}{a}$$

Substituting this value of v_2 in equation 5.8.7 we get

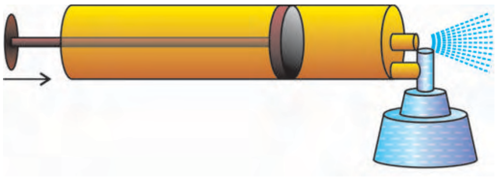
$$(\rho_2 - \rho_1)gh = \frac{1}{2}\rho_1(\frac{A^2}{a^2}\ v_1^2 - v_1^2)$$

$$\therefore\ v_1^2 = \frac{2(\rho_2 - \rho_1)gh}{\rho_1} \cdot \frac{a^2}{A^2 - a^2}$$

$$\therefore\ v_1 = a\sqrt{\frac{2(\rho_2 - \rho_1)gh}{\rho_1(A^2 - a^2)}} \qquad (5.8.8)$$

To find the volume flux or the rate of flow, $R = v_1 A$ or $v_2 a$ should be found.

Air flows through a venturie channel of a carburator in automobiles. At throat, pressure being low, the fluid is sucked in and proper mixture of air and fuel is made available for combustion.



Spray pump

Figure 5.16

The same principle is used in a spray pump, as shown in Figure 5.16. On pushing the piston in air comes out of the hole with high velocity. As a result of this, pressure near the hole is reduced and liquid is raised up in a capillary and it is sprayed along with air.

(2) The change in pressure with depth :

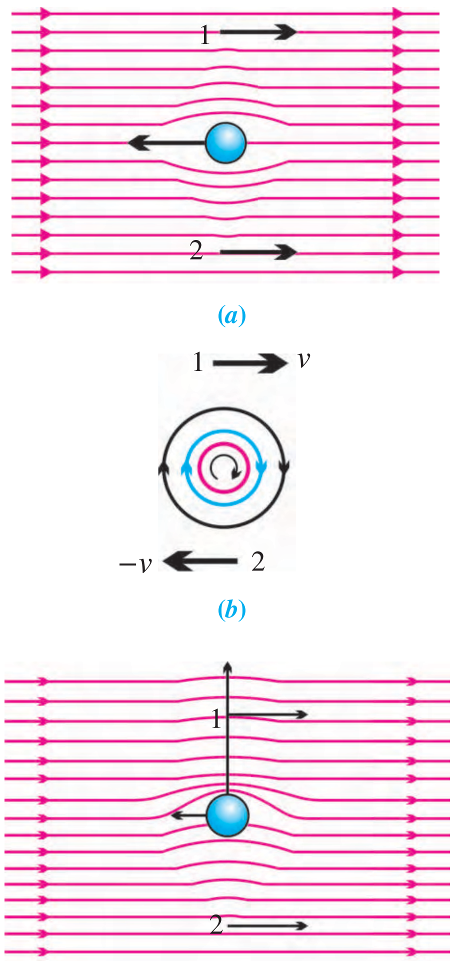
Earlier we have seen that $P - P_a = h\rho g$. This equation can be arrived at as a special case of Bernoulli's equation. If the fluid is stationary $v_1 = v_2 = 0$, and $P_2 = P_a$ (Pressure on free surface of liquid, see Figure 5.6). If the difference in height is taken to be $y_2 - y_1 = h$, from Bernoulli's equation we get $P_1 = P_a + \rho gh$.

(3) Dynamic Lift and Swing Bowling :

Figure 5.17(*a*) shows a ball moving in air. The streamlines are symmetric (w.r.t. the ball) around the ball (because the ball is symmetric). The velocity of air at points 1 and 2 is the same. According to the Bernoulli's equation the pressures at 1 and 2 would also be the same. Hence, the dynamic lift on the ball is zero.

Now, as shown in Figure 5.17(*b*) suppose the ball is having spin motion about an axis passing through its centre and perpendicular to the plane of the figure. As the surface of the ball is not quite smooth, it drags some air along with it. The streamlines produced due to such motion is shown in the figure.

Figure 5.17(*c*) shows the streamline pattern of air when the ball performs linear as well as spin motions. The crowding of streamlines at point 1 indicates high velocity and low pressure, while the sparse streamlines at point 2 indicates low velocity and high pressure. So a ball thrown with such a spin will move up w.r.t. its trajectory. (Now, you might have understood why the bowlers play the michief with a ball to make its surface rough).



(*a*)

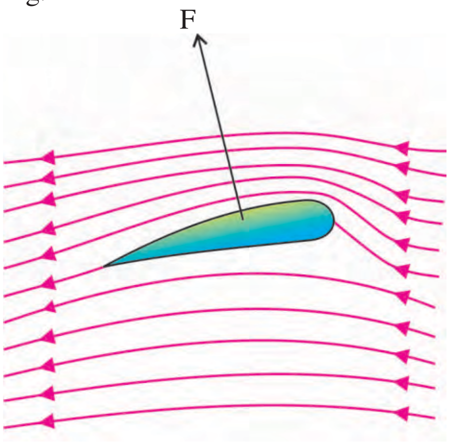
(*b*)

(*c*)

Swing Bowling

Figure 5.17

Now, if the ball is thrown making it spin about an axis lying in the plane of the figure and perpendicular to the direction of its motion, it may deviate towards the off stump or leg stump. This is the main reason of the swing of a ball in fast bowling.



Aerofoil

Figure 5.18

(4) Aerofoils : Figure 5.18 shows an aerofoil which is a solid piece shaped to provide an upward lift when it moves horizontally through air and hence it can float in air.

The shape of the wings of an aeroplane (shape of the cross-section perpendicular to the length of the wings) is like an aerofoil. As shown in the figure, the air has streamline flow about the wings. (Only when the angle between the wing and the direction of motion – called an angle of attack – is small, the streamline flow is possible). In Fig. 5.18, the streamlines are shown arround the wing. The crowded streamlines over the wings indicate high velocity and low pressure, while the sparse streamlines below the wings indicate low velocity and high pressure.

Due to this pressure difference an aeroplane experiences the upward thrust. Thus, the aeroplane in motion may float in air due to the dynamic lift.

Illustration 3 : The diameter of one end of a tube is 2 cm and that of another end is 3 cm. Velocity and pressure of water at narrow end are 2 ms^{-1} and 1.5×10^5 Nm^{-2} respectively. If the height difference between narrow and broad ends is 2.5 m, find the velocity and pressure of water at the broad end. (Density of water is 1×10^3 kg m^{-3}). The narrow end is higher.

Solution :

The narrow end of the flow tube

$d_1 = 2$ cm .

$\therefore\ r_1 = 1 \text{ cm} = 1 \times 10^{-2}$ m

$v_1 = 2\ ms^{-1}$

$P_1 = 1.5 \times 10^5\ Nm^{-2}$

The broad end of the flow tube

$d_2 = 3$ cm

$\therefore\ r_2 = 1.5 \text{ cm} = 1.5 \times 10^{-2}$ m

$v_2 = ?$

$P_2 = ?$

$A_1 v_1 = A_2 v_2$

$$\therefore\ v_2 = \frac{A_1}{A_2} \cdot v_1$$

$$= \frac{\pi r^2{}_1}{\pi r^2{}_2} \cdot v_1 = \frac{r^2{}_1}{r^2{}_2} \cdot v_1$$

$$= \frac{(1 \times 10^{-2})^2}{(1.5 \times 10^{-2})^2} \times 2$$

$$= 0.89\ ms^{-1}$$

According to Bernoulli's equation

$$P_1 + \frac{1}{2}\rho v_1{}^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_1{}^2 + \rho g y_2$$

$$\therefore\ P_2 = P_1 + \frac{1}{2}\rho(v_1{}^2 - v_2{}^2) + \rho g(y_1 - y_2)$$

$$= (1.5 \times 10^5) + \frac{1}{2} \times 1 \times 10^3 \times [(2)^2 - (089)^2] + 1 \times 10^3 \times 9.8 \times 2.5$$

$$P_2 = 1.76 \times 10^5\ Nm^{-2}$$

Illustration 4 : Figure 5.19 shows a cylindrical vessel having cross-sectional area A_1 in which liquid of density ρ is filled. At the bottom of the container there is a small hole of cross-section A_2. Find the velocity of liquid coming out of the hole when the height of the liquid column is *h* from the hole. (Here, $A_1 >> A_2$)

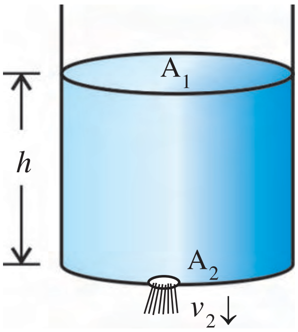


Figure 5.19

Solution : Suppose, the velocity of the liquid at cross-section A_1 and A_2 are v_1 and v_2 respectively. Both the cross-sections being open in the atmosphere, the pressure at the cross-sections is same as the atmospheric pressure P_a.

$$\therefore\ P_a + \frac{1}{2}\rho v_1{}^2 + \rho g h = P_a + \frac{1}{2}\rho v_2{}^2 \quad (1)$$

According to the equation of continuity,

$$A_1v_1 = A_2v_2$$

$$\therefore \; v_1 = \frac{A_2v_2}{A_1} \qquad (2)$$

Substituting the value of v_1 from eqn. (2) in eqn. (1),

$$\frac{1}{2}\left(\frac{A_2}{A_1}\right)^2 v_2^2 + gh = \frac{1}{2}v_2^2$$

$$\therefore \; v_2^2 = \frac{2gh}{\left[1-\left(\frac{A_2}{A_1}\right)^2\right]} \cong 2gh \; (\because A_2{<<}A_1)$$

$$\therefore \; v_2 = \sqrt{2gh}$$

Note : The velocity of the liquid coming out of a hole at the depth h from the free surface of the liquid is the same as the final velocity of a particle falling freely from the same height. **This statement is called Torricelli's law.**

Illustration 5 : Water is filled in a container upto height H as shown in the Figure 5.20. A small hole is bored on the surface of a container at the depth h from the surface of water. What will be the distance of a point along the horizontal where the jet of the water strikes the ground ? For which value of h will this distance be maximum ? Also find this maximum distance.

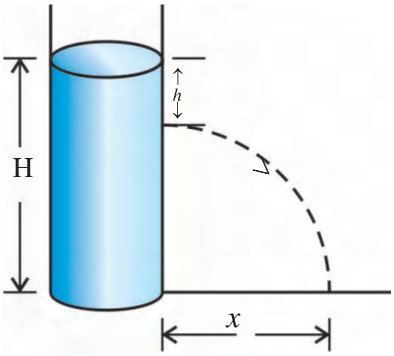


Figure 5.20

Solution : The velocity of water coming out of the hole in horizontal direction is

$$v = \sqrt{2gh} \qquad (1)$$

As the acceleration of water coming out of the hole is downward, it moves with constant velocity in horizontal direction while it moves with constant acceleration downward. (It is like projectile motion).

From the equations of motion, displacement in downward direction,

$$H - h = \frac{1}{2}gt^2 \qquad (2)$$

where t = time taken by the water to fall on the ground.

The distance travelled along horizontal,

$$x = vt \qquad (3)$$

Substituting the values of v and t from equations (1) and (2) in equation (3),

$$x = \sqrt{2gh}\left(\frac{2(H-h)}{g}\right)^{\frac{1}{2}}$$

$$= (4hH - 4h^2)^{\frac{1}{2}}$$

$$= [H^2 - (H - 2h)^2]^{\frac{1}{2}} \qquad (4)$$

Equation (4) shows that x is maximum if $H = 2h$, i.e. $h = \frac{H}{2}$

$$\therefore \; h = \frac{H}{2}$$

Also, for $h = \frac{H}{2}$ equation (4) gives

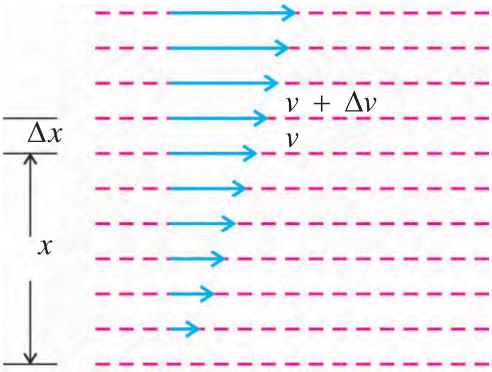
$$\therefore \; x = H$$

5.9 Viscosity

We know that the liquids like water and kerosene flow easily. While the liquids like honey, castor oil cannot flow easily. If we consider Bernoulli's equation for horizontal flow, i.e. $y_1 = y_2$, we have

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

This equation suggests that to maintain horizontal fluid flow with constant speed ($v_1 = v_2$), no pressure difference is required i.e. $P_1 = P_2$. But this is not found in reality. We need some pressure difference to maintain fluid flow with constant speed. This means that there must be some force opposing the motion of fluid. This force is due to viscosity. To understand this let us consider the steady flow of liquid on some horizontal stationary surface.



Laminar flow

Figure 5.21

The layer of liquid which is in contact with the surface, remains stuck to it due to adhesive force acting between the molecules of the liquid and molecules of the surface. The velocity of the layer at the top is the maximum.

In Figure 5.21, some of the layers are shown along with their velocity vectors. Thus in a steady flow different layers slide over each other without getting mixed up. This kind of flow is called **laminar flow.**

In a laminar flow any two consecutive layers of fluid have relative velocity between them. As a result, resistive force is produced tangentially at the surface of layers in contact. This internal force of friction is called **viscous force. The property of fluid due to which relative motion between two consecutive layers in opposed is known as viscosity of the fluid.** In order to maintain the relative velocity of the layers the minimum external force to be applied must balance the viscous force. In absence of such external force the relative motion between the layers decreases with time due to viscous force and fluid comes to rest. This is the reason why milk in cup comes to rest after sometimes after it has been stirred.

Velocity gradient : In a laminar flow the difference in velocity between two layers of liquid per unit perpendicular distance, in the direction perpendicular to the direction of flow is called velocity gradient.

As shown in the Figure 5.21 difference in velocity of two layers having separation Δx is Δv. So, $\frac{\Delta v}{\Delta x}$ is velocity gradient. If Δx is very small velocity gradient becomes

$$\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = \frac{dv}{dx}$$

In case of laminar flow velocity gradient is constant for all layers.

Its SI unit is s^{-1}.

Now let us come back to viscosity. Here viscosity is the force opposing the motion. According to Newton's experimental work at constant temperature viscous force is given by

$$F = \eta A \frac{dv}{dx} \tag{5.9.1}$$

where F is the viscous force, A is contact-area between two layers and η is the constant of proportion known as co-efficient of viscosity. Value of η depends on type of fluid and its temperature.

If the value of η is larger, the viscous force is larger, and hence fluid can flow slowly and vice versa. Thus co-efficient of viscosity is the measure of viscosity of fluid. Also, value of η decreases with temperature of liquid but increase with increase in temperature of gases. From equation 5.9.1,

$$\eta = \frac{F}{A\frac{dv}{dx}}$$

If we take A = 1 unit and $\frac{dv}{dx}$ = 1 unit

$\eta = F$

Thus **"the viscous force acting per unit surface area of contact and per unit velocity gradient between two adjacent layers in a laminar flow, is known as the co-efficient of viscosity."**

CGS Unit of co-efficient of viscosity is dyne s cm^{-2}, and is called 'poise' in honour of Jean Lois Poiseuille, a French physician and physicist. Its SI unit is N s m^{-2} or Pa s. Its dimensional formula is $M^1L^{-1}T^{-1}$.

Value of co-efficient of viscosity for some fluids are given in Table 5.2.

Co-efficient of viscosity of some fluids

(For Information Only)

Table 5.2

Fluid	Temperature	Co-efficient of Viscosity (N s m^{-2})
Water	20ºC	1×10^{-3}
	100ºC	2.8×10^{-4}
Air	0ºC	1.71×10^{-5}
	340ºC	1.9×10^{-5}
Blood	38ºC	1.5×10^{-3}
Seasem oil		4.0×10^{-2}
Engine oil	16ºC	1.13×10^{-1}
	38ºC	3.4×10^{-2}
Honey		2.0×10^{-1}
Water vapour	100ºC	1.25×10^{-5}
Glycerine	20ºC	8.30×10^{-1}
Aceton	25ºC	3.6×10^{-4}

Illustration 6 : A disc of area 10^{-2} m^2 is placed over a layer of oil having thickness 2×10^{-3} m. If the co-efficient of viscosity of the oil is 1.55 N s m^{-2}, find the horizontal (tangential) force required to move the disc with the velocity of 3×10^{-2} ms^{-1}.

Solution :

$A = 10^{-2}$ m^2

$\Delta v = 3 \times 10^{-2}$ ms^{-1}

$\Delta x = 2 \times 10^{-3}$ m

$\eta = 1.55$ N s m^{-2}

Since,

$$F = \eta A \frac{\Delta v}{\Delta x}$$

$$= 1.55 \times 10^{-2} \times \frac{3 \times 10^{-2}}{2 \times 10^{-3}}$$

$\therefore$ $F = 2.32 \times 10^{-1}$ N

Illustration 7 : The velocities of cylindrical layers of liquid flowing through a tube, situated at distances 0.8 cm and 0.82 cm from the axis of the tube are 3 cm s^{-1} and 2.5 cm s^{-1} respectively. Find the viscous force acting between these layers if the length of the tube is 10 cm and the co-efficient of viscosity of the liquid is 8 poise.

Solution :

$r_1 = 0.8$ cm

$r_2 = 0.82$ cm

$\Delta v = 3 - 2.5 = 0.5$ cm s^{-1}

Δx = distance between the layers

= 0.02 cm

L = 10 cm

A = Area of contact of two layers

$$= 2\left(\frac{r_1 + r_2}{2}\right)L$$

η = 8 poise

$$F_v = \eta A \frac{\Delta v}{\Delta x}$$

$$= \eta\left[2\pi\left(\frac{r_1 + r_2}{2}\right)L\right]\frac{\Delta v}{\Delta x}$$

$$= 8\left[2 \times 3.14\left(\frac{0.8 + 0.82}{2}\right)10\right]\frac{0.5}{0.02}$$

$$= 16 \times 3.14 \times 0.81 \times 10 \times \frac{0.5}{0.02}$$

= 10173.6 dyne

5.10 Stokes' Law

When a body moves through a viscous medium, the layer of the medium in contact with the body drifts along with it. Hence, this layer moves with the velocity same as that of the body. But the distant layer remains stationary. Thus, laminar flow is produced between the body and distant stationary layer. Hence, viscous force acts between two adjacent layers of the medium and as a result a resistive force acts on the body moving through the medium.

The resistive force (viscous force) on a small, smooth, spherical, solid body of radius r moving with velocity v through a viscous medium, of large dimension, having co-efficient of viscosity η is given by,

$$F(v) = 6\pi\eta r v \qquad (5.10.1)$$

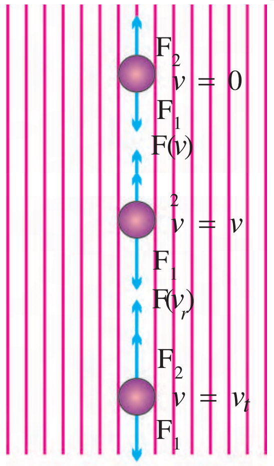
This equation is called Stokes' Law

Stokes' Law is an interesting example of a velocity dependent force in which the force acting on a body opposing its motion, is proportional to its velocity.

Motion of a sphere in a fluid and its terminal velocity :

Suppose, a small, smooth, solid sphere of radius r of material having density ρ starts motion with initial velocity zero in a fluid as shown in Figure 5.22. Let the density of the fluid be ρ_o and its co-efficient of viscosity be η. Here, $\rho > \rho_o$.

In Figure 5.22 forces acting on the sphere at three different instants are shown. These forces are : (1) weight of the sphere F_1 (downward) (2) buoyant force of the fluid F_2 (upward) (3) viscous force $F(v)$ (upward).



Free fall of a small, smooth, spherical body in a viscous medium

Figure 5.22

(1) Volume of the sphere, $V = \frac{4}{3}\pi r^3$

$\therefore$ Mass of the sphere, $m = V\rho = \frac{4}{3}\pi r^3\rho$

$\therefore$ Weight of the sphere, $F_1 = mg = \frac{4}{3}\pi r^3\rho g$

(2) The buoyant force exerted by the fluid is equal to the weight of the fluid displaced by the sphere. Volume of the fluid displaced by the sphere of volume

$$V = \frac{4}{3}\pi r^3$$

$\therefore$ Mass of the fluid displaced by the sphere,

$$m_o = V\rho_o = \frac{4}{3}\pi r^3\rho_o$$

$\therefore$ Weight of the fluid displaced by the sphere, $m_o g = \frac{4}{3}\pi r^3\rho_o g$

$\therefore$ The buoyant force, $F_2 = \frac{4}{3}\pi r^3\rho_o g$ (5.10.3)

(3) Viscous force opposing the motion, according to Stokes' Law $F(v) = 6\pi\eta rv$ (5.10.4)

$\therefore$ The resultant force acting on fluid sphere is,

$$F = F_1 - F_2 - F(v)$$

$$\therefore F = \frac{4}{3}\pi r^3\rho g - \frac{4}{3}\pi r^3\rho_o g - 6\pi\eta rv \quad (5.10.5)$$

Equation 5.10.5 represents the equation of motion of the sphere in the fluid.

At $t = 0$, when the motion of the sphere starts in the fluid, its velocity $v = 0$.

Hence, the viscous force on the sphere $F(v) = 0$.

$$\therefore F = \frac{4}{3}\pi r^3\rho g - \frac{4}{3}\pi r^3\rho_o g = \frac{4}{3}\pi r^3 g(\rho - \rho_o) \quad (5.10.6)$$

If the acceleration of sphere is a_o at $t = 0$,

$$F = ma_o = \frac{4}{3}\pi r^3\rho a_o \quad (5.10.7)$$

Comparing equations (5.10.6) and (5.10.7)

$$\frac{4}{3}\pi r^3\rho a_o = \frac{4}{3}\pi r^3 g(\rho - \rho_o)$$

$$a_o = \frac{\rho - \rho_o}{\rho} \quad (5.10.8)$$

The sphere is accelerated in the fluid. As the velocity of the sphere increases gradually with time, the viscous force acting on the sphere in upward direction also increases. Thus the velocity of the sphere increases while its acceleration decreases.

When $F_1 = F_2 + F(v)$, the resultant force on the sphere becomes zero. Hence, the acceleration also becomes zero. Now onwards, the sphere travels with constant velocity. This velocity is known as the terminal velocity v_t of the sphere. When the sphere acquires terminal velocity, $F = 0$ and $v = v_t$ and from equation (5.10.8)

$$\therefore\ 0 = \frac{4}{3}\pi r^3\rho g - \frac{4}{3}\pi r^3\rho_o g - 6\pi\eta r v$$

$$\therefore\ 6\pi\eta r v_t = \frac{4}{3}\pi r^3 g(\rho - \rho_o)$$

$$\therefore\ v_t = \frac{2}{9}\,\frac{r^2 g}{\eta}\,(\rho - \rho_o) \qquad (5.10.9)$$

With the help of terminal velocity coefficient of viscosity of the fluid can be determined using equation. (5.10.9)

A bubble formed in a liquid may be considered to be a sphere of air. In this case $\rho_o > \rho$. Hence, from the beginning F_1 being less than F_2 the bubble is accelerated upward. Therefore, the bubble rises up and acquires terminal velocity after some time. This terminal velocity can be obtained using eqn. (5.10.9). Here, v_t is negative which shows that the bubble has terminal velocity in upward direction. You might have observed the bubble rising up in a bottle of soda water.

Illustration 8 : Two rain drops of equal volume, falling with terminal velocity 10 cm s^{-1}, merge while falling and forms a larger drop. Find the terminal velocity of the larger drop.

Solution :

Let the radius and volume of each drop be r and V respectively. When they merge and form a larger drop, its volume V' will be double the volume of each one of them. (As the mass and density remain constant).

Let the radius of the bigger drop so formed, be R.

Now, $V' = 2V$

$$\frac{4}{3}\pi R^3 = 2\left(\frac{4}{3}\pi r^3\right)$$

$$R^3 = 2r^3$$

$$\therefore\ R = (2^{\frac{1}{3}})r$$

Let the terminal velocity of the smaller drop be v and that of the larger drop be v',

$$v = \frac{2}{9}\,\frac{r^2 g}{\eta}\,(\rho - \rho_o) \text{ and}$$

$$v' = \frac{2}{9}\,\frac{R^2 g}{\eta}\,(\rho - \rho_o)$$

$$\therefore\ \frac{v'}{v} = \frac{R^2}{r^2}$$

$$\therefore\ v' = v\frac{R^2}{r^2} = 10\,(2^{\frac{1}{3}})^2 = 15.87 \text{ cm s}^{-1}$$

5.11 Reynold's Number and Critical Velocity

The flow of a fluid through a given tube may be streamline, turbulent or of mixed type. In almost all experiments designed for the measurement of the co-efficient of viscocity the flow must be streamline. Therefore, it becomes necessary to know the conditions in which the flow becomes streamline.

Osborne Reynolds (1842–1912), a British mathematician and physicist, has shown that the type of flow through a tube depends on (1) the co-efficient of viscosity (η) of fluid, (2) the density (ρ) of the fluid, (3) average velocity (v) of the fluid and (4) the diameter (D) of the tube.

The number (N_R) formed by the combination of these four physical quantities is called Reynold's number.

$$\text{Reynolds number, } N_R = \frac{\rho v D}{\eta} \qquad (5.11.1)$$

The magnitude of N_R depends on the type of the flow.

N_R is dimensionless. Experiment shows that, if $N_R < 2000$ the flow is streamline and if $N_R > 3000$ the flow is turbulent. For $2000 < N_R < 3000$, the flow is unstable and its type keeps changing.

Critical Velocity : It is clear from equation 5.11.1 that with increase in velocity Reynolds number also increases. **The maximum velocity for which the flow remains streamline is called critical velocity.** The corresponding Reynolds Number is called critical Reynolds number.

It should be noted that if $\eta = 0$ (i. e. if fluid is non-viscous) N_R becomes infinite So, non-viscous flow can never be streamline.

Illustration 9 : As shown in Figure 5.23 laminar flow is obtained in a tube of internal radius r and length l. To maintain such flow, the force balancing the viscous force is obtained by producing the pressure difference

(p) across the ends of the tube. Derive the equation of velocity $v = \frac{p}{4\eta l}(r^2 - x^2)$ of a layer situated at distance 'x' from the axis of the tube.

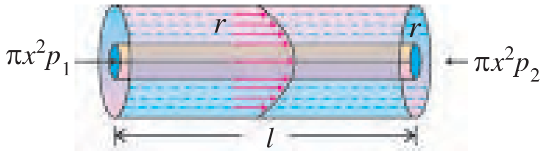


Figure 5.23

Solution : Consider a cylindrical layer of liquid of radius x as shown in Figure 5.23. The forces acting on it are as follows :

(1) The force due to the difference in pressure p is, $F_1 = \pi x^2 p$

(2) Viscous force, $F_2 = \eta A \frac{dv}{dx}$

$$= \eta(2\pi x l)\left(-\frac{dv}{dx}\right)$$

where, A = area of the curved surface of the cylinder = $2\pi x l$.

Here, v decreases with increase in x. Hence the velocity-gradient is taken negative. For the motion of the cylindrical layer with a constant velocity,

$$F_1 = F_2$$

$$\therefore \pi x^2 p = -\eta \cdot 2\pi x l \cdot \frac{dv}{dx}$$

$$\therefore -dv = \frac{p}{2\eta l}\ x\ dx$$

At $x = r$, $v = 0$ and at $x = x$, $v = v$ and so integrating the above equation in these limits we get,

$$-\int_v^0 dv = \int_x^r \frac{p}{2\eta l}\ x\ dx$$

$$\therefore -[v]_v^0 = \frac{p}{4\eta l}\left[x^2\right]_x^r$$

$$\therefore -[0 - v] = \frac{p}{4\eta l}[r^2 - x^2]$$

$$\therefore v = \frac{p}{4\eta l}(r^2 - x^2)$$

Illustration 10 : Find the volume of the liquid passing through the tube in one second in the above example. [**Hint :** Take the average of the velocities at the wall and the axis of the tube as the velocity of the flow.]

Solution :

$$v = \frac{p}{4\eta l}(r^2 - x^2)$$

$\therefore$ At the axis ($x = 0$), velocity $v = \frac{pr^2}{4\eta l}$

At the wall ($x = r$), velocity $v = 0$

$\therefore$ Average velocity $= \frac{pr^2}{8\eta l}$

Now, the volume of the liquid passing through the tube per second is,

V = (velocity) (area of cross-section)

$$= \left(\frac{pr^2}{8\eta l}\right)(\pi r^2)$$

$$\therefore V = \frac{\pi p r^4}{8\eta l}$$

[**Note :** This equation is called **Poiseiulle's Law**]

Illustration 11 : The radius of a pipe decreases according to $r = r_0 e^{-\alpha x}$; where $\alpha = 0.50\ \text{m}^{-1}$ and x is the distance of a cross-section from the first end ($x = 0$). Find the ratio of Reynolds number for two cross-sections lying at the distance of 2 m from each other. (take $e = 2.718$)

Solution : Reynolds number $N_R = \frac{\rho v D}{\eta}$

$\therefore$ For a given liquid $N_R \propto vD$

$$\therefore \frac{(N_R)_1}{(N_R)_2} = \frac{v_1}{v_2} \times \frac{D_1}{D_2} \quad (1)$$

According to the equation of continuity,

$$A_1 v_1 = A_2 v_2$$

$$\therefore \pi r_1^{\ 2} v_1 = \pi r_2^{\ 2} v_2$$

$$\therefore \frac{v_1}{v_2} = \left(\frac{r_2}{r_1}\right)^2 = \left(\frac{D_2}{D_1}\right)^2$$

From eqns. (1) and (2),

$$\frac{(N_R)_1}{(N_R)_2}=\left(\frac{D_2}{D_1}\right)^2\times\frac{D_1}{D_2}=\frac{D_2}{D_1}=\frac{r_2}{r_1}=\frac{r_0e^{-\propto x_2}}{r_0e^{-\propto x_1}}$$

$$\frac{(N_R)_1}{(N_R)_2}=e^{-\propto(x_2-x_1)}=e^{-(0.5)(2)}=e^{-1}$$

$$=0.368$$

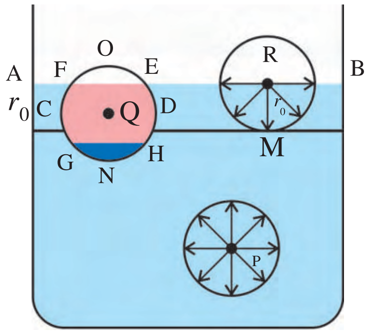
5.12 Surface Energy and Surface Tension

You must have noted that glass becomes wet due to water but lotus or lotus-leaves do not get wet by water. In lamp oil rises up against gravitational force. Some insects can walk on the water surface. If a needle is carefully placed horizontally, on the water surface, it floats. For such phenomena the property called surface tension of liquid is responsible. Due to surface tension liquid surface behaves like a stretched membrane. It is an exclusive property of liquids.

5.12.1 Surface energy :

The inter-molecular attractive force between the molecules of the same substance is called **cohesive force.** The attractive force between the molecules of different substances is known as **adhesive force.**

The maximum distance upto which two molecules can exert attractive force on each other is called the '**range of inter-molecular force (r_0).** An imaginary sphere of radius r_0, drawn by taking any molecule as the centre, is called the **sphere of the molecular action of the molecule at the centre.** Only the molecules inside this sphere can exert attractive forces on the molecule at the centre. The molecules outside this sphere will not exert forces of attraction on the molecule at the centre.



Spheres of molecular action

Figure 5.24

To understand the surface effect produced due to the inter-molecular forces, consider three molecules P, Q and R of a liquid along with their spheres of molecular action as shown in Figure 5.24.

Suppose the range of inter-molecular force is r_0. AB shows the free surface of the liquid. The sphere of action of molecule P is completely immersed in the liquid. Therefore, it is fully occupied uniformly with the molecules of the liquid. As a result P is acted upon by equal forces of attraction from all sides. The resultant force on P will thus be zero and it remains in equilibrium. All molecules at depths more than r_0 from the free surface of the liquid will be in similar situation.

The depth of molecule Q is less than r_0. Part FOEF of its sphere of action is outside the liquid and this part contains the molecules of both air and liquid vapour. The densities of air and vapour are much less than that of the liquid. Moreover, the adhesive forces acting between the molecules of air and liquid are comparatively feeble. Hence the resultant force due to the molecules in the GNHG part is more than the resultant upward force due to the molecules of air and vapour in the similar region FOEF. The number of molecules of the liquid in the regions CDHG and CDEF is equal. Hence the resultant force on Q due to molecules in these regions is zero. Thus, molecule Q is under the influence of resultant downward force. **A layer of thickness r_0 below the free surface of a liquid is called the surface of the liquid.** Thus, the resultant force on the molecules of the liquid in its surface is in vertically downward direction. As we move upwards in the surface, the magnitude of the downward resultant force keeps on increasing. The resultant force on the molecules on the free surface AB is maximum. Hence the molecules of liquid lying in the surface have a tendency to go inside the body of the liquid.

In these circumstances, some of the molecules do go down and as a result of this the density below he surface of the liquid increases. Hence, more than a certain number of molecules will not be able to go down. As a result the density of the liquid below the surface is more and it decreases gradually as we move upwards in the surface. In other words, the inter-molecular distances between the molecules are less below the surface while within the surface these distances are more. Taking the inter-molecular forces as a function of inter-molecular distances it can be proved that inter-molecular distance being more in the surface, the molecules lying in it experience force of tension parallel to the surfce.

Thus, the surface of a liquid has a tendency to contract like stretched elastic membrane and as a result tension prevails in the surface (parallel to the surface). The magnitude of this tension is given by a physical quantity known as surface tension.

"The force exerted by the moleculs lying on one side of an imaginary line of unit length, on the molecules lying on the other side of the line, which is perpendicular to the line and parallel to the surface is defined as the surface tension (T) of the liquid."

$$\therefore \text{ Surface tension } T = \frac{F}{L} \qquad (5.12.1)$$

$$\therefore \; F = TL \qquad (5.12.2)$$

The SI unit of surface tension is N m^{-1}.

It should be noted that the force of surface tension is not the resultant cohesive force between the molecules on the surface of a liquid. In fact, the resultant cohesive force on the molecules acts in a direction perpendicular to the surface and towards the inside of the liquid, while the force of surface tension is parallel to the surface.

For a line (imaginary) of unit length in the middle of the surface of a liquid, the molecules on both the sides of it exert forces which are equal in magnitude and opposite in direction. Hence, the effect of force of surface tension is not felt in the middle of the surface. At the edge of the surface there are no molecules on the other side of the edge. Hence, surface tension manifests there, parallel to the surface and perpendicular to the edge in the inward direction.

Surface tension in context of potential energy :

We have seen that the molecules in the surface of a liquid have a tendency to go down inside the liquid. This behaviour can be explained on the basis of potential energy of the molecules. If a molecule like P, in Fig 5.24 is to be brought up in the surface, work has to be done on it against the downward force acting on it during this. Hence, when such a molecule reaches the surface it acquires potential energy. This fact shows that the potential energy of the molecules in the surface is more than that of the molecules beneath the surface. **Now a system always tries to remain in such a state where its potential energy is minimum.** Therefore, molecules in the surface of a liquid have a tendency to reduce their potential energy and so the surface of a liquid has a tendency to contract in such a way that its area becomes minimum.

The magnitude of the surface tension can also be given in the context of the potential energy of the molecules. We have noted that work has to be done in bringing the molecules from within the liquid to the surface, which is stored in the form of its potential energy. An important point to be noted is that the molecule thus coming to the surface does not occupy a place between two molecules already present in the surface. The molecules reaching to the surface generate new surface, which means that the surface gets expanded. The whole surface of a liquid can be considered to have been generated in this way. Thus, the molecules in the surface of a liquid possesses potential energy equal to the work done on them in bringing them to the surface.

"The potential energy, stored in the surface of a liquid, per unit area, is known as surface tension (T) of the liquid."

According to this definition the unit of

$$\text{surface tension } T = \frac{E}{A}$$

According to this definition the unit of surface tension is J m^{-2}.

$$\text{Now, } \frac{\text{joule}}{m^2} = \frac{\text{newton meter}}{\text{meter}^2} = \frac{\text{newton}}{\text{meter}}$$

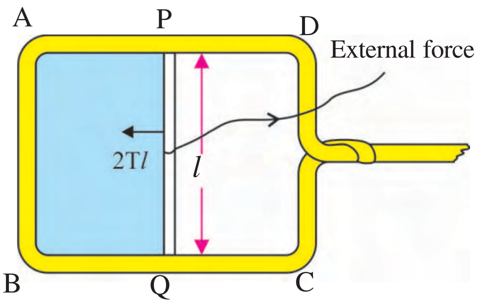
Thus, the unit obtained from both the definitions are the same. Surface tension of a liquid depends on the type of the liquid and its temperature. The surface tension decreases with increace in temperature and at critical temperature surface tension becomes zero. Also, the surface tension of a liquid depends on the type of the medium it is in contact with.

Surface-energy : Suppose, surface tension of a given liquid at a given temperature is T. If the surface of liquid is to be increased by unit at constant temperature, the work required to be done on it is T. But we know that when a surface expands, its temperature decreases. Hence, if surface is to be expanded at constant temperature, heat should be supplied from outside during its expansion. Thus, by increasing the surface of liquid by unit, the new surface so formed gets thermal energy over and above the potential energy (=T).

∴ Total surface-energy per unit area = Potential energy (surface tension) + Heat energy.

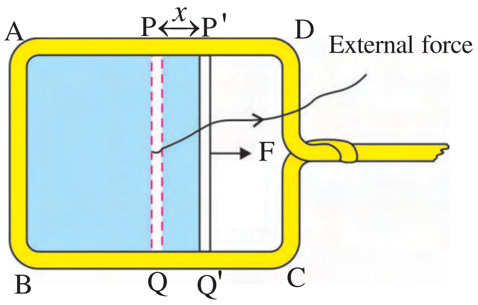
Thus, at any temperature the value of surface energy is more than surface tension. The surface tension and surface energy both decrease with increase in temperature and at critical temperature they become zero.

Our discussion so far have been phenomenological. Now, we experimentally verify the conclusions of this discussion. For this, concentrate on a rectangular frame ABCD made from a wire as shown in Fig. 5.25. The wire PQ is able to slide without friction over the sides AD and BC of the frame. A light string is tied to PQ.



(*a*)

A thin film of liquid formed on a rectangular frame



(*b*)

Expansion of the film

Figure 5.25

If the frame is dipped and taken out of the soap solution then, pulling the wire PQ properly with the help of the string, a thin film ABQP is formed on the frame. If the string is released, PQ is found to slide towards AB, and the film contracts.

This experiment shows that the surface tension manifests itself on the edge of the surface of the liquid, perpendicular to the edge and parallel to the surface.

Again prepare the film ABQP. Pull the wire (with the help of the string) by a force, which is slightly more than the force of surface tension acting on it and displace it by x. The work done for these can be calculated as under.

Suppose the surface tension of the solution is T and the length of wire PQ is l.

Hence, the force acting on the wire due to surface tension is $= 2Tl$.

As the film has two free surfaces, '2' appears in the equation of force.

Applied external force $F = 2Tl$ (5.12.4)

Now, work W = external force × displacement.

$\therefore\ W = 2Tlx$

But, increase in the area of the surface of the film $= \Delta A = 2lx$. (5.12.5)

$\therefore\ W = T\Delta A$

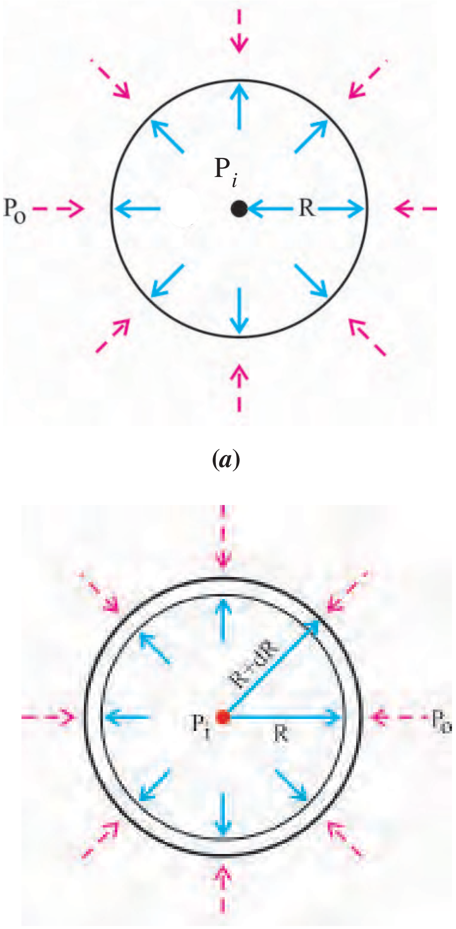
If $\Delta A = 1$ unit, $W = T$

∴ Work done to increase the area of the surface by 1 unit is equal to the measure of surface tension.

5.13 Drops and Bubbles

Small drops of liquid or bubbles are always spherical. Obvious question coming to mind is that why they should be of spherical in shape only ? Due to surface tension free surface of liquids have a tendency to make its surface area as small as possible. Since spherical surface has minimum area for a given volume, small drops of liquid are always spherical.

The surface of a drop or a bubble are curved. The pressure on a concave surface is always more than that on the convex surface. Hence, the pressure inside a drop or a bubble is always more than the pressure outside.



(a)

(b)

Bubble

Figure 5.26

Consider a bubble of radius R, in air as shown in Figure 5.26. The pressures inside and outside are P_i and P_0 respectively. Here, $P_i > P_o$. Let the surface tension of the liquid (solution) forming the wall of the bubble be T.

Suppose, by blowing the bubble its radius increases from R to (R + *dR*). (Figure 5.26). Hence, the area of its free surface increases from S to S + *dS*. The work done in this process can be calculated in two different ways :

(1) While blowing a bubble, the force exerted on its surface of area $4\pi R^2$, due to the pressure difference $(P_i - P_0)$, is $(P_i - P_0)\ 4\pi R^2$. And the surface displaces by an amount *d*R under the influence of this force.

∴ Work done on the surface is

$$W = \text{force} \times \text{displacement}$$

$$= (P_i - P_0)\ 4\pi R^2 \cdot dR \qquad (5.13.1)$$

(2) The surface area of the bubble of radius R is, $S = 4\pi R^2$

Now, when the radius becomes (R + *d*R), the increase in the surface area is,

$$dS = 8\pi R dR$$

But the bubble in air has two free surfaces.

∴ Total increase in its area $= 2 \times 8\pi R dR$

$= 16\pi R dR$

Hence, the work required to be done on the surface is,

W = surface tension × total increase in area.

$$\therefore\ W = 16\pi T R dR \qquad (5.13.2)$$

Comparing equation (5.13.l) with equation (5.13.2),

$$4\pi(P_i - P_0)R^2\ dR = 16\pi T R dR$$

$$\therefore\ P_i - P_0 = \frac{4T}{R} \qquad (5.13.3)$$

If a bubble is formed in a liquid, it has only one free surface.

$$\therefore\ P_i - P_0 = \frac{2T}{R} \qquad (5.13.4)$$

Note : A drop of liquid has one free surface and so the pressure difference for the drop can be obtained from equation (5.13.4).

Illustration 12 : Find the pressure in a bubble of radius 0.2 cm formed at the depth of 5 cm from the free surface of water. The surface tension of water is 70 dyn cm^{-1} and its density is 1 g cm^{-3}. Atmospheric pressure is 10^6 dyn cm^{-2}. The gravitational acceleration is 980 cm s^{-2}.

Solution :

h = 5 cm

R = 0.2 cm

T = 70 dyne cm^{-1}

ρ = 1 g cm^{-3}

P = atmospheric pressure

= 10^6 dyne cm^{-2}

g = 980 cm s^{-2}

If the pressures inside and outside of the air bubble formed in water are P_i and P_0 respectively,

$$P_i - P_0 = \frac{2T}{R}$$ (A bubble in water has one free surface only)

$$\therefore\ P_i = P_0 + \frac{2T}{R} \qquad (1)$$

But P_0 = atmospheric pressure + pressure due to water column of height h.

$\therefore P_0 = P + h\rho g$ (2)

from equations (1) and (2),

$$P_i = P + h\rho g + \frac{2T}{R}$$

$$= 10^6 + (5 \times 1 \times 980) + \frac{2 \times 70}{0.2}$$

$$= 10^6 + 4900 + 700$$

$$P_i = 1.0056 \times 10^6 \text{ dyn cm}^{-2}$$

Illustration 13 : When a hollow sphere having a hole in it is taken to the depth of 40 cm from the surface of water, water starts entering into the sphere. If the surface tension of water is 70 dyn cm^{-1}, find the radius of the hole. Take $g = 10$ ms^{-2}.

Solution : Let the radius of the hole be r. Here, the depth of the sphere is $h = 40$ cm. At this depth pressure of water is $= h\rho g = 40 \times 1 \times 1000 = 40000$ dyn cm^{-2}.

When water enters into the sphere, bubble having the radius same as the radius of the hole comes out of it. The excess pressure inside the bubble $= \frac{2T}{r} = \frac{2 \times 70}{r}$

$\therefore$ In the state of equilibrium, $h\rho g = \frac{2T}{r}$

$$\therefore \quad 40000 = \frac{2 \times 70}{r}$$

$$\therefore \ r = 3.5 \times 10^{-3} \text{ cm}$$

Illustration 14 : n droplets, each of radius r, merge to form a bigger drop of radius R. If the surface tension of the liquid is T, find the energy released.

Solution : Total volume of n droplets of radius r = Volume of a drop of radius R.

$$\therefore \left(n\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi R^3$$

$$\therefore \ nr^3 = R^3 \qquad (1)$$

Total surface area of n drops, $A_1 = n(4\pi r^2)$

and the area of one large drop $A_2 = 4\pi R^2$

$\therefore$ The decrease in the area = ΔA

$$= A_1 - A_2 = n \cdot 4\pi r^2 - 4\pi R^2$$

$$= 4\pi(nr^2 - R^2)$$

$\therefore$ Energy released, $W = T\Delta A = 4\pi T (nr^2 - R^2)$ (2)

(To obtain result (2) it is not necessary to obtain result (1), but to represent the result (2) in a following specific form result (1) is necessary.)

$$W = T\Delta A = 4\pi TR^3\left(\frac{nr^2 - R^2}{R^3}\right)$$

$$= 4\pi TR^3\left(\frac{nr^2}{nr^3} - \frac{R^2}{R^3}\right)$$

$$= 4\pi TR^3\left(\frac{1}{r} - \frac{1}{R}\right) \quad (3)$$

Illustration 15 : Two soap bubbles of radii R_1 and R_2 merge to form a bubble of radius R. If the pressure of atmosphere is P and surface tension of the soap solution is T, prove that,

$$P(R_1^3 + R_2^3 - R^3) = 4T(R^2 - R_1^2 - R_2^2)$$

Assume that the temperature remains constant during this process.

Solution :

Pressure inside the first bubble $= P_1$

$$= P + \frac{4T}{R_1}$$

Pressure inside the second bubble $= P_2$

$$= P + \frac{4T}{R_2}$$

And the pressure inside the compound bubble

$$= P_3 = P + \frac{4T}{R}$$

Here, P = pressure outside each bubble = atmospheric pressure (which is the same for all)

If the volumes of these bubbles are V_1, V_2 and V_3 respectively,

$$V_1 = \frac{4}{3}\pi R_1^3;\ V_2 = \frac{4}{3}\pi R_2^3;\ V_3 = \frac{4}{3}\pi R^3$$

As temperature is constant, according to Boyle's Law,

$$P_1V_1 + P_2V_2 = P_3V_3$$

$$\therefore \left(P + \frac{4T}{R_1}\right)\left(\frac{4}{3}\pi R_1^3\right) + \left(P + \frac{4T}{R_2}\right)\left(\frac{4}{3}\pi R_2^3\right)$$

$$= \left(P + \frac{4T}{R}\right)\left(\frac{4}{3}\pi R^3\right)$$

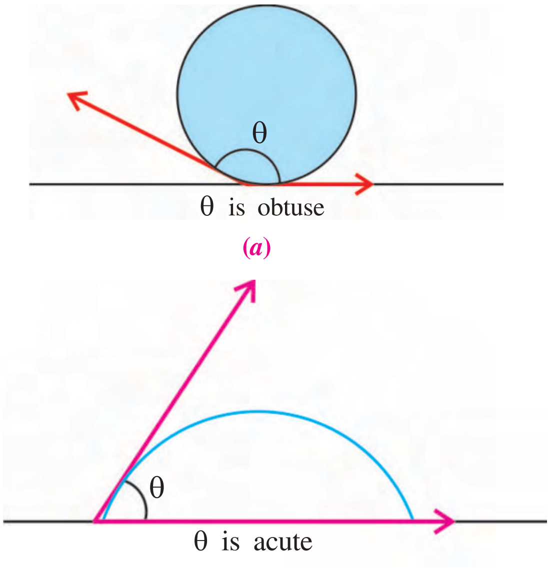
$$\therefore \frac{4}{3}\pi P(R_1^3 + R_2^3 - R^3) = \frac{4}{3}\pi \times 4T$$

$$(R^2 - R_1^2 - R_2^2)$$

$$P\ (R_1^3 + R_2^3 - R^3) = 4T(R^2 - R_1^2 - R_2^2)$$

5.14 Angle of contact

You must have observed a dew-drop. It is spherical. Any liquid when comes in contact with another medium its surface is curved. Consider a liquid drop as shown in Figure 5.27(*a*) and 5.27(*b*) to understand it better.



(*a*)

(*b*)

Angle of contact

Figure 5.27

The angle between tangent to the liquid surface at the point of contact and solid surface inside the liquid is called the angle of contact. Angle of contact depends on types of liquid and solid which are in contact.

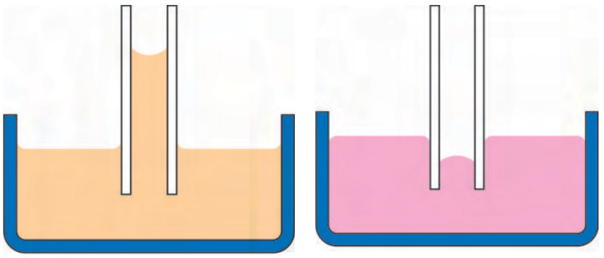
If the angle of contact is less than 90°, the liquids wet the solid, sticks to the solid and rises up in the capillary of that solid.

If the angle of contact is more than 90°, the liquid does not wet the solid, does not stick to the solid and falls in the capillary of that solid.

If a water droplet is in contact with lotus-leaf (Figure 5.27(*a*)) angle of contact is obtuse. If water is in contact with glass (Figure 5.27(*b*)) angle of contact is acute.

5.15 Capillarity

The phenomenon of rise or fall of a liquid in a capillary (held vertical in a liquid) is called capillarity. In this phenomenon, the surface tension of the liquid plays an important role.

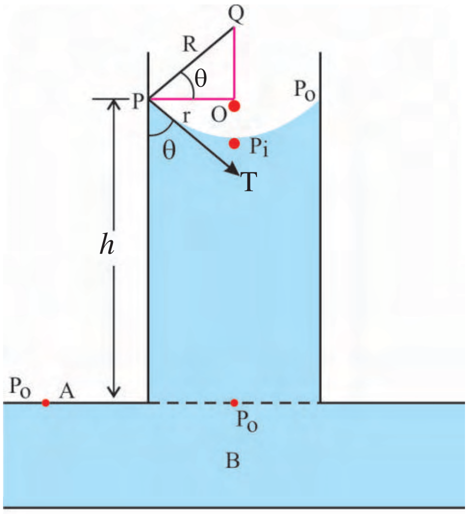


Water (*a*) **Mercury (*b*)**

Phenomenon of capillarity

Figure 5.28

As shown in Figure 5.28(*a*) when a glass capillary (having a small bore) is held vertical in water, water rises in the capillary. Whereas (as shown in Figure 5.28(*b*) when a capillary is held vertical in mercury, mercury falls in the capillary. Also, note that water wets the glass while mercury does not. If you observe attentively you will find that the free surface (meniscus) of the water rising in the capillary is concave, while the free surface of mercury falling in the capillary is convex.



Column of liquid in a capillary

Figure 5.29

As shown in Figure 5.29 suppose a capillary of radius *r* is held vertical in liquid and liquid rises to height *h* in the capillary. The radius of the concave meniscus of the liquid in capillary is R.

The relation between the radius of curvature of meniscus (R) and the radius of capillary (*r*) may be obtained as follows :

From the geometry of Figure 5.29 ∠OPQ = θ in ΔOPQ,

$$\therefore \cos\theta = \frac{OP}{PQ}$$

$$= \frac{\text{Radius of the capillary } (r)}{\text{Radius of the meniscus (R)}}$$

$$\therefore R = \frac{r}{\cos\theta} \qquad (5.15.1)$$

Now, the liquid shown in the figure is in equilibrium. Let the pressure on the concave surface of the meniscus be P_o and that on its convex surface be P_i. Here, $P_o > P_i$ and $P_o - P_i = \frac{2T}{R}$ ($\because$ The liquid has one free surface.) (5.15.2)

Note that P_o is atmospheric pressure. The same pressure acts on plain surface of the liquid at point A and also at point B in the same horizontal level, with A.

Pressure at point B is, $P_o = P_i + h\rho g$

Here, ρ is the density of the liquid and *g* is gravitational acceleration.

$$\therefore P_o - P_i = h\rho g \qquad (5.15.3)$$

Comparing eqns. (5.15.2) and (5.15.3)

$$\frac{2T}{R} = h\rho g$$

$$\therefore T = \frac{Rh\rho g}{2}$$

Substituting the value of R from eqn. (5.15.1),

$$T = \frac{rh\rho g}{2\cos\theta} \qquad (5.15.4)$$

Using this equation T can be found.

From this equation $h = \frac{2T\cos\theta}{r\rho g}$

(i) If θ < 90°, cosθ is positive and this equation gives h as positive. ∴ The liquid rises up in the capillary (e.g. glass-water)

(ii) If θ < 90°, cosθ is negative and this equation gives h as negative. ∴ The liquid falls in the capillary. (e.g. glass-mercury)

In this case meniscus is convex. Also, $P_i > P_o$. Thus in eqn. (5.15.2) $P_i - P_o = \frac{2T}{R}$ should be taken. As $P_i - P_o = h\rho g$, the final result in eqn. (5.15.4) will not change.

When a detergent or soap dissolves in water, surface tension of the solution becomes lesser than that of water. Due to this washing ability increases.

Illustration 16 : Radius of a glass capillary is 0.5 mm. Find the height of the column of water when it is held vertical in water. The density of water is 10^3 kg m^{-3} and the angle of contact between glass and water is 0°, $g = 9.8$ ms^{-2} and the surface tension of water is T = 0.0727 Nm^{-1}.

Solution :

$r = 0.5 \text{ mm} = 5 \times 10^{-4}$ m

$\rho = 103$ kg m^{-3}

$\theta = 0^o \;\; \therefore \cos 0^o = 1$

$g = 9.8$ ms^{-2}

T = 0.0727 Nm^{-1}

$$T = \frac{rh\rho g}{2cos\theta}$$

$$\therefore\ h = \frac{2Tcos\theta}{r\rho g}$$

$$= \frac{2 \times 0.0727 \times 1}{5 \times 10^{-4} \times 10^{3} \times 9.8}$$

$$...\ h = 0.0296 \text{ m} = 2.96 \text{ cm}$$

Illustration 17 : Two rectangular slides of glass are kept 1 mm apart. They are partially immersed in water in such a way that air column (as well as that of water) may remain vertical between them as shown in Figure 5.30. What is the height of the water which rises between the plates ?

$T = 70$ dyn cm^{-1}.

Solution : Suppose that the breadth of the glass slides is l. In this state total length at which water and glass are in contact is $2l$. The angle of contact between water and glass is zero. Suppose water rises to height h cm.

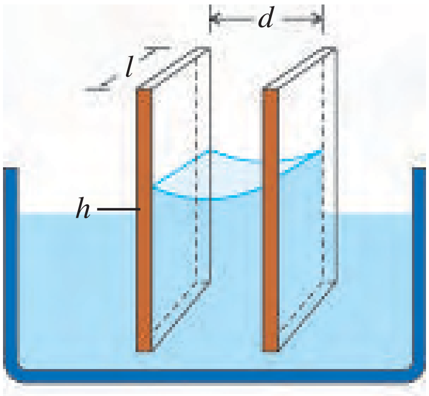


Figure 5.30

$\therefore$ Volume of the column of water rising up $= ldh$.

where d = distance between two plates.

If the density of water is ρ and gravitational acceleration is g, the weight of water in downward direction = $(ldh)\ \rho g$. This force should be equal to the force of surface tension acting on length $2l$.

$$\therefore\ 2Tl = (ldg)h\rho$$

$$h = \frac{2T}{dg\rho} = 1.43 \text{ cm}$$

SUMMARY

1. A substance that can flow is known as a fluid.
2. Magnitude of force acting perpendicularly to a surface of unit area is called pressure. It is a scalar quantity. Its unit is Nm^{-2} or (P_a).
3. If force acting on surface makes angle θ with the normal drawn to a surface, the component $Fcos\theta$ is taken into account for the pressure. Thus pressure $P = \frac{Fcos\theta}{A}$.
4. Ratio of mass of the body to its volume is called density. Its unit is kg m^{-3}.
5. Ratio of density of a substance and density of water at 277K is called specific density. It is dimensionless.
6. **Pascal's Law :** If the effect of gravity is neglected, pressure in fluid is same everywhere.
7. **Pascal's Law of transmission of pressure :** In an enclosed liquid, if pressure is changed in any part of the liquid, the change is transmitted equally to all the parts of the liquid.
8. Hydraulic lift, hydraulic brake, door closer and shockabsorbers of automobiles work on the Pascal's Law.
9. In a fluid rate of change in pressure with depth is ρg.

10. Pressure at the bottom of incompressible fluid column is $h\rho g$.

11. Pressure due to fluid column does not depend on shape and area of the container.

12. **Archimedes' Principle :** When a body is partially or completely immersed in a liquid, the buoyant force acting on it is equal to the weight of liquid displaced by it and it acts in the upward direction at the centre of mass of the displaced liquid.

13. **Law of Floatation :** When the weight of a body is equal to the weight of the liquid displaced by the part of the body immersed in it the body floats on the surface of the liquid.

14. **Steady Flow :** If in a fluid flow velocity of fluid particle remains constant with time, fluid flow is called steady flow.

15. **Turbulent flow :** If in a flow of fluid velocity of fluid particle changes in an irregular manner from time to time and from point to point, the flow is known as turbulent flow.

16. **Irrotational flow :** If an element of a fluid at each point has no net angular velocity about that point, the fluid flow is called irrotational.

17. **Incompressible flow :** If density of fluid remains constant with time everywhere, the flow is said to be incompressible.

18. **Non-viscous flow :** The flow of a fluid having small co-efficient of viscosity is known as non-viscous flow.

19. Flow of an ideal fluid is steady, irrotational, incompressible and non-viscous.

20. **Line of flow :** Path along which particle moves in a fluid is called a line of flow.

21. **Streamline :** The curve for which tangent drawn at any point shows the direction of velocity of fluid particle is known as streamline.

22. **Tube of flow :** An imaginary tube formed by a bundle of streamlines is called tube of flow.

23. **Volume flux :** Volume of fluid flowing through any cross section in unit time is called volume flux. It is equal to the product of area of cross section and velocity.

24. **Dynamic lift :** When an object undergoes relative motion with respect to fluid, a force arises which diverts the object from its original path. This phenomenon is known as dynamic lift.

25. **Aerofoil :** An object when travels horizontally experiences force in upward direction due to its shape is called an aerofoil.

26. **Force of viscosity :** In a laminar flow, any two consecutive layers of fluid have relative velocity between them. As a result, a resistive force is produced tangentially at the surfaces of the layers in contact. This force is known as viscous force.

27. **Velocity gradient :** In a laminar flow, the difference in velocity between two layers of liquid per unit perpendicular distance, in the direction perpendicular to the direction of flow, is called velocity gradient. Its unit is s^{-1}.

28. **Co-efficient of viscosity :** The viscous force acting per unit surface area of contact and per unit velocity gradient between two adjacent layers in a laminar flow is known as the co-efficient of viscosity.

29. **Stokes' Law :** The viscous force on a small spherical solid body of radius r and moving with velocity v through a viscous medium of large dimensions having co-efficient of viscocity η is $6\eta\pi rv$.

30. When a fluid flows through a tube, type of flow depends on density of fluid (ρ), velocity of fluid (v), diameter of tube (D) and viscosity (η) of fluid. Type of flow can be decided by Reynolds number

Reynolds number, $N_R = \dfrac{\rho D v}{\eta}$ ....

If $N_R < 2000$ flow is streamline, if $N_R > 3000$ is turbulent and if $2000 < N_R < 3000$ type of flow is uncertain.

31. The maximum velocity for which flow can be streamline flow is known as critical velocity.

32. **Adhesive force :** Attractive force between the molecules of different matter is known as adhesive force.

33. **Cohesive force :** The inter-molecular attractive force between molecules of the same matter is called cohesive force.

34. The maximum distance up to which a molecule can exert attractive force on the other is called range of inter-moleculer force (r_0). An imaginary sphere of radius r_0 with a molecule at its centre is called sphere of action.

35. Work required to be done to increase the surface area by one unit at constant temperature is called surface tension. Also, the force exerted by the molecules lying on one side of an imaginary line of unit length on the molecules lying on the other side of the line which is perpendicular to the line and parallel to the surface, is defined as the surface tension (T) of the liquid.

36. Shape of the free surface of liquid depends on pressures acting on two sides. If the pressure is larger on the outer side of the surface, then the surface is concave and vice versa.

37. If for a bubble in air, inner pressure is P_i and outer pressure is P_o.

$P_i - P_o = \dfrac{4T}{R}$. Where T is surface tension and R is the radius of bubble.

For a drop of liquid or the bubble formed inside the liquid $P_i - P_o = \dfrac{2T}{R}$.

38. When liquid comes in contact with a solid, its surface becomes curved. Angle between the tangent to liquid surface at point of contact and the solid surface inside the liquid is called angle of contact.

39. The phenomenon of rise or fall of a liquid in a capillary held vertical in liquid is called capillarity.

40. When a soap or detergent dissolves in water, surface tension of the solution is less than that of water. Due to this, washing ability of water increases.

EXERCISES

Choose the correct option from the given options :

1. The speed of air above the wings of an aeroplane is 120 ms^{-1} and below the wings it is $90ms^{-1}$. If density of air is 1.3 kgm^{-3} the pressure difference is .... (Neglect the thickness of wings.)

(A) 156 P_a (B) 39 P_a (C) 4095 P_a (D) 6300 P_a

2. A small sphere of mass m and radius r falls through a viscous medium. Its terminal velocity is proportional to ....

(A) only $\frac{1}{r}$ (B) only m (C) $\sqrt{\frac{m}{r}}$ (D) $\frac{m}{r}$

3. A plate of area 10 cm^2 is placed over a plate. A layer of glycerine of 1 mm thickness is lying between two plates. To move the upper plate with velocity $10ms^{-1}$ the required external force is ....($\eta_{\text{glycerine}}$ = 20 poise)

(A) 80 dyne (B) 200 × 10^3 dyne

(C) 800 dyne (D) 2000 × 10^3 dyne

4. As small sphere falls through a viscous medium. Curve .... of the graph in Figure 5.31 represents its motion.

(A) A (B) B

(C) C (D) D

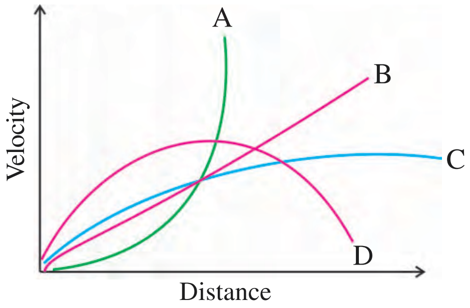


Figure 5.31

5. Reynolds number is small for a liquid with ....

(A) low velocity (B) low density (C) high viscosity (D) all of these

6. With reference to Reynolds number in which of the following cases flow is more likely to be streamline ?

(A) low ρ (B) high ρ, high η

(C) high ρ, low η (D) low ρ, high η

7. Surface tension of soap solution is 1.9 × 10^{-2} Nm^{-1}. The work done to blow a bubble of diameter 2.0 cm is ....

(A) 17.6 × $10^{-6}\pi$ J (B) 15.2 × $10^{-6}\pi$ J

(C) 19 × $16^{-6}\pi$ J (D) $10^{-4}\pi$ J

8. Excess pressure inside two bubbles are 1.01 atm and 1.02 atm respectively. The ratio of their surface areas is ....

(A) 4 : 1 (B) 1 : 26 (C) 8 : 1 (D) 1 : 8

9. Liquid rises in a capillary to height h. In which of the following case will water rise more than h ?

(A) In an elevator accelerated downwards
(B) In an elevator accelerated upwards
(C) On poles
(D) Height will remain constant

10. The type of a flow of water having velocity 10 cms^{-1} through a tube of radius 0.5 cm is .... (η_{water} = 0.1 poise, ρ_{water} = 1g cm^{-3})

(A) streamline (B) unstable (C) turbulent (D) none of these

11. A ring of radius 4 cm is dipped into glycerine. (T = 63 dyne cm^{-1}) and pulled out of the liquid keeping it horizontal at the surface of liquid. The force required to detach the ring from the surface over and above its weight is .... dyne.

(A) 63 π (B) 504 π (C) 1008 π (D) 1512 π

12. A film of soap solution is made in rectangular frame of length 10 cm and breadth 4 cm. The force of surface tension on the smaller edge of the frame is .... dyne. (Surface tension of soap solution = 30 dyne cm^{-1})

(A) 60 (B) 120 (C) 300 (D) 240

13. To form the film described in the above question, amount of mechanical work done against the force of surface tension is .... erg.

(A) 1200 (B) 2400 (C) 2600 (D) 4800

14. When an air bubble rises from the bottom of a lake to the surface its radius doubles. If 10 *m* of water column produces pressure equal to 1 atm, depth of tube is .... *m*. (Take g = 10 m s^{-2})

(A) 10 (B) 20 (C) 70 (D) 80

15. An incompressible fluid flows through a cylindrical horizontal pipe of radius x at point A and $\frac{x}{2}$ at another point B. Ratio of velocities at point A and point B is ....

(A) 2 : 1 (B) 1 : 2 (C) 1 : 4 (D) 4 :1

16. The rate of flowing water from an orifice in a wall of tank will be more if the orifice is ....

(A) closer to top (B) closer to bottom

(C) in the middle (D) none of these

17. Particles of liquid P, Q and R are on free surfaces within the surface and bellow the surface respectively. If their potential energies are U_P, U_Q and U_R then.

(A) $U_P < U_Q < U_R$ (B) $U_P < U_R < U_Q$

(C) $U_R < U_P < U_Q$ (D) $U_R < U_Q < U_P$

18. A small ball is released in a viscous liquid, velocity of ball in liquid ....

(A) keeps increasing (B) keeps decreasing

(C) remains constant

(D) first increases and then remains constant

ANSWERS

1. (C) **2.** (D) **3.** (D) **4.** (C) **5.** (D) **6.** (D)

7. (B) **8.** (A) **9.** (A) **10.** (A) **11.** (C) **12.** (D)

13. (B) **14.** (C) **15.** (C) **16.** (B) **17.** (D) **18.** (D)

Answer the following questions in short :

1. State the Pascal's Law for pressure transmission.
2. Which has more pressure ? 75 cm of mercury or column 10 m of water column ? (Specific gravity of mercury is 13.6)
3. What is the principle behind the working of a water sprinkler ?
4. Bernoulli's equation for fluid flow is an alternative statement of law of conservation of energy. True or false ?
5. What are the units of pressure head, velocity head and elevation head ?
6. Standing on a railway platform close to the tracks, why do we feel being dragged toward the train, passing by very fast ?
7. What is an aerofoil ?
8. How does viscosity of fluids vary with temperature ?
9. Water flowing in a broad pipe enters into a narrow pipe. How does its Reynolds' number change ? (Pipe is horizontal)
10. Why are some insects able to walk on water ?
11. Will the angle of contact for water and material of raincoat be acute or obtuse ?
12. Define surface tension. Write its unit and dimensions.
13. A larger bubble and smaller bubble are formed in air at two ends of a thin tube. What will happen to bubbles ?

Answer the following questions :

1. State and prove Pascal's law.
2. Deduce the formula of pressure due to a liquid column of height h and density ρ.
3. What is a steramline flow. Derive the equation of continuity for steady incompressible flow.
4. Derive Bernoulli's equation for steady, incompressible, irrotational, non-viscous flow of fluid.
5. Explain the working of a venturie meter with neat diagram and necessary formula.
6. What is a laminar flow ? Explain viscous force in such a flow.
7. State Stokes' Law and deduce the expression for initial acceleration for a small, smooth sphere falling in a viscous liquid.
8. Write a short note on Reynolds' number.
9. Derive the formula for excess pressure inside the bubble in case of bubble in air.
10. What is capillary action ? Derive the formula for rise of liquid in a capillary tube immersed vertically in liquid.

Solve the following problems :

1. The piston and nozzle of a syringe kept horizontal have diameters 5mm and 1mm. The piston is pushed with constant velocity of 0.2 m s^{-2}. Find the horizontal distance travelled by water jet before touching water (g = 10 m s^2). Height of syringe from ground is 1 m. [**Ans. :** $\sqrt{5}$ m]

2. Water is partially filled in a U shaped tube held vertical. When another immiscible liquid is poured in one of the arms of the tube, water rises by '*d*' unit in the other arm. If the free surface of the liquid is heigher by '*h*' unit compared to water, find density of the liquid. Take density of water to be ρ unit. [**Ans. :** $\left(\frac{2d}{2d+h}\right)\rho$]

3. Water is flowing through a horizontal pipe of irregular cross section. If the pressure at a point where the velocity is 0.2ms^{-1} is 30mm Hg, what will be the pressure at a point where the velocity is 1.2 ms^{-1} ? (Density of mercury = 13.6 g cm^{-3}, g = 1000 cm s^{-2}, Density of water = 1 g cm^{-3})

[**Ans. :** 24.85 mm Hg]

4. Find the work required to be done to increase the volume of a bubble of soap solution having a radius 1 mm to 8 times. (Surface tension of soap solution is 30 dyne cm^{-1}). [**Ans. :** 2261 erg.]

5. Diameters of the arms of a U tube are 10 mm and 1 mm. It is partially filled with water and it is held in a vertical plane. Find the difference in heights of water in both the arms. (Surface tension of water = 70 dyne cm^{-1}. Angle of contact = 0^o. $g = 980\text{ cm s}^{-2}$) [**Ans. :** 2.8571 cm]

6. A bubble of air of diameter 0.2 cm rises up uniformly in water with a velocity of 200 cm/s. If the density of water is 1 g cm^{-3}, find co-efficient of viscocity of water. Neglect density of air w.r.t. that of water.

($g = 4.8\text{ m s}^{-2}$) [**Ans. :** 0.0109 poise)

7. The velocity of a cylindrical layer of liquid at a distance 0.4 cm from the axis of a tube of radius 0.5 cm is 3.6 cm/s. Find the velocity of a layer lying at a distance of 0.3 cm from axis. [**Hint :** $v = \frac{P}{4\eta l}(r^2 - x^2)$]

[**Ans. :** 6.4 cm/s]

8. What should be the difference in pressure across the ends of a 4km long horizontal pipe line of diameter 8 cm to make water flow at the rate 20 litre/s. $\eta_{water} = 10^{-2}$ MKS. Neglect all forces other than viscous force.

(**Hint :** $V = \frac{\pi p r^4}{8\eta l}$) [**Ans. :** 7.96×10^5 P*a*]

9. A bubble of soap solution of radius 2.4×10^{-4}m is lying in a cylinder containing air. Keeping temperature constant when air is compressed radius of bubble reduces to half. Find the new pressure of air in the cylinder. (Surface tension of soap solution : 0.03 Nm^{-1}) [**Ans. :** 8.03×10^5 P_a]

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CHAPTER 6

THERMODYNAMICS

6.1 Introduction

Whether it is frozen night of winter or a sweltering noon of summer, our body needs to maintain its normal temperature to be nearly constant at 98.60 °F or 37.00 °C. Our body has certain inbuilt temperature control mechanisms (biological) that help to maintain its temperature, upto a certain extent. But in extreme cold or hot weather situations, we have to provide external protection to our body.

You would have observed that when a cup of cold coffee and a cup of hot tea are kept open for some time, the coffee becomes warmer and the tea becomes cool until they attain the room temperature. This type of phenomena leads to the Zeroth law of thermodynamics.

The existence of different phases of matter at certain temperature and pressure conditions are discussed in terms of phase diagram in this chapter.

The terms **temperature** and **heat** are often used with same meaning in every day life. But in physics, these two terms have very different meanings. In this chapter, the definition of temperature as a function of some physical (thermal) properties of matter is given along with the different scales of measurement and their inter-relationships. The heat which is the transfer of heat energy related with the temperature difference between two bodies, is also discussed.

The first law of thermodynamics is the extension of the principle of conservation of energy. It broadens this principle by including the energy exchange in terms of heat transfer, mechanical energy in terms of work and the internal energy of the system.

The specific heat as well as heat capacity of water and oil are also discussed in this chapter.

Now a days we see the star ratings on home appliances, like refrigerator and air-conditioner, the fuel efficiency of vehicles in km/litre of petrol or diesel is provided by the manufacturers.

These instruments represent their efficiency of conversion of one type of energy into the other. The second law of thermodynamics defines the limitations of these processes.

The working of heat engine and Carnot engine is also explained in this chapter.

6.2 Concept of Thermodynamic System and Environment

In thermodynamics, normally, we use the word 'system' in lieu of 'a body'. Thermodynamic system is a part of the universe under thermodynamic study. A system can be one dimensional, two dimensional or three dimensional. It may consist of only one object or many objects. The objects comprising the system are called **components** of the system. A system may be made up of **radiation** or radiation may be a component of the system.

The remaining part of the universe surrounding the system and having a direct impact on the behaviour of the system is known as its **environment**. The boundary separating the system and its environment is called the **wall** of the system. The type of interaction taking place between the system and its environment depends on the nature of its wall.

In all branches of physics, the **macroscopic** description of any system is done in terms of certain measurable properties. For instance, while studying the kinematics of rotational motion of a rigid body, the macroscopic properties like position and velocity of its centre of mass with respect to a system of co-ordinate axes are studied at different moments, without worrying about its internal aspects. Such quantities are called **mechanical co-ordinates.** Values of potential and kinetic energy and hence mechanical energy of a rigid body are determined using such mechanical co-ordinates with respect to some system of co-ordinates.

In thermodynamics, the macroscopic quantities having direct effect on the internal state of the system are taken into consideration. Such quantities are called **thermodynamic co-ordinates.** The system represented by thermodynamic co-ordinates is called a **thermodynamic system.**

The thermodynamic state of a system is determined from the values of mechanical and thermal properties of the system. For example, the mechanical properties like pressure of a gaseous system and its volume, and thermal properties like temperature and quantity of heat energy, determine the thermodynamic state of the system. The interaction between a system and its environment is called a thermodynamic process.

If a system does not interact with its surrounding then it is called an **isolated system.** Thermal and mechanical properties of such a system remain constant and this system is said to be in a definite thermodynamic equilibrium state.

Due to interactions with the surrounding, a system exchanges heat energy and / or mechanical energy, and hence the thermal and mechanical properties of the system change continuously. After passing through many thermodynamic states like this, finaly the system attains another definite thermodynamic equilibrium state. The amount of heat energy exchanged during the interaction of system with environment is called **heat** (Q), and the mechanical energy exchanged is called **work.** (W).

The state of equilibrium of a thermodynamic system is determined in terms of some variable quantities known as **thermodynamic variables** or **state variables.** The mathematical relation between state variables is called the **equation** of state. e.g. in the chapter "Kinetic Theory of Gases" you have learnt that the equation relating the pressure, volume, temperature and the quantity of an ideal gas is $PV = \mu RT$, which is the equation of state for the ideal gas.

Thermodynamic state variables are of two types :

(i) Extensive Variables : The variables depending on the dimensions of the system are called extensive variables. For example, mass, volume, internal energy etc.

(ii) Intensive Variables : The variables independent of the dimensions of the system are called intensive variables. For exmaple, pressure, temperature etc.

6.3 Thermal Equilibrium and Definition of Temperature (Zeroth Law of Thermodynamics)

When two systems having different temperatures are brought in thermal contact with each other, the heat flows from the system at higher temperature to that at lower temperature. When both the systems attain equal temperature, the net heat exchange between them becomes zero. In this state they are said to be in **thermal equilibrium** with each other.

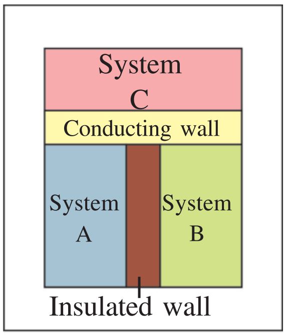
When the **wall,** separating the system and its surroundings, both having different temperatures, is thermal insulator (adiabatic wall) then there does not occur any heat exchange between them. But when the wall separating the system and its surroundings is a good conductor of heat (diathermic wall), then the exchange of heat takes place between them. After the temperatures of the system and its surroundings become equal, the exchange of heat between them becomes zero.

When there is no imbalanced force acting between a system and its environment, the system is said to be in **mechanical equilibrium.** When there is no chemical reaction taking place in a system or there is no motion of any chemical component from one part of the system to the other, the system is said to be in **chemical equilibrium**. If a system is simultaneously in thermal, mechanical and chemical equilibria, it is said to be in **thermodynamic equilibrium**.

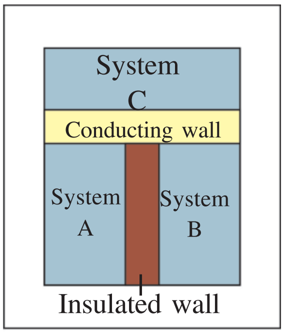
6.3.1 Zeroth Law of Thermodynamics :

To know, whether a system and its surrounding or two systems are in thermal equilibrium with each other, a third body (e.g. thermometer) can be used. [Ideally, the third body, should not exchange (absorb or emit) heat with these systems.]

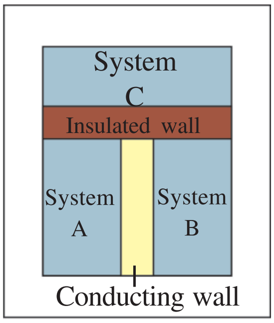
As shown in figure (6.1–*a*), suppose two systems A and B are separated from each other by a thermally insulated wall, and both of them are in contact with a third system C through a conducting wall. As shown in Figure 6.1(*b*), after sometime the two systems A and B attain thermal equilibrium with system C.



(*a*) Before thermal equilibrium



(*b*) State of thermal equilibrium



(*c*) State of thermal equilibrium

The establishment of thermal equilibrium among systems A, B and C

Figure 6.1

Now, the insulated wall separating A and B is replaced by a conducting wall, and system C is isolated from A and B by insulating wall as shown in Figure 6.1(*c*), then also there will be no change observed in them. Now instead of allowing the systems A and B to attain thermal equilibrium with C simultaneously, if they are allowed to attain thermal equilibrium with C one after another and then A, B and C are brought in contact through conducting wall, then also the thermal equilibrium is not affected. Thus,

"If the systems A and B are in thermal equilibrium with a third system C, then A and B are also in thermal equilibrium with each other."

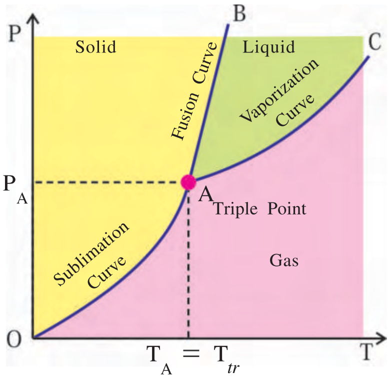
This statement is known as **zeroth law of thermodynamics**.

In practice, we use the concept of temperature to express the degree of hotness or coldness of a body. In reference to this, the zeroth law of thermodynamics indicates that the temperature is a property of the system. When bodies in thermal contact with each other attain thermal equilibrium, their temperatures become equal. Thinking in broad sense it can be written from the zeroth law, that "There exists an important physical quantity called temperature."

6.4 Phase Diagram

The phase (solid, liquid or gaseous state) in which matter will exist depends on the factors like pressure and temperature. In some situations, two or three states of matter may co-exist in equilibrium. The graph of pressure against temperature indicating the phase of matter at given temperature (T) and pressure (P) is called **phase diagram of that matter**. Figure 6.2 represents the phase diagram of some substance.

For the values of pressure and temperature corresponding to points on the curve AB, the solid and liquid phases of the matter co-exist in equilibrium state. Hence the curve **AB** is called **fusion curve.**



Phase Diagram

Figure 6.2

Similarly, for the values of temperature and pressure corresponding to points on the curve OA, the solid and gas phases of matter co-exist in equilibrium. Hence the curve **OA** is called **sublimation curve**.

For all values of pressure and temperature corresponding to all points on the curve AC the liquid and gaseous states of matter co-exist in equilibrium. Hence the curve **AC** is known as **vaporization curve.**

The point A at which the vaporization curve, the fusion curve and sublimation curve meet, i.e., the values of pressure and temperature at which all the three states of matter co-exist in equilibrium is called **triple point of the matter**. In the figure, point **A** is the triple point of the given matter (substance).

For different substances, the co-existence of two or three states in equilibrium can be obtained at definite specific values of pressure and temperature. **The triple point of water is obtained at the pressure of 4.58 mm Hg and 273.16 K temperature.** The triple point of water is used to fix the scale of thermometer.

6.4.1 Measurement of Temperature Thermometry :

Whether a substance is cold or hot that cannot be judged by mere sense of touch only. For example, if one keeps his left hand in hot water and right hand in cold water for some time, and then both hands are immersed in lukewarm water, it appears cold to the left hand and hot to the right hand. Moreover, the results obtained by the sense of touch are also subjective.

For a body in some definite thermal equilibrium, if we assign an appropriate real number to its temperature, and this way assign appropriate real numbers to the temperature of the body during its different thermal equilibrium states, then the function defined on such thermal equilibrium states is called **temperature function.**

Zeroth law of thermodynamics ensures that such a function is one-one function.

A device used to measure a unique real number (i.e. temperature) related with given thermal equilibrium state is called thermometer.

To prepare a thermometer the commonly used property is variation of the volume of a liquid with temperature. Mercury and alcohol are the liquids used in most liquid-in-glass thermometers.

Thermometers are calibrated so that a numerical value may be assigned to a given temperature. For calibration of any standard scale, two fixed (known) reference temperatures are required. The freezing point of water (32°F or 0°C) and boiling point of water (212°F or 100°C) at 1 atmospheric pressure are two convenient fixed points.

The liquid–in–glass thermometers show different readings for temperatures other than the fixed points because of differing expansion properties of liquids. But a thermometer using gas, at low enough pressure and constant volume for the measurement of temperature gives the same readings regardless of, which gas is used.

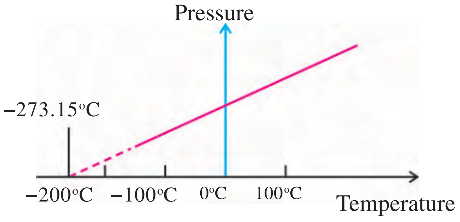
A given quantity of gas having low enough pressure satisfies the **ideal gas state equation.**

$PV = \mu RT$

where μ = number of moles of the gas,

and $R = 8.31 \text{ J mol}^{-1} \text{ K}^{-1}$

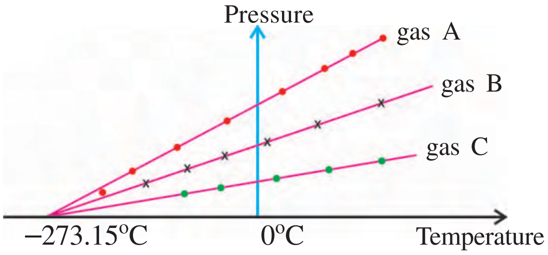
Thus keeping the volume of the gas to be constant, it gives $P \propto T$. Thus, with a constant–volume gas thermometer, the temperature is read in terms of its pressure. A plot of P – T gives a straight line in this case (See Figure 6.3).



Graph of Pressure versus temperature of a low density gas kept at constant volume

Figure 6.3

At low temperature, the measurements of temperature on real gases deviate from the values predicted by the ideal gas law. But the relationship is linear over a temperature range. It looks like that the pressure approaches zero with decreasing temperature if the gas continues to be a gas. The absolute minimum temperature for an ideal gas inferred by extrapolating the straight line to the temperature axis approaches **–273.15°C** and is designated **as absolute zero** (See Figure 6.4).



A plot of P – T and extrapolation of lines of low density gases indicates the same absolute zero temperature

Figure 6.4

It can be seen from Figure 6.4 that for different gases of low density, and having different thermal expansion, the same absolute zero temperature is obtained. Absolute zero is the foundation of the kelvin temperature scale or absolute scale temperature and is taken as 0 K.

In practice, for the measurement of temperature the celsius scale and Fahrenheit scale are used.

Celsius scale : If the temperature on the Celsius scale is represented by T_C and on the kelvin scale by T, then

$T_C = T - 273.15^\circ$

Measuring the temperature of triple point of water on Celsius Scale

$T_C = 273.16^\circ - 273.15^\circ = 0.01^\circ C$

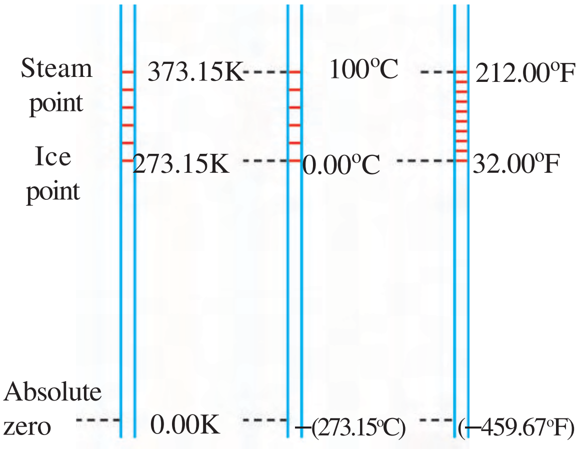
When equilibrium is established between pure water and its vapour at atmospheric pressure, the value of temperature is taken as 100°C. In Kelvin scale its value is

$T = 100 + 273.15 = 373.15 \text{ K}$

Fahrenheit Scale : The relation between the temperature T_F on Fahrenheit scale and the temperature T_C on Celsius scale is

$$T_F = \frac{9}{5} T_C + 32^\circ$$

If the boiling point and freezing point of water are known on one temperature scale, then the measured value of temperature can be easily represented in terms of the other scale. Figure 6.5 shows the comparison among kelvin, Celsius and Fahrenheit scales.



Comparison among Kelvin, Celsius and Fahrenheit scale for water

Figure 6.5

To represent the temperature in Celsius and Fahrenheit scales, the letters C and F are used, respectively, such as

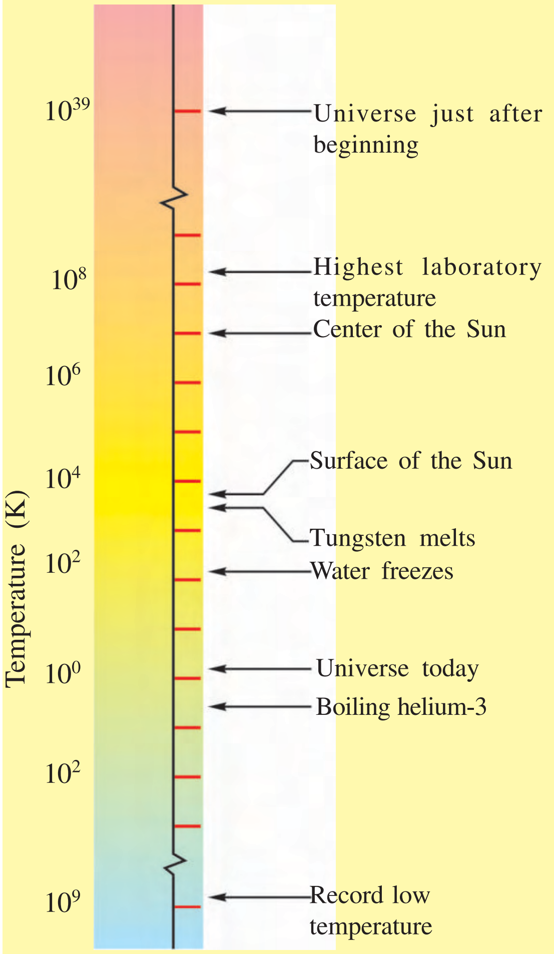
0°C = 32°F

means that 0° on the Celsius scale measures the same temperature as 32° on the Fahrenheit scale, whereas

5 C° = 9 F°

means that a temperature difference of 5 Celsius degrees **(note the degree symbol appears after C)** is equivalent to a temperature difference of 9 Fahrenheit degrees.

Only For Information :



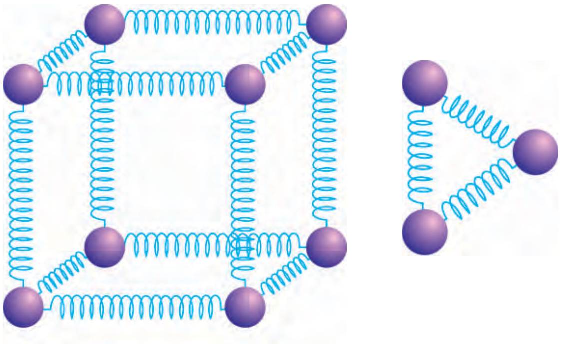
Some values of temperature on kelvin scale

Figure 6.6

6.5 Thermal Expansion

We know that the dimensions of most of the substances increase with increase in their temperature (by absorbing heat) and decrease with decrease in temperature (by releasing heat). **The increase in dimensions of a substance due to absorption of heat is called thermal expansion and decrease in dimensions of the substance by releasing the heat is called thermal contraction.**

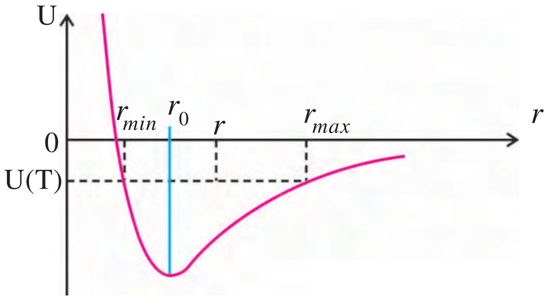
The constituent particles (atoms, molecules, or ions) of solids are arranged in a definite manner in the internal structure of solids. They exert attractive and repulsive forces on each other and execute oscillations about their mean positions. Thus, we can think of constituent particles to be connected by imaginary springs (See Figure 6.7). With rise in temperature, the amplitude of oscillations increases, as a result of which the average intermolecular distances become larger. Hence the size of the body increases as temperature increases.



Constituent particles shown to be connected by imaginary springs

Figure 6.7

Figure 6.8 shows the graph of intermolecular potential energy versus distance, and it is found that the curve is not symmetric about interatomic equilibrium distance (r_0). For distance more than r_0, the attractive potential energy does not increase at the same rate as the increase in repulsive potential energy for distance less than r_0.



Graph of Potential energy versus Intermolecular distance

Figure 6.8

At a given temperature (for a given value of potential energy U(T)) the constituent particles are vibrating between r_{min} and r_{max} (See Figure 6.8). If r is the average distance between two consecutive constituent particles at that temperature, then

$$r = \frac{r_{\text{min}} + r_{\text{max}}}{2}$$

It is clear from the asymmetry of the curve that these average distances increase with rise in temperature. This asymmetry is responsible for thermal expansion.

Linear Expansion :

The increase in the length of a body with increase in temperature is called linear expansion. For small changes in temperature, the increase in length (Δl) is directly proportional to original length 'l' and increase in temperature 'ΔT', i.e.

$$\Delta l \propto l, \text{ and}$$

$$\Delta l \propto \Delta \text{T}$$

$$\therefore\ \Delta l \propto l\Delta \text{T}$$

$$\therefore\ \Delta l = \alpha l \Delta \text{T} \qquad (6.5.1)$$

Here 'α' is a constant of proportionality **called coefficient of linear expansion of material of the body**. The value of 'α' depends on the type of material of the body and its temperature. If the temperature interval is not very large, then 'α' does not depend on the temperature.

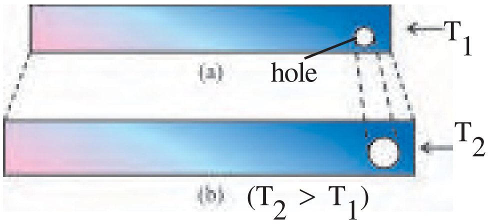
The unit of α is $(^{\circ}\text{C})^{-1}$ or K^{-1}. The coefficients of linear expansion of some substances are given in Table 6.1 (for information only).

Table 6.1

Some Coefficients of Linear Expansion (For Information Only)

Substance	α $(10^{-6}\ \text{C}^{o-1})$	γ $(10^{-5}\text{C}^{o-1}$ or $\text{K}^{-1})$
Aluminium	29	8.7
Brass	23	6.9
Concrete	12	3.6
Steel	11	3.3
Glass (ordinary)	9	2.7
Glass (Pyrex)	3.2	0.96
Ice (at 0°C)	51	15.3

Some substances exhibit uniform thermal expansion in all directions. Such substances are called **isotropic substances**. In such substances, the proportionate changes in length, breadth and thickness are the same with the change in temperature. Hence the expansion of isotropic substances looks like a **photographic magnification**. (See Figure 6.9)



Isotropic expansion (exaggerated) of a steel ruler after increase in its temperature

Figure 6.9

Thus, increase in area $\Delta A = 2\ \alpha\ A\Delta T$,

and

Increase in volume $\Delta V = 3\alpha V\Delta T$

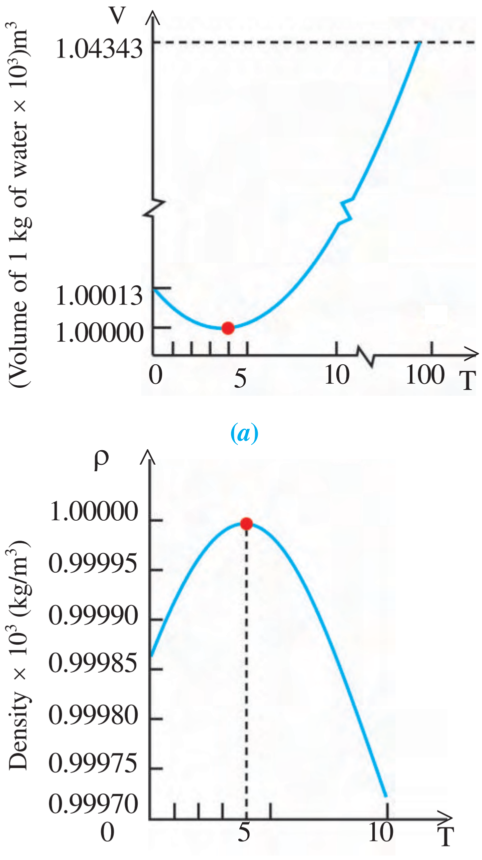
The coefficients of volume expansion ($\gamma=3\alpha$) of some substances are given in Table 6.1 (for information only)

Thermal expansion is more in liquids than in solids, and it is maximum in gases.

Anomalous Thermal Expansion of Water

The thermal expansion with temperature appears to be anomalous in case of water. When temperature of water is reduced upto 4 °C, it contracts, but when the temperature is decreased further from 4 °C to 0 °C, water expands (See Figure 6.10 *a*). Thus, for given quntity of water, the volume of water is minimum at 4 °C and hence, the density of water is maximum at 4 °C. (See Figure 6.10 *b*)

Due to this type of behaviour of water, lakes freeze from top to down rather than bottom to top.



(*a*)

(*b*)

Variation of (*a*) volume, and (*b*) density, of 1 kg of water in the temperature range from 0°C to 10°C.

Figure 6.10

As the temperature of upper layer of water decreases (say from 10°C) towards freezing point, it becomes denser than water in the lower region and moves towards the lower region. This process continues till the entire water of the lake reaches 4°C. Now, when the temperature of upper surface of water decreases below 4°C, its density decreases (See Figure 6.5 *b*), and hence it remains there on the upper surface and continues to cool down further. This way the upper surface freezes while the lower surface water is still liquid.

Due to this anomalous behaviour of water, the aquatic life in water is survived even at very low temperature of the atmosphere.

Illustration 1 : A blacksmith fixes iron ring on the rim of the wooden wheel of a bullock cart. The diameter of the rim and the iron ring are 1.5 m and 1.495 m respectively, at 27°C. To what temperature should the ring be heated so as to fit the rim of the wheel ? For steel $\alpha = 12 \times 10^{-6}\ K^{-1}$. (Neglect thermal expansion of wooden wheel).

Solution :

Given $T = 27^{\circ}C = 273 + 27 = 300\ K$

$T' = ?$

$\alpha = 12 \times 10^{-6}\ K^{-1}$

Diameter of rim $d_1 = 1.5$ m

Diameter of iron ring $d_2 = 1.495$ m

Hence total length of rim $l_1 = \pi d_1$

Total length of ring $l_2 = \pi d_2$

$\therefore\ \Delta l = l_1 - l_2 = \pi d_1 - \pi d_2$

Now $\Delta l = \alpha l \Delta T$

$\therefore\ \pi(d_1 - d_2) = \alpha\pi d_2(T' - T)$

$$\therefore\ T' - T = \frac{d_1 - d_2}{\alpha d_2}$$

$$\therefore\ T' = \frac{d_1 - d_2}{\alpha d_2} + T$$

$$= \frac{1.5 - 1.495}{12 \times 10^{-6} \times 1.495} + 300$$

$= 278.7 + 300$

$\therefore\ T' = 578.7\ K$

$\therefore\ T' = 578.7 - 273 = 305.7^{o}C$

Thus the ring should be heated up to 305.7°C (in practice slightly more than this).

Illustration 2 : What should be the lengths of a brass and an aluminium rod at 0°C, if the difference between their lengths is to be maintained equal to 5 cm at any temperature ?

(For brass $\alpha = 18 \times 10^{-6}\ C^{o-1}$ and for aluminium $\alpha = 24 \times 10^{-6}\ C^{o-1}$)

Solution : Suppose l_1 and l_2 are the lengths of aluminium and brass rods at 0°C respectively. At any temperature, difference of their lengths remains the same. Hence, increase in their lengths with increase in temperature must also be the same.

$\therefore\ \Delta l_1 = \Delta l_2$

$\therefore\ \alpha_1\, l_1\, \Delta T = \alpha_2\, l_2\, \Delta T$

$$\therefore\ \frac{l_1}{l_2} = \frac{\alpha_2}{\alpha_1} = \frac{24 \times 10^{-6}}{18 \times 10^{-6}} = \frac{4}{3} \qquad (1)$$

Now, according to the given condition

$l_1 - l_2 = 5\ cm \qquad (2)$

From equations (1) and (2),

$$\frac{l_1}{l_1 - 5} = \frac{4}{3}$$

$\therefore\ 3l_1 = 4l_1 - 20$

$\therefore\ l_1 = 20\ cm$, and $l_2 = 15\ cm$

Thus, at 0°C lengths of brass and aluminium rods should be 20 cm and 15 cm respectively.

Illustration 3 : Density of the material of a body of volume V is ρ at temperature T. Show that density of the material decreases by $\gamma\rho dT$ for a very small rise (dT) in temperature. (**Hint :** $\frac{dx^n}{dx} = n\ x^{n-1}$)

Solution :

$$\text{Density } \rho = \frac{M}{V}, \qquad (1)$$

where M = mass, and V = volume of the body.

Volume of the body depends on temperature. If volume of the body increases by dV on increasing the temperature by dT then,

$\therefore\ dV = \gamma V dT \qquad (2)$

It is clear that density decreases as volume increases. Suppose decrease in density is $d\rho$.

$\therefore$ From equation (1),

$$d\rho = -\frac{M}{V^2}\ dV \qquad (3)$$

$$= -\frac{M}{V^2} \cdot \gamma V dT$$

$$= -\frac{M}{V}\ \gamma \cdot dT$$

$\therefore\ d\rho = -\rho\gamma dT \qquad (4)$

Here, $-ve$ sign shows that ρ decreases as temperature increases.

Illustration 4 : Prove that the co-efficient of volume expansion of an ideal gas at constant pressure decreases with increase in temperature. What is the co-efficient of volume expansion of an ideal gas at 0°C ?

Solution : For an ideal gas, $PV = \mu RT \qquad (1)$

Suppose increase in volume is ΔV corresponding to increase in temperature ΔT at constant pressure.

$\therefore\ P\Delta V = \mu R\Delta T \qquad (2)$

Dividing equation (2) by equation (1),

$$\frac{\Delta V}{V} = \frac{\Delta T}{T}$$

$$\therefore\ \frac{\Delta V}{V\Delta T} = \frac{1}{T}$$

$$\therefore\ \gamma = \frac{1}{T}\ (\because \Delta V = \gamma V\Delta T) \qquad (3)$$

Equation (3) shows that the co-efficient of volume expansion decreases as temperature increases. At T = 0°C = 273.15 K.

$$\gamma = \frac{1}{273.15} = 3.66 \times 10^{-3}\ K^{-1}$$

Illustration 5 : Co-efficient of volume expansion of glycerine is $49 \times 10^{-5}\ C^{o-1}$. Find percentage decrease in its density on increasing its temperature by 30 C°.

Solution : $V = V_0\ (1 + \gamma\Delta T)$

$$\text{Now, } V = \frac{M}{\rho},\ V_0 = \frac{M}{\rho_0}$$

$$\therefore\ \frac{M}{\rho} = \frac{M}{\rho_0}\ (1 + \gamma\Delta T)$$

$$\therefore \frac{\rho_0}{\rho} = 1 + \gamma\Delta T$$

$$\therefore \frac{\rho}{\rho_0} = \frac{1}{1+\gamma\Delta T}$$

$$\therefore \frac{\rho - \rho_0}{\rho_0} = \frac{-\gamma\Delta T}{1+\gamma\Delta T}$$

$$= -\frac{(49)(10^{-5})(30)}{1+(49)(10^{-5})(30)}$$

$$= -0.0145$$

∴ Percentage decrease in density = 1.45 %

Note : Since value of γ is very small, this problem can also be solved by using formula obtained in Illustration 3.

Illustration 6 : Average temperature of the Earth was 300 K when the Earth came into existence. At present its average temperature is 3000 K. (This is due to the heat evolving from the disintegration of radioactive substances at the core of the Earth.) What would be the radius of the Earth at the time of its birth ? For the material of the Earth $\gamma = 3 \times 10^{-5}$ K^{-1}. At present, radius of the Earth = 6400 km.

Solution :

$$V = V_0 (1 + \gamma\Delta T)$$

$$\therefore \frac{4}{3}\pi R^3 = \frac{4}{3}\pi R_0^{\,3}(1 + \gamma\Delta T)$$

$$\therefore R = R_0(1+\gamma\Delta T)^{\frac{1}{3}}$$

$$\therefore R_0 = \frac{R}{(1+\gamma\Delta T)^{\frac{1}{3}}}$$

$$= \frac{6400}{[1+(3\times10^{-5})(2700)]^{\frac{1}{3}}}$$

$$= 6236 \text{ km}$$

6.6 Heat of Transformation (Latent Heat)

When heat is given to a solid or liquid substance, it is not necessary that its temperature will increase. Sometimes by absorbing heat, the substance may change its state or phase to another.

To melt a solid substance to a liquid state, i.e. to make the molecules free from the rigid structure of the solid, more heat energy is required (e.g. conversion of ice into water). In the same manner, when a liquid freezes to form a solid, the energy is released from the liquid.

To vaporize a liquid, the heat energy is given to it (e.g. transformation of water to vapour). Similarly when gas molecules condense to form a liquid, the heat energy is released (reduced) from the gas.

"The amount of energy per unit mass that must be transferred as heat when a substance completely undergoes a phase change (from one state to other) is called the heat of transformation (Latent heat) L."

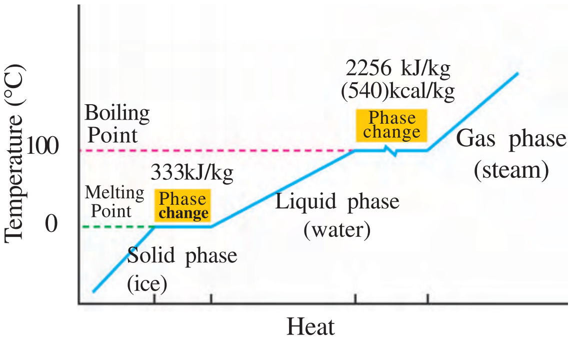
The total energy Q provided as a heat to a substance of mass m to transform it from one state (phase) to completely in other is Q = Lm.

The necessary amount of heat required for transformation of a liquid into gas (vapour), or a gas (vapour) into liquid, is called **heat of vaporization L_V**. For water L_V = 2256 kJ/kg.

When a solid substance of unit mass is transformed into liquid (then the substance gains heat), or when the liquid is transformed into solid (then it loses heat) the heat of transformation is called **heat of fusion L_F.**

For water L_F = 333 kJ/kg

A plot of temperature versus heat for a quantity of water is shown in Figure 6.11.



Temperature versus heat for water at 1 atm pressure (not to scale)

Figure 6.11

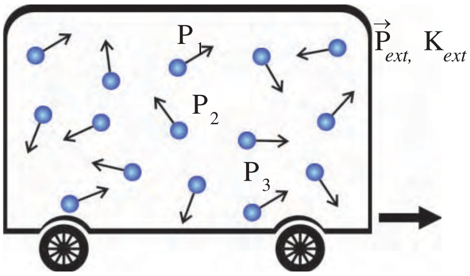
Figure 6.11 shows that when heat is added (or removed) during a change of state, the temperature remains constant. The slopes of phase lines are not all the same, which indicates that specific heats of various states are not equal. For water L_F = 333 kJ/kg represents that 333 kJ of heat is needed to melt 1 kg of ice at O°C, and L_V = 2256 kJ/kg represents that 2256 kJ heat is needed to convert 1 kg of water to steam at 100°C. So, steam at 100°C carries 2256 kJ/kg more heat than water at 100°C. This is why burns from steam are usually more serious than those from boiling water.

6.7 Heat, Internal Energy and Work

For a vessel containing a gas, due to the random motion of gas molecules about their centre of mass in the vessel, they possess momentum and kinetic energy. As the probability of random motion of gas molecules is same in all directions, the total momentum associated with the random motion of molecules is zero ($\vec{P}_{int} = 0$). But the kinetic energy associated with the random motion of these molecules is not zero. ($K_{int} \neq 0$)

The total kinetic energy associated with the random motion of molecules of the gas (such that the total momentum of motion is zero) is called heat (or thermal) energy possessed by the gas.

If the gas molecules are interacting with each other, then the molecules possess the potential energy (U_{int}) associated with these interactions. Further, if an external agency (e.g. gravitation) is interacting with the gas, then the gas as a whole can possess additional potential energy U_{int}.



Motion of vessel containing gas

Figure 6.12

As shown in Figure 6.12, suppose the vessel containing gas is in motion. In this case, the gas also moves with the vessel. Hence along with random motion, the gas molecules possess average momentum $\vec{P}_{ext}$ and kinetic energy K_{ext}.

Thus, a gas can possess following four types of energy :

(1) K_{int} , (2) U_{int} , (3) K_{ext} , (4) U_{ext}

The sum of first two energies ($K_{int} + U_{int}$) is called the internal energy (E_{int}) of the gas, whereas the sum of last two energies (K_{ext} + U_{ext}) is called mechanical energy of the gas.

This discussion of energy of gases is also valid for other phases of matter.

We know that when two bodies at different temperature are brought in thermal contact with each other, the temperature of the hotter body decreases and that of the colder body increases. This means that exchange of heat energy takes place between the two bodies. The exchange of heat energy is called heat. Thus, we can conclude that **the energy exchanged between a system and its environment, only due to the difference of temperature between them, is called heat.**

It is clear that a system can possess heat energy but cannot possess the heat, because heat is a process.

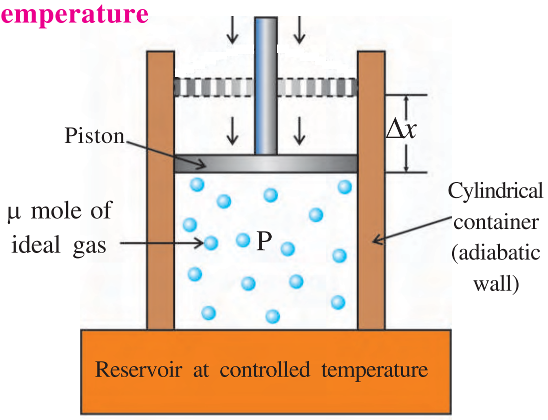
The heat absorbed by a system is considered positive and heat lost by the system is considered negative.

6.7.1 Work in Thermodynamics

The amount of mechanical energy exchanged between two bodies during mechanical interaction is called work. Thus work is a quantity related to mechanical interaction. **A system can possess mechanical energy, but cannot possess work.**

Previously you studied about work, according to which the work done by a system against a force is considered negative, and work done on the system is considered positive. But in thermodynamics the work done by the system is considered positive and the work done on the system is considered negative. The reason behind such a sign convention is due to the mode of working of a heat engine in which the engine absorbs heat from the environment and converts it into work W means the energy of the system reduces by W.

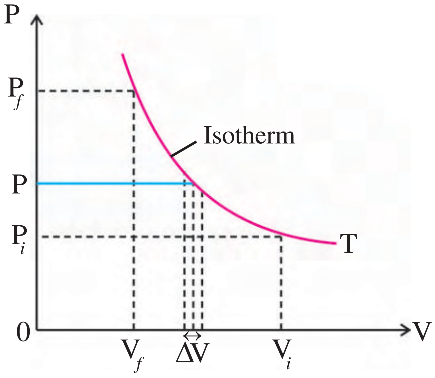
6.7.2 Formula for the work done during the compression of gas at constant temperature



μ mole of ideal gas confined in a cylindrical container

Figure 6.13

As shown in Figure 6.13, μ mole of an ideal gas having enough low density is filled in a cylindrical container, and an air–tight piston of cross-sectional area A, capable to move without friction is provided. The conducting bottom of the cylinder is placed on an arrangement whose temperature can be controlled. At constant temperature, measuring the volume of the gas for different values of pressure, the graph of P – V can be plotted as shown in Figure 6.14. These types of processes are called isothermal processes and the curve of P – V is called isotherm.



P – V curve for given gas
(at constant temperature)
Figure 6.14

Suppose that in initial state *i*, the pressure and volume of the gas are P_i and V_i respectively. Keeping the temperature T of the gas to be constant, the volume of the gas is decreased slowly and slowly by increasing a force on the piston. Let final pressure of the gas is P_f and final volume is V_f.

During this process, at one moment when the pressure of the gas is P and volume is V, at that time, let the piston moves inward by Δx. Then the volume of the gas decreases by ΔV. This displacement is so small that there is no apparent change in pressure. Hence, the work done on the gas by the external force is

$$\Delta W = F\Delta x \quad (6.7.1)$$

$$= PA\Delta x \quad (\because F = PA)$$

$$\therefore \Delta W = P\Delta V \quad (\because A\Delta x = \Delta V)$$

If the volume of the gas is decreasing from V_i to V_f through such small changes, then the total work done on the gas.

$$W = \Sigma\Delta W = \sum_{V_i}^{V_f} P\Delta V \quad (6.7.2)$$

In this summation, taking $\lim_{\Delta V \to 0}$ the summation results in integration.

$$\therefore W = \int_{V_i}^{V_f} P dV \quad (6.7.3)$$

But the ideal gas state equation for μ mole of gas at constant temperature is

$$PV = \mu RT$$

$$\therefore P = \frac{\mu RT}{V}$$

Substituting the value of pressure in equation (6.7.3)

$$W = \int_{V_i}^{V_f} \frac{\mu RT}{V} dV \quad (6.7.4)$$

$$\therefore W = \mu RT \int_{V_i}^{V_f} \frac{dV}{V}$$

$$= \mu RT [\, ln\, V\,]_{V_i}^{V_f}$$

$$= \mu RT\, [\, lnV_f - lnV_i\,]$$

$$\therefore W = \mu RT\, ln \frac{V_f}{V_i} \quad (6.7.5)$$

In equation (6.7.5) we have $V_f < V_i$, hence $ln \frac{V_f}{V_i} < 0$. Thus we get negative value of work, which represents that **during the compression of gas at constant temperature, the work is done on the gas**.

If the gas is expanded at constant temperature (volume is increasing), then $V_f > V_i$. Hence in equation (6.7.5), we get $ln \frac{V_f}{V_i} > 0$. Thus we get positive value of W. This shows that **during isothermal expansion of gas, the work is done by the gas**.

6.7.3 Work done at constant volume and at constant pressure :

Equation (6.7.5) does not give the work W done by an ideal gas during every thermodynamic process, but it gives the work done only for a process in which the temperature is held constant. If the temperature varies, then the symbol T in equation (6.7.4) cannot be taken outside the integral, and hence we could not get equation (6.7.5).

In equation (6.7.3), if the volume V of the gas is kept constant, then ($dV = \Delta V = 0$)

$$\therefore \; W = 0 \text{ (for constant volume)} \qquad (6.7.6)$$

Similarly, if the volume is changing while the pressure P is held constant, then from equation (6.7.3)

$$W = P\int_{V_i}^{V_f} dV = P[V]_{V_i}^{V_f}$$

$$= P[V_f - V_i]$$

$$\therefore \; W = P\Delta V \text{ (for constant pressure)} \quad (6.7.7)$$

Illustration 7 : (*a*) During the expansion of one mole of oxygen (considering it as an ideal gas) at constant temperature of 310 K, its volume increases from $V_i = 12$ L to $V_f = 19$ L. What will be the work done by the gas ? (*b*) Keeping this temperature to be constant, if the volume of 1 mole of oxygen is decreased from 19 L to 15 L, then how much work is required to be done on oxygen by an external force ?

($R = 8.31$ J mol^{-1} K^{-1})

Solution :

$\mu = 1$ mol $T = 310$ K

$V_i = 12$ L $V_f = 19$ L

Here, the expansion of oxygen is isothermal.

$$\therefore \; W = \mu RT \, ln \frac{V_f}{V_i}$$

$$= 1 \times 8.31 \times 310 \times ln\left(\frac{19}{12}\right)$$

$$\therefore \; W = 1183.6 \text{ J}$$

Hence, during isothermal expansion, the work done by oxygen will be 1183.6 joule.

(*b*) In second case,

$\mu = 1$ mol $T = 310$ K

$V_i = 19$L $V_f = 15$ L

Here, the compression of oxygen is isothermal.

$$\therefore \; W = \mu RT \, ln \frac{V_f}{V_i}$$

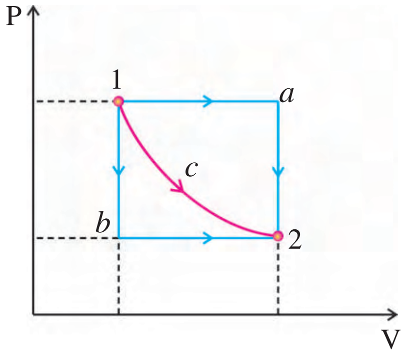
$$\therefore \; W = 1 \times 8.31 \times 310 \times ln\left(\frac{15}{19}\right)$$

$$\therefore \; W = -\ 608.7 \text{ J}$$

Hence, the work done by oxygen during isothermal compression is –608.7 J. This means that the work done by the external force on oxygen for its compression (from 19 L to 15 L) will be 608.7 Joule.

6.7.4 More Understanding of Heat and Work :

Suppose a system is slowly and slowly carried from initial state 1 to final state 2 (in such a way that at every stage the thermal equilibrium between the system and its environment is maintained). Different paths for this process are shown in Figure 6.15.



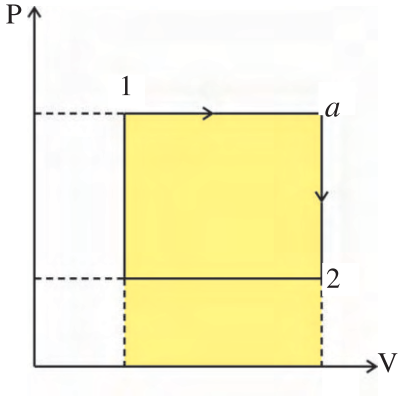
Different ways for carrying the system from initial state to the final state

Figure 6.15

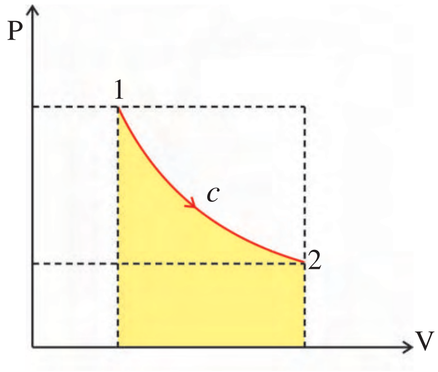
During these processes, the work done is calculated from equation (6.7.3) as

$$W = \int_1^2 PdV$$

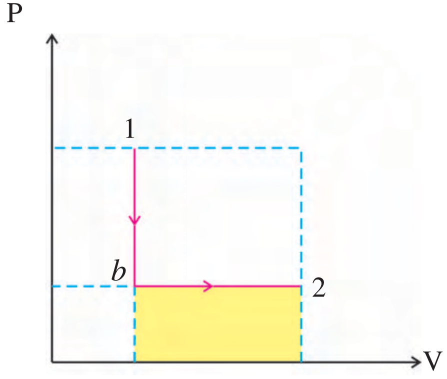
The value of this integral is equal to the area covered by the path joining states 1 and 2 with the V axis. Thus the work done in carrying the system from initial state 1 to final state 2 along 1*a*2, 1*c*2 and 1*b*2 paths is shown in Figure 6.16 by the area covered by the process paths.



(*a*) 1 *a* 2 path



(*b*) 1 *c* 2 path



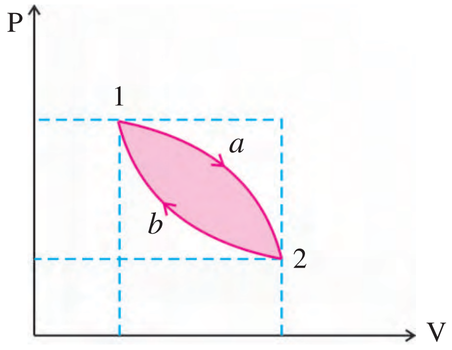
(*c*) 1 *b* 2 path

Work done on different paths

Figure 6.16

Figure 6.16 shows that while carrying the system from state 1 to state 2, the maximum work done by the system is along 1*a*2 path (maximum area covered) whereas minimum work is done along 1*b*2 path (minimum area covered).

If the system is carried from state 2 to state 1 along 2*a*1, 2*c*1 or 2*b*1 path then the work done will be negative (since there is a decrease in volume, ΔV will be negative), which shows that work is done on the system by external force.



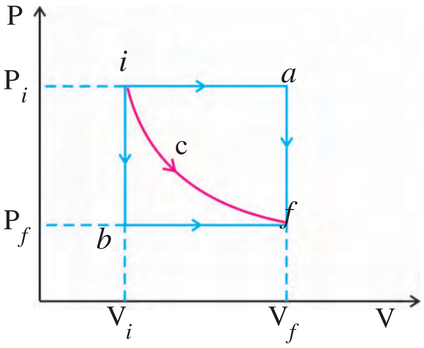
Total work done along 1*a*2*b*1 path of cyclic process of the system

Figure 6.17

As shown in Figure 6.17 when a system is carried from state 1 to state 2 by 1*a*2 path and then back to state 1 by 2*b*1 path, then during this cyclic process 1*a*2*b*1, the total work done by the system is equal to the area covered by closed loop. (The work done along 1*a*2 path is positive, whereas the work done along 2*b*1 path is negative, hence the total work done along 1*a*2*b*1 path is equal to the area covered by closed loop).

6.8 First Law of Thermodynamics

Suppose a system absorbs heat and as a result work is done by it (by the system). We can think of different paths (processes) through which the system can be taken from the initial state *i* to the final state *f*.



Different paths for carrying a system from initial state *i* to final state *f*

Figure 6.18

For the processes iaf, ibf, icf shown in Figure 6.18. Suppose the heat absorbed by the system are Q_a, Q_b, Q_c respectively and the values of the work done are respectively W_a, W_b, W_c. Here, $Q_a \neq Q_b \neq Q_c$ and $W_a \neq W_b \neq W_c$, but the difference of heat and work done turns out to be the same, i.e.

$$Q_a - W_a = Q_b - W_b = Q_c - W_c$$

Thus, when a system is taken from the initial state *i* to the final state *f*, the values of heat Q and work done W depend on the type of process (path), but the value of Q − W does not depend on the path. The value of Q − W depends only on the initial and final states of the system.

It can be concluded from this discussion that for different thermodynamic states of a system, a thermodynamic state function can be defined such that the difference between any two states is equal to Q − W. Such a function is called internal energy E_{int} of the system.

The system gains energy Q in the form of heat energy and spends energy W to do work. Hence the internal energy of the system changes by Q − W.

If the internal energies of the system in initial state *i* and final state *f* are respectively E_i and E_f, then

$$E_f - E_i = \Delta E_{int} = Q - W \qquad (6.8.1)$$

which is **the first law of thermodynamics.**

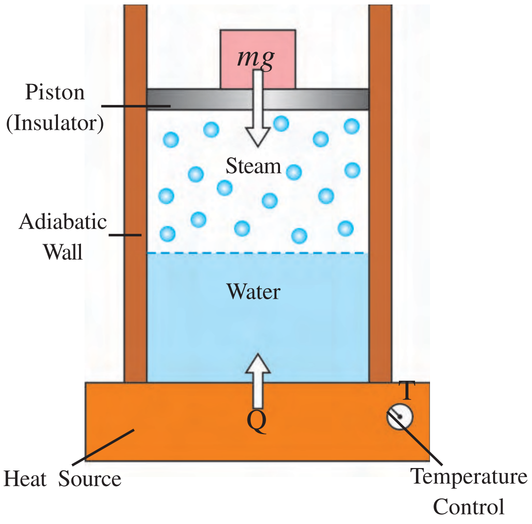
When a system gains heat Q, its internal energy E_{int} increases, but when the work W is done by system then its internal energy decreases.

The first law of thermodynamics is obeyed in all the changes occurring in nature.

Illustration 8 : As shown in Figure 6.19, 1.00 kg of liquid water at 100 °C is converted to steam at 100°C by boiling at standard atmospheric pressure of 1.00 atm. During this process the volume of water increases from an initial value of 1.00×10^{-3} m^3 as a liquid to 1.671 m^3 as steam.

(*a*) How much work is done by the system during this process ? (*b*) How much energy is transferred as heat during the process ? (*c*) What is the change in the internal energy of the system during the process ?

For water $L_V = 2256 \dfrac{kJ}{kg}$



Boiling water at constant pressure

Figure 6.19

Solution :

(*a*) $V_i = 1.00 \times 10^{-3}\ m^3$, $V_f = 1.671\ m^3$

$P = 1.00$ atm $= 1.01 \times 10^5$ *Pa*

Here, the volume increases at constant pressure. Hence the work done by the system will be positive, having value

$$W = \int_{V_i}^{V_f} P dV = P \int_{V_i}^{V_f} dV$$ (P is constant, so can be taken outside integral)

$$= P[V]_{V_i}^{V_f} = P[V_f - V_i]$$

$$\therefore W = 1.01 \times 10^5 \times [1.671 - 1.00 \times 10^{-3}]$$

$$= 1.69 \times 10^5$$

$$\therefore W = 169 \text{ kJ} \qquad (1)$$

(*b*) Since the liquid water at 100°C is converted to steam at 100°C by boiling, the heat energy gained by the system

$$Q = L_V m$$

$$= (2256 \times 1.00)$$

$$\therefore Q = 2256 \text{ kJ} \qquad (2)$$

(*c*) According to the first law of thermodynamics the change in internal energy of the system

$$\Delta E_{int} = Q - W = 2256 - 169$$

$$= 2087 \text{ kJ} \qquad (3)$$

ΔE_{int} is positive, which shows that the internal energy of the system increases. This energy goes into liberating water molecules from the surface of liquid water for vaporization.

6.9 Heat Capacity and Specific Heat

As we add more and more heat to a body, its temperature goes on increasing. The amount of heat required for the same change in temperature, is different for different bodies. Scientists have defined **the amount of heat required to increase the temperature of one kilogram of pure water from 14.5°C to 15.5°C as one kilocalorie. One thousandth part of one kilocalorie is called one calorie.**

The ratio of the heat Q supplied to a body to a change in its temperature ΔT is called heat capacity H_C of the body.

$$H_C = \frac{Q}{\Delta T} \qquad (6.9.1)$$

The SI unit of H_C is J K^{-1} or cal/K.

Heat capacity of a body depends on the material of the body as well as on its mass. Different bodies of the same material but different mass have different values of heat capacity.

Heat capacity does not mean like the capacity of a bucket which can hold (contain) certain quantity of water. It also does not mean about how much heat a substance can hold or absorb. The gain or loss of heat continues until the required temperature difference is maintained. During this process the substance (body) may melt or vaporize.

The quantity of heat required per unit mass for unit change in temperature of a body is called the specific heat of the material of the body. The unit of specific heat is cal g^{-1} K^{-1} or J kg^{-1} K^{-1}. Thus,

$$\text{Specific heat} = \frac{\text{Heat capacity}}{\text{Mass}}$$

$$\therefore\ C = \frac{Q/\Delta T}{m} = \frac{Q}{m\Delta T} \qquad (6.9.2)$$

Remember that in case of coin of copper, we can talk about heat capacity of the coin, but specific heat is that of copper only. None of the two quantities, heat capacity or specific heat, are constant, as their values depend on the temperature at which the temperature interval ΔT is considered. Equations (6.9.1) and (6.9.2) give their average values over that interval of temperature. From equation (6.9.2)

$$Q = mC\Delta T \qquad (6.9.3)$$

Table 6.2 shows the specific heats of some substances at room temperature **for information only.**

Table 6.2

Specific heats of some substances at room temperature (For Information Only)

Substance	Specific Heat		Molar Specific Heat
	Cal $g^{-1}K^{-1}$	J $kg^{-1}K^{-1}$	J $mol^{-1}K^{-1}$
Silver	0.0564	236	25.5
Copper	0.0923	386	24.5
Aluminium	0.215	900	24.4
Ice (−10°C)	0.530	2220	–
Water	1.00	4190	–
Sea Water	0.93	3900	–

6.9.1 Specific Heats of Gases

In the chapter of Kinetic Theory of Gases in semester I, you have studied the specific heat and molar specific heat of gases. Recalling these definitions we will establish the relation among the specific heats of gases.

Molar Specific Heat : The amount of heat required to change the temperature of one mole of a gas by 1 K (or 1°C) is called molar specific heat of the gas.

Molar specific heats of some substances are given in Table 6.2 for knowledge.

Specific Heat at Constant Volume (C_V)

The amount of heat required to change the temperature of 1 mole of gas by 1 K, keeping its volume constant, is called the specific heat C_V of the gas at constant volume.

Specific Heat at Constant Pressure (C_P)

The amount of heat required to change the temperature of 1 mole of gas by 1 K, keeping its pressure constant, is called the specific heat C_P of the gas at constant pressure.

Relation between C_P and C_V :

According to the first law of thermodynamics, for infinitesimal changes

$$dE_{int} = dQ - dW$$

$$\therefore\ dQ = dE_{int} + dW$$

$$\therefore\ dQ = dE_{int} + PdV \qquad (6.9.4)$$

But at constant volume, $dV = 0$

$$\therefore\ dQ = dE_{int}$$

$$\therefore\ \left(\frac{dQ}{dT}\right)_V = \left(\frac{dE_{int}}{dT}\right)_V$$

In Kinetic Theory of Gases (Semester–I) you have learnt that, if the degrees of freedom of 1 mole of a gas are f, then the internal energy of the gas is

$$E_{int} = \frac{fRT}{2} \qquad (\mu = 1) \qquad (6.9.5)$$

Hence,

$$\left(\frac{dQ}{dT}\right)_V = C_V = \left(\frac{dE_{int}}{dT}\right)_V = \frac{fR}{2} \qquad (6.9.6)$$

Similarly, when heat is supplied to one mole of gas at constant pressure

$$(dQ)_P = dE_{int} + PdV$$

But for 1 mole of (ideal) gas

$$PV = RT \quad (\mu = 1)$$

$$\therefore\ PdV = RdT$$

Hence,

$$(dQ)_P = dE_{int} + RdT$$

$$\therefore\ \left(\frac{dQ}{dT}\right)_P = \left(\frac{dE_{int}}{dT}\right)_P + R$$

Here using equation (6.9.5)

$$\left(\frac{dQ}{dT}\right)_P = C_P = \frac{fR}{2} + R \qquad (6.9.7)$$

From equations (6.9.6) and (6.9.7)

$$C_P - C_V = R \qquad (6.9.8)$$

The ratio of specific heat C_P at constant pressure to the specific heat C_V at constant volume, is represented by γ. Hence,

$$\gamma = \frac{C_P}{C_V} = \frac{\frac{fR}{2} + R}{\frac{fR}{2}} = \frac{fR + 2R}{fR}$$

$$\therefore\ \gamma = \frac{f+2}{f} = 1 + \frac{2}{f} \qquad (6.9.9)$$

The degrees of freedom of monoatomic molecules of gas are $f = 3$. Hence for monoatomic gas,

$$C_V = \frac{3R}{2},\ C_P = \frac{5R}{2},\ \gamma = \frac{5}{3}$$

For the diatomic molecules (rigid rotator) of the gas f = 5. Hence,

$$C_V = \frac{5R}{2},\ C_P = \frac{7R}{2},\ \gamma = \frac{7}{5}$$

and for diatomic molecules (with vibrations) $f = 7$.

$$\therefore\ C_V = \frac{7R}{2},\ C_P = \frac{9R}{2},\ \gamma = \frac{9}{7}$$

For diatomic and polyatomic gases the values of specific heat are relatively high. The specific heat of gas increases with increase in number of atoms in a gas molecule. This means that, to increase the temperature of polyatomic molecules, more heat is required, which is due to the following reason :

Monoatomic molecules have only the translational kinetic energy. Hence their kinetic energy increases as heat energy is given to them. But polyatomic molecules possess rotational kinetic energy and energy of vibrations (oscillations) in addition to their translational kinetic energy. Therefore, when heat energy is given to such gases, it is utilized in increasing the translational kinetic energy, rotational kinetic energy and vibrational kinetic energy of the gas molecules, and hence more heat is required. This way polyatomic molecules possess more specific heat.

Illustration 9 :

(*a*) How much heat should be provided to ice of 720 g mass, lying at −10°C to melt it to water at 0°C ?

(*b*) How much heat should be provided to water at 0ºC to increase its temperature to 100ºC ?

(*c*) How much heat should be given to water at 100ºC to transform it completely into water ?

(*d*) Totally, how much heat should be given to ice of 720 g at −10ºC to convert it completely into vapour ?

(C_{ice} = 2220 J kg^{-1} K^{-1},

C_{water} = 4190 J kg^{-1} K^{-1}, L_F = 333 kJ/kg, L_V = 2256 kJ/kg)

Solution : (*a*) The temperature of ice will not increase until it melts completely. Thus the heat to be given to ice, to carry its temperature from $T_i = -10$ºC to $T_f = 0$ºC (thereafter it will start to melt), is

$$Q_1 = C_{ice}\ m\ (T_f - T_i)$$

where,

C_{ice} = Specific heat of ice at −10ºC

$$= 2220\ \frac{J}{kg\ K}$$

$$\therefore\ Q_1 = 2220 \times 0.720 \times [0-(-10)]$$

$$= 15{,}984\ J$$

$$\therefore\ Q_1 = 15.98\ kJ \qquad (1)$$

Until the ice melts completely, its temperature does not increase above 0ºC. Hence, the heat to be given to ice to melt it completely is

$$Q_2 = L_F m = (333\ kJ/kg)(0.720\ kg)$$

$$\therefore\ Q_2 = 239.8\ kJ \qquad (2)$$

(*b*) Now to increase the temperature of 0.720 kg water from T_i = 0ºC to T_f = 100ºC, the required amount of heat is

$$Q_3 = C_{water}\ m(T_f - T_i)$$

$$\therefore\ Q_3 = 4190 \times 0.720 \times [100-0]$$

$$\therefore\ Q_3 = 301680\ J$$

$$\therefore\ Q_3 = 301.68\ kJ \qquad (3)$$

(*c*) The heat to be given to water at 100ºC to transform it completely into vapour is

$$Q_4 = L_V\ m$$

$$= 2220 \times 1 \times [0-(-10)]$$

$$\therefore\ Q_4 = 1624.32\ kJ \qquad (4)$$

(*d*) The total amount of heat to be given to 720 g of ice at −10ºC, to transform it completely into vapour is

$$Q = Q_1 + Q_2 + Q_3 + Q_4$$

$$\therefore\ Q = 2181.78\ kJ \qquad (5)$$

Illustration 10 : What will be the mass and temperature of water obtained by giving 210 kJ heat to ice of 1 kg, lying at −10ºC ? (C_{ice} = 2220 J kg^{-1} K^{-1})

Solution : Mass of ice *m* = 1 kg

To raise the temperature of ice from

T_i = − 10ºC to

T_f = 0ºC, the required amount of heat is

$$Q_1 = C_{ice}\ m(T_f - T_i)$$

$$= 2220 \times 1 \times [0-(-10)]$$

$$= 22200\ J$$

$$\therefore\ Q_1 = 22.2\ kJ \qquad (1)$$

Until the ice melts completely, its temperature will not increase above 0ºC. The heat given to ice is 210 kJ, out of which Q_1 = 22.2 kJ of heat is utilized to increase the temperature of ice from −10ºC to 0ºC. Hence the net amount of heat gained by ice after attaining 0ºC temperature is,

$$Q' = Q - Q_1 = 210\ kJ - 22.2\ kJ$$

$$\therefore\ Q' = 187.8\ kJ \qquad (2)$$

The quantity (mass) of ice melts by this much amount of heat is

$$m = \frac{Q'}{L_F} = \frac{187.8}{333}$$

$$\therefore\ m = 0.564\ kg \qquad (3)$$

This shows that out of 1 kg ice, 0.564 kg of ice is melted (means 0.564 kg is converted into water), and remaining ice which is not melted is

1 − 0.564 = 0.436 kg

Therefore, after melting of ice, the mass of the water obtained will be

$$m = 0.564\ kg \qquad (4)$$

and its temperature will be

$$T = 0^{o}C \qquad (5)$$

Some thermodynamic processes :

In thermodynamics we can achieve the same result through different methods. For example, temperature of a gas filled in a cylindrical container with an air tight smooth piston can be increased by suddenly increasing pressure on it or temperature can also be increased by heating the gas externally by means of a flame. Thus in thermodynamics the conditions on interaction taking place between a system and its environment are very important and accordingly the process is identified with a specific name. Let us study such a few processes.

Isobaric process : "The process during which pressure of the system remains constant is called an isobaric process."

Thermodynamic equilibrium state of a system goes on changing during such a process. The thermodynamic functions of the system possess definite values in intermediate states. The graph of P – V for such a process is a straight line parallel to the V-axis.

From equation (6.7.3), $W = \int_{V_i}^{V_f} P\,dV$

As P is constant, $W = P\int_{V_i}^{V_f} dV$

$$= P(V_f - V_i) \qquad (6.10.1)$$

Isochoric process : Volume of a system remains constant during this process. Since no work is done during this process, $Q = \Delta E_{int}$ from the first law of thermodynamics. Thus, during an isochoric process the change in internal energy of the system is equal to the amount of heat exchanged between the system and its environment.

Adiabatic process : No exchange of heat takes place between a system and its environment in this process. This is possible when (1) walls of a system are thermal insulator or (2) process is very rapid.

During propagation of sound waves the process of formation of condensation and rarefaction is very rapid and hence it can be considered as an adiabatic process. Now you can understand why the air pump (used to fill air in bicycles) gets heated on pumping rapidly. Since $\Delta Q = 0$ for an adiabatic process, it follows from the first law of thermodynamics that $\Delta E_{int} = -W$. If work is done by the system ($W > 0$), internal energy of system decreases and if work is done on the system, its internal energy increases. The relation between pressure and volume for an ideal gas (do not worry about its derivations, leave it for the future) is,

$$PV^{\gamma} = \text{constant, where } \gamma = \frac{C_P}{C_V}$$

Isothermal process : "A thermodynamic process during which temperature of a system remains constant is called an isothermal process."

Work done during an isothermal process (expansion)

Suppose, volume of μ mole ideal gas changes from V_1 to V_2 through a series of small changes in volume. Then the total work done is (according to equation (6.7.3))

$$W = \int_{V_1}^{V_2} P\,dV$$

From the equation of state for an ideal gas, $PV = \mu RT$

$$\therefore\ P = \frac{\mu RT}{V} \qquad (6.10.2)$$

$$\therefore\ W = \int_{V_1}^{V_2} \frac{\mu RT}{V}\,dV$$

$$= \mu RT \int_{V_1}^{V_2} \frac{1}{V}\,dV$$

(Since temp. remains constant during isothermal process, T has been taken out of integration sign)

$$= \mu RT \left[ln\ V \right]_{V_1}^{V_2}$$

$$= \mu RT\ [ln\ V_2 - ln\ V_1]$$

$$\therefore\ W = \mu RT\ ln \left(\frac{V_2}{V_1} \right) \qquad (6.10.3)$$

Internal energy of an ideal gas depends only on its temperature, hence the change in internal energy of an ideal gas during an isothermal process, is zero. Therefore, taking $\Delta E_{int} = 0$ in the first law of thermodynamics ($Q = W + \Delta E_{int}$), we get $Q = W$. Hence equation (6.10.3) can be expressed as,

$$W = Q = \mu RT\ ln\left(\frac{V_2}{V_1}\right) \qquad (6.10.4)$$

Cyclic process : "A thermodynamic process in which system undergoes a series of processes, starting from some thermodynamic equilibrium state, and finally the system is brought back to its original (initial) state is called a cyclic process."

In a cyclic process initial and final states of a system are the same and hence there is no change in its internal energy (i.e. $\Delta E_{int} = 0$) and so from the first law of thermodynamics $Q = W$. Thus, the net amount of heat exchanged between a system and its environment is equal to the net amount of work done by the system at the end of a cyclic process.

6.11 Reversible and irreversible processes

Suppose a gaseous system filled in a cylinder is in some initial equilibrium state *i* in which its pressure, temperature and volume are P, T and V respectively. Now, suppose we wish to reduce its volume to half at constant temperature and carry the system to some final equilibrium state *f*. This can be achieved in many different ways.

In one such process, the piston may be rapidly pushed down and then we can wait till the gas attains its initial temperature T by attaining equilibrium with the surrounding. When a gas is rapidly compressed in this way, many effects are produced creating inequilibrium in the gaseous system. Hence the system passes quickly through many inequilibrium states while going from state *i* to the state *f*. Though, as mentioned ealier it may return to the equilibrium state *f* after a long wait.

Now, if the process is reversed by moving piston rapidly upward to restore its initial volume V, the intermediate states of inequilibrium through which the system will pass, may not be the same as those through which it had passed during its transition from state *i* to state *f* (during compression). Such **process is called an irreversible process.**

Now, we think of another process in which the gas is compressed to half of its original volume through infinitesimally small (and slow) changes. On reducing the volume slowly in this manner, the system does experience a little, momentary inequilibrium and temperature also rises slightly. But as the process is very slow, the system releases additional heat in its surroundings and regains equilibrium. Temperature of the system remains constant during all intermediate states. Thus, the system can be considered to be passing through equilibrium states during every stage of compression. **Such a process is called quasi-static process.** In this manner volume of the system can be reduced to half of its original volume at constant temperature. If the process is reversed in the same manner by reducing pressure on the gas, very slowly, so that its volume increases extremely slow, the system returns to original state *i* following the same path (i.e. passing through the same intermediate states of the earlier process of compression). Such a process is **called a reversible process.** Here, we must remember that in the present example we have considered a reversible isothermal process and a piston moving without friction so that there is no dissipation of energy. When the process is reversed, not only the system but its environment also return to its original state. From the above discussion, it is clear that the absence of the factors responsible for the dissipation of energy is an ideal situation and hence a completely reversible process cannot be realized in practice. All the processes taking place in nature (i.e., processes occurring on their own) are irreversible e.g. rusting of iron, erosion of rocks, ageing of all animals etc.

Illustration 11 : Prove that the work done by an ideal gas during an adiabatic process, when it goes from initial state (P_1, V_1, T_1) to the final state (P_2, V_2, T_2) is

$$W = \frac{P_1V_1 - P_2V_2}{\gamma - 1} = \frac{\mu R(T_1 - T_2)}{\gamma - 1}$$

[For aidabatic process $PV^{\gamma} = A$ (constant)]

Solution : For an adiabatic process

$$W = \int_{V_1}^{V_2} P\,dV$$

$$= A\int_{V_1}^{V_2} \frac{1}{V^{\gamma}}\,dV \qquad (\because P = \frac{A}{V^{\gamma}})$$

$$\therefore\ W = A\int_{V_1}^{V_2} V^{-\gamma}\,dV$$

$$= A\left[\frac{V^{-\gamma+1}}{-\gamma+1}\right]_{V_1}^{V_2}$$

$$= A\left[\frac{V_2^{-\gamma+1} - V_1^{-\gamma+1}}{(1-\gamma)}\right]$$

$$= \frac{AV_2^{-\gamma+1} - AV_1^{-\gamma+1}}{(1-\gamma)}$$

$$= \frac{P_2V_2^{\gamma}V_2^{-\gamma+1} - P_1V_1^{\gamma}V_1^{-\gamma+1}}{(1-\gamma)}$$

$$= \frac{P_2V_2 - P_1V_1}{(1-\gamma)} \qquad (1)$$

$$\therefore\ W = \frac{P_1V_1 - P_2V_2}{\gamma - 1} \qquad (2)$$

For ideal gas $PV = \mu RT$

$$\therefore\ W = \frac{\mu RT_1 - \mu RT_2}{\gamma-1} = \frac{\mu R(T_1 - T_2)}{\gamma - 1} \qquad (3)$$

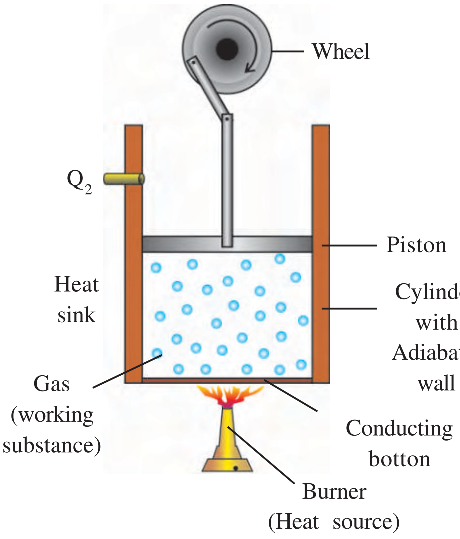
6.12 Calorimetry

Calorimetry means measurement of heat. When a body at highter temperature is brought in contact with another body at lower temperature, the heat lost by the hot body is equal to the heat gained by the colder body (provided no heat is allowed to escape to the surroundings). This is possible only if the system is isolated i.e., no exchange or transfer of heat occurs between the system and its surroundings.

A device which measures the heat is called calorimeter. It consists of a metallic vessel and stirrer of the same material like copper or aluminium. The vessel is kept inside a wooden jacket which contains heat insulating materials like glass, wool etc. The outer jacket acts as a heat shield and reduces the heat loss from the inner vessel. There is an opening in the outer jacket through which a mercury thermometer can be inserted into the calorimeter.

6.13 Heat Engine and its Efficiency

A device converting heat energy into mechanical work is called heat engine.

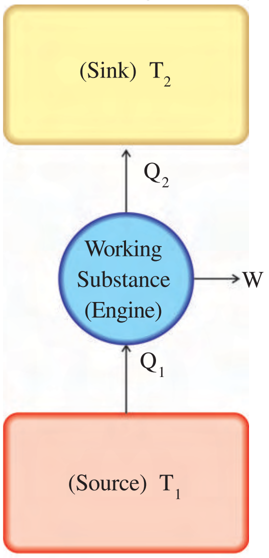


A simple heat engine
Figure 6.20

A simple heat engine is shown in Figure 6.20. The gas enclosed in a cylinder with a piston receives heat from the flame of a burner. On absorbing heat energy the gas expands and pushes the piston upwards. So the wheel starts rotating. To continue the rotations of the wheel, an arrangement is done in the heat engine so that the piston can move up and down periodically. For this, when the piston moves more in upward direction, then the hot gas is released from the hole provided on upper side.

Here, the gas is called a working substance. The flame of the burner is called heat source, and the arrangement in which the gas is released after expansion is called heat sink.

Figure 6.21 shows the working of the heat engine by line diagram. In the heat engine, the working substance undergoes a cyclic process.



Working of heat engine
Figure 6.21

For this the working substance absorbs heat Q_1 from the heat source at higher temperature T_1, out of which a part of heat energy is converted to mechanical energy (work, W) and the remaining heat Q_2 is released into the heat sink.

Hence, the net amount of heat absorbed by the working substance is

$$Q = Q_1 - Q_2 \qquad (6.13.1)$$

But for a cyclic process, the net heat absorbed by the system is equal to the net work done.

$$\therefore\ Q = W$$

$$\therefore\ Q_1 - Q_2 = W \qquad (6.13.2)$$

In the cyclic process, the ratio of the network (W) obtained during one cycle to the heat (Q_1) absorbed during the cycle is called the efficiency (η) of the heat engine. That is,

$$\text{Efficiency, } \eta = \frac{\text{Net work obtained per cycle}}{\text{Heat absorbed per cycle}}$$

$$\therefore\ \eta = \frac{W}{Q_1} = \frac{Q_1 - Q_2}{Q_1}$$

$$\therefore\ \eta = 1 - \frac{Q_2}{Q_1} \qquad (6.13.3)$$

From equation (6.13.3) it can be said that if $Q_2 = 0$, then the efficiency of heat engine is $\eta = 1$. This means that the efficiency of heat engine becomes 100% and total heat supplied to the working substance gets completely converted into work. In practice, for any engine $Q_2 \neq 0$, means that some heat Q_2 is always wasted. Hence $\eta < 1$.

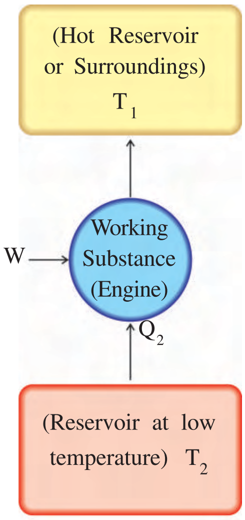
Usually heat engines are of two types :

(1) External combustion engine such as steam engine.

(2) Internal combustion engine such as diesel and petrol engines.

6.14 Refrigerator / Heat Pump and Coefficient of Performance

If the cyclic process performed on the working substance in heat engine is reversed, then the system works as a refrigerator or heat pump. Figure 6.22 shows the block diagram of refrigerator / heat pump.



Working of refrigerator

Figure 6.22

In the refrigerator, the working substance absorbs heat Q_2 from the cold reservoir at lower temperature T_2, external work W, is performed on the working substance and the working substance releases heat Q_1 into the hot reservoir at higher temperature T_1.

The ratio of the heat Q_2 absorbed by the working substance to the work W performed on it, is called the coefficient of performance (α) of the refrigerator. That is,

$$\alpha = \frac{Q_2}{W} \qquad (6.14.1)$$

Here the heat released in surrounding (reservoir) is

$$Q_1 = W + Q_2$$

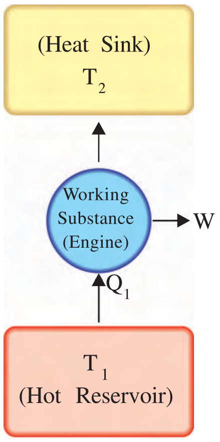
$$\therefore\ W = Q_1 - Q_2 \qquad (6.14.2)$$

$$\therefore\ \alpha = \frac{Q_2}{Q_1 - Q_2} \qquad (6.14.3)$$

Here, the value of α can be more than 1 ($\because\ Q_2 > Q_1 - Q_2$), but it cannot be infinite.

6.15 Second Law of Thermodynamics

The statements made by different scientists regarding the heat engine and refrigerator are called statements of second law of thermodynamics, which are as follows :



An ideal engine which is impossible (Q_1=W)

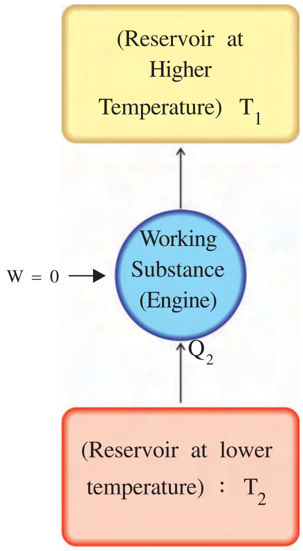
Figure 6.23

Statement of Kelvin–Planck

It is impossible to construct an engine which converts the heat absorbed from the reservoir completely into equal amount of work during each cycle of the cyclic process. (See Figure 6.23)

Statement of Rudolf Clausius

It is impossible to construct a cyclic machine in which continuous transfer of heat occurs from the reservoir at lower temperature to the reservoir at higher temperature without the input of energy by work. (See Figure 6.24)



An ideal refrigerator (Q1 = Q2 and W = 0)

Figure 6.24

Illustration 12 : A heat engine absorbs 360 J of energy and performs 25 J of work in each cycle. Find (*a*) the efficiency of the engine, and (*b*) the energy expelled to the cold reservoir in each cycle.

Solution : Here Q_1 = 360 J, W = 25 J

(*a*) Efficiency of heat engine

$$\eta = \frac{W}{Q_1} = \frac{25 \text{ J}}{360 \text{ J}} = 0.07 = 7\%$$

(*b*) The heat energy expelled to the cold reservoir in each cycle

$$Q_2 = Q_1 - W = 360 - 25 = 335 \text{ J}$$

Illustration 13 : The energy absorbed by an engine is three times greater than the work it performs.

(*a*) What is its thermal efficiency ?

(*b*) What fraction of energy absorbed is expelled to the cold reservoir ?

Solution : Here Q_1 = 3W

$$(a)\ \eta = \frac{W}{Q_1} = \frac{W}{3W} = \frac{1}{3} = 0.333$$

Hence, thermal efficiency η = 33.3%

(*b*) The heat expelled by the engine to the cold reservoir.

$$Q_2 = Q_1 - W = 3W - W = 2W$$

$$\text{Hence, } \frac{Q_2}{Q_1} = \frac{2W}{3W} = \frac{2}{3}$$

Hence, $\frac{2}{3}$rd part of the energy absorbed by the engine will be expelled to the cold reservoir.

Illustration 14 : A refrigerator has a coefficient of performance equal to 5. Assuming that the refrigerator absorbs 120 J of energy from a cold reservoir in each cycle, find (*a*) the work required in each cycle, (*b*) the energy expelled to the hot reservoir.

Solution : Here α = 5, Q_2 = 120 J

$$(a)\ \alpha = \frac{Q_2}{W}$$

$$\text{Hence, } W = \frac{Q_2}{\alpha} = \frac{120\text{J}}{5} = 24 \text{ J}$$

(*b*) The energy expelled to the hot reservoir

$$Q_1 = W + Q_2 = 24 \text{ J} + 120 \text{ J} = 144 \text{ J}$$

6.16 Carnot Cycle and Carnot Engine

In Semester-I, we have studied the behaviour of real gases by anlayzing an ideal gas which obeys simple law PV = μRT. Although an ideal gas does not exist but the real gas approaches an ideal gas behaviour when its density is low enough.

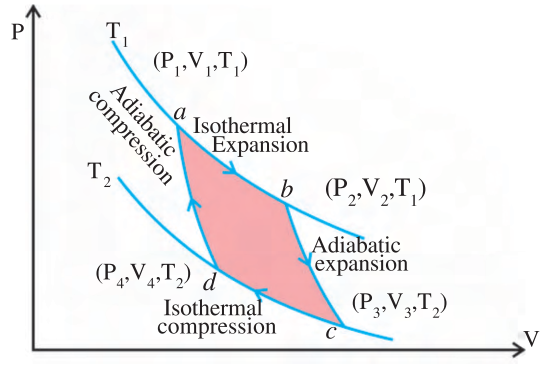
In an ideal engine, all processes are reversible and no wasteful energy transfers occur (due to friction, terbulence, etc).

In this topic we will study Carnot engine after the French scientist and engineer Sadi Carnot (pronounced "car–no"), who first proposed the engine's concept in 1824.

The Carnot engine converts heat energy into mechanical energy using a reversible cyclic process consisting of two isothermal and two adiabatic processes. Thus, a reversible heat engine operating between two temperatures is called carnot engine.

Carnot engine consists of a cylinder whose sides are perfect insulators of heat except the bottom and a piston sliding without friction. The working substance in this engine is μ mole of a gas at low enough pressure (behaving as an ideal gas). During each cycle of the engine, the working substance absorbs energy as heat from a heat source at constant temperature T_1 and releases energy as heat to a heat sink at a constant lower temperature $T_2 < T_1$.

The cyclic process, shown by P – V graph in Figure 6.25, is completed in four stages. The Carnot engine and its different stages are shown in Figure 6.26.



P – V graph of carnot cycle

Figure 6.25

(i) First stage : Isothermal expansion of the gas ($a \rightarrow b$)

Initially the working substance is in equilibrium state a (P_1, V_1, T_1), as shown in Figure 6.26(*a*).

Now, the conducting bottom of the cylinder is brought in contact with the heat source at temperature T_1 and the gas is slowly allowed to expand isothermally and is brought to equilibrium state *b* (P_2, V_2, T_1) [See Figure 6.26(*b*)]. Suppose the gas absorbs heat Q_1 during the process $a \rightarrow b$. Hence according to equation (6.10.4), the work done by the gas is

$$W_1 = Q_1 = \mu RT_1 \, ln\left(\frac{V_2}{V_1}\right) \qquad (6.16.1)$$

Further, for the isothermal process

$$P_1V_1 = P_2V_2 \qquad (6.16.2)$$

(ii) Second stage : Adiabatic expansion of the gas ($b \rightarrow c$)

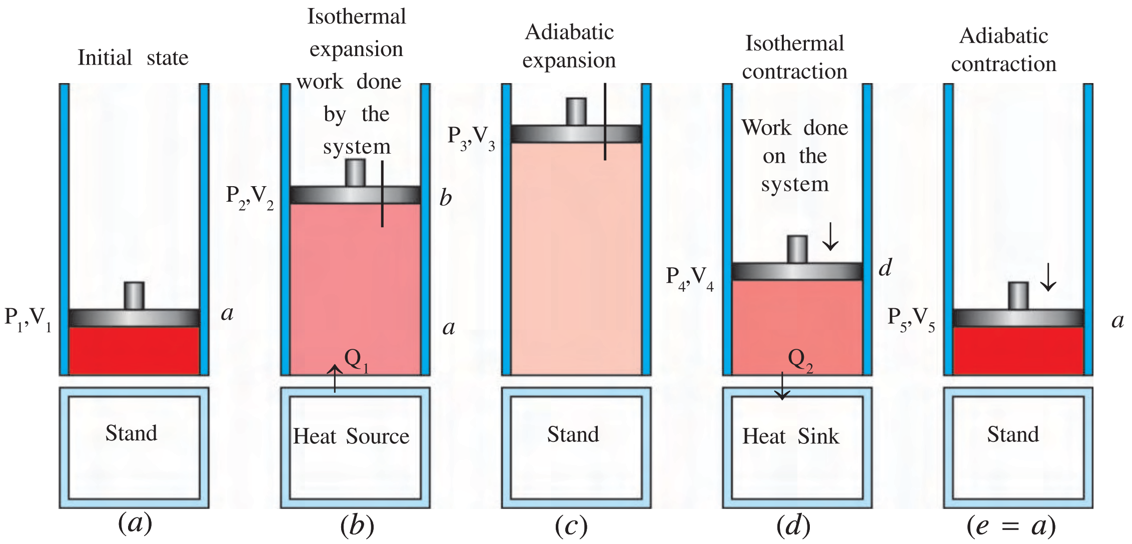
Now, the cylinder is placed on a thermally insulated stand and the gas is adiabatically expanded to attain the state *c* (P_3, V_3, T_2). See Figure 6.26(*c*).

During this (adiabatic) process the gas does not absorb any heat but does work while expanding, so its temperature decreases. For this process.

$$P_2V_2^{\gamma} = P_3V_3^{\gamma} \qquad (6.16.3)$$

(iii) Third Stage : Isothermal compression of the gas ($c \rightarrow d$)

Now, the cylinder is brought in contact with the heat sink at temperature T_2 and isothermally compressed slowly to attain an equilibrium state *d* (P_4, V_4, T_2) (See Figure 6.26(*d*)). Work done on the gas during this process of isothermal compression from state $c \rightarrow d$ is



Different stages of carnot engine

Figure 6.26

$$W_2 = Q_2 = -\ \mu RT_2\ ln\left(\frac{V_4}{V_3}\right)$$

(Here, negative sign is used as the work is done on the system).

$$\therefore\ W_2 = Q_2 = \mu RT_2\ ln\left(\frac{V_3}{V_4}\right) \qquad (6.16.4)$$

Here, Q_2 = heat released by the gas into heat sink.

Further, for this isothermal process

$$P_3V_3 = P_4V_4 \qquad (6.16.5)$$

(iv) Fourth Stage : Adiabatic compression of the gas (*d* → *a*)

Now, the cylinder is placed on a thermally insulated stand and compressed adiabatically to its original state *a* (P_1, V_1, T_1) as shown in Figure 6.16(*e*). This process is adiabatic, therefore, there is no exchange of heat with surroundings, but the work is done on the gas and hence its temperature increases from T_2 to T_1.

For this adiabatic process

$$P_4V_4^{\gamma} = P_1V_1^{\gamma} \qquad (6.16.6)$$

Note that over the whole cycle, the heat absorbed by the gas is Q_1 and the heat given out by the gas is Q_2. Hence the efficiency η of the carnot engine is

$$\eta = 1 - \frac{Q_2}{Q_1}$$

$$\therefore\ \eta = 1 - \frac{T_2\ ln\left(\frac{V_3}{V_4}\right)}{T_1\ ln\left(\frac{V_2}{V_1}\right)} \qquad (6.16.7)$$

Multiplying equations (6.16.2), (6.16.3) (6.16.5) and (6.16.6), we get

$$P_1V_1P_2V_2^{\gamma}\ P_3V_3P_4V_4^{\gamma} = P_2V_2P_3V_3^{\gamma}\ P_4V_4P_1V_1^{\gamma}$$

$$\therefore\ (V_2V_4)^{\gamma - 1} = (V_3V_1)^{\gamma - 1}$$

$$\therefore\ V_2V_4 = V_3V_1$$

$$\therefore\ \frac{V_2}{V_1} = \frac{V_3}{V_4} \qquad (6.16.8)$$

$$\therefore\ ln\left(\frac{V_2}{V_1}\right) = ln\left(\frac{V_3}{V_4}\right) \qquad (6.16.9)$$

Using this result in (6.16.7), we get the efficiency of carnot engine as

$$\eta = 1 - \frac{T_2}{T_1} \qquad (6.16.10)$$

Equation (6.16.10) shows that **the efficiency of the Carnot engine depends only on the temperatures of the source and the sink. Its efficiency does not depend on the working substance (if it is ideal gas).** If the temperature of the source (T_1) is infinite or the temperature of the heat sink (T_2) is absolute zero (which is not possible) then only, the efficiency of carnot engine will be 100%, which is impossible.

Illustration 15 : The temperature of a heat sink in a Carnot engine is 280 K and its efficiency is 40%. How much should the temperature of heat source be increased, at constant temperature of sink so that efficiency of the engine becomes 50% ?

Solution : $T_2 = 280$ K, $\eta_1 = 0.4$, $\eta_2 = 0.5$

$$\eta_1 = 1 - \frac{T_2}{T_1}$$

$$\therefore \frac{T_2}{T_1} = 1 - \eta_1 = 1 - 0.4 = 0.6 \qquad (1)$$

$$\therefore T_1 = \frac{T_2}{0.6} = \frac{280}{0.6} = 466.6 \text{ K}$$

$\eta_2 = 1 - \dfrac{T_2}{T_1 + x}$ (where x = increase in temperature of the source)

$$\therefore \frac{T_2}{T_1 + x} = 1 - \eta_2 = 1 - 0.5 = 0.5 \qquad (2)$$

Taking the ratio of equations (1) and (2),

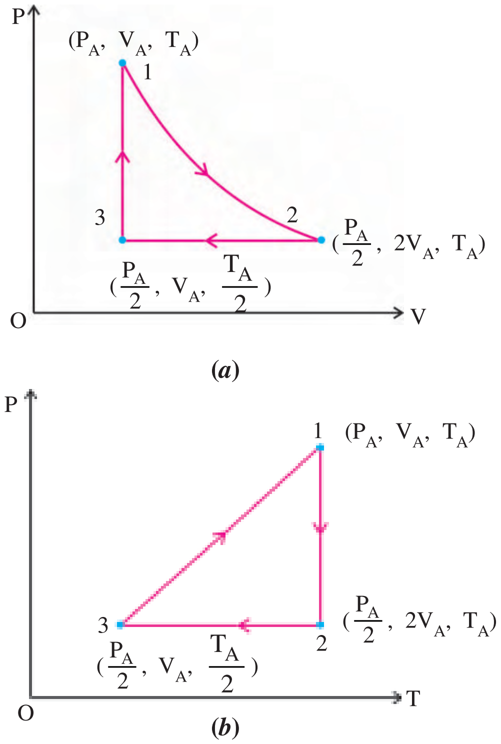
$$\frac{T_1 + x}{T_1} = \frac{0.6}{0.5}$$

$$\therefore 5T_1 + 5x = 6T_1, \ \therefore T_1 = 5x$$

$$\therefore x = \frac{T_1}{5} = \frac{466.6}{5} = 93.32 \text{ K}$$

Illustration 16 : The pressure of 1 mole of an ideal gas is P_A and temperature is T_A. First its volume is doubled through isothermal expansion. Then, its volume is restored to original by compressing it at constant pressure and then at constant volume its pressure is made again P_A. Sketch the graphs of P – V and P – T for the whole process.

Solution :



(*a*)

(*b*)

Figure 6.27

SUMMARY

1. **System :** Thermodynamic system is a part of the universe under thermodynamic study.
2. **Environment :** The remaining part of the universe surrounding a system and having a direct impact on the behaviour of the system is known as its environment.
3. **Wall :** The boundary separating the system and its environment is called the wall of the system.
4. **Thermodynamic Process :** The interaction between a system and its environment is called a thermodynamic process.
5. **Isolated System :** If a system does not interact with its surroundings then it is called isolated system.
6. **Zeroth Law of Thermodynamics :** If the systems A and B are in thermal equilibrium with a third system C, then A and B are also in thermal equilibrium with each other.
7. **Phase Diagram :** The graph of pressure against temperature indicating the phase (solid, liquid or gaseous state) of matter at given temperature (T) and pressure (P) is called phase diagram of that matter.

8. **Triple Point :** The values of pressure and temperature at which all the three states of matter co-exist in equilibrium is called triple point of the matter.

9. **Thermal Expansion / Contraction :** The increase in dimensions of a substance due to absorption of heat is called thermal expansion, and decrease in dimensions of the substance by releasing the heat is called thermal contraction.

10. **Linear Expansion :** The increase in the length of the body with increase in temperature (by absorbing heat) is called **linear expansion.** Substances exhibiting uniform thermal expansion in all directions are called **isotropic substances.**

11. **Heat Energy :** The total kinetic energy associated with the random motion of molecules of the gas (such that the total momentum of motion is zero) is called heat (or thermal) energy possessed by the gas.

12. **Heat :** The energy exchanged between a system and its environment, only due to the difference of temperature between them is called heat.

13. **Thermodynamic Work :** The amount of mechanical energy exchanged between two bodies during mechanical interaction is called thermodynamic work.

14. **First Law of Thermodynamics :** When a system is taken from initial state i to the final state f, then the change in its internal energy (ΔE_{int}) is equal to the difference between heat Q absorbed (gained) by it to the work **W** done by it. That means

    $$\Delta E_{int} = Q - W$$

15. **Adiabatic Process :** When there is no exchange of heat ($Q = 0$) between a system and its environment, the process is called adiabatic process.

16. **Constant – Volume Process (Isochoric Process) :** If the volume of a system is kept constant during thermodynamic process, then it is called constant – volume process.

17. **Cyclic Process :** A thermodynamic process in which a system at one thermodynamic equilibrium state is carried to another equilibrium state through a series of processes and then brought back to its original (initial) state is called a cyclic process.

18. **Calorie :** The amount of heat required to increase the temperature of one kilogram of pure water from 14.5 °C to 15.5 °C is called one kilocalorie. Its one thousandth part is called Calorie.

19. **Heat Capacity :** The ratio of the heat Q supplied to a body, to a change in its temperature ΔT, is called heat capacity H_C of the body.

20. **Specific Heat :** The quantity of heat required per unit mass for unit change in temperature of a body is called the specific heat of the material of the body.

21. **Molar Specific Heat :** The amount of heat required to change the temperature of one mole of a gas by 1 K (or 1 °C) is called molar specific heat of the gas.

22. **Specific Heat at Constant Volume (C_V) :** The amount of heat required to change the temperature of 1 mole of gas by 1 K, keeping its volume constant, is called the specific heat of the gas at constant volume C_V.

23. **Specific Heat at Constant Pressure (C_P) :** The amount of heat required to change the temperature of 1 mole of gas by 1 K, keeping its pressure constant, is called the specific heat of the gas at constant pressure C_P.

24. **Heat of Transformation (Latent Heat) :** The amount of energy per unit mass that must be transferred as heat when a substance completely undergoes a phase change (from one state to other) is called the heat of transformation (or Latent Heat, L)

25. **Heat of Fusion (L_F) :** When a solid substance of unit mass is transformed into liquid (then the substance gains heat) or when the liquid is transformed into solid (then it loses heat), the heat of transformation is called heat of fusion L_F.

26. **Irreversible Process :** If a process is reversed such that the intermediate states of inequilibrium through which the system passes are not the same through which it has passed during its transition from initial state to the final state, then such a process is called an Irreversible process.

27. **Reversible Process :** If a process is reversed very slowly such that it returns to the initial state following the same path (passing through the same intermediate states of earlier process from initial state to the final state) then, such a process is called a reversible process.

28. **Heat Engine :** A device converting heat energy into mechanical work is called heat engine.

29. **Efficiency of heat engine :** In the cyclic process, the ratio of the work (W) obtained during one cycle to the net heat (Q_1) absorbed during the cycle is called the efficieney (η) of the heat engine.

30. **Refrigerator :** If the cyclic process performed on the working substance in heat engine is reversed, then the system works as a refrigerator or heat pump.

31. **Coefficient of Performance (α) of the refrigerator :** The ratio of the heat Q_2 absorbed by the working substance to the work W performed on it is called the coefficient of performance (α) of the refrigerator.

32. **Second Law of Thermodynamics**

    Statement of Kelvin–Planck : It is impossible to construct an engine which converts the heat absorbed from the reservoir, completely into equal amount of work during each cycle of the cyclic process.

    Statement of Rudolf Clausius : It is impossible to construct a cyclic machine in which continuous transfer of heat occurs from the reservoir at lower temperature to a reservoir at higher temperature, without the input of energy by work.

33. **Calorimetry :** Calorimetry means measurement of heat.

34. **Calorimeter :** A device that measures the heat is called calorimeter.

35. **Carnot engine :** The Carnot engine converts heat energy into mechanical energy using a reversible cyclic process consisting of two isothermal and two adiabatic processes.

36. **Efficiency of Carnot engine :** The efficiency of Carnot engine is given by $\eta = 1 - \frac{T_2}{T_1}$. This shows that the efficiency of Carnot engine depends only on the temperature (T_1) of the source and temperature (T_2) of the sink. Its efficiency does not depend on the working substance.

EXERCISES

Choose the correct option from the given options :

1. An ideal gas has an initial pressure of 3 pressure units and an initial volume of 4 volume units. The table gives the final pressure and volume of the gas (in those same units) in five processes. Which processes start and end on the same isotherm ?

	i	*ii*	*iii*	*iv*	*v*
P	12	6	5	4	1
V	1	2	7	3	12

(A) *i, ii, iii, iv* (B) *ii, iii, iv, v* (C) *i, iii, iv, v* (D) *i, ii, iv, v*

2. A certain amount of heat Q increases the temperature of 1 g of material 'A' by 3C^{o} and 1 g of material B by 4 C^{o}. Which material has greater specific heat ?

(A) A (B) B

(C) A and B (D) Neither A nor B.

3. The measurement of temperature of triple point of water gives .... ^{o}C temperature.

(A) 0 (B) −273.16 (C) 100 (D) 0.01

4. When equilibrium is established between pure water and its vapour at atmospheric pressure, the temperature is taken as ....K.

(A) 100 (B) 273.15 (C) 373.15 (D) 273.16

5. The value of absolute zero temperature on a Fahrenheit scale is taken as .... ^{o}F.

(A) 0 (B) −273.15 (C) −459.67 (D) −356.67

6. On the temperature scales of ^{o}C and ^{o}F, which value of temperature is the same ?

(A) 0 (B) 40 (C) −40 (D) 32

7. A gas system absorbs 450 cal of heat and the work done by the system is 200 cal. Then the change in the internal energy of the system is .... cal.

(A) 250 (B) 850 (C) 325 (D) zero

8. A system can possess ...., but cannot possess................

(A) heat, heat energy (B) heat energy, heat

(C) heat, mechanical energy (D) work, heat energy.

9. Heat capacity of a body depends on the .... as well as on ....

(A) material of the body, its mass

(B) material of the body, its temperature

(C) mass of the body, its temperature

(D) volume of the body, its mass.

10. Given figure shows P – V diagram of one complete cycle of a cyclic process. Affter one cycle, (*a*) the internal energy of the gas ΔE_{int} and (*b*) the net transfer of heat energy will be.

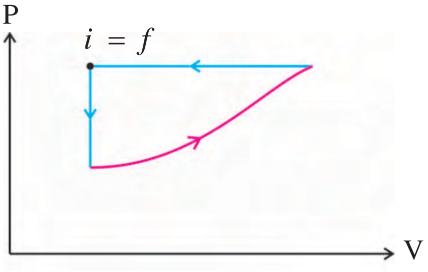


Figure 6.28

(A) positive, negative (B) positive, zero

(C) zero, negative (D) zero, positive

11. In thermodynamics, the work done by the system is considered ....and the work done on the system is considered.............

(A) positive, zero (B) positive, negative

(C) negative, positive (D) zero, infinite

12. The density of water at 20°C is 998 kg/m^3 and it is 992 kg/m^3 at 40°C. The co-efficient of volume expansion of water is..........C^{o-1}.

(A) $\frac{998}{992 \times 20}$ (B) $\frac{992}{998 \times 20}$ (C) $\frac{6}{998 \times 20}$ (D) $\frac{6}{992 \times 20}$

13. The relation between pressure and volume of an ideal gas during isothermal process is

(A) $P^{1-\gamma} T^{\gamma}$ = constant, (B) $P^{\gamma-1} T^{\gamma}$ = constant

(C) $P^{\gamma} T^{1-\gamma}$ = constant, (D) $P^{\gamma} T^{\gamma-1}$ = constant

14. The net heat absorbed by the system for one cycle of the cyclic process given in the figure is .... J.

(A) 400

(B) 900

(C) 200

(D) 300

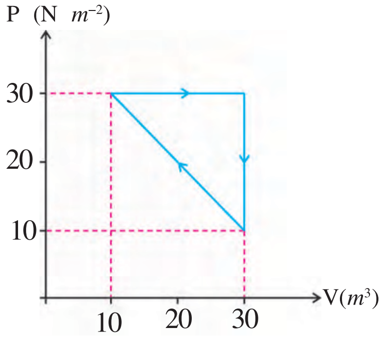


Figure 6.29

15. An ideal gas goes from state A to state B through three different processes 1, 2, and 3 as shown in figure. If the work done during these processes is respectively W_1, W_2 and W_3 then.

(A) $W_1 > W_2 > W_3$

(B) $W_1 = W_2 = W_3$

(C) $W_1 < W_2 < W_3$

(D) $W_1 > W_3 > W_2$

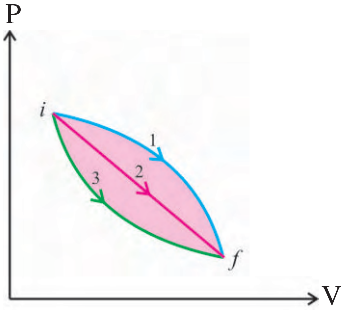


Figure 6.30

16. 100 g ice at 0°C is placed in 100 g water at 100°C. The final temperature of the mixture will be .... (Latent heat of melting of ice is 80 cal/g, and specific heat of water is 1 cal/g C°).

(A) 10°C (B) 20°C (D) 30°C (D) 50°C

17. During some process on an ideal gas $dW = 0$ and $dQ < 0$. Then for this gas ....

(A) temperature will increase. (B) volume will increase.

(C) pressure will remain constant. (D) temperature will decrease.

18. The work done in increasing the temperature of 1 mole ideal gas from 0°C to 100°C at constant pressure is ....

(A) 8.3×10^{-3} J (B) 8.3×10^{-2} J (C) 8.3×10^{2} J (D) 8.3×10^{3} J

19. The volume of an ideal gas increases by 24% during an adiabatic process. Its pressure will decrease by ....($\gamma = \frac{5}{3}$)

(A) 24% (B) 76% (C) 48% (D) 30%

20. The pressure of 10 mole gas changes from 8 atm to 4 atm during an isothermal expansion at 27°C. The amount of heat absorbed by the gas is .... J.

(A) 2079 R (B) 903 R (C) 187 R (D) 81.3 R

21. If a heat engine absorbs 50 kJ heat from a heat source and has efficiency of 40%, then the heat released by it in heat sink (environment) is................

(A) 40 kJ (B) 20 J (C) 30 kJ (D) 20 kJ

22. The efficiency of a heat engine is 30%. If it gives 30 kJ heat to the heat sink, then it should have absorbed ....kJ heat from heat source.

(A) 9 (B) 39 (C) 29 (D) 42.8

23. If a heat engine absorbs 2 kJ heat from a heat source and releases 1.5 kJ heat into cold reservoir, then its efficiency is...........

(A) 25% (B) 50% (C) 75% (D) 0.5%

24. For which value of the temperature will the values of Fahrenheit scale and Kelvin scale be equal ?

(A) 459.67, (B) 574.32 (D) −32 (E) 100

25. A diatomic (rigid rotator) ideal gas is used in a Carnot engine as the working substance. If during the adiabatic expansion part of the cycle, the volume of the gas increases from V to 32 V, the efficiency of the engine is

(A) 0.35 (B) 0.25 (C) 0.5 (D) 0.75

26. On a hot day at Ahmedabad a trucker loaded 37,000 L of diesel fuel. He delivered the diesel at Shrinagar (Kashmir), where the temperature was lower than that of Ahmedabad by 23 K. How many liters did he deliver ? For diesel $\gamma = 3\alpha = 9.50 \times 10^{-4}\ C^{o-1}$

(Neglect the thermal expansion / contraction of steel tank of the truck).

(A) 808 L (B) 36,190 L (C) 37808 L (D) 37,000 L

27. Figure 6.32 shows four rectangular plates of equal thickness and made from the same material. If their temperature is increased from T to T + ΔT, rank the plates according to increase in (*a*) their height, and (*b*) their area, greatest first.

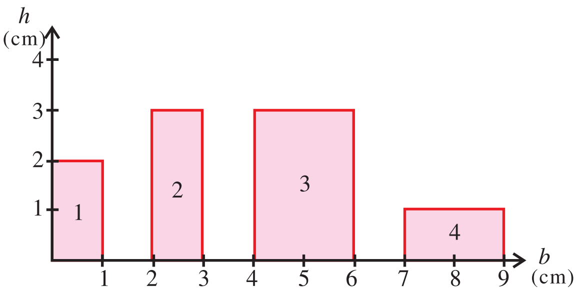


Figure 6.31

(A) 2, 3, 1, 4 (B) 1, 2, 3, 4 (C) 4, 1, 2, 3 (D) 3, 2, 1, 4

ANSWERS

1.	(D)	**2.**	(A)	**3.**	(D)	**4.**	(C)	**5.**	(C)	**6.**	(C)
7.	(A)	**8.**	(B)	**9.**	(A)	**10.**	(C)	**11.**	(B)	**12.**	(D)
13.	(A)	**14.**	(C)	**15.**	(A)	**16.**	(A)	**17.**	(D)	**18.**	(C)
19.	(D)	**20.**	(A)	**21.**	(C)	**22.**	(D)	**23.**	(A)	**24.**	(B)
25.	(D)	**26.**	(B)	**27.**	(D)						

Answer the following in short :

1. What is a phase diagram ?

2. What do you mean by one kilocalorie ?

3. What is an irreversible process ?

4. What do you mean by Isotropic substance ?

5. Why are the burns due to steam (vapour) more dangerous than the boiling water ?

6. What is a quasi–static process ?

7. In which situation does the efficiency of Carnot engine become 100% ?

8. When can you say that two systems are in thermodynamic equilibrium ?

9. What is adiabatic process.

10. What is cyclic process.

11. Why does the polyatomic molecules have more specific heat ?

12. What is coefficient of performance of a refrigerator ?

13. What is the isobaric process ?

14. Given figure shows the cyclic process 1-2-1 along different paths on P – V diagram (Such that each time the thermal equilibrium is established between the system and its environment).

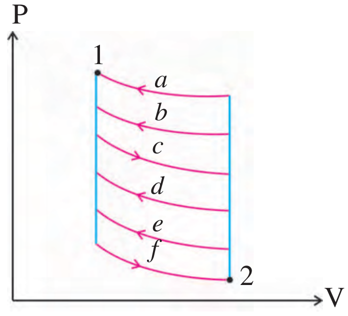


Figure 6.32

For which closed path the total work done by the system has maximum positive value ?

15. At what temperature is the Fahrenheit scale reading equal to

(*a*) Twice that of the celsius scale reading ?

(*b*) Half that of the celsius scale reading ?

Answer the following questions :

1. Explain the zeroth law of thermodynamics.

2. State and explain the first law of thermodynamics.

3. Explain the working of heat engine and its efficiency.

4. Obtain an expression for the work done on a gas during its compression at constant temperature.

5. When a system is taken from an initial state to a final state by different paths, explain the work done using P – V graphs. Explain the work done during cyclic process.

6. Explain the reversible and irreversible processes.
7. Give only the statements of second law of thermodynamics.
8. Given figure shows four paths for taking a system from initial state *i* to final state *f*.
   (*a*) On which path will the change in internal energy ΔE_{int} be maximum ?
   (*b*) On which path will the work W done by the system be maximum ?
   (*c*) On which path will the transfer of heat be maximum ?

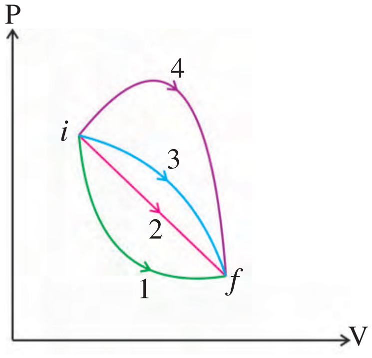


Figure 6.33

Solve the following Problems :

1. How much heat should be given to an Aluminium sphere of 200 *g* to carry it from 26°C to 66°C temperature ? What will be the heat capacity of the Aluminium sphere ? C = 0.215 cal g^{-1} C^{o-1}

   [**Ans. :** 1720 cal, 43 cal C^{o-1}]

2. The pressure and temperature of 10 g of O_2 are 3 × 10^5 N m^{-2} and 10°C respectively. On heating the gas at constant pressure, its volume becomes 10 L. Calculate
   (*a*) heat absorbed by the gas,
   (*b*) change in internal energy of the gas, and
   (*c*) the work done by the gas during expansion. (Take R = 8.3 J mol^{-1} K^{-1})
   Consider O_2 as diatomic rigid rotator.

   [**Ans. :** (*a*) 7929 J (*b*) 5664 J (*c*) 2265 J]

3. The temperature of the sink of a Carnot engine is 300 K and its efficiency is 40%. Find the decrease in temperature of the sink required to increase the efficiency of the engine to 50%, keeping temperature of the source to be constant. [**Ans.** : 50 K]

4. In a Carnot engine, temperature of the source is 500 K and that of the sink is 375 K. If the engine absorbs 600 k cal heat from the source per cycle, find (i) its efficiency (ii) work done per cycle, (iii) heat released in the sink. (J = 4.2 J/cal) [**Ans. :** (i) 25% (ii) 6.3 × 10^5 J (iii) 450 k cal]

5. 1 mole ideal gas at 27°C temperature and 2 atm pressure is compressed adiabatically to one eighth of its original volume. Find the final pressure and temperature of the gas. Take γ = 1.5 [**Ans. :** 45.2 atm, 848 K]

6. Calculate the work done on the gas in problem 5 given above. R = 8.3 J mol^{-1} K^{-1} [**Ans.** : 9097 J]

7. An ideal gas is enclosed in a closed container of 0.0083 m^3 at 300 K temperature and pressure of 1.6 × 10^6 P*a*. Find final temperature and pressure of the gas if 2.49 × 10^4 J heat is supplied to the gas. Neglect expansion of the container. R = 8.3 J mol^{-1} K^{-1} [**Ans. :** 675 K, 3.6 × 10^6 P*a*]

8. Calculate the work required to be done to increase the temperature of 1 mole ideal gas by 30°C. Expansion of the gas takes place according to the relation $V \propto T^{\frac{2}{3}}$. Take $R = 8.3\ \text{J mol}^{-1}\ \text{K}^{-1}$ [**Ans. :** 166 J]

9. For an adiabatic process PV^{γ} = constant. Evaluate this "constant" for an adiabatic process in which 2 mol of an ideal gas is filled at 1.0 atm pressure and 300 K temperature. Consider the ideal gas to be diatomic rigid rotator.

[**Ans. :** 1.48×10^3]

10. One mole of an ideal monoatomic gas traverses the cycle of given Figure 6.34. Process 1 → 2 occurs at constant volume, process 2 → 3 is adiabatic, and process 3 → 1 occurs at constant pressure. Compute the required heat Q, change in internal energy ΔE_{int}, and the work done W for the processes. 1 → 2 and 3 → 1. **Take :** $R = 8.314\ \text{J mol}^{-1}\ \text{K}^{-1}$

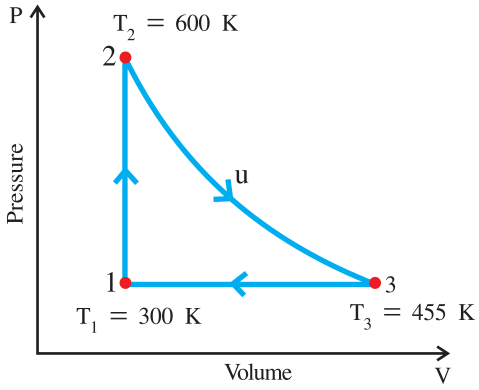


Figure 6.34

Answer :

Process	Q	ΔE_{int}	W
1 → 2	3741 J	3741 J	0
3 → 1	–3221.7 J	–1933 J	–1288.7 J

11. A heat engine has an efficiency of 22%. If the difference between the heat absorbed and heat lost by it in one cycle remains 75 J, then calculate the heat obtained by the engine from heat source and heat lost in the heat sink during each cyle. [**Ans. :** 341 J, and 266 J]

12. A heat engine getting energy from gasoline takes in 10,000 J of heat and converts 2000 J into work. The heat of combustion (transformation) is $L_C = 5.0 \times 10^4$ J/g.

(*a*) What is the efficiency of the heat engine ?

(*b*) During each cycle, how much heat will be released by the engine in the heat sink ?

(*c*) How much gasoline is burnt in each cycle ?

(*d*) If the engine performs 25 cycles per second, how much gasoline is burnt in 1 hour ?

(*e*) How much power is generated by the engine per second ? In horsepower ? (1 hp = 746 W)

[**Ans. :** (*a*) 20% (*b*) 8000 J (*c*) 0.2 g (*d*) 18 kg/h (*e*) 50 kW, 67 hp]

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CHAPTER 7

OSCILLATIONS

7.1 Introduction

Dear students, while studying the circular motion and the projectile motion you have learnt that how the forces acting in specific manners affect the trajectory of the particle. Even in Std. IX, you have learnt about the wave motion, concept of periodic motion (harmonic motion) and oscillatory motions, their characteristics such as frequency, periodic time, amplitude etc.

The study of periodic motion is very much important in physics. This motion plays fundamental role in understanding generation and propagation of sound waves and electromagnetic waves. You know that constituent particles like atoms, molecules and ions too have oscillatory motion.

In this chapter first we refresh our concept of periodic (harmonic) and oscillatory motion, and study such motion under a position dependent forces. The mathematical formulations for potential energy, kinetic energy and total mechanical energy will also be seen. We shall also study the damping of oscillation, forced oscillation and phenomenon of resonance.

7.2 Periodic Motion and Oscillatory Motion

If a body repeats its motion along a certain path, about a fixed point, at a definite interval of time, it is said to have a periodic motion.

The motion of hands of a clock, motion of moon around the earth, and the revolution of earth around the sun are the best examples of periodic motion.

If a body repeatedly moves to and fro, back and forth, or up and down about a fixed point in a definite interval of time, such a motion is called an oscillatory motion. The body performing such motion is called an **oscillator.**

The motion of the bob of the pendulum and the motion of a loaded spring are the known examples for oscillatory motion.

All oscillatory motions are periodic motions but all periodic motions may not be oscillatory. Like the motion of hands of a clock, motion of the earth around the sun are periodic but not an oscillatory. The concept of to and fro, back and forth or up and down about some fixed point is not present in these cases.

We will see that the oscillatory motion can be expressed in terms of sine and cosine functions. The sine and cosine trigonometric functions are periodic functions having period of 2π rad. In mathematics, these functions are known as harmonic functions. Hence, oscillatory motion is also called as harmonic motion.

7.3 Simple Harmonic Motion (SHM)

Simple harmonic motion is a simplest type of periodic motion.

The periodic motion of a body about a fixed point, on a linear path, under the influence of the force acting towards the fixed point and proportional to displacement of the body from the fixed point is called a simple harmonic motion (SHM).

A body performing simple harmonic motion is known as **simple harmonic oscillator (SHO).**

Let us consider a massless spring obeying Hook's Law. It is suspended vertically from a rigid support as shown in Figure 7.1 and an object of mass *m* is tied at its lower end. When we pull an object downward and release, it will perform (almost) simple harmonic motion.

Now use Figure 7.1 to understand some basic terms associated with simple harmonic motion.

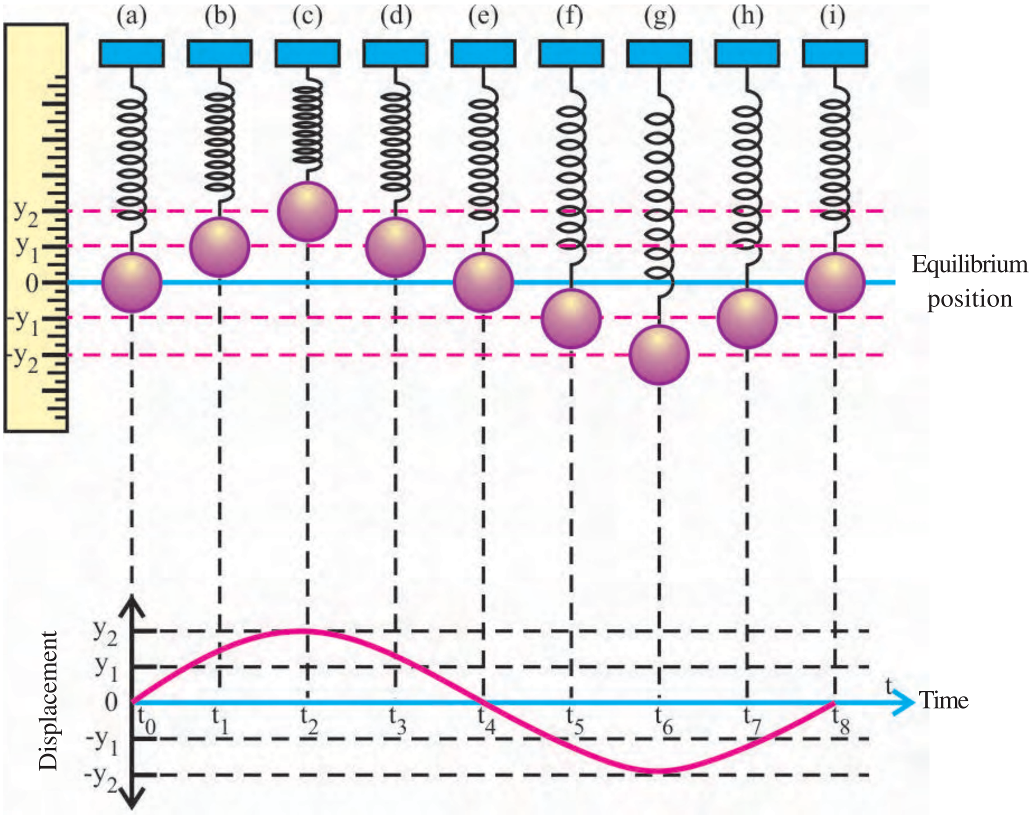
Equilibrium position (Mean position)

The point about which the simple harmonic oscillator performs simple harmonic motion is known as **equilibrium position** or **mean position.**

In the Figure 7.1 (*a*), (*e*) and (*i*), the object is at mean position.

Displacement

The distance of the oscillator at any instant from the equilibrium position is known as the **displacement** of the oscillator at that instant.



Simple harmonic motion of a massive object attached to a spring and displacement–time graph

Figure 7.1

In the Figure 7.1 (*b*) the displacement of the oscillator at $t = t_1$ is y_1. The displacement of the oscillator at $t = t_5$ is $-y_1$ Figure 7.1(*f*).

Amplitude :

The maximum displacement of the oscillator on either side of mean position is called **amplitude** of the oscillator.

As shown in the Figure 7.1 (*c, g*), the maximum displacement achieved by the oscillator is y_2. Hence for this, y_2 be the amplitude of the oscillator.

Periodic Time (Time period or period) :

The time required to complete one oscillation is known as **periodic time** (T) of the oscillator.

In other words, the least interval of time after which the periodic motion of an oscillator repeat itself is called as periodic time of the oscillator.

SI unit of periodic time is second (s).

For the oscillator of Figure 7.1, $t_8 - t_0$ is the periodic time.

Ferquency :

Frequency of simple harmonic oscillator is defined as the number of oscillations completed by the simple harmonic oscillator in one second.

Its SI unit is s^{-1} or hertz (H_z)

It is denoted by f, and $f = 1/T$.

Angular frequency :

2π times the frequency of an oscillator is called the **angular frequency** of that oscillator.

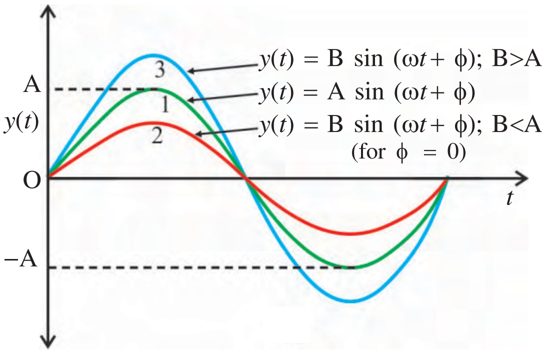
It is denoted by ω $(= 2\pi f)$

Its SI uint is rad s^{-1}.

If we draw a graph of displacement of SHO against time, it appears as shown in the lower part of the Figure 7.1. Such motion can be represented by mathematically as a function of time as under :

$$\underset{\text{Displacement at time } t}{y(t)} = \underset{\text{Amplitude}}{A}\sin\overbrace{(\underset{\text{Angular frequency}}{\omega}\underset{\text{Time}}{t} + \underset{\text{Initial phase (epoch)}}{\phi})}^{\text{Phase}} \qquad (7.3.1)$$

We know that the span of the sine function is $[-1, 1]$. Hence the displacement $y(t)$ of the SHM varies between $\pm$A (See Figure 7.2). If another SHM is represented by $y(t) = B\sin(\omega t + \phi)$ with $B < A$ than it will be of type -2 curve shown in Figure 7.2. And if $B > A$, the curve will be of type -3.



Displacement of SHM as a function of time

Figure 7.2

The quantity $(\omega t + \phi)$ is known as phase of the SHM at time t and represents the state of motion at that time.

The phase at $t = 0$ is called initial phase, phase constant or epoch (ϕ) of SHM.

For one complete oscillation, phase of SHM increases by 2π rad and hence after n oscillations its phase is increased by $2n\pi$ rad.

As the motion is periodic with time period T, the displacement of the oscillator at $(t + T)$ must be the same for any t.

i.e.

$$y(t) = y(t + T)$$

$$A\sin(\omega t + \phi) = A\sin[\omega(t + T) + \phi]$$

$$\sin(\omega t + \phi + 2\pi) = \sin(\omega t + \omega T + \phi)$$

$$\omega t + \phi + 2\pi = \omega t + \omega T + \phi$$

$$\omega T = 2\pi$$

$$\therefore\ \omega = \frac{2\pi}{T} = 2\pi f \ (\because\ T = \frac{1}{f}) \qquad (7.3.2)$$

Velocity :

Now the velocity of an oscillator is

$$v(t) = \frac{dy(t)}{dt}$$

$$v(t) = \omega A\cos(\omega t + \phi) \qquad (7.3.3)$$

From equation (7.3.3)

$$v = \pm A\omega\sqrt{1 - \sin^2(\omega t + \phi)}$$

$$v = \pm\omega\sqrt{A^2 - A^2\sin^2(\omega t + \phi)}$$

$$v = \pm\omega\sqrt{A^2 - y^2} \qquad (7.3.4)$$

At $y = 0$, $v = \pm A\omega = \pm v_m$

This is the maximum velocity or velocity amplitude (v_m) of the SHM.

At $y = \pm A$ (end point of the SHM), $v = 0$

Acceleration

The acceleration of SHO is

$$a(t) = \frac{dv(t)}{dt} = \frac{d^2y(t)}{dt^2}$$

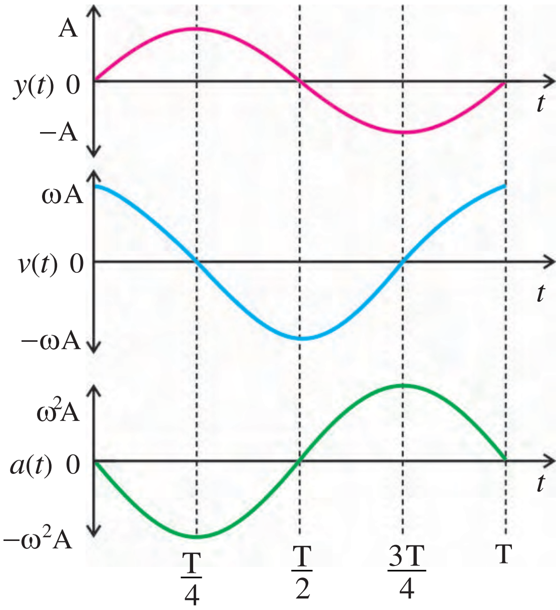
$$a(t) = -\omega^2 A\sin(\omega t + \phi)$$

$$a(t) = -\omega^2\, y(t) \qquad (7.3.5)$$

At $y = 0$, $a(t) = 0$ and

at $y = \pm A$, $a(t) = \mp\,\omega^2 A$

The graphs of particle displacement $y(t)$, velocity $v(t)$ and acceleration $a(t)$ against time t of a SHM are shown in Figure 7.3.



Graphs of Displacement, velocity and acceleration of SHO against time (ϕ = 0)

Figure 7.3

The variation of $y(t)$, $v(t)$ and $a(t)$ with time are also summerized in Table 7.1.

Table 7.1

Values of $y(t)$, $v(t)$ and $a(t)$

t	0	$\frac{T}{4}$	$\frac{T}{2}$	$\frac{3T}{4}$	T
Displacement $y(t)$	0	A	0	−A	0
Velocity $v(t)$	ωA	0	$-\omega$A	0	ωA
Acceleration $a(t)$	0	$-\omega^2$A	0	ω^2A	0

Illustration 1 : The displacement of a simple harmonic oscillator is given by

$$y = 0.40\ \sin(440t + 0.61).$$

For this, what are the values of

(i) Amplitude, (ii) angular frequency, (iii) time period and (iv) initial phase ?

Here, y is in metre and t is in second.

Solution :

Compare $y = 0.40\ \sin(440t + 0.61)$

with $y = A\ \sin(\omega t + \phi)$

(i) The amplitude is $A = 0.40\ m$

(ii) Angular frequency $\omega = 440$ rad/s

(iii) Periodic time $T = \dfrac{2\pi}{\omega} = 2 \times \dfrac{22}{7} \times \dfrac{1}{440}$

$= 0.0143$ s

(iv) Initial phase $\phi = 0.61$ rad

7.4 The Force Law for Simple Harmonic Motion

It is seen from equation (7.3.5) that the acceleration of simple harmonic oscillator is a function of time. Hence we can use Newton's second law of motion to answer a question : How much force is needed for this acceleration ?

We know that

$$F = ma$$

$$\therefore\ F = -m\omega^2 y(t) \qquad (7.4.1)$$

This is restoring force.

According to Hook's Law, the restoring force is given by

$$F = -ky(t) \qquad (7.4.2)$$

with k as spring constant

Comparing equations (7.4.1) and (7.4.2),

$k = m\omega^2$

$\therefore$ The angular frequency

$$\omega = \sqrt{\frac{k}{m}} \qquad (7.4.3)$$

and frequency of oscillation

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad (7.4.4)$$

The period of oscillation

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}} \qquad (7.4.5)$$

In many cases, the simple harmonic motion can also occur even without spring. In that case k is called the force constant of SHM and it is restoring force per unit displacement ($k = -\frac{F}{y}$).

7.5 Differential Equation of Simple Harmonic Motion

According to Newton's second law of motion,

$$F = ma = m\frac{dv(t)}{dt} = m\frac{d^2y(t)}{dt^2} \qquad (7.5.1)$$

Comparing this with $F = -ky(t)$

$$m\frac{d^2y(t)}{dt^2} = -ky(t)$$

$$\therefore \frac{d^2y(t)}{dt^2} = -\frac{k}{m}y(t)$$

$$\frac{d^2y(t)}{dt^2} = -\omega^2 y(t) \qquad (\because\ 7.4.3)$$

$$\therefore \frac{d^2y(t)}{dt^2} + \omega^2 y(t) = 0 \qquad (7.5.2)$$

This is the second order differential equation of the simple harmonic motion. The solution of this equation is of the type,

$y(t) = A \sin \omega t$

or

$y(t) = B \cos \omega t$

or any linear combination of sine and cosine functions,

$y(t) = A \sin \omega t + B \cos \omega t$

Illustration 2 : Length of an elastic spring increases by 9 cm on suspending a body of mass 14.4 g at its lower end. Now this body is pulled down by 3 cm and released, so that it starts executing SHM.

Find (1) amplitude and initial phase (epoch),

(2) angular frequency and period,

(3) phase at $t = 3$ s,

(4) Equation of displacement and

(5) displacement at $t = 1.5$ s

Take $g = 100\pi^2$ cm s^{-2}.

Solution :

(1) Body is pulled down by 3 cm

Hence its amplitude is 3 cm

As the motion begins from the lower end, at $t = 0$, $y = -A$

$\therefore y = A \sin(\omega t + \phi)$ gives

$-A = A \sin\phi$

$\therefore \sin\phi = -1$

$$\therefore \phi = \frac{3\pi}{2} \text{ rad}$$

$$(2)\ \omega = \sqrt{\frac{k}{m}}$$

$$= \sqrt{\frac{mg}{\Delta l} \times \frac{1}{m}} = \sqrt{\frac{g}{\Delta l}}$$

$$= \sqrt{\frac{100\pi^2}{9}}$$

$$= \frac{10\pi}{3} \text{ rad s}^{-1}$$

$$\text{As } T = \frac{2\pi}{\omega}$$

$$\therefore T = \frac{2\pi}{\left(10\frac{\pi}{3}\right)}$$

$$= \frac{3}{5} \text{ s}$$

(3) We know the phase is given by

$$\theta = \omega t + \phi$$

$$= \frac{10\pi}{3} \times 3 + \frac{3\pi}{2}$$

$$\theta = \frac{23\pi}{2} \text{ rad}$$

(4) For the displacement at time t,

$$y = \text{A} \sin(\omega t + \phi)$$

$$= 3 \sin\left(\frac{10\pi}{3}t + \frac{3\pi}{2}\right) \text{(in cm)}$$

(5) at $t = 1.5$ sec

$$y = 3 \sin\left(\frac{10\pi}{3} \times 1.5 + \frac{3\pi}{2}\right)$$

$$= 3 \sin(5\pi + \frac{3\pi}{2})$$

$$y = 3 \text{ cm}$$

Illustration 3 : The SHM is represented by $y = 3 \sin 314\, t + 4 \cos 314\, t$. y in cm and t in second. Find the amplitude, epoch the periodic time and the maximum velocity of SHO.

Solution :

$$y = \text{A} \sin(\omega t + \phi)$$

$$= \text{A} \cos\phi \sin\omega t + \text{A} \sin\phi \cos\omega t$$

Comparing $y = 3\sin 314t + 4\cos 314t$

with above equation

$$3 = \text{A} \cos\phi \text{ and}$$

$$4 = \text{A} \sin\phi$$

$$\therefore\ \text{A}^2\cos^2\phi + \text{A}^2\sin^2\phi = 3^2 + 4^2$$

$$\therefore\ \text{A}^2 = 25$$

$$\text{A} = 5 \text{ cm}$$

The initial phase (epoch) is obtained as

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{4}{3}$$

$$\therefore\ \phi = \tan^{-1}\left(\frac{4}{3}\right)$$

$$\therefore\ \phi = 53^\circ 8'$$

$$\text{Now T} = \frac{2\pi}{\omega}$$

$$= \frac{2\pi}{314} = 0.02 \text{ s}$$

The maximum velocity

$$v_{\text{max}} = \omega\text{A}$$

$$= 314 \times 5$$

$$= 1570 \text{ cm/s}$$

Illustration 4 : A particle executes SHM on a straight line path. The amplitude of oscillation is 2 cm. When the displacement of the particle from the mean position is 1 cm, the magnitude of its acceleration is equal to that of its velocity. Find the time period, maximum velocity and maximum acceleration of SHM.

Solution :

Here A = 2 cm

When $y = 1$ cm,

$$\begin{Bmatrix}\text{The magnitude} \\ \text{of velocity}\end{Bmatrix} = \begin{Bmatrix}\text{The magnitude} \\ \text{of acceleration}\end{Bmatrix}$$

$$\therefore\ \omega\sqrt{\text{A}^2 - y^2} = \omega^2 y$$

$$\text{A}^2 - y^2 = \omega^2 y^2$$

$$2^2 - 1^2 = \omega^2 \times 1^2$$

$$\therefore\ \omega = \sqrt{3} \text{ rad/s}$$

$$\therefore\ \text{Period T} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{3}} \text{ s}$$

Now the maximum velocity

$$v_m = \omega\text{A}$$

$$= \sqrt{3} \times 2 = 2\sqrt{3} \text{ cm s}^{-1}$$

$$\text{The maximum acceleration} = \text{A}\omega^2$$

$$= 2 \times 3$$

$$= 6 \text{ cm s}^{-2}$$

Illustration 5 : A spring balance has a scale that reads 50 kg. The length of the scale is 20 cm. A body suspended from this spring, when displaced and released, oscillates with period of 0.6 s. Find the weight of the body.

Solution :

Here $m = 50$ kg

Maximum extension of spring

$$y = 20 - 0$$

$$= 20 \text{ cm}$$

$$= 0.2 \text{ m}$$

$$\text{Periodic time T} = 0.6 \text{ s}$$

$$\text{Maximum force F} = mg$$

$$= 50 \times 9.8$$

$$= 490 \text{ N}$$

$$\therefore\ k = \frac{\text{F}}{y} = \frac{490}{0.2}$$

$$= 2450 \text{ N m}^{-1}$$

$$\text{As T} = 2\pi\sqrt{\frac{m}{k}}$$

$$m = \frac{\text{T}^2 k}{4\pi^2}$$

$$= \frac{(0.6)^2 \times 2450}{4 \times (3.14)^2} = 22.36 \text{ kg}$$

$$\therefore\ \text{weight of a body} = mg = 22.36 \times 9.8$$

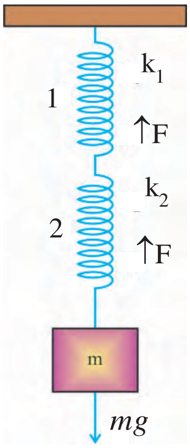
$$= 219.1 \text{ N} = 22.36\ kgf$$

[1 *kgf* (kilogram force) = g N;

where g = gravitational acceleration]

7.6 Oscillations in Loaded Springs

(i) As shown in the Figure 7.4, consider a series combination of two massless springs of spring constant k_1 and k_2 suspended vertically with one of its end fixed to a rigid support and mass m is attached to the free end. Let the body is pulled downward through small distance y and left free to oscillate vertically.



Series combination of two springs

Figure 7.4

The increase in the length of spring 1 is y_1 and that of spring 2 is y_2.

Then

$$y = y_1 + y_2$$

But the restoring force (= mg) acting on each spring is same.

$$\therefore \ \mathrm{F} = -k_1y_1 \text{ and}$$

$$\mathrm{F} = -k_2y_2$$

As $y = y_1 + y_2$

$$\therefore \ y = \frac{-\mathrm{F}}{k_1} + \frac{-\mathrm{F}}{k_2}$$

$$y = -\mathrm{F}\left(\frac{k_1 + k_2}{k_1k_2}\right)$$

$$\therefore \ \mathrm{F} = -y\left(\frac{k_1k_2}{k_1 + k_2}\right) \qquad (7.6.1)$$

Hence the equivalent force constant for series combination of two springs is

$$k = \frac{k_1k_2}{k_1 + k_2} \qquad (7.6.2)$$

Now period of oscillation,

$$\mathrm{T} = 2\pi\sqrt{\frac{m}{k}}$$

$$\therefore \ \mathrm{T} = 2\pi\sqrt{m\left(\frac{k_1 + k_2}{k_1k_2}\right)} \qquad (7.6.3)$$

If $k_1 = k_2 = k'$

then $k = \dfrac{k'k'}{k' + k'}$

so the equivalent spring constant is

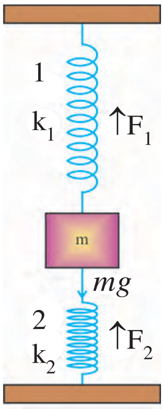
$$k = \frac{k'}{2}$$

and period of oscillation

$$\mathrm{T} = 2\pi\sqrt{\frac{2m}{k'}}$$

(ii) Now consider a situation as shown in Figure 7.5, where a body of mass m is attached in between the two massless springs of spring constant k_1 and k_2. Let body is left free for SHM in vertical plane after pulling mass m.

In this situation when body is pulled to one side through a small displacement y, one spring gets compressed by length y and the second spring gets stretched by y. Hence, the restoring forces F_1 and F_2 set up in both these springs will act in the same direction.



Mass loaded two springs

Figure 7.5

The net restoring force will be

$$\begin{aligned} \mathrm{F} &= \mathrm{F}_1 + \mathrm{F}_2 \\ &= -k_1y - k_2y \\ &= -(k_1 + k_2)y \\ &= -ky \end{aligned}$$

Hence, the equivalent spring constant in this case will be,

$$k = k_1 + k_2 \tag{7.6.4}$$

Now the periodic time

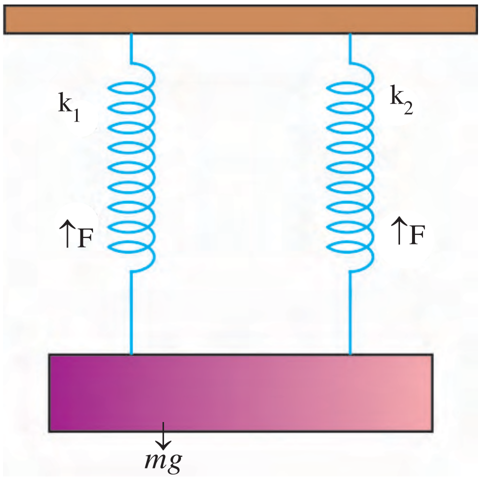
$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{k_1 + k_2}} \tag{7.6.5}$$

If $k_1 = k_2 = k'$ then

$k = 2k'$

and $T = 2\pi\sqrt{\frac{m}{2k'}}$

(iii) Two massless springs of equal lengths and having force constants k_1 and k_2 respectively are suspended vertically from a rigid support as shown in Figure 7.6. At their free ends, a block of mass m having non-uniform density distribution is suspended so that springs undergo equal extension.



Parallel combination of two springs

Figure 7.6

In this situation two bodies are pulled down through a small distance y and the system is made free to perform SHM in vertical plane.

Here, the springs have different force constants. Moreover, the increase in their length is the same. Therefore, the load is distributed unequally between the springs. Hence, the restoring force developed in each spring is different.

If F_1 and F_2 are the restoring forces set up due to extension of springs, then

$F_1 = -k_1 y$ and

$F_2 = -k_2 y.$

Also the total restoring force (= mg)

$$F = F_1 + F_2$$
$$= -k_1 y - k_2 y$$
$$-ky = -(k_1 + k_2)y$$

Where k is the equivalent spring constant of the parallel combination of two springs.

$$\therefore\ k = k_1 + k_2 \tag{7.6.6}$$

The time period of oscillation

$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$T = 2\pi\sqrt{\frac{m_1}{k_1 + k_2}} \tag{7.6.7}$$

If $k_1 = k_2 = k'$, then

$k = 2k'$

and $T = 2\pi\sqrt{\frac{m}{2k'}}$

Illustration 6 : A spring compressed by 0.1 m developes a restoring force 10 N. A body of mass 4 kg is placed on it. Calculate (i) the force constant of the spring, (ii) the depression of the spring under the weight of two bodies and (iii) the time period of oscillation, if the body performs SHO. ($g = 10$ N/kg)

Solution :

Here F = 10 N

Displacement $\Delta y = 0.1\ m$

$m = 4$ kg

We know

(i) $k = \frac{F}{\Delta y}$

$= \frac{10}{0.1}$

$k = 100\ \text{Nm}^{-1}$

(ii) $y = \frac{mg}{k} = \frac{4 \times 10}{100} = 0.4\text{m}$

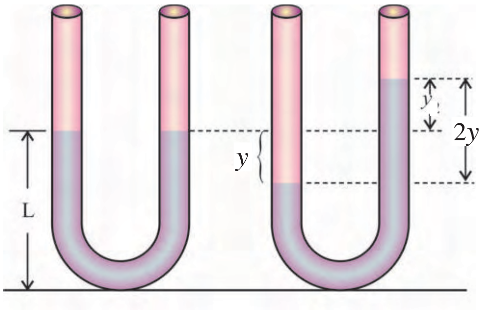
(iii) $T = 2\pi\sqrt{\frac{m}{k}}$

$= 2\pi \times \sqrt{\frac{4}{100}}$

$= \frac{4\pi}{10}$

$T = 0.4\pi$ s

Illustration 7 : A U-tube is partially filled with a liquid of density ρ. The length of the liquid column in each arm of the U-tube is L. Now the free-surface of the liquid in one arm is given a diplacement y in vertical direction and allowed to oscillate. Prove that these oscillations are simple harmonic. Also obtain the period of SHM.



U–tube filled with liquid

Figure 7.7

Solution :

When the liquid is displaced in one arm by y unit in the downward direction the liquid in the second arm is displaced by y unit in the upward direction.

$\therefore$ As shown in Figure 7.7 the difference in the heights of the free-surfaces of liquids in the two arms = $2y$

$\therefore$ Pressure created due to liquid column of height $2y$, $P = 2y\rho g$

where ρ = density of liquid; g = acceleration due to gravity.

The force developed due to this pressure, F = PA

$\therefore F = 2y\rho gA = (2\rho gA)y = ky$

$\therefore F \propto y$

Since, this force acts in a direction opposite to that of y, $F \propto - y$

$\therefore$ The oscillations are simple harmonic.

The period of oscillation

$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$= 2\pi\sqrt{\frac{m}{2\rho gA}}$$

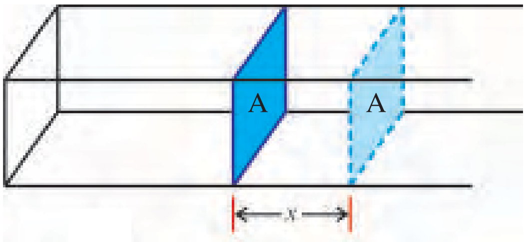
As mass of liquid $m = LA\rho = 2yA\rho$

$$\therefore T = 2\pi\sqrt{\frac{2yA\rho}{2\rho gA}}$$

$$T = 2\pi\sqrt{\frac{y}{g}}$$

Illustration 8 : A rectangular pipe having cross-sectional area A is closed at one end and at its other end a block having same cross-section is placed so that the system is airtight. In the equilibrium position of the block, the pressure and volume of air enclosed in the pipe are P and V respectively. Prove that the block performs SHM when it is given a small displacement 'x' inward and released. Also find the period of this SHM. Assume the walls to be frictionless and compression of air to be isothermal.

Solution :



Rectangle pipe

Figure 7.8

Due to small compression, suppose, increase in pressure = ΔP and decrease in volume = ΔV

For an isothermal compression,

$(P + \Delta P)(V - \Delta V) = PV$ (From Boyle's Law, PV = constant)

$\therefore PV - P\Delta V + V\Delta P - \Delta P\Delta V = PV$

Now ΔPΔV is very small compared to the other terms. Hence, neglecting ΔPΔV and taking ΔP as the subject to formula,

$$\Delta P = \frac{P\Delta V}{V} = \frac{PAx}{V} \quad (\because \Delta V = Ax)$$

Restoring force acting on the block in a direction opposite to the displacement due to this excess pressure,

$F = A\Delta P.$

Substituting ΔP from eqn. (1) into eqn. (2),

$$F = \left(\frac{PA^2}{V}\right)x = kx$$

$$\text{where } k = \frac{PA^2}{V} = \text{constant}$$

This force is opposite to the displacement and is directly proportional to it. Hence, the block performs SHM.

$$\text{Now, period, } T = 2\pi\sqrt{\frac{m}{k}}$$

$$\therefore T = 2\pi\left(\frac{mV}{PA^2}\right)^{\frac{1}{2}}$$

Illustration 9 : A tunnel is dug in the earth as shown in the figure and a body is allowed to fall freely into it. Prove that the body performs SHM. Assume the earth to be a sphere of uniform density ρ. What is the periodic time of the SHO ?

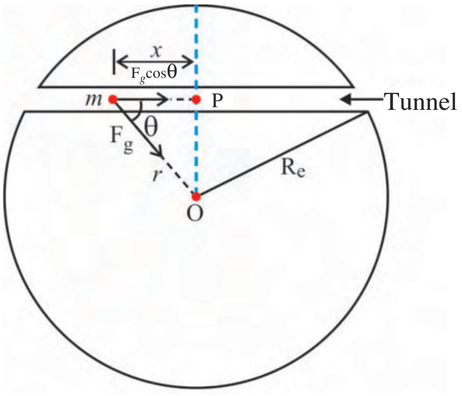


Figure 7.9

Solution : Suppose the body of mass m is at a distance r from the centre O of the earth, as shown in the Figure 7.9. In this position, it will experience a gravitational force F_g due to the mass of spherical part of the earth of radius r. The cosine component of F_g is responsible for the motion of the body in the tunnel.

$$\therefore \ F = F_g\cos\theta$$

$$= \frac{Gm\left(\frac{4}{3}\pi r^3\rho\right)}{r^2}\cos\theta \qquad (1)$$

When the body is at a distance r from the centre of the earth, suppose its distance is x from the midpoint P of the tunnel.

$$\therefore \ \cos\theta = \frac{x}{r} \qquad (2)$$

From equations (1) and (2)

$$F = \left(\frac{4}{3}\pi G\rho m\right)x$$

$$\Rightarrow F \propto x \text{ and } k = \frac{4}{3}\pi G\rho m$$

Also, this force is acting towards the mid-point P.

∴ Hence the body performs SHM.

$$\text{Now period, } T = 2\pi\sqrt{\frac{m}{k}}$$

$$T = 2\pi\sqrt{\frac{m\times 3}{4\pi G\rho m}}$$

$$T = 2\pi\sqrt{\frac{3}{4\pi G\rho}}$$

7.7 Total Mechanical Energy in Simple Harmonic Oscillator

A particle executing SHM possesses two types of energy :

(i) kinetic energy (KE) due to the velocity of the particle and

(ii) potential energy (PE) due to the position of the particle.

Dear students, you are knowing that the kinetic energy of the particle is

$$K = \frac{1}{2}mv^2$$

Using equation,

$$v = \pm\omega\sqrt{A^2 - y^2} \text{ we get,}$$

$$K = \frac{1}{2}m\omega^2(A^2 - y^2) \qquad (7.7.1)$$

If the displacement of the particle is $y = A\sin(\omega t + \phi)$ then $v = A\omega\cos(\omega t + \phi)$

$$\therefore \ K = \frac{1}{2}m\omega^2A^2\cos^2(\omega t + \phi) \qquad (7.7.2)$$

In the present case, the force on the oscillator is $F = -ky$ (which is called restoring force). In such a case the potential energy is given by (as you studied in Semester-I).

$$U = \frac{1}{2}ky^2 \qquad (7.7.3)$$

∴ The potential energy for a particle executing SHM is

$$U = \frac{1}{2}kA^2\sin^2(\omega t + \phi) \qquad (7.7.4)$$

Now, the total mechanical energy of an oscillator

$$E = K + U$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}ky^2$$

$$= \frac{1}{2}m\omega^2(A^2 - y^2) + .\frac{1}{2}m\omega^2y^2$$

$$(\because \ k = m\omega^2)$$

$$E = \frac{1}{2}m\omega^2A^2 \qquad (7.7.5)$$

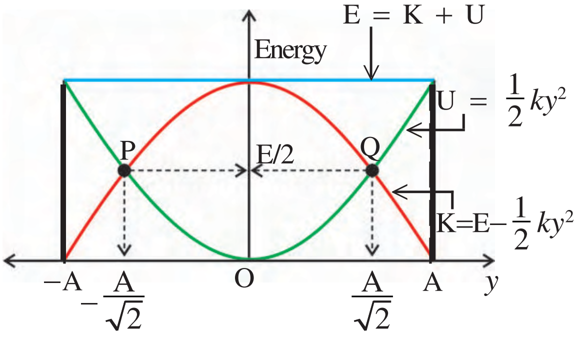
or

$$E = \frac{1}{2}kA^2 \qquad (7.7.6)$$

These equations (7.7.5) and (7.7.6) suggest that the total mechanical energy of a linear simple harmonic oscillator is constant and independent of time t as well as displacement y. But $E \propto A^2$.

Figure 7.10 shows the graphs of kinetic energy, potential energy and mechanical energy of an SHO as a function of displacement.

Use equations (7.7.1), (7.7.3) and (7.7.6).



Energies of SHO against displacement

Figure 7.10

Following points are to be noted from the Figure 7.10 :

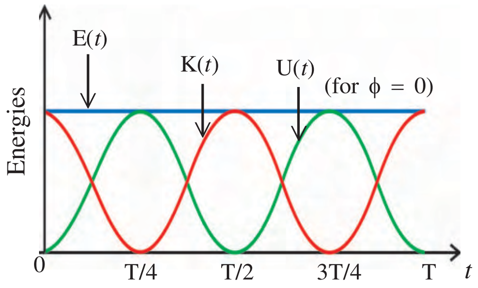
(i) At mean position $y = 0$, the potential energy is minimum (U = 0) and the kinetic energy is maximum ($K = \frac{1}{2}kA^2 = E$).

(ii) At $y = \underline{+}A$ (end points of path), the potential energy is maximum ($U = \frac{1}{2}kA^2 = E$) and the kinetic energy is minimum (K = 0).

(iii) The points P and Q at which the graphs of U and K intersect, the value of $U = K = \frac{1}{2}E$.

(iv) The co-ordinates of P and Q are $(\mp \frac{A}{\sqrt{2}}, \frac{E}{2})$.

Figure 7.11 shows the graph of kinetic energy, potential energy and mechanical energy of SHO as a function of time. (Use equations (7.7.2), (7.7.4) and (7.7.6))



Energies of SHO as a function of time

Figure 7.11

It is seen from the graphs 7.11 that K and U completes two vibrations in time during which SHO completes one oscillation. Thus the frequency of kinetic energy and potential energy is double than that of SHM.

Illustration 10 : A body of mass 10 kg has a velocity of 6 ms^{-1}, after one second of starting from the mean position. If the time period of SHO is 6 s, find the kinetic energy, potential energy and total mechanical energy of SHO.

Solution :

Here, $m = 10$ kg, $\phi = 0$,

$v = 6\ ms^{-1}$,

$T = 6$ s

$$\text{Now } K = \frac{1}{2}mv^2 = \frac{1}{2} \times 10 \times 36 = 180 \text{ J}$$

$$v = \omega A \cos \omega t = \omega A \cos\left(\frac{2\pi}{T} \cdot t\right)$$

$$6 = A\omega \cos\left(\frac{2\pi}{6} \times 1\right)$$

$$= A\omega/2$$

$$\therefore A\omega = 12$$

$$\text{Now } E = \frac{1}{2}mA^2\omega^2$$

$$= \frac{1}{2} \times 10 \times 144$$

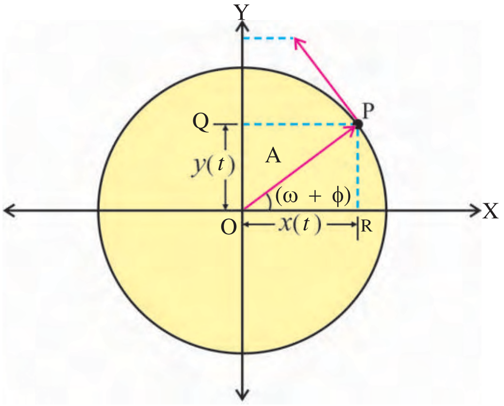
$$E = 720 \text{ J}$$

$$\text{As } U = E - K = 720 - 180$$

$$\therefore U = 540 \text{ J}$$

7.8 Simple Harmonic Motion and Uniform Circular Motion

Consider a particle P moving with a constant angular speed ω in an anti–clockwise direction on a circular path having centre O and radius A (See Figure 7.12). Here, particle is referred as **reference particle** and the circle as **reference circle.**



Uniform circular motion

Figure 7.12

At time t, the angular position of the particle is given by $\omega t + \phi$, with ϕ be the initial phase relative to the reference line OX. Now the projection of P on Y-axis is Q, which gives projection of position vector OP = OQ = $y(t)$ at time t.

From the geometry of the Figure 7.12,

$$\sin(\omega t + \phi) = \frac{OQ}{OP}$$

$$\therefore\ y(t) = A \sin(\omega t + \phi) \qquad (7.8.1)$$

This equation (7.8.1) represents the displacement of particle executing SHM along Y-axis.

If the projection of OP is taken on the X-axis as OR then

$$\cos(\omega t + \phi) = \frac{OR}{OP}$$

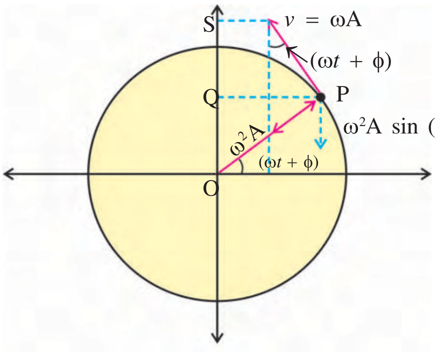
$$x(t) = A \cos(\omega t + \phi) \qquad (7.8.2)$$

This equation (7.8.2) represents the displacement of a particle executing SHM along X-axis.

Thus we conclude that :

Simple harmonic motion is the projection of uniform circular motion on a diameter of the reference circle.

Now the magnitude of velocity $\vec{v}$ of the reference particle P moving on the circle of radius A with angular speed ω is $v = \omega A$. The projection of v on the Y-axis at time t is shown in Figure 7.13.



Velocity and acceleration of uniform circular motion

Figure 7.13

From the geometry of figure 7.13,

$$\cos(\omega t + \phi) = \frac{SQ}{\omega A}$$

$$\therefore\ v(t) = \omega A \cos(\omega t + \phi) \qquad (7.8.3)$$

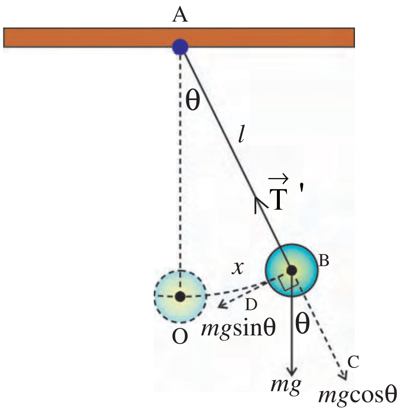
When the oscillator is moving in the positive y-direction, v is positive and when the oscillator is moving along the negative y-direction, v is negative.

Similarly the component of centripetal acceleration $\omega^2 A$ of the reference particle in the y-direction is $\omega^2 A \sin(\omega t + \phi)$.

7.9 Simple Pendulum

A system of small massive body suspended by a light inextensible and twistless string from a fixed (rigid) support is called a simple pendulum.

Consider Figure 7.14. Entire mass of the simple pendulum is supposed to be concentrated at the centre of mass of the suspended bob. The distance from the point of suspension to the centre of mass of the bob is called (effective) length (l) of the simple pendulum.



Simple pendulum

Figure 7.14

When the bob of the pendulum is displaced to a point B, through a small angle θ from its equilibrium position O and then released, it performs oscillations in a vertical plane. The forces acting on the bob of mass are as follows.

(1) Weight of the bob (= mg) acting in the downward direction and

(2) Tension in the string $\vec{T}'$ acting in the direction $\overrightarrow{BA}$.

The component of force mg are

(i) mg $\cos\theta$ acting along $\overrightarrow{BC}$, and

(ii) mg $\sin\theta$ acting along $\overrightarrow{BD}$

As thread remains taut,

$$T' = mg \cos\theta. \qquad (7.9.1)$$

The second component of the force, mg $\sin\theta$

brings the bob back to its equilibrium position O. Hence, this is the restoring force acting on the bob.

$$F = -mg\ \sin\theta \qquad (7.9.2)$$

If the angular displacement of the bob is smaller than

$$F = -mg\theta \qquad (\text{as } \theta \to 0,\ \sin\theta \approx \theta\ )$$

$$= -mg\frac{\text{arc OB}}{l}$$

$$= -mg\frac{x}{l} \qquad (\because\ \text{arc OB} \cong x)$$

$$\therefore F = -\left(\frac{mg}{l}\right)x \qquad (7.9.3)$$

Since m, g and l are constants,

$$\therefore\ F = -kx$$

$$\text{with } k = \frac{mg}{l} \qquad (7.9.4)$$

Equation (7.9.4) gives the force constant for simple pendulum.

Now the periodic time of simple pendulum,

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{mg\,/\,l}}$$

$$\therefore\ T = 2\pi\sqrt{\frac{l}{g}} \qquad (7.9.5)$$

The frequency of oscillation

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{g}{l}} \qquad (7.9.6)$$

and the angular frequency

$$\omega = 2\pi f = \sqrt{\frac{g}{l}} \qquad (7.9.7)$$

Dear students, remember that for small angle θ, the periodic time of the simple pendulum is

(i) independent of the mass of the bob,

(ii) independent of the amplitude of the oscillation,

(iii) depends on the length of the pendulum, $T \propto \sqrt{l}$, and

(iv) depends on the acceleration due to gravity, $T \propto \frac{1}{\sqrt{g}}$.

From equation (7.9.5) following types of graphs can be plotted (Figure 7.15).

Dear students, following points are also to be noted :

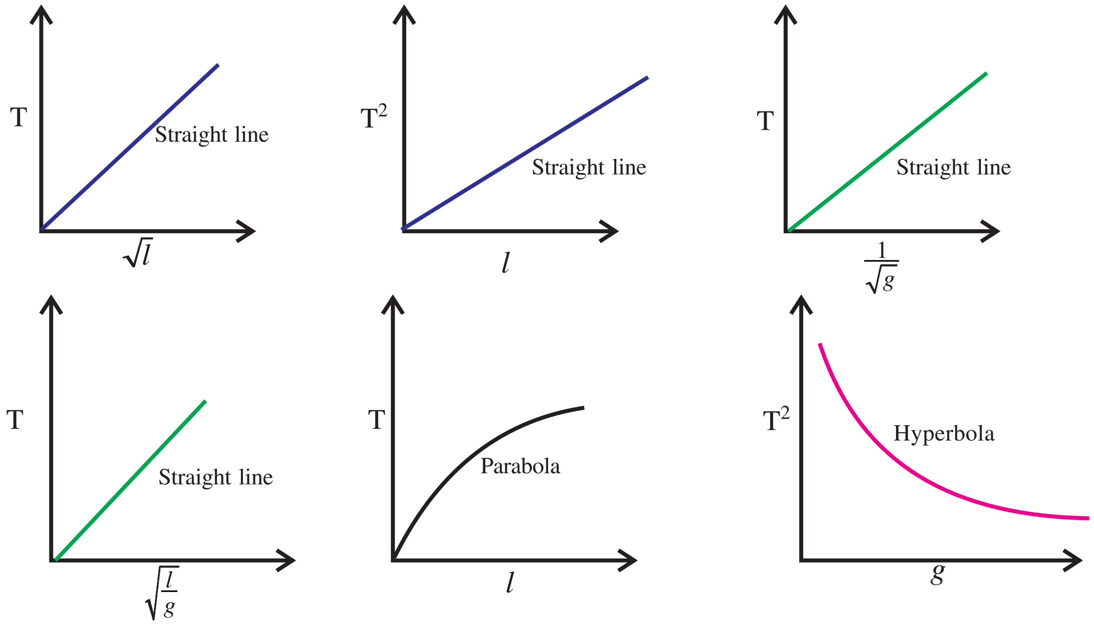
(i) $T \propto \sqrt{l}$ does not mean that as $l \to \infty$, $T \to \infty$.

This relation is not valid for $l \geq$ Radius of the earth.

(ii) Instead of cotton thread, if the bob is suspended by a metallic wire then the length of pendulum will increase with the increase in temperature and vice versa.

It means that the periodic time of the pendulum increases or decreases depending on the increase or decrease of the temperature. This is the reason that pendulum clock slows down in summer and moves faster in the winter.

(iii) The value of g is less at mountains or in mines than that of surface of earth. Therefore, the periodic time of simple pendulum increase at mountains or inside the mines, in principle



For simple pendulum $T - \sqrt{l}$, $T^2 - l$, $T - \frac{1}{\sqrt{g}}$, $T - \sqrt{\frac{l}{g}}$, $T - l$ and $T^2 - g$ graphs.

Figure 7.15

(A) Simple pendulum in a lift :

If the simple pendulum is oscillating in a lift moving with acceleration a, then the effective g of the pendulum is

$$g_{eff} = g \pm a$$

+ sign is taken when the lift is moving upward and

− sign is taken when the lift is moving downward.

Hence, the periodic time for simple pendulum in a lift is

$$\mathrm{T} = 2\pi\sqrt{\frac{l}{g \pm a}}.$$

Now, suppose that the lift is falling freely,

$$\therefore \; a = g$$

$$\text{and } \mathrm{T} = 2\pi\sqrt{\frac{l}{g - g}} = \infty.$$

That is pendulum does not oscillate.

(B) Simple pendulum in the compartment of a train :

If the simple pendulum is oscillating in a compartment of a train accelerating or retarding horizontally at the rate '*a*' then the effective value of *g* is

$$g_{eff} = \sqrt{g^2 + a^2}$$

$$\therefore \; \mathrm{T} = 2\pi\sqrt{\frac{l}{(g^2 + a^2)^{\frac{1}{2}}}}$$

(C) Second's Pendulum :

The pendulum having the time-period of two seconds, is called the second pendulum. It takes one second to go from one end to the other end during oscillation. It also crosses the mean position at every one second.

Illustration 10 : What will be the time period of second's pendulum if its length is doubled ?

Solution :

We know for second's pendulum

$$\mathrm{T} = 2\pi\sqrt{\frac{l}{g}} = 2 \text{ s}$$

$$\therefore \; \mathrm{T}' = 2\pi\sqrt{\frac{2l}{g}}$$

$$= \sqrt{2} \times 2\pi\sqrt{\frac{l}{g}}$$

$$= \sqrt{2} \times 2$$

$$\mathrm{T}' = 2.828 \text{ s}$$

Illustration 11 : Length of a second's pendulum on the surface of the earth is l_1 and l_2 at a height '*h*' from the surface of the earth. Prove that the radius of the earth is given by $\mathrm{R}_e = \dfrac{h\sqrt{l_2}}{\sqrt{l_1} - \sqrt{l_2}}$.

Solution :

Period of the second's pendulum is 2 seconds.

For the second's pendulum on the surface of the earth, $2 = 2\pi\sqrt{\dfrac{l_1}{g_1}}$,

where g_1 = acceleration due to gravity on earth's surface.

For the second's pendulum at a height '*h*' from the earth's surface,

$$2 = 2\pi\sqrt{\frac{l_2}{g_2}},$$

where g_2 = acceleration due to gravity at a height *h* from earth's surface

$$\therefore \; \frac{l_1}{g_1} = \frac{l_2}{g_2} \Rightarrow \frac{g_2}{g_1} = \frac{l_2}{l_1} \qquad (1)$$

But acceleration due to gravity, $g = \dfrac{\mathrm{GM}_e}{r^2}$ (A)

where *r* = distance of the point from the centre of the earth.

Now, $r_1 = \mathrm{R}_e$ = Radius of the earth

$$r_2 = \mathrm{R}_e + h$$

From equations (A)

$$\frac{g_2}{g_1} = \frac{\mathrm{R}_e^{\;2}}{(\mathrm{R}_e + h)^2} \qquad (2)$$

From equations (1) and (2),

$$\sqrt{\frac{l_2}{l_1}} = \frac{\mathrm{R}_e}{\mathrm{R}_e + h}$$

$$\sqrt{l_2}\,\mathrm{R}_e + \sqrt{l_2}\,h = \sqrt{l_1}\,\mathrm{R}_e$$

$$(\sqrt{l_1} - \sqrt{l_2})\mathrm{R}_e = \sqrt{l_2}\,h$$

$$\therefore \; \mathrm{R}_e = \frac{h\sqrt{l_2}}{\sqrt{l_1} - \sqrt{l_2}}$$

7.10 Damped Simple Harmonic Motion

The simple harmonic motion represents a highly idealized situation. A mechanical system execute SHM only when no resistive or frictional forces is acting on it.

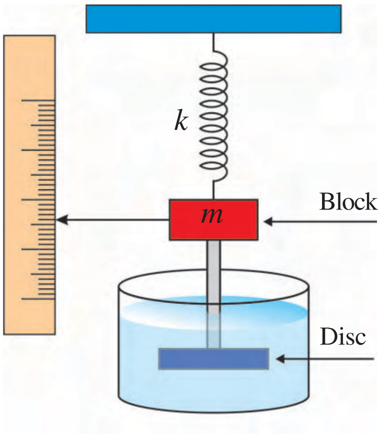
In practice, the oscillations of any mechanical system take place in some medium which offers resistance. Also there may be internal frictional forces in the mechanical system. Because of this the mechanical energy of the oscillating system is dissipated in the form of heat energy as it has to do work against resistive force.

Equation $E = \frac{1}{2}kA^2$ for mechanical energy of SHO suggest that as mechanical energy decreases, the amplitude of oscillator also decreases. Hence eventually motion dies out.

Thus, **when a simple harmonic system oscillates with a decreasing amplitude with time, its oscillations are called damped oscillations.**

A simple pendulum experiences resistive force of air, while oscillating in air. When a tuning fork vibrates, the internal frictional forces are acting in its metal.

As shown in the Figure 7.16, a block of mass m oscillates vertically on a spring having spring constant *k*. Attach a disc to a rod from the lower end of the block and submerged into the liquid, in the vessel. As the disc moves up and down, the liquid exerts an inhibiting drag force (resistive force) on the entire oscillating system. Because of this the mechanical energy of the oscillating system decreases.



A damped simple harmonic oscillator

Figure 7.16

Experimental studies have shown that the resistive force acting on the oscillator in a fluid medium depends upon the velocity of the oscillator.

Thus the resistive force or damping force acting on the oscillator (when the velocity is not too large) is

$$F_d \propto v$$

$$\therefore \; F_d = -bv \qquad (7.10.1)$$

Here, b is **damping constant** and has the SI unit of kg / second. The negative sign indicates that the force F_d opposes the motion (velocity).

Thus, a damped oscillator will oscillate under the influence of the following forces :

(i) restoring force $F_y = -ky$ and

(ii) resistive force $F_d = -bv$

$\therefore$ The net force $F = F_y + F_d$

According to Newton's Second Law of Motion,

$$ma = -ky - bv$$

$$m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dt}$$

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0 \qquad (7.10.2)$$

This is the second order differential equation for damped harmonic oscillation and the solution of this equation is

$$y(t) = A\, e^{-bt/2m} \sin(\omega' t + \phi) \qquad (7.10.3)$$

OR $y(t) = A(t) \sin(\omega' t + \phi)$ (7.10.4)

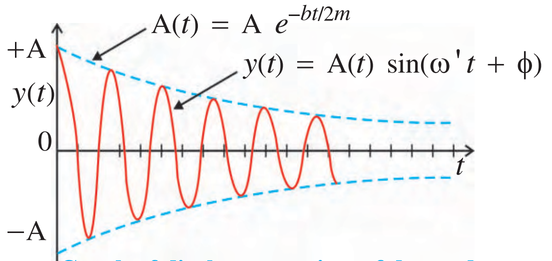
Here, $A(t) = A\, e^{-bt/2m}$ is the amplitude of the damped oscillation at time t and it decreases exponentially with time.

The angular frequency ω' of the damped oscillator is given by

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \qquad (7.10.5)$$

If $b = 0$, $\omega' = \sqrt{\frac{k}{m}}$ represents ideal SHM.

The graph of $y(t) - t$ for damped oscillator is shown in Figure 7.17.



Graph of displacement-time of damped oscillation (For $\phi = \frac{\pi}{2}$)

Figure 7.17

We know that the mechanical energy of the oscillator

$$E = \frac{1}{2}kA^2$$

$$\therefore\ E(t) = \frac{1}{2}kA^2(t)$$

$$E(t) = \frac{1}{2}kA^2\ e^{-bt/m} \qquad (7.10.6)$$

It is also clear from equation (7.10.6) that the mechanical energy of damped oscillation decreases exponentially with time.

The equation (7.10.6) is valid only for small damping, $b << \sqrt{km}$.

Illustration 12 : A simple pendulum is made by suspending a small sphere of brass at the end of a string. When it is oscillated in air its period is T. Now this sphere is immersed in a liquid of density 1/2 times that of brass and oscillated. Prove that its new period is $\sqrt{2}$ T. Neglect all resistive damping force.

Solution :

When the sphere is immersed in liquid, the buoyant force acting on it in upward direction = m_0g, where m_0 is the mass of liquid displaced by the sphere.

If weight of the sphere in air is mg,

its effective weight in liquid = $mg - m_0g$

Here, $m_0 = V\rho_0 = \frac{V\rho}{2} = \frac{m}{2}$;

where V = volume of the sphere = volume of liquid displaced by the sphere and ρ_0 = density of liquid, ρ = density of brass.

$$\therefore \text{ Effective weight in liquid} = mg - \frac{mg}{2}$$

$$= \frac{1}{2}mg$$

∴ Effective acceleration due to gravity in liquid

$$= g' = \frac{1}{2}g$$

Now, $T \propto \sqrt{\frac{1}{g}}$ from $T = 2\pi\sqrt{\frac{l}{g}}$

$$\therefore\ \frac{T'}{T} = \sqrt{\frac{g}{g'}} = \sqrt{\frac{2g}{g}}$$

$$\therefore\ T' = \sqrt{2}\,T$$

Illustration 13 : Calculate the time during which the amplitude becomes $\frac{A}{2^n}$ in case of damped oscillations, where A = initial amplitude .

Solution : $A(t) = Ae^{-bt/2m}$

But, $A(t) = \frac{A}{2^n}$

$$\therefore\ \frac{A}{2^n} = Ae^{-bt/2m}$$

∴ Taking log with base e on both the sides,

$$\therefore\ \frac{bt}{2m} = n\ ln2$$

(natural log is written as ln)

$$\therefore\ t = \frac{2mn}{b}(2.303)\log_{10}(2)$$

$$(\because\ ln\ x = 2.303\ \log_{10}x)$$

$$= \frac{2mn}{b}(2.303)(0.3010)$$

$$\therefore\ t = \frac{2mn}{b}(0.693)$$

7.11 Natural Oscillations, Forced Oscillations and Resonance

When a system capable of oscillating is given some initial displacement from its equilibrium position and left free it begins to oscillate. Thus the oscillations performed by it in absense of any type of resistive forces are known as **natural (free) oscillations**. The frequency of natural oscillations is known its **natural frequency**, f_o.

E.g. when a bob of the simple pendulum is slightly displaced and then released, it performs its natural oscillations with natural frequency $f_o = \frac{1}{2\pi}\sqrt{g/l}$. (Here, we have neglected the resistive force of air).

Dear students, you must have enjoyed swinging in a swing. You must have also experienced that if you want to swing continuously you have to apply push by pressing your feet against the ground, repeatedly or someone has to push you repeatedly (Figure 7.18). Hence under the influence of external periodic force swinging is continued.

A child enjoying swing

Figure 7.18

In most of the cases, the damping forces are present and eventually oscillations die out with time. Therefore to sustain the oscillations external periodic forces are required.

Thus oscillations of the **system under the influence of an external periodic force are forced oscillations.**

Consider an external periodic force $F = F_0 \sin\omega t$ acting on the system which is capable to oscillate.

Hence, equation (7.10.2) is written as

$$m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dt} + F_0 \sin\omega t$$

$$\therefore \frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{ky}{m} = \frac{F_0}{m}\sin\omega t \qquad (7.11.1)$$

This is second order differential equation of forced oscillation. The solution of equation (7.11.1) is given by

$y = A \sin(\omega t + \phi)$

Here, A and ϕ are the constants of the solution.

They are found as,

$$A = \frac{F_0}{[m^2(\omega_0{}^2 - \omega^2)^2 + b^2\omega^2]^{\frac{1}{2}}} \qquad (7.11.2)$$

and $$\phi = \tan^{-1}\frac{\omega y_0}{v_0} \qquad (7.11.3)$$

Here, m is the mass of the oscillator, v_0 and y_0 are the velocity and displacement respectively of the oscillator when periodic force is applied.

The oscillator initially oscillates with its natural frequency. When we apply external periodic force, the oscillations with natural frequency die out, and then body oscillates with the frequency of the external periodic force.

From equation (7.11.2) it is seen that the amplitude of forced oscillations depends inversely on (*i*) the difference $(\omega_0{}^2 - \omega^2)$ and (*ii*) damping coefficient b.

For small damping factor, $b\omega << m(\omega_0{}^2 - \omega^2)$ hence equation (7.11.2) is written as

$$A = \frac{F_0}{m(\omega_0{}^2 - \omega^2)} \qquad (7.11.4)$$

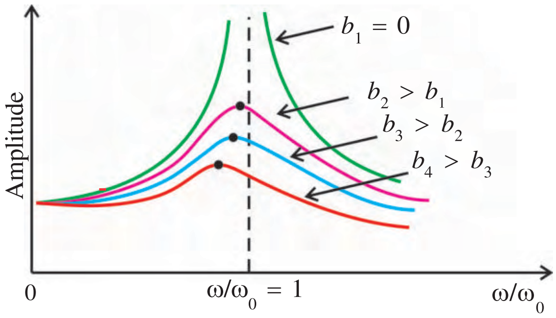
For $\omega \approx \omega_0$

$m(\omega_0{}^2 - \omega^2) << b\omega$, hence

$$A = \frac{F_0}{b\omega} \qquad (7.11.5)$$

As the value of ω approches to ω_0, the amplitude increases and for some characteristic values of ω the amplitude becomes maximum. This phenomenon is known as **resonance.** The value of ω for which resonance occures is known as the **resonant frequency.**

The graph of amplitude $-\omega/\omega_0$ corresponding to different values of damping factor b are shown in Figure 7.19.



Resonance curves

Figure 7.19

For $b = 0$, the amplitude becomes infinite when $\omega = \omega_0$.

As damping increases, peak value in the curve shifts towards left.

Mechanical systems found in practice have got one or more natural frequencies of their oscillations. If the frequency of the external periodic force acting on a system become equal (or nearely equal) to the natural frequency of that system, the system oscillates with a very large amplitude and the system may break or collapse.

Hence, soldiers marching on a suspended bridge are advised to go out of steps. When a bridge is designed, care is taken so that the frequency of the external force due to the gusts of wind and natural frequencies of that bridge do not become equal or nearly equal. It is seen that sometimes, in an earthquake, low and high rise structures remain less affected while medium high structures fall down. This is because the natural frequencies of low rise structures are to be higher and those of high structures are to be lower than the frequency of the seismic waves.

SUMMARY

1. If a body repeats its motion along a certain path, about a fixed point, at a definite interval of time, it is said to have periodic motion.
2. If a body moves to and fro, back and forth, or up and down about a fixed point in a fixed interval of time, such a motion is called an oscillatory motion.
3. When a body moves to and fro repeatedly about an equilibrium position under a restoring force, which is always directed towards equilibrium position and whose magnitude at any instant is directly proportional to the displacement of the body from the equilibrium position of that instant then such a motion is known as simple harmonic motion.
4. The maximum displacement of the oscillator on either side of mean position is called amplitude of the oscillator.
5. The time taken by the oscillator to complete one oscillation is known as periodic time or time period or period (T) of the oscillator.
6. The number of oscillation completed by the simple harmonic oscillator in one second is known as its frequency (f).
7. 2π times the frequency of oscillator is the angular frequency ω of that oscillator.
8. $\mathrm{T} = \frac{1}{f} = \frac{2\pi}{\omega}$ or $f = \frac{1}{\mathrm{T}}$ or $\omega = \frac{2\pi}{\mathrm{T}}$
9. For simple harmonic motion, the displacement $y(t)$ of a particle from its equilibrium position is represented by sine, cosine or its linear combination like

   $y(t) = \mathrm{A}\ \sin(\omega t + \phi)$

   $y(t) = \mathrm{B}\ \cos(\omega t + \phi)$

   $y(t) = \mathrm{A}'\ \sin\omega t + \mathrm{B}'\ \cos\omega t.$

   where $\mathrm{A}' = \mathrm{A}\cos\phi$ and $\mathrm{B}' = \mathrm{B}\sin\phi$
10. The velocity of SHO is given by $v = \pm\ \omega\ \sqrt{\mathrm{A}^2 - y^2}$
11. The acceleration of SHO is given by $a = -\omega^2 y$

12. A particle of mass m oscillating under the influence of Hook's Law exhibits simple harmonic motion with

$$\omega = \sqrt{\frac{k}{m}};$$

$$T = 2\pi\sqrt{\frac{m}{k}}$$

13. Differential equation for SHM is $\frac{d^2y}{dt^2} + \omega^2 y = 0$.

14. For series combination of n spring of spring constants k_1, k_2, k_3, ..., k_n, the equivalent spring constant is

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots + \frac{1}{k_n} \text{ and the periodic time } T = 2\pi\sqrt{\frac{m}{k}}$$

15. For parallel combination of n springs of spring constants k_1, k_2, k_3, ..., k_n, the equivalent spring constant is

$$k = k_1 + k_2 + k_3 + \ldots\ldots + k_n \text{ and period } T = 2\pi\sqrt{\frac{m}{k}}$$

16. The kinetic energy of the SHO is $K = \frac{1}{2}m\omega^2 (A^2 - y^2)$

17. The potential energy of the SHO is $U = \frac{1}{2}ky^2$

18. The total mechanical energy of SHO is $E = K + U = \frac{1}{2}m\omega^2 A^2 = \frac{1}{2}kA^2$

19. For SHO, at $y = 0$, the potential energy is minimum ($U = 0$) and the kinetic energy is maximum ($K = \frac{1}{2}kA^2 = E$)

20. For SHO, at $y = \pm A$, the potential energy is maximum ($U = \frac{1}{2}kA^2 = E$) and the kinetic energy is minimum ($K = 0$)

21. Simple harmonic motion is the projection of uniform circular motion on a diameter of the reference circle.

22. For simple pendulum, for small angular displacement

$$T = 2\pi\sqrt{\frac{l}{g}} \text{ and}$$

$$\omega = 2\pi f = \frac{2\pi}{T} = \sqrt{\frac{g}{l}}$$

23. For simple pendulum, T is independent of the mass of the bob as well as the amplitude of the oscillaions.

24. The differential equaiton for damped harmonic oscillation is

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

with the displacement

$$y(t) = Ae^{-bt/2m} \sin(\omega' t + \phi)$$

and angular frequency $\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$

25. $E(t) = \frac{1}{2}kA^2 e^{-bt/m}$ gives the mechanical energy of damped oscillation at time t.

26. A system oscillates under the influence of external periodic force are forced oscillations.

27. The differential equation for forced oscillations is

$$\frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{F_0}{m}\sin\omega t$$

28. The amplitude for forced oscillation is

$$A = \frac{F_0}{[m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2]^{\frac{1}{2}}}$$

EXERCISES

Choose the correct option from the given options :

1. In SHM, the acceleration of the particle is zero when its
(A) Velocity is zero.
(B) displacement is zero.
(C) both velocity and displacement are zero.
(D) both velocity and displacement are maximum.

2. The maximum acceleration of a body moving in SHM is a_{max} and maximum velocity is V_{max}. Then its amplitude is
(A) v^2_{max} / a_{max} (B) a^2_{max} / v_{max}
(C) $v^2_{max}\ a^2_{max}$ (D) v_{max} / a_{max}

3. Which of the following is an essential condition for the motion to be simple harmonic ?
(A) constant force.
(B) Force proportional to displacement.
(C) Force opposite to displacement.
(D) Force proportional and opposite to displacement.

4. The graph between time period T and length of a simple pendulum l is
(A) straight line. (B) ellipse.
(C) parabola. (D) hyperbola.

5. Periods of SHO are T and $\frac{5T}{4}$. They begin their motion simultaneously from the mean position. What is the difference between their phases when 1 oscillation of the oscillator having period T is completed.
(A) 45^o (B) 72^o (C) 90^o (D) 112^o

6. Period of an oscillator is T. In what time it completes $\frac{3}{8}^{th}$ of its oscillation, if motion begins from the fixed point.
(A) $\frac{3}{8}$T (B) $\frac{5}{8}$T (C) $\frac{5}{12}$T (D) $\frac{8}{3}$T

7. Velocity of a bob of a simple pendulum of length 0.5 m is 3 m/s when it passes from the equilibrium position. When the pendulum makes an angle of 60^o with the vertical, its velocity is .... (Take g = 10 m/s^2)
(A) $\frac{1}{3}$ m/s (B) $\frac{1}{2}$ m/s (C) 2 m/s (D) 3 m/s

8. A mass m is attached to two springs of same force constants as shown in Figure 7.20. What is $\frac{T_1}{T_2}$?

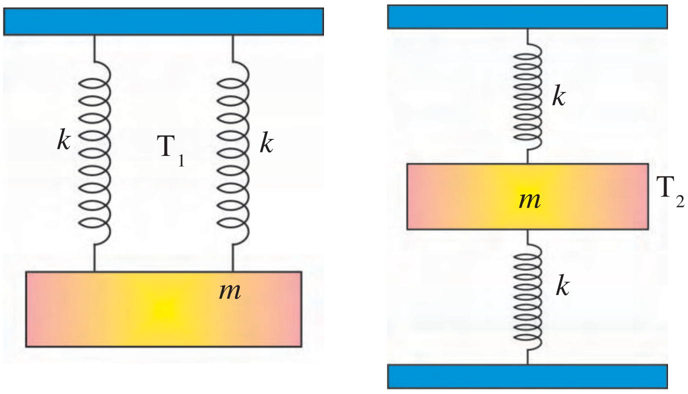


Figure 7.20

(A) 1 (B) 2 (C) 3 (D) 4

9. Three springs are connected to a mass m as shown in Figure 7.21. What is T ?

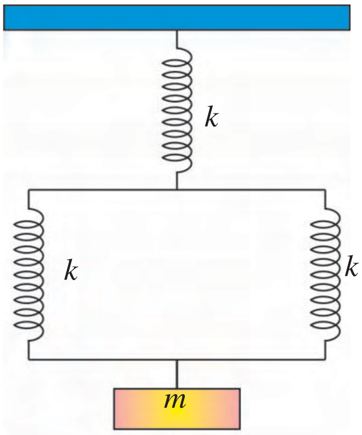


Figure 7.21

(A) $2\pi\sqrt{\frac{m}{k}}$

(B) $2\pi\sqrt{\frac{m}{3k}}$

(C) $2\pi\sqrt{\frac{3m}{2k}}$

(D) $2\pi\sqrt{\frac{2k}{3m}}$

10. If F is the restoring force in the spring and k is spring constant, then what is the mechanical energy stored in the spring when it is stretched by y on loading ?

(A) $\frac{F^2}{2y}$ (B) $\frac{F^2}{2k}$ (C) $\frac{2y}{F^2}$ (D) $\frac{2k}{F^2}$

11. The amplitude of an oscillator performing damped oscillation becomes 1/e times the initial amplitude in time ....

(A) $\frac{m}{2b}$ (B) $\frac{2m}{b}$ (C) $e^{-bt/2m}$ (D) $e^{2m/b}$

12. An SHO begins its motion from the lowest point on its path of oscillation. Its phase after 10 oscillations will be .... Motion is along Y–axis and positive X–axis is the reference line.

(A) $\frac{1}{2}\pi$ rad (B) 5π rad (C) 10π rad (D) $\frac{43}{2}\pi$ rad

13. Let $F = F_0 \sin \omega t$ is the external periodic force acting on the oscillator. If amplitude of the oscillator is maximum for $\omega = \omega_1$ and the energy is maximum for $\omega = \omega_2$, then (ω_0 is the natural angular frequency)

(A) $\omega_1 = \omega_0$ and $\omega_2 \neq \omega_0$ (B) $\omega_1 \neq \omega_0$ and $\omega_2 = \omega_0$

(C) $\omega_1 \neq \omega_0$ and $\omega_2 \neq \omega_0$ (D) $\omega_1 = \omega_0$ and $\omega_2 = \omega_0$

14. A mass of 1 kg attached to the bottom of a spring has certain frequency of vibration. What mass is to be added in order to reduce the frequency by half ?

(A) 1 kg (B) 2 kg (C) 3 kg (D) 4 kg

15. A pendulum suspended from the ceiling of a compartment of a train has periodic time 2 s. When the train is accelerating at 10 ms^{-2}. What will be its time period when the train retards at 10 m s^{-2}.

(A) 2 s (B) $\sqrt{2}$ s (C) $2\sqrt{2}$ s (D) $\frac{2}{\sqrt{2}}$ s

ANSWERS

1. (B)	**2.** (A)	**3.** (D)	**4.** (C)	**5.** (B)	**6.** (C)
7. (C)	**8.** (A)	**9.** (C)	**10.** (B)	**11.** (B)	**12.** (D)
13. (D)	**14.** (C)	**15.** (A)			

Answer the following questions in short :

1. What is the work done by simple pendulum in one complete oscillation ?

2. What is the periodic time of a pendulum in freely falling lift ?

3. Write equation for peiodic time of oscillation of the liquid in U-tube.

4. What is an epoch ? In which unit it is measured ?

5. Amplitude of an SHO is 4 cm. At what distance from the equilibrium position, the kinetic energy and potential energy becomes equal ?

6. What is the SI unit of force constant ?

7. Write the relation between the acceleration amplitude (a), the displacement amplitude (A) and the angular frequency (ω) for SHM.

8. Why does a simple pendulum eventually stop ?

9. Write expression of the mechanical energy of damped harmonic oscillation for $b << \sqrt{km}$.

10. Write general form of the second order differential equation for forced oscillation.

Answer the following questions :

1. Define periodic motion and oscillatory motion. Give proper examples of it.

2. Deduce an expression for the time period of a simple pendulum.

3. What are damped oscillations ? What are the factors affecting its motion ?

4. Deduce the relation for the total energy of damped harmonic oscillator.

5. Explain forced oscillations and resonance.

6. Show that for a particle in linear SHM the average KE over a period of oscillation equals the average PE over the same period.

7. Obtain the co-ordinates of the points where the KE and PE against displacement graphs intersect.

8. What is the nature of acceleration against displacement curve of SHM ? What is the slope of this curve ?

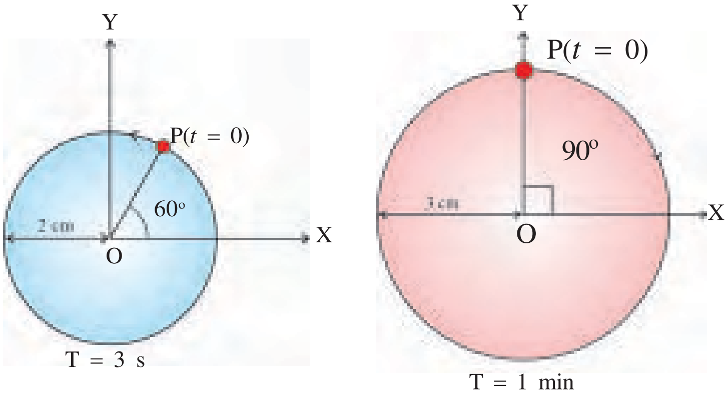
9. Periodic time of the particle excluding SHM is T = $2\pi\sqrt{\frac{m}{k}}$. Explain why then the periodic time of a simple pendulum is independent of a mass of the pendulum ?

10. What provides the restoring force for simple harmonic oscillator in the following cases ?

    (i) Simple pendulum (ii) Spring (iii) Column of mercury in U-tube.

Solve the following problems :

1. Obtain the equation for SHM of the Y-projection of the radius vector of the revolving particle P in case (*a*) and (*b*) of Figure 7.22.



(*a*) (*b*)

Figure 7.22

[**Ans.** : (*a*) $y = 2\sin\left(\frac{2\pi t}{3}+\frac{\pi}{3}\right)$ (*b*) $y = 3\cos\left(\frac{\pi}{30}t\right)$]

2. Three springs are connected to a mass $m = 80\ g$ as shown in Figure 7.23. What is the effective spring content and periodic time, if $k = 2\ \text{N m}^{-1}$?

   [**Ans.** : $k = 8\ \text{Nm}^{-1}$, T = 0.628 s]

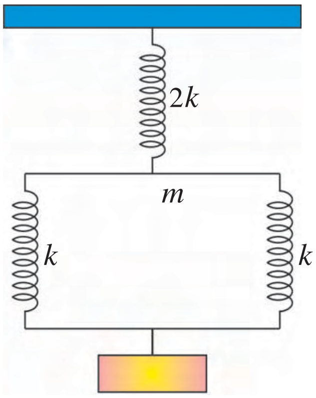


Figure 7.23

3. A spring of length l and force constant k is cut into two parts of length l_1 and l_2. Here $l_1 = nl_2$. Obtain force constants k_1 and k_2 respectively of these parts in terms of n and k. [**Ans.** : $k_1 = \left(1+\frac{1}{n}\right)k,\ k_2 = (n+1)k$]

4. An oscillator of mass 100 *g* is performing damped oscillations. Its amplitude becomes half of its initial amplitude after 100 oscillations. If its period is 2 s find the damping co-efficient. [**Ans. :** 0.693 *dyn* s cm^{-1}]

5. Amplitude of an SHO is A. When it is at a distance *y* from the mean position of the path of its oscillation, the SHO receives blow in the direction of its motion which doubles its velocity instantaneously. Find the new amplitude of its oscillations. [**Ans. :** $\sqrt{4A^2 - 3y^2}$]

6. For an SHM prove that $a^2T^2 + 4\pi^2v^2$ = constant, where *a* and *v* are acceleartion and velocity respectively at any instant. T is periodic time.

7. A simple pendulum has a length L and a bob of mass *m*. The bob is oscillating with amplitude A. Show that the maximum tension (T) in the string is (for small angular displacement). $T_{max} = mg\left[1 + \left(\frac{A}{L}\right)^2\right]$.

8. Two simple harmonic motions are represented by $y_1 = 10 \sin\frac{\pi}{4}(12t + 1)$ and $y_2 = 5(\sin 3\pi t + \sqrt{3}\cos 3\pi t)$. Find out the ratio of their amplitudes. What are the time period of two motions ? [**Ans. :** $\frac{A_1}{A_2} = 1$, $T_1 = T_2 = \frac{2}{3}$ s]

9. For a linear harmonic oscillator the force constant is 2×10^6 N/m and total mechanical energy is 160 J. At some instant of time, its displacement is 0.01 m. Find its potential energy and kinetic energy at this position.

[**Ans. :** 100 J, 60 J]

10. For a linear SHM, when the distance of the oscillator from the equilibrium postion has values y_1 and y_2, the velocities are v_1 and v_2. Show that the time period of oscillation is $T = 2\pi\left[\frac{y_2^2 - y_1^2}{v_1^2 - v_2^2}\right]^{\frac{1}{2}}$.

●

CHAPTER 8

WAVES

8.1 Introduction

Friends, earlier we have studied that the universe is made up of matter and radiation. This radiation propagates in the form of waves. Waves have basic importance in almost every branch of physics. Light and sound energy also propagate in the form of waves. Different types of radiant energy emitting from the sun also reaches us in the form of waves. Music produced from musical instruments also reaches us in the form of waves. Communication done through radio, television and mobile is due to the waves. In 20th century, concept of matter wave was introduced due to which importance of the waves also increase.

In this chapter we will learn about waves, types of waves, speed of waves in different medium, reflection of waves, superposition of waves, beats and Doppler effect.

8.2 Waves

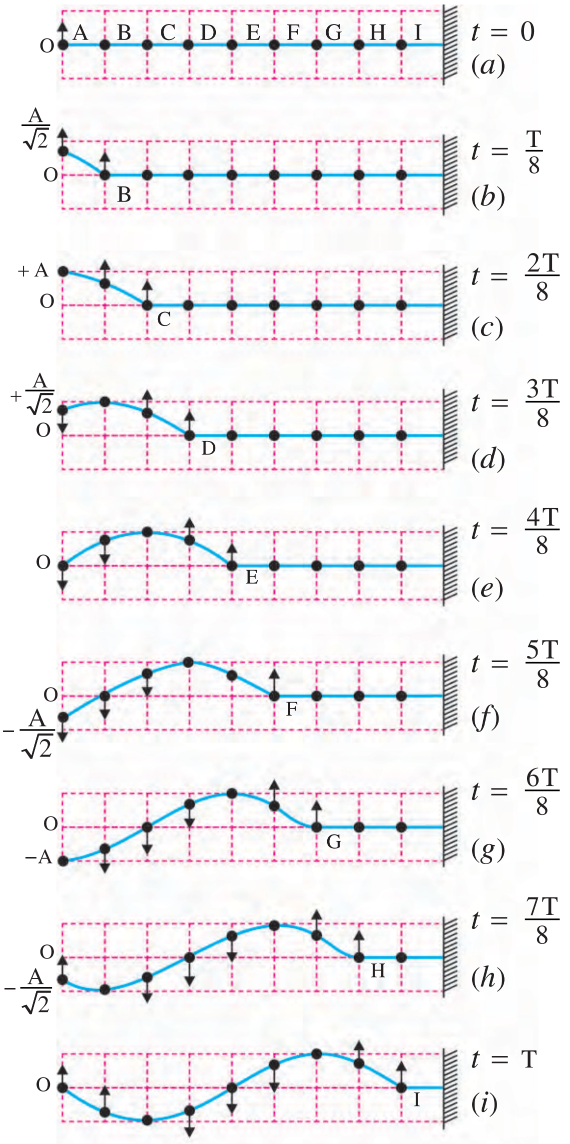
When a particle moves in space it carries the kinetic energy associated with it. There is another way to transport energy in which the particle oscillates near its position and yet the energy reaches too far from it. They transport their associated energy to the far distance without leaving their position. Sound is transmitted in air in this manner. When you say ‘Hello’ to your friend, no material particle is ejected from your lips and reaches to your friend’s ear. You create some disturbance in the air close to your lips which propagates as a wave and reaches to ear of your friend.

To understand clearly the concept of a wave, consider a long and elastic string with one end fixed to rigid support and other held by a person. The person pulls on the string keeping it tight. Here, string is a one dimensional elastic medium. As shown in Figure 8.1, suppose that A, B, C, ... I are the particles of a string. At time $t = 0$ all the particles of the medium are in the equilibrium state. (See Figure 8.1*a*)

(i) Suppose, at $t = 0$ a disturbance is produced by the person in the particle A so that it starts simple periodic motion according to $y = A \sin\omega t$. The period of this oscillation is T.

(ii) Because of elastic property of the medium, suppose the effect of disturbance produced at A is transmitted to particle B in time $\frac{T}{8}$. During this time $\frac{T}{8}$, the displacement of a particle A would be $y = A\sin\left(\frac{2\pi}{T}\right)\left(\frac{T}{8}\right) = \frac{A}{\sqrt{2}}$ and particle B is on the verge of starting its simple periodic oscillation, (See Figure 8.1 *b*)

(iii) Now, during an additional time period of $\frac{T}{8}$, that is at $\frac{T}{8} + \frac{T}{8} = \frac{T}{4}$, the disturbance produced at A reaches C and at that moment C is on the verge of starting its oscillation. During this time $\frac{T}{4}$, the displacement of particle A would be $y = A\sin\left(\frac{2\pi}{T}\right)\left(\frac{T}{4}\right) = A$ i.e. Displacement of A is equal to its amplitude and that of B is equal to $\frac{A}{\sqrt{2}}$. (See Figure 8.1(*c*)).



Wave generation on a string

Figure 8.1

(iv) Thus, due to the disturbance produced at A, the subsequent particles start oscillations and transmit the effect of their oscillation to the subsequent particles and the disturbance propagates in the medium.

(v) In this way the propagating disturbance reaches the particle D at $t = \frac{3T}{8}$, the particle E at $\frac{4T}{8}$ ...... and the particle I at time T. At time T one oscillation of A is completed and the particle I is just to start its oscillation.

This entire situation is shown in Figure 8.1. Remember that the particles of the medium were in the equilibrium position. At time $t = 0$, we produced a simple periodic disturbance at the particle A. This disturbance has travelled in the medium and reached the particle I at time $t = T$.

(vi) Here, disturbance considered is such that it produces a simple periodic motion in the particle A and hence the shape produced in the string is like a sine curve. If the displacement or oscillation of particle A had been of some other type, the shape formed in the string would also have been accordingly of the other type. **Thus, the shape formed in the string gives an idea of the type of disturbance.** For example, if the free end of the string snappad one, then the shape formed in string is shown in Figure 8.2, which is known as a pulse.

Shape formed on the string according to the type of disturbance

Figure 8.2

As the time elapses this disturbance (or shape in the string) passes over the particles J, K, L,..... etc. Here, the shape is that of the sine curve is lying between the particles A and I at time $t = T$. This shape proceeds further along the string and comes between particles I and Q at time $t = 2T$ as shown in Figure 8.3. During this time the particles between A and I stop oscillating and string comes back to its original position.

Shape of the string at t = 2T

Figure 8.3

Thus, by producing a disturbance at any particle in the string, a shape corresponding to the disturbance is produced and that shape (without alteration) moves along the string, which means that the disturbance propagates in the medium of the string. **The motion of the disturbance in the medium (or in free space) is called a wave disturbance or generally a wave.**

Remember that here the particles of the string A, B, C... are not moving as a 'single unit' in the medium but they only displace or oscillate about their equilibrium positions. Thus, wave is not physical 'body' travelling in the medium. As the effect of disturbance produced in any part of the medium is being experienced by the subsequent particles of the medium, the wave is said to propagate. **After the disturbance has passed through any particle, it comes back to its equilibrium position.**

When the engine of a railway train joins the coaches, in the beginning the first coach vibrates, then the second coach and then the third coach and so on. Thus, the effect of vibration moves from the first to the last coach. This phenomenon is the propagation of the wave in the medium 'made up of railway coaches.'

Wavetrain

In the above discussion, if the particle continues to oscillate in its simple harmonic motion, the second waveform generated after the first one follows it and so on. Thus, a series of waveform appears to move ahead as a continuous chain. Such a series of propagating waveform is called a wave train.

We discussed a situation in which particles participating in wavemotion are executing a simple harmonic oscillation, here the wave shape formed in medium is of the nature of a sine (or equivalent cosine) curve. Such a wave is called a **harmonic wave.**

If the waves are continuously moving ahead in the medium, they are called **travelling or progressive waves.**

8.3 Classification of Waves

(i) Mechanical waves : The waves which require elastic medium for their transmission are called mechanical waves. Such a wave propagates due to the elastic properties of the medium. For example, waves on a string, ripples on the water suface, sound waves and seismic waves. All these waves have the characteristics that they are governed by Newton's laws.

(ii) Electromagnetic waves : For the propagation of electromagnetic waves no material medium is essential. They can propagate in the vacuum also. In this types of waves the disturbance in the electric and magnetic fields that propagates. Here, instead of particles, the vectors of the electric and magnetic field intensities are oscillating.

Light waves, radio waves, microwave, X−ray etc. are the examples of the electromagnetic waves. (More information you will get in Std. 12)

(iii) Matter waves : Matter waves are associated with moving electrons, protons, neutrons and other fundamental particles and even atoms and molecules. These particles constituting matter, therefore, such waves are called matter waves. The concept of these types of wave you will learn in Std. 12. From the concept of these waves scientific instruments are developed in modern technology. The matter waves associated with electron are employed in the electron microscope.

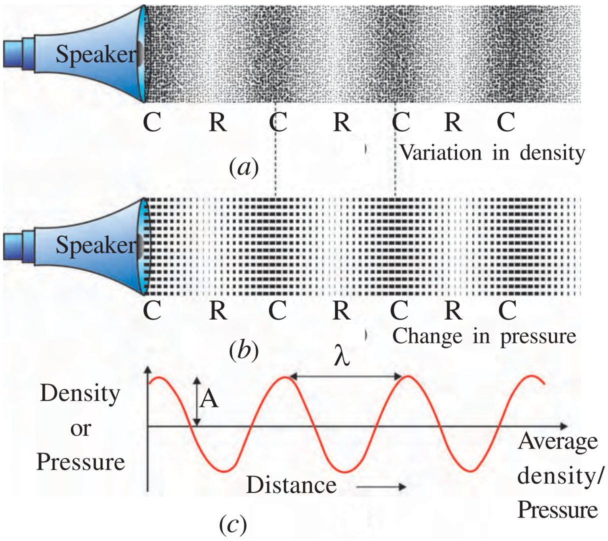
In this chapter we will study only the mechanical waves.

Transverse wave : Waves in which the oscillations of the particles are in a direction perpendicular to the direction of wave propagation is called the **transverse wave.** The waves along a string discussed in article 8.2 is an example of transverse wave. Electromagnetic waves (e.g. light waves) are also a transverse waves. In such waves the locations of the maximum displacement of the particle in one direction are called the **'crests'** and locations of maximum displacement in the opposite direction are called **'trough'**.

These waves propagate through a medium in the form of crests and troughs.

Longitudinal wave : Waves in which the oscillations of the particles of medium are along the direction of wave propagation are called **longitudinal waves.** Sound waves propagating in air are longitudinal. Such waves propagate through a medium by forming condensations and rarefaction in the medium. When waves propagate in medium, the particles at medium oscillate about their equilibrium position, in the direction of propagation of waves.

For simplicity, the positions of the particles of air at some instant of time in case of longitudinal waves is shown in Figure 8.4.



Longitudinal wave in air

Figure 8.4

When sound waves pass through that region of air, the air molecules in certain region are pushed very close to each other during their oscillations. Hence, both density and pressure of air increase in such regions. In such region **condensation** is said to be formed. In the regions between consecutive condensations, the air molecules are found to be quite separated. In such regions density and pressure of air decreases and here **rarefaction** is said to be formed. (See Figure 8.4)

Thus, during the propagation of sound the layers of medium perform oscillation about their mean positions and during this the condensations and rarefactions are alternatingly formed. As the effect of such oscillations reaches one layer after the other, the condensations and rarefactions propagate further and further in the medium. In this way the sound propagates in a medium. During the propagation of sound the pressure in different region of the medium changes with time and position. Hence, such waves are also called the **pressure waves.**

The direction of the oscillations of the particles of the medium is perpendicular to the direction of propagation of transverse waves in the medium. Hence during the propagation of the transverse waves every element of the medium experience shearing strain. But shearing stress is possible only in solid medium. So, the transverse waves can propagate in solid medium like string, wire, rod but they cannot propagate in a fluid medium.

During the propagation of the longitudinal waves, the oscillations of the particles of the medium are in the direction of propagation of the waves. Hence, compressive strain is produced during the propagation of these waves. Now, compressive stress is possible in solids, liquids and gases. So, the longitudinal waves can propagate in any medium.

Thus, in a solid medium both types of mechanical waves, transverse waves and longitudinal waves can propagate whereas in a fluid medium only longitudinal waves can propagate.

[During an earthquake two types of waves, transverse and longitudinal are produced on the earth. They are known as S-wave (secondary wave) and P-wave (primary wave) respectively. Longitudinal wave (P-wave) is similar to sound waves produced in the earth's interior. The speed of P-wave is approximately 4 – 8 km/s and that of S-wave is approximately 2 – 5 km/s. In an S-wave, particles in the earth's interior vibarate at right angles to the direction of the wave propagation. By measuring the time interval between the first arrivals of P-wave and S-wave, the origin of earthquake (epicentre) can be determined.

8.4 Amplitude of A Wave, Propagation of Energy in a Wave, Wavelength And Frequency

Amplitude of a wave :

Amplitude of wave is the amplitude of

oscillation of particle of the medium. As shown in Figure 8.5 amplitude of the wave is A.

Propagation of energy in a wave :

A particle has to be displaced from its mean position in order to produce a wave. Hence some work has to be done on the particle. This energy imparted to the particle will be in the form of kinetic and potential energy of its oscillations. As the successive particles experience the disturbance, this energy is communicated to them. Thus, energy is propagated in a wave. If the medium has some internal friction, energy is dissipated in the form of heat and hence, the wave weakens on propagation.

Energy passing through a unit area taken in the direction normal to the propagation of the wave in one second is called intensity of wave.

$$\text{Wave Intensity (I)} = \frac{\text{Energy / Time}}{\text{Area}}$$

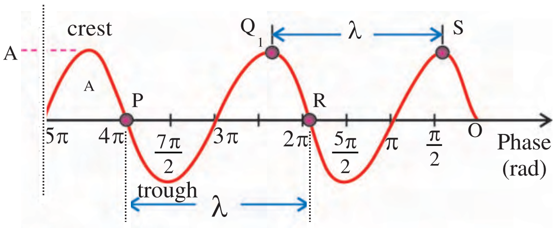
SI unit of intensity of wave is $\frac{\text{J/s}}{\text{m}^2}$ or $\frac{\text{W}}{\text{m}^2}$.

Its dimensional formula is $M^1L^0T^{-3}$.

Energy of an oscillatory particle is $E = \frac{1}{2}kA^2$, hence the intensity of wave is directly proportional to the square of its amplitude. $(I \propto A^2)$.

Wavelength :

The linear distance between any two points or particles having phase difference of 2π rad is called the wavelength (λ) of the wave. Its SI unit is m.



Amplitude and wavelength of a wave

Figure 8.5

As shown in Figure 8.5 the phase difference of oscillation between particles P and R is $4\pi - 2\pi = 2\pi$ rad. Hence, the distance between P and R represent the wavelength (λ) of a wave. From the figure it is clear that phase difference between consecutive crests or consecutive trough is 2π rad. Therefore, the distance between consecutive crests/trough is also a wavelength of a wave. Same way, in case of the sound waves the distance between consecutive condensations or consecutive rarefactions also represents the wavelength.

Wave number and wave vector :

Number of waves per unit distance is called wave number $\left(\frac{1}{\lambda}\right)$. The SI unit of wave number is m^{-1}.

In the wave propagation the particles at a distance of λ has the phase difference of 2π rad. Hence, the particle at a unit distance has phase difference of $\frac{2\pi}{\lambda}$. $\frac{2\pi}{\lambda}$ is known as wave vector or angular wave number or propagation constant (k).

$$k = \frac{2\pi}{\lambda}$$

The SI unit of k is rad/m. Its dimensional formula is $M^0L^{-1}T^0$. Wave vector is in the direction of wave propagation.

Frequency of a wave :

The number of oscillations performed by the particle of medium in one second is known as the frequency of oscillation of particle. Frequency (f) of the wave is just the frequency of oscillation of the particles of the medium. The number of the wave passing through point in one second is called frequency of the wave.

Its SI unit is s^{-1} or Hz (Hertz).

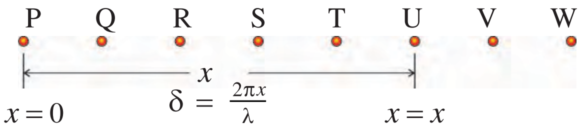
$\omega = 2\pi f$ is called angular frequenecy of the wave. $T = \frac{1}{f}$ is the periodic time of the wave.

8.5 Wave Equation

A complete description for a wave propagation can be obtained if we know displacements of all the particles of medium participating in the wave motion at any time. For this purpose we shall derive the wave equation for wave in one dimension, which gives displacement of a particle having coordinate x at time t. From such an equation, substituting the appropriate values for x and t, we get the displacement for any particle at a required time and thus obtain a description of the wave motion.

Such an equation is called **wave equation.** (Here, we shall discuss wave equation only for one dimension).

Here we shall obtain an equation for travelling wave or progressive harmonic waves. To obtain the wave equation of a wave propagating in positive x direction, consider particles of a medium as shown in Figure 8.6.



Wave equation

Figure 8.6

Suppose, at $t = 0$, simple harmonic oscillations of the particle P start with zero intial phase i.e. wave originates at P at time $t = 0$. The x-coordinate of the particle P is zero as well as initial phase is also zero ($\phi = 0$). The equation for the displacement of this particle would be,

$$y = A \sin\omega t \quad (8.5.1)$$

Now when the wave originating at P travels through a distance x, the medium particle (U) lying at a distance x from P starts its simple harmonic motion and the phase of its oscillation would be less than that of P. Let the phase of this particle (U) be δ less than that of P. Hence the equation for the displacement of particle at distance x from P, would be,

$$y = A \sin(\omega t - \delta) \quad (8.5.2)$$

Let the wavelength of wave be λ. We know that the phase of the particle at a distance λ from P, is less than that of P by 2π. Hence, the phase of the particle at a distance x from P would be less than that of P by $\frac{2\pi x}{\lambda}$.

$$\therefore \delta = \frac{2\pi x}{\lambda} \quad (8.5.3)$$

Substituting δ in equation (8.5.2)

$$y = A \sin\left(\omega t - \frac{2\pi x}{\lambda}\right)$$

But $\frac{2\pi}{\lambda} = k$

$$\therefore y = A \sin(\omega t - kx) \quad (8.5.4)$$

Here, $(\omega t - kx)$ is known as the phase of the wave at distance x from the origin at time t. The direction of wave vector k is taken along the direction of propagation of the wave.

Equation (8.5.4) is the wave equation for the progressive harmonic wave travelling in the direction of the increasing value of x. **If the wave is travelling in the direction of decreasing value of x then $\omega t - kx$ is replaced by $\omega t + kx$.**

$$y = A \sin(\omega t + kx) \quad (8.5.5)$$

substituting $\omega = \frac{2\pi}{T}$ and $k = \frac{2\pi}{\lambda}$ in equation (8.5.4)

$$y = A \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) \quad (8.5.6)$$

If the velocity of wave is v, then substituting $\lambda = vT$ in above equation

$$y = A \sin 2\pi\left(\frac{t}{T} - \frac{x}{vT}\right)$$

$$y = A \sin 2\pi f\left(t - \frac{x}{v}\right) \quad (\because \frac{1}{T} = f) \quad (8.5.7)$$

Now,

$$y = A \sin 2\pi \frac{f}{v}(vt - x)$$

$$\therefore y = A \sin \frac{2\pi}{\lambda}(vt - x) \quad (\because v = f\lambda) \quad (8.5.8)$$

The above equations (8.5.6), (8.5.7) and (8.5.8) are the different forms of wave equation for the progressive harmonic wave.

If particle P is oscillating with initial phase ϕ, then the wave equation (8.5.4) will be as follows :

$$y = A \sin(\omega t - kx + \phi) \quad (8.5.9)$$

8.6 Wave speed and phase speed

Wave travells a distance λ in periodic time T.

$$\therefore \text{ wave speed } v = \frac{\text{Distance}}{\text{Time}} = \frac{\lambda}{T}$$

But $\frac{1}{T} = f$

$$\therefore v = f\lambda \quad (8.6.1)$$

$$= \frac{\lambda(2\pi f)}{2\pi}$$

But, $2\pi f = \omega$ and $\frac{2\pi}{\lambda} = k$

$$\therefore v = \frac{\omega}{k} \quad (8.6.2)$$

So far in the discussion of wave motion we have seen that amplitude, period of oscillation and frequency of oscillation (f) of the particles of medium are the amplitude, periodic time and frequency of the wave respectively.

But the velocity of oscillating particle and velocity of wave are not the same.

Note that, **the frequency of the wave is the property of the source of the wave while the wavelength is the property of the medium in which the wave propagates.**

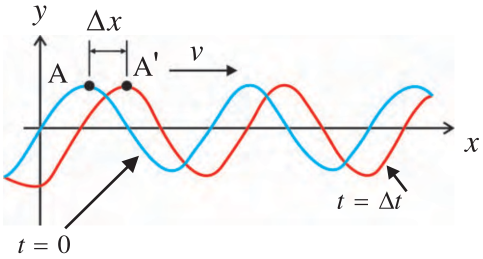
In the different mediums the wave speed is different. Therefore, the wavelength of the wave is also different in different type of medium. But in a given medium the wave speed is constant.

Phase Speed

As shown in Figure 8.7 the wave is travelling in the direction of increasing value of x. The entire wave pattern is moving a distance Δx in that direction during the interval Δt. As the wave moves, each point of the moving wave form (such as point A) retains its displacement. (Remember that points on the string do not retain their displacement but point on the wave forms do). For each point on the wave pattern phase must be constant. In Figure 8.7 phase at point A and A$'$ is same.

$$\therefore \ \omega t - kx = \text{constant} \qquad (8.6.3)$$

Here, both x and t are changing. As t increases, x must also increase to keep the $\omega t - kx$ constant. This confirms that the wave pattern is moving towards increasing x.



Wave motion

Figure 8.7

Differentiating above equation with respect to t.

$$\frac{d}{dt}(\omega t - kx) = 0$$

$$\therefore \ \omega - k\frac{dx}{dt} = 0$$

$$\therefore \ \frac{dx}{dt} = v = \frac{\omega}{k} \qquad (8.6.4)$$

Here, v is the phase speed of the wave.

Above equation (8.6.4) is similar to the equation (8.6.2). So, the wave speed which we find is the phase-speed of the waves in reality.

Illustration 1 : The frequency of the radio-waves broadcast by Ahmedabad Vividhbharati is 96.7 MHz. Find the wavelength, wave vector and angular frequency of these waves. Speed of radio waves in air is 3×10^8 m/s.

Solution :

$f = 96.7$ MHz $= 96.7 \times 10^6$ Hz

$v = 3 \times 10^8$ m/s

Wave speed, $v = f\lambda$

$$\therefore \ \lambda = \frac{v}{f} = \frac{3 \times 10^8}{96.7 \times 10^6} = 3.102 \text{ m}$$

$$\text{Wave vector, } k = \frac{2\pi}{\lambda} = \frac{2 \times 3.14}{3.102} = 2.024 \text{ rad/m}$$

Angular frequency, $\omega = 2\pi f$

$= 2 \times 3.14 \times 96.7 \times 10^6$

$= 6.07 \times 10^8$ rad/s

Illustration 2 : The wave equation of a propagating wave is given by $y = 0.5\sin(x - 60t)$ cm. Find, (*i*) amplitude of the wave (*ii*) wave vector (*iii*) wavelength (*iv*) angular frequency and frequency of wave (*v*) periodic time and (*vi*) wave speed.

Solution : Compare the equation

$y = 0.5 \ \sin(x - 60t) = -0.5\sin(\ 60t - x\ )$

with wave equation

$y = \text{A} \sin(\omega t - kx)$

(i) Amplitude of a wave A $= -0.5$ cm

(ii) wave vector $k = 1$ rad/cm

$$\text{(iii) wavelength } \lambda = \frac{2\pi}{k} = \frac{2 \times 3.14}{1} = 6.28 \text{ cm}$$

(iv) Angular frequency of a wave $\omega = 60$ rad/s

Now, from $\omega = 2\pi f$, the frequency of wave,

$$f = \frac{\omega}{2\pi} = \frac{60}{2 \times 3.14} = 9.55 \text{ Hz}$$

(v) Periodic time $T = \frac{1}{f} = \frac{1}{9.55} = 0.105$ s

(vi) Wave speed $v = \frac{\omega}{k} = \frac{60}{1} = 60$ cm/s

Illustration 3 : How far does sound travel in air when a tuning fork of frequency 250 Hz completes 50 vibrations ? The speed of sound in air is 340 m/s.

Solution : Wavelength of the wave produced from tuning fork $\lambda = \frac{v}{f} = \frac{340}{250} = 1.36$ m.

One wavelength is the distance travelled by the wave in one complete vibration of tuning fork.

$\therefore$ Distance travelled by the sound in 50 vibrations.

$= 50 \times \lambda$

$= 50 \times 1.36 = 68$ m

Illustration 4 : A stone dropped from the top of a tower of height 100 m high splashed into the water of a pond near the tower. When is the splash heard at the top ? The speed of sound in air is 340 m/s. At what time the splash is heared at the top, after it is dropped ?

Solution : Suppose t_1 is the time taken by the stone to reach the surface of water and t_2 is the time taken by the splash to reach from water surface to the top. The splash will be heard at the top of the tower after time $t = t_1 + t_2$.

Now, time t_1 taken by stone to reach water surface can be determine as follows :

$$s = v_0 t_1 + \frac{1}{2} g\, t_1^2$$

$s = 100$ m, $v_0 = 0$, $g = 9.8$ m/s^2

$$\therefore\ 100 = 0 + \frac{1}{2}(9.8)t_1^2$$

$$\therefore\ t_1 = 4.52 \text{ s}$$

Now, time t_2 taken by the splash to reach water surface to the top is,

$$t_2 = \frac{\text{Distance}}{\text{Sound speed}} = \frac{100}{340} = 0.29 \text{ s}$$

$$\therefore\ t = t_1 + t_2 = 4.52 + 0.29 = 4.81 \text{ s}$$

Illustration 5 : Equation of a one dimensional propagating wave is,

$$y = 5\sin 30\pi\left(t - \frac{x}{240}\right).$$

Here, y is in metre and t is in second.

(i) Is the particle of medium moving in positive Y or negative Y direction at the origin at time $t = 0$? i.e. what will be produced first-crest or trough ?

(ii) Find the displacement, velocity of the particle and the slope of the wave at 480 m away from the origin at time $t = 2$ s.

(iii) Find the speed of wave.

Solution :

(i) At $x = 0$, starting from $t = 0$, if y increases in the negative direction then the trough will be produced and if y increases in the positive direction then the crest will be produced.

Here, at $x = 0$, $y = 5\ \sin 30\pi t$. Hence, starting from $t = 0$, here y increases in the positive direction. Hence, first a crest will be produced at the origin.

(ii) Displacement at $t = 2$ s for a particle at $x = 480$ m

$$y = 5\sin 30\pi\left(2 - \frac{480}{240}\right)$$

$$= 5\sin 30\pi(0) = 0 \text{ m}$$

Velocity of the particle,

$$v = \frac{dy}{dt} = 150\pi\cos 30\pi\left(t - \frac{x}{240}\right)$$

$$= 150\pi\cos 30\pi\left(2 - \frac{480}{240}\right)$$

$$= 150\pi \text{ m/s}$$

Slope of the wave,

$$\frac{dy}{dx} = -\frac{5\pi}{8}\cos 30\pi\left(t - \frac{x}{240}\right)$$

$$= -\frac{5\pi}{8}\cos 30\pi\left(2 - \frac{480}{240}\right)$$

$$= -\frac{5\pi}{8}$$

(iii) Compare the given equation with,

$$y = A\ \sin 2\pi f\left(t - \frac{x}{v}\right)$$

$\therefore$ Wave speed $v = 240$ m/s

Here, note that the wave speed and the magnitude of velocity of the particle taking part in the wave propagation are not equal.

8.7 Wave Speed in Medium

8.7.1 Speed of Transverse Wave on Stretched String :

Earlier we have seen that the particles of the string come back to their original position after the disturbance (or wave) has passed through those particles. In order that particles come back to their original positions, restoring force and hence elasticity in medium are essential. Moreover, the inertia of the medium plays a role in deciding the displacement of the oscillatory particles. **Thus, the elasticity and intertia of medium are necessary for the propagation of the mechanical waves.** From these two properties of medium, the speed of wave in a meduim is determined.

It is found that the speed of transverse wave in a medium like a string kept under the tension, depends on (i) linear mass density (μ) and (ii) tension T in the string.

Here, we will obtain the speed of wave on a string using dimensioned analysis.

Linear density of a string means mass per unit length (μ) of the string.

Dimensional formula of

$$\mu = \frac{[\text{Total mass of string}]}{[\text{Total length of string}]} = \frac{M^1}{L^1}$$

$$= M^1L^{-1}T^0$$

Dimension of Tension $T = M^1L^1T^{-2}$

Suppose, wave speed

$$v = k\ \mu^a\ T^b \qquad (8.7.1)$$

Here, k = dimensionless constant and $[a, b] \in R$.

Substituting dimensions on both the sides,

$$M^0L^1T^{-1} = [M^1L^{-1}T^0]^a\ [M^1L^1T^{-2}]^b$$

$$= M^{a+b}\ L^{-a+b}\ T^{-2b}$$

Comparing dimensions of both the sides, $a + b = 0$, $-a + b = 1$ and $-2b = -1$

$$\therefore\ a = -\frac{1}{2} \text{ and } b = \frac{1}{2}$$

Substituting value of a and b in equations (8.7.1)

$$v = k\ \mu^{-\frac{1}{2}}T^{\frac{1}{2}}$$

From the experimental and other studies, $k = 1$

$$\therefore\ v = \sqrt{\frac{T}{\mu}} \qquad (8.7.2)$$

Above equation shows that wave speed is independent of frequency of a wave and amplitude of a wave.

Illustration 6 : A long wire PQR is made by joining two wires PQ and QR of equal radii. The wire PQ has length 4.8 m and mass. 0.06 kg. The wire QR has length 2.56 m and mass 0.2 kg. The wire PQR is under the tension of 80 N. Find the time taken by a wave produced at the end P to reach the other end R.

Solution :

Mass per unit length for the wire PQ,

$$\mu_1 = \frac{0.06}{4.8} = \frac{1}{80}\ \frac{kg}{m}$$

Mass per unit length for the wire QR,

$$\mu_2 = \frac{0.2}{2.56} = \frac{10}{128}\ \frac{kg}{m}$$

∴ Speed of wave in the wire PQ,

$$v_1 = \sqrt{\frac{T}{\mu_1}} = \sqrt{\frac{80}{\frac{1}{80}}} = 80 \text{ m/s}$$

∴ Speed of wave in the wire QR,

$$v_2 = \sqrt{\frac{T}{\mu_2}} = \sqrt{\frac{80}{\frac{1}{128}}} = 32 \text{ m/s}$$

∴ Time taken by the wave to reach R from P, $t = t_1 + t_2$

$$= \frac{PQ}{v_1} + \frac{QR}{v_2}$$

$$= \frac{4.8}{80} + \frac{2.56}{32}$$

$$= 0.14 \text{ s}$$

Illustration 7 : A uniform rope of length 12 m and mass 6 kg hangs vertically from a rigid support. A block of mass 2 kg is attached to the free end of the rope. A transverse pulse of wavelength 0.06 m is produced at the lower end of the rope. What is the wavelength of the pulse when it reaches the top of the rope ?

Solution :

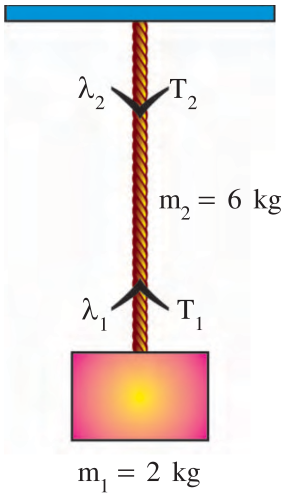


Figure 8.8

As the rope is heavy, its tension will be different at different points.

mass of rope $m_2 = 6$ kg

mass a block $m_1 = 2$ kg

Tension at the lower end of rope,

$T_1 = m_1 g = 2g$

Tension of the upper end of rope,

$T_2 = (m_1 + m_2)g$

$= (6 + 2)g = 8g$

$$\text{wave speed in a string } v = \sqrt{\frac{T}{\mu}}$$

$$\therefore f\lambda = \sqrt{\frac{T}{\mu}}$$

$(\because v = f\lambda)$

The frequency of the wave pulse will be the same everywhere on the rope and μ is also the same throughout the rope as it is uniform. Therefore,

$\lambda \propto \sqrt{T}$

Wavelength of wave at lower end of rope,

$\lambda_1 \propto \sqrt{T_1}$

Wavelength of wave at upper end of rope,

$\lambda_2 \propto \sqrt{T_2}$

$$\therefore \frac{\lambda_2}{\lambda_1} = \sqrt{\frac{T_2}{T_1}}$$

$$\text{and } \lambda_2 = \lambda_1\sqrt{\frac{T_2}{T_1}}$$

$$= (0.06)\sqrt{\frac{8g}{2g}}$$

$$= 0.12 \text{ m}$$

Illustration 8 : The speed of transverse wave going on a wire having length 50 cm and mass 5.0 g is 80 m/s. The area of cross-section of the wire is 1.0 mm^2 and its Young's modulus is 16×10^{11} N/m^2. Find the extension of the wire over its natural length.

Solution :

Length of the wire $L = 50$ cm $= 50 \times 10^{-2}$ m

mass of wire $m = 5$g $= 5 \times 10^{-3}$ kg

cross sectional area of wire

$A = 1\text{mm}^2 = 1 \times 10^{-6}$ m^2

Young's modulus of wire $Y = 16 \times 10^{11}$ N/m^2

wave speed in a wire $v = 80$ m/s.

mass per unit length of wire,

$$\mu = \frac{m}{L} = \frac{5 \times 10^{-3}}{50 \times 10^{-2}} = 1 \times 10^{-2} \text{ kg/m}$$

$$\text{The wave speed in wire } v = \sqrt{\frac{T}{\mu}}$$

$\therefore$ Tension in wire $= T = F = \mu v^2$

$= (1 \times 10^{-2})(80)^2$

$= 64$ N

$$\text{Now, Young's modulus } Y = \frac{F/A}{\Delta L/L}$$

$\therefore$ Extension in the length of wire,

$$\Delta L = \frac{FL}{AY}$$

$$= \frac{(64)(50 \times 10^{-2})}{(1 \times 10^{-6})(16 \times 10^{11})}$$

$$= 0.02 \text{ mm}$$

8.7.2 Speed of sound waves (longitudinal wave) in a medium :

It is found that the speed of longitudinal waves like sound waves in a medium depends on (*i*) the elastic constant E and (*ii*) density ρ of the medium.

Using these facts, we can obtain the speed of the longitudinal waves using dimensional analysis as follows.

Wave speed $v = kE^a\rho^b$

Here, k is dimensionless constant and $[a, b] \in R$.

Now, $[E] = M^1L^{-1}T^{-2}$, $[\rho] = M^1L^{-3}T^0$

Writing dimensional formula on both the sides,

$$M^0L^1T^{-1} = [M^1L^{-1}T^{-2}]^a\ [M^1L^{-3}T^0]^b$$

$$= M^{a+b}\ L^{-a-3b}\ T^{-2a}$$

Comparing dimensions on both the sides,

$a + b = 0$, $-a - 3b = 1$ and $-2a = -1$

$$\therefore\ a = \frac{1}{2} \text{ and } b = -\frac{1}{2}$$

$$\therefore\ v = kE^{\frac{1}{2}}\rho^{-\frac{1}{2}}$$

From the experimental and other studies $k = 1$,

$$\therefore\ v = \sqrt{\frac{E}{\rho}} \qquad (8.7.3)$$

The propagation of longitudinal waves in fluid is in the form of condensations and rarefactions. In such a situation due to the variation in pressure of different regions of the medium bulk modulus (B) is taken as the elastic constant.

$$\therefore\ v = \sqrt{\frac{B}{\rho}} \qquad (8.7.4)$$

During the propagation of longitudinal waves in a linear medium like a rod, the longitudinal strain is produced. Hence in such a situation Young's modulus is taken as the elastic constant.

$$\therefore\ v = \sqrt{\frac{Y}{\rho}} \qquad (8.7.5)$$

Table 8.1 gives the speed of sound in various media.

Table 8.1 Speed of sound in some media (Only For Information)

Medium	Speed (m/s)
Gases	
Air (0°C)	331
Air (20°C)	343
Helium	965
Hydrogen	1284
Liquids	
Water (0°C)	1402
Water (20°C)	1482
Seawater	1522
Solids	
Aluminium	6420
Copper	3560
Steel	5941
Rubber	54

It is clear from the Table (8.1) that although the densities of liquids and solids are much greater than those of the gases, the speed of sound in them is higher. It is because liquids and solids are less compressible than gases. i.e have much greater bulk modulus.

Newton's Formula :

Newton assumed that the process of propagation of sound in gas (or air) is isothermal. Hence, the isothermal bulk modulus is to be used in the equation (8.7.4).

For an isothermal process PV = constant

(Taking T = constant, PV = μRT = constant)

Differentiating with respect to V,

$$P\frac{dV}{dV} + V\frac{dP}{dV} = 0$$

$$\therefore\ P = -V\frac{dP}{dV} = -\frac{dP}{dV/V} = \text{Bulk modulus B}$$

Thus, isothermal bulk modulus B = Pressure P.

$$(\because\ B = -\frac{dP}{dV/V})$$

$$\therefore\ \text{Wave speed } v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{P}{\rho}} \qquad (8.7.6)$$

This formula is called Newton's formula for the speed of sound in air.

Illustration 9 : Obtain the speed of sound in air at STP using Newton's formula. Mass of 1 mole of air = 29.0×10^{-3} kg.

$P = 1.01 \times 10^5\ P_a)$

Solution : Volume of 1 mole of air at STP = 22.4 L = 22.4×10^{-3} m^3

$$\text{Density of air at STP } \rho = \frac{\text{Mass}}{\text{Volume}}$$

$$\therefore\ \rho = \frac{29.0 \times 10^{-3}}{22.4 \times 10^{-3}} = \frac{29.0}{22.4}$$

$\therefore$ Speed of sound in air at STP,

$$v = \sqrt{\frac{P}{\rho}}$$

$$= \sqrt{\frac{1.01 \times 10^5 \times 22.4}{29.0}} = 279.3 \text{ m/s}$$

Laplace's Correction :

The speed of sound according to Newton's formula is 279.3 m/s while its experimental value is 332 m/s at STP. This suggests that there is some defect in the formula (8.7.6)

Laplace suggested that the temperature of the region where condensation is formed increases and that of the region of rarefaction decreases. Hence, the process of propagation of sound in a medium cannot be considered isothermal.

The process of formation of condensation and rarefaction in the medium is so quick that the heat produced during the condensation, is absorbed at the same place during rarefaction before being dissipated outside. Relatively small thermal conductivity of gas also helps in not allowing the heat to be dissipated outside. Thus, **the process of sound propagation in the gas is adiabatic and not isothermal.** Hence, adibatic bulk modulus of the gas should be used in place of isothermal bulk modulus.

For an adiabatic process of an ideal gas,

PV^{γ} = constant

Where γ is the ratio of two specific heats C_P and C_V.

Differentiating the equation with respect to V.

$$P \cdot \gamma V^{\gamma-1} + V^{\gamma}\frac{dP}{dV} = 0$$

$$\therefore \gamma P + V\frac{dP}{dV} = 0$$

$$\therefore \frac{-dP}{dV/V} = \gamma P$$

$$\therefore B = \gamma P$$

Thus, for an adiabatic process bulk modulus $B = \gamma$ P.

Using this value of B in equation (8.7.4)

$$\text{wavespeed } v = \sqrt{\frac{\gamma P}{\rho}} \qquad (8.7.7)$$

For air γ is 1.41. Speed of sound comes out 331.6 m/s at STP on taking this value of v. This agrees very well with the experimental value. To determine speed of wave in ideal gas Laplace equation (8.7.7) should use instead of Newton's formula.

Various factors affecting speed of sound waves : The equation of state for 1 mole of ideal gas is.

$$PV = RT \quad (\mu = 1 \text{ mol})$$

$$\therefore P = \frac{RT}{V}$$

Substituting value of P in $v = \sqrt{\frac{\gamma P}{\rho}}$

$$\therefore v = \sqrt{\frac{\gamma RT}{V\rho}}$$

But, ρV = mass of one mole gas = molecular mass M of gas

$$\therefore \text{Speed } v = \sqrt{\frac{\gamma RT}{M}} \qquad (8.7.9)$$

From this expression it is clear that the speed of sound in a gas is directly proportional to the square root of its absolute temperature (T).

i.e. $v \;\alpha\; \sqrt{T}$

If pressure (P) of the gas is changed keeping its temperature constant, $\frac{P}{\rho}$ remains constant as the density ρ of the gas directly varies as the pressure P. Therefore, the **speed of sound in a gas does not depend on the pressure of the gas** at constant temperature and constant humidity.

Density of water vapour is less than the density of dry air at same pressure. Hence, the **speed of sound increases with increasing humidity** as per $v = \sqrt{\frac{\gamma P}{\rho}}$.

Illustration 10 : Show that the velocity of sound in a gas at temperature t is given by,

$$v_t = v_0\left(1 + \frac{t}{546}\right)$$

Where, v_0 is speed of sound in air at 0° C ($t << 273$)

Solution : The speed of the wave in gas is

$$v = \sqrt{\frac{\gamma RT}{M}}$$

i. e. $v \propto \sqrt{T}$

If, v_t = speed of sound in gas at t° C

v_0 = speed of sound in gas at 0° C

$$\therefore \frac{v_t}{v_0} = \sqrt{\frac{273+t}{273}}$$

$$(\because \text{T(K)} = t(^\circ\text{C}) + 273)$$

$$\therefore v_t = v_0\left(1+\frac{t}{273}\right)^{\frac{1}{2}}$$

Using binomial expansion and neglecting higher order terms,

$$v_t = v_0\left(1+\frac{1}{2}\times\frac{t}{273}\right)$$

$$v_t = v_0\left(1+\frac{t}{546}\right)$$

[**Note :** If the speed of sound in air at 0° C is 332 m/s, than speed at 1° C will be,

$$v_t = v_0\left(1+\frac{t}{546}\right)$$

$$= 332\left(1+\frac{1}{546}\right) = 332.61 \text{ m/s}$$

Thus, the velocity of sound in air increases by 332.61 − 332 = 0.61 m/s for every 1° C rise in temperature.]

Illustration 11 : If the velocity of sound in air at 27° C and 76 cm of mercury is 345 m/s. Find the velocity at 127° C and 75 cm of mercury.

Solution : Remember that there is no effect of change of pressure on the velocity of sound.

If v_1 and v_2 be the velocities of sound at 27° C and 127° C, then we have

$$\frac{v_2}{v_1} = \sqrt{\frac{T_2}{T_1}} = \sqrt{\frac{273+127}{273+27}} = \sqrt{\frac{4}{3}}$$

∴ Speed of sound at 127° C,

$$v_2 = v_1 \times \sqrt{\frac{4}{3}} = 345 \times \sqrt{\frac{4}{3}} = 398.4 \text{ m/s}$$

Illustration 12 : The speed of sound in dry air at STP is 332 m s^{-1}. Assume air as composed of 4 part of nitrogen and one part of oxygen. Calculate speed of sound in oxygen under similar condition when the density of oxygen and nitrogen at STP are in the ratio of 16:14.

Solution : Density of air = $\frac{\text{Total mass}}{\text{Total volume}}$

$$\rho_a = \frac{\text{(Mass of oxygen)+(Mass of nitrogen)}}{\text{(Volume of oxygen)+(Volume of nitrogen)}}$$

$$\rho_a = \frac{(V\times\rho_0)+(4V\times\rho_N)}{V + 4V}$$

$$= \frac{\rho_0 + 4\rho_N}{5}$$

$$= \frac{\rho_0\left(1+4\times\frac{\rho_N}{\rho_0}\right)}{5}$$

$$= \frac{\rho_0\left(1+4\times\frac{14}{16}\right)}{5}$$

$$= 0.9\rho_0$$

Speed of sound $v \propto \frac{1}{\sqrt{\rho}}$ $\left(\because v = \sqrt{\frac{\gamma P}{\rho}}\right)$

∴ Speed of sound in oxygen, $v_0 \propto \frac{1}{\sqrt{\rho_0}}$

Speed of sound in air $v_a = \frac{1}{\sqrt{\rho_a}}$

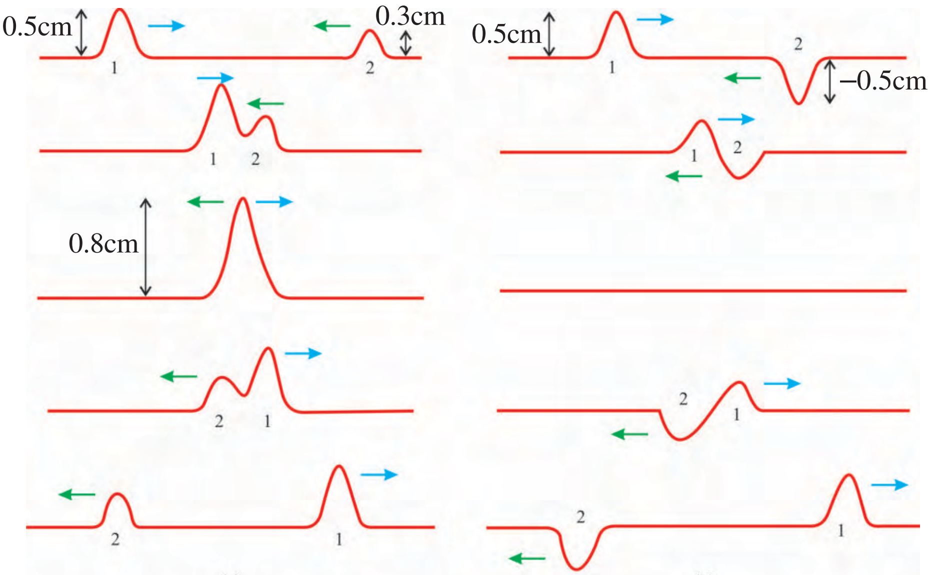
$$\therefore \frac{v_0}{v_a} = \sqrt{\frac{\rho_a}{\rho_0}} = \sqrt{\frac{0.9\rho_0}{\rho_0}} = 0.9487$$

$$\therefore v_0 = v_a \times 0.9487 = 332 \times 0.9487$$

$$= 314.77 \text{ m/s}$$

8.8 Superposition Principle and Reflection of the Wave

So far we have discussed a single wave propagating on a string. Suppose two persons holding the string at the two ends snap their hands once, then two wave pulses will be produced and move towards each other as shown in figure 8.9(*a*). The pulses travel at same speed because the medium is same.



Principle of superposition

Figure 8.9

Suppose the maximum displacement of particle in first wave is 0.5 cm and in second wave is 0.2 cm. As the two wave approaching each other, at any instant both the waves will overlap in some region of the string. Then after they move with their original shape and in their original direction. The net maximum displacement of the particle of string in overlaped region would be 0.5 cm + 0.3 cm = 0.8 cm.

Suppose the two persons snap the end of the string such that wave pulse generates at both the end of the string as shown in Figure 8.9(*b*).

In the first wave pulse the maximum displacement of particle is 0.5 cm in upward and in the other wave pulse the maximum displacement of particle is 0.5 cm in downward direction.

When the wave pulses approach each other, at some instant they overlay on the string and displacement of all the particles will be 0.5cm + (–0.5 cm) = 0. However, the velocities of the particles will not be zero. In this situation, string becomes straight everywhere than both the wave pulses will emerge and move in their original direction.

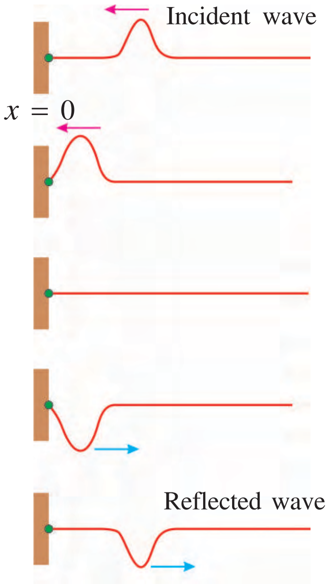
From the above observation principle of superposition can be given as follows.

"When a particle of medium comes under the influence of two or more waves simultaneously, its net displacement is the vector sum of displacement that would occur under the influence of the individual waves."

Reflection of Waves :

(*a*) Reflection of waves from a rigid support :

Suppose a wave propagating in the direction of decreasing value of x, represented by the equation $y = A \sin (\omega t + kx)$ reaches a point $x = 0$, when the wave arrives at the rigid end it exerts a force on the support (wall). By Newton's third law, the support exerts an equal but opposite reaction force on the string. This reaction force generates a wave at the support which travels back along the string in the direction opposite that of incident wave. This wave is known as **reflected wave.**



Reflection of wave from rigid support

Figure 8.10

Oscillation of the particle at point $x = 0$, due to wave $y = A \sin(\omega t + kx)$, can be represented as, $y_i = A \sin\omega t$ (8.8.1)

But, the support at $x = 0$ is fixed, the displacement at $x = 0$ must always be zero. According to principle of superposition, the displacement at $x = 0$ due to reflected wave. It can be give as,

$$y_r = -A \sin\omega t \quad (8.8.2)$$

Equation (8.8.2) can be represented as follows :

$$y_r = A\sin(\omega t + \pi) \quad (8.8.3)$$

This shows that as the wave reflected from a fixed support, its phase is increased by π. Thus, the 'shape' of the waveform is inverted on reflection. i.e. Crest becomes a trough and trough becomes a crest.

The reflected wave is travelling in the direction of increasing value of x. So the equation can be written as,

$$y_r = A \sin(\omega t + \pi - kx)$$

$$\therefore\ y_r = -A \sin(\omega t - kx) \quad (8.8.4)$$

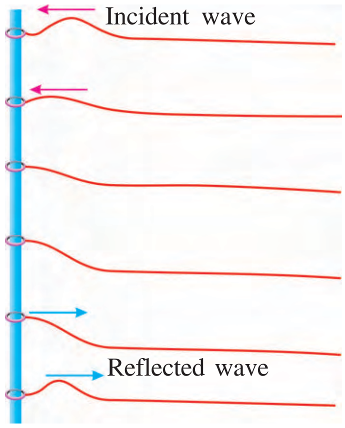
If the incident wave is travelling in the direction of increasing value of x, then $y_i = A \sin(\omega t - kx)$ (8.8.5)

And equation of reflected wave can be given as,

$$y_r = -A \sin(\omega t + kx) \quad (8.8.6)$$

(*b*) Reflection of waves from a free end :

As shown in Figure 8.11 suppose one end of a string is tied to a very light ring which can slide or move without any friction on a vertical rod. Such an end of the ring is said to be free end and here, we will understand the reflection of waves from such a free end.



Reflection of a wave from a free end

Figure 8.11

Suppose the crest like shape of the wave produced from the other end of the string reaches the ring. The ring is then pushed upwards as it is not fixed. Hence the string tied to the ring is also pulled up. As a result of this, now, a reflected wave pulse is generated from this end of the string. Phase of this reflected waves is equal to the phase of the incident wave. So, in this situation the shape is not inverted and a crest is reflected as a crest and a trough as a trough only. Moreover, during such a reflection both the waves are simultaneously present on the ring in same phase and hence the displacement of the ring on the rod is twice the amplitude of the incident wave.

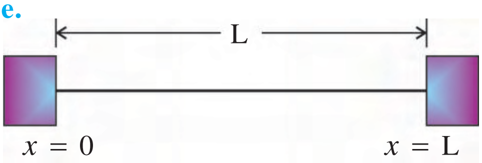
From this discussion it is clear that if the equation of the incident wave is $y_i = A \sin(\omega t + kx)$ then the equation of its reflected wave from a free end will be

$$y_r = A \sin(\omega t - kx) \quad (8.8.7)$$

Thus, a travelling wave at a rigid boundary or a closed end, is reflected with a phase reversal of π but the reflection at an open boundary takes place without any phase change.

8.9 Stationary Waves

When two waves having the same amplitude and frequency (i.e. wavelength) and travelling in mutually opposite directions are superposed, the resultant wave formed loses the property of propagation and a stationary pattern is created in the medium. Such a wave is called a **stationary wave.**



A string fixed at both the ends with rigid supports

Figure 8.12

To understand stationary waves, consider a string of length L, kept under a suitable tension, fixed at its two ends. The harmonic waves produced in this string will be reflected from rigid supports repeatedly so that each element of string is under the influence of "incident" and "reflected" waves.

Let the wave propagating in the direction of increasing x be

$$y_1 = A \sin(\omega t - kx) \qquad (8.9.1)$$

The reflected wave propagating in the direction of decreasing x will then be,

$$y_2 = -A \sin(\omega t + kx) \qquad (8.9.2)$$

According to principle of superposition, the displacement of a particle of a string is given by,

$$y = y_1 + y_2$$

$$= A \sin(\omega t - kx) - A \sin(\omega t + kx)$$

Now,

$$\therefore y = -2A\cos\omega t \sin kx \text{ (see foot note)}$$

$$= -2A \sin kx \cos\omega t \qquad (8.9.3)$$

The functional form of this wave is not of type $f(\omega t \pm kx)$ which means that it is not a travelling or progressive wave. Equation (8.9.3) is an equation of a stationary wave. **Energy does not propagate in this type of a wave and hence it is named as a stationary wave.**

The term 'cos ωt' of the equation (8.9.3) shows that each particle of the string is executing a simple harmonic motion and their amplitudes depends upon position x according to $2A\sin kx$. Here, amplitudes of all the particles are not same.

The location of particles for which $\sin kx = 0$, have zero amplitude and these points remains stationary. These points are called **'Nodes'.**

The positions in a stationary wave where the amplitude always remains zero are called the 'Nodes'.

$$\text{Now } \sin kx = 0$$

$$\therefore kx = n\pi \qquad \text{where } n = 1, 2, 3 \ldots\ldots$$

$$\therefore x = \frac{n\pi}{k} = \frac{n\pi}{2\pi/\lambda}$$

$$\therefore x = \frac{n\lambda}{2} \qquad (8.9.4)$$

This shows that the nodes are located at a distance $x = \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \ldots\ldots\ \frac{n\lambda}{2}$ from the end $x = 0$. **The distance between the successive node is $\frac{\lambda}{2}$.**

Maximum amplitudes occur at points for which $\sin kx = \pm 1$. These points are called "Antinodes".

The positions in a stationary wave where the amplitude always remains maximum are called the 'Antinodes'.

$$\sin kx = \pm 1$$

$$\therefore kx = (2n - 1)\frac{\pi}{2} \quad \text{where, } n = 1, 2, \ldots\ldots$$

$$\therefore x = \frac{(2n - 1)\pi}{2k}$$

$$= (2n - 1)\frac{\lambda}{4} \qquad (8.9.5)$$

Thus, the antinodes are located at $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots\ldots$ from the end $x = 0$. **Distance between successive antinode is also $\frac{\lambda}{2}$. Distance between a node and an adjacent antinode is $\frac{\lambda}{4}$.**

In Figure 8.13 the antinodes are shown as A and the nodes are shown as N.

The displacement of string at the end $x = 0$ and at the end $x = L$ is always zero because the string is fixed to a rigid suppot at $x = L$.

$$\therefore \sin kL = 0$$

$$\therefore kL = n\pi \qquad \text{where } n = 1, 2, 3, \ldots\ldots$$

$$\therefore \frac{2\pi}{\lambda}L = n\pi$$

$$\therefore \lambda_n = \frac{2L}{n} \qquad (8.9.6)$$

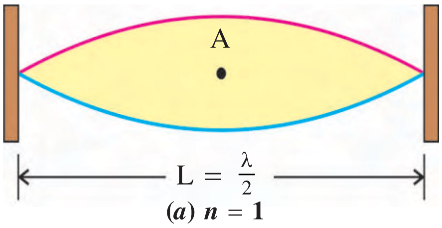
This equation shows that for a string at given length L, stationary wave can be formed only with waves having specific discrete values for their wave length like 2L, L, $\frac{2L}{3}, \frac{L}{2}, \ldots\ldots$ appropriate to different values of n. Thus, waves with arbitrary wavelength cannot form stationary waves on a string of a given length.

The frequency of the standing waves produced on a string will have corrosponding to its restricted wavelength. It is given by,

$$f_n = \frac{v}{\lambda_n}$$

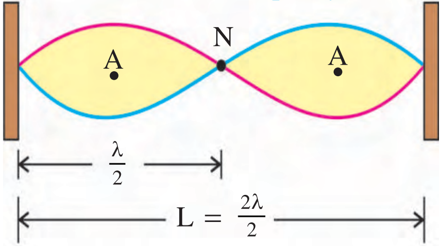
$$\therefore f_n = \frac{nv}{2L} \quad \text{(from equation 8.9.6)} \qquad (8.9.7)$$

Foot note : $\sin C - \sin D = 2\cos\left(\frac{C + D}{2}\right)\sin\left(\frac{C - D}{2}\right)$



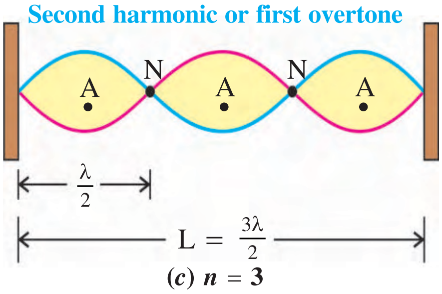
(*a*) $n = 1$

Fundamental frequency



(*b*) $n = 2$

Second harmonic or first overtone



(*c*) $n = 3$

Third harmonic or second overtone

Stationary waves on a string

Figure 8.13

$$\text{or } f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}} \qquad (8.9.8)$$

Where, v = speed of wave on string = $\sqrt{\frac{T}{\mu}}$

Substituting $n = 1$ in equation (8.9.7)

$$f_1 = \frac{v}{2L}$$

Here, f_1 is called **fandamental frequency** or **first harmonic.**

Taking $n = 2$,

$$f_2 = \frac{2v}{2L} = 2f_1$$

f_2 is called **second harmonic** or **first overtone.**

Taking $n = 3$,

$$f_3 = \frac{3v}{2L} = 3f_1$$

f_3 is called **third harmonic** or **second overtone**

In this way taking successive integral values of n, all possible oscillation of the string are obtained and corresponding frequencies of the fourth, fifth etc. harmonics are obtained.

Figure 8.13 shows oscillation of string with first, second and third harmonics. Form figure it is clear that number of loops produced on the string is same as value of n.

These oscillations with discrete frequencies in various harmonics are called the **'Normal Modes of Oscillation of a system.**

Frequencies appropriate to the different normal modes of vibrations can be obtained from the following equation.

$$f_n = \frac{nv}{2L} = nf_1 \qquad \text{where } n = 1, 2, 3....$$

Here, f_n is the frequency of wave produced on a string. It is also called nth harmonic or $(n - 1)$th overtone. The integer n indicates the number of loops on the string.

Illustration 13. : The stationary waves produced in a 60 cm long string tied at both the ends with rigid support are represented by $y = 4\sin\left(\frac{\pi x}{15}\right)\cos(96\pi t)$. Here, x and y are in cm and t is second. Find out,

(1) position of nodes,

(2) positions of anti-nodes,

(3) maximum displacement of the particle at $x = 5$ cm

(4) the equation of the component waves.

Solution : Comparing

$$y = 4\sin\left(\frac{\pi x}{15}\right)\cos(96\pi t) \text{ with}$$

$$y = 2A\sin(kx)\cos(\omega t),$$

$$A = 2 \text{ cm}, k = \frac{\pi}{15}\frac{\text{rad}}{\text{cm}} \text{ and } \omega = 96\pi \text{ rad/s}$$

$$\text{But, } k = \frac{2\pi}{\lambda}$$

$$\therefore \frac{2\pi}{\lambda} = \frac{\pi}{15} \Rightarrow \lambda = 30 \text{ cm}$$

(1) Positions of nodes

$$= \frac{n\lambda}{2}, \qquad \text{where } n = 1, 2,...$$

$$= 15 \text{ cm}, 30 \text{ cm}, 45 \text{ cm}$$

(The particles at 0 cm and 60 cm are tied to the rigid supports and hence they are not considered here.)

(2) Positions of antinodes,

$$= (2n - 1)\frac{\lambda}{4}, \quad \text{where } n = 1, 2, 3, ...$$

$$= 7.5 \text{ cm}, 22.5 \text{ cm}, 37.5 \text{ cm}, 52.5 \text{ cm}$$

(3) Maximum displacement of the particle at a distance

$$x = 2A\sin kx$$

$$= 4\sin\left(\frac{\pi x}{15}\right)$$

$$= 4\sin\left(\frac{\pi}{3}\right) \quad (\because x = 5 \text{ cm})$$

$$= 4\frac{\sqrt{3}}{2}$$

$$= 2\sqrt{3} \text{ cm}$$

(4) $y = 4\sin\left(\frac{\pi x}{15}\right) \cos(96\pi t)$

$$= 2\sin\left(\frac{\pi x}{15} + 96\pi t\right) + 2\sin\left(\frac{\pi x}{15} - 96\pi t\right)$$

∴ Component of waves are

$$y_1 = 2\sin\left(\frac{\pi x}{15} + 96\pi t\right) \text{cm and,}$$

$$y_2 = 2\sin\left(\frac{\pi x}{15} - 96\pi t\right) \text{cm}$$

Illustration 14 : The equation of a progressive, harmonic waves travelling in a medium is given by an equation $y_i = A\cos(ax + bt)$; where A, a and b are positive constants. This wave is reflected from a rigid support kept at $x = 0$. The intensity of the reflected wave is 0.64 times that of the incident wave.

(*a*) What are wavelength and frequency of the incident wave ?

(*b*) Write the equation of the reflected wave.

(*c*) Express the resultant wave in the form of progressive and stationary waves.

Solution :

(*a*) Incident wave is $y_i = A\cos(ax + bt)$

Comparing this equation with the wave-equation $y = A\cos(kx + \omega t)$,

∴ wave-vector $k = a$

$$\therefore \frac{2\pi}{\lambda} = a$$

$$\therefore \lambda = \frac{2\pi}{a}$$

Angular frequency $\omega = 2\pi f = b$

$$\therefore f = \frac{b}{2\pi}$$

(*b*) Intensity $I \propto A^2$, where A = amplitude.

Suppose amplitudes of the incident and the reflected waves are A_1 and A_2 respectively and I_1 and I_2 are their intensities respectively.

$$\therefore \frac{I_2}{I_1} = \frac{(A_2)^2}{(A_1)^2}$$

$$\therefore \frac{A_2}{A_1} = \left(\frac{I_2}{I_1}\right)^{\frac{1}{2}} = (0.64)^{\frac{1}{2}}$$

∴ $A_2 = 0.8$ A (∵ A_1 = Amplitude of the incident wave = A)

∴ Amplitude of the reflected wave $A_2 = 0.8$ A

Equation of the reflected wave

$$y_r = -A_2 \cos(bt - ax)$$

$$\therefore y_r = -0.8 A\cos(bt - ax)$$

(*c*) Resultant wave $y = y_i + y_r$

$$= A\cos(bt + ax) - 0.8 A\cos(bt - ax)$$

$$= 0.8 A[\cos(bt + ax) - \cos(bt - ax)] + 0.2 A\cos(bt + ax)$$

$$= -1.6 A\sin(ax) \cdot \sin(bt) + 0.2 A\cos(bt + ax),$$

where, stationary wave,

$y_s = -1.6 A\sin(ax) \cdot \sin(bt)$ and

progressive wave $y_p = 0.2 A\cos(bt + ax)$

Illustration 15 : A block is attached to the free end of a sonometer wire. The wire has fundamental frequency f_1 Hz in this situation. Now the block is immersed in water and it is found that the wire has a fundamental frequency f_2 Hz. When the block is immersed in some liquid, the fundamental frequency of the wire is f_3 Hz. Find the specific gravity of the material of the block and that of the liquid.

Solution : The force of buoyancy is different when the block is in air, in water and in liquid. So the effective weight is different in these cases. Hence, tension in the wire is also different and as a result the frequency is also different for the wire of same length and same material.

Suppose, the weight of block in air is W_1, in water W_2 and in liquid W_3.

Fundamental frequency, $f = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$

Here, L and μ being constant,

$f \propto \sqrt{T}$

$\therefore$ $T = kf^2$ where, k = constant of proportionality.

But, tension T = Weight W

$\therefore$ $W = kf^2$

$\therefore$ $W_1 = kf_1^2$; $W_2 = kf_2^2$; $W_3 = kf_3^2$

According to Archimedes' principle,

Specific gravity of block

$$= \frac{\text{Weight of block in air}}{\text{Loss of weight of block in water}}$$

$$= \frac{W_1}{W_1 - W_2} = \frac{f_1^2}{f_1^2 - f_2^2}$$

Specific gravity of liquid

$$= \frac{\text{Loss of weight of block in liquid}}{\text{Loss of weight of block in water}}$$

$$= \frac{W_1 - W_3}{W_1 - W_2} = \frac{kf_1^2 - kf_3^2}{kf_1^2 - kf_2^2}$$

$$= \frac{f_1^2 - f_3^2}{f_1^2 - f_2^2}$$

8.10 Stationary Wave in Pipes

As stationary waves are formed on a string due to superposition of incident and reflected transverse waves of definite frequencies, stationary waves are also formed due to reflection of longitudinal waves of definite frequencies in the air column, from the end of a pipe. The flute trumpet, clarionet etc. are the musical instrument that are organ pipes in which stationary longitudinal waves are formed. Such pipes are of two types : (1) an open pipe in which both ends are open e.g. flute. (2) a closed pipe in which one end is closed, e.g. clarionet.

Just as in case of string, a node obtained at the fixed end, for a closed pipe a **node** is always formed at the closed end because longitudinal waves are reflected from closed end. If the pipe is narrow compared to the wavelength of wave, an **antinode** is formed at the open end (slightly outside). The situation is slightly complicated for the reflection of longitudinal waves at the open end of the pipe.

Stationary Waves in a Closed Pipe :

For stationary waves to be formed in a closed pipe the wavelength (λ) of the wave should be such that a node is formed at the closed end of the pipe and an antinode at its open end. In stationary waves the distances between nodes and antinodes are $\frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, ...(2n-1)\frac{\lambda}{4}$, where n = 1, 2, 3,....

Similarly, in general stationary waves is produced in a pipe of length L for wavelength λ only when,

$$L = (2n-1)\frac{\lambda}{4} \qquad (8.10.1)$$

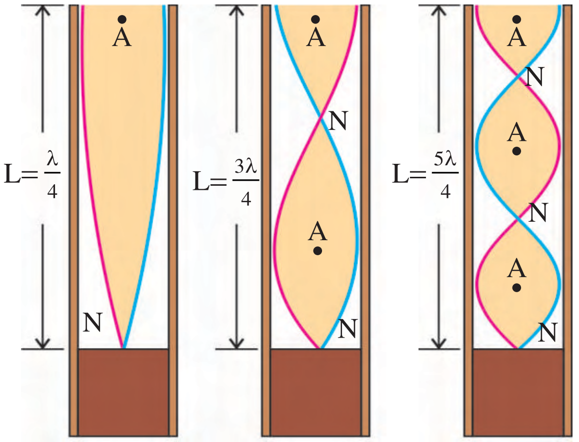
where n = 1, 2, 3, ...

In a closed pipe the value of possible wavelengths required for stationary waves are given by,

$$\lambda_n = \frac{4L}{(2n-1)} \qquad (8.10.2)$$

The frequency of stationary waves in pipe will be,

$$f_n = \frac{v}{\lambda_n}$$



n = 1 (*a*) n = 2 (*b*) n = 3 (*c*)

n = 1 Fundamental frequency (first harmonic)

n = 2 Third harmonic (first overtone)

n = 3 Fifth harmonic (second overtone)

Stationary waves in a closed pipe

Figure 8.14

$$\therefore\ f_n = \frac{v}{4L}(2n - 1) \qquad (8.10.3)$$

where, v is a speed of wave.

(i) Taking $n = 1$,

$$f_1 = \frac{v}{4L}$$

f_1 is known as **fundamental frequency** or the **first harmonic.**

(ii) Taking $n = 2$,

$$f_2 = \frac{3v}{4L} = 3f_1 \qquad (\because\ f_1 = \frac{v}{4L})$$

f_2 is known as **third harmonic** or **first overtone.**

(iii) Similarly for $n = 3$,

$$f_3 = \frac{v}{4L}(2(3) - 1) = \frac{5v}{4L} = 5f_1$$

f_3 is known as fifth harmonic or second overtone.

In general, the frequency of n^{th} mode of normal oscillation in closed pipe is given by,

$$f_n = \frac{v}{4L}(2n - 1)$$
$$= (2n - 1)f_1 \qquad (8.10.4)$$

where $n = 1, 2, 3,....$

Here, f_n represents $(2n - 1)$ harmonic or $(n - 1)^{\text{th}}$ overtone.

Thus, in the closed pipe all the harmonics are not possible. The harmonics are possible only for odd multiples of fundamental frequency (f_1, $3f_1$, $5f_1$, ...).

[In this reference, equation (8.10.3) can be written as,

$$f_n = nf_1 = \frac{nv}{4L} \quad \text{where } n = 1, 3, 5....$$

where, f_n represent nth harmonic or $\left(\frac{n-1}{2}\right)$th overtone]

The frequencies for which stationary waves are formed are called natural or characteristic frequencies of the given pipe.

Stationary waves in an open pipe :

In an open pipe, antinodes are formed at both the ends. We know that in stationary waves distances of antinodes are $\frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}\ .....\ \frac{n\lambda}{2}$.

Where $n = 1, 2, 3,.......$

Therefore, in an open pipe of length L, the stationary waves can be produced only of those wavelength λ for which,

$$L = \frac{n\lambda}{2}$$

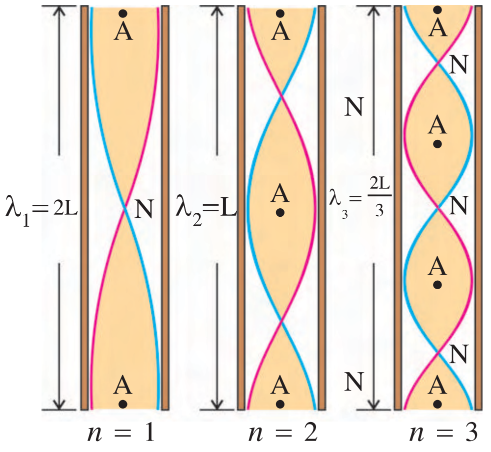
So, possible wavelenghts in pipe will be,

$$\lambda_n = \frac{2L}{n} \qquad (8.10.5)$$

The frequency of a stationary waves in open pipe will be,

$$f_n = \frac{v}{\lambda_n} = \frac{nv}{2L} \qquad (8.10.6)$$

where, v is speed of a wave.



(*a*) Fundamental frequency or first harmonic

(*b*) Second harmonic (first overtone)

(*c*) Third harmonic (second overtone)

Stationary waves in open pipes

Figure 8.15

(i) Substituting $n = 1$ in equation (8.10.6)

$$f_1 = \frac{v}{2L} \qquad (8.10.7)$$

Here, f_1 is called the **fundamental frequency** or the **first harmonics** (See Figure 8.15*a*) which is double than the fundamental frequency of a closed pipe. ($\because\ f_1 = \frac{v}{4L}$).

(ii) Taking $n = 2$,

$$f_2 = \frac{2v}{2L} = \frac{v}{L} = 2f_1$$

f_2 is called the **second harmonics** or **first overtone.**

Thus, taking different values of n in equation (8.10.6) third, fourth ..... harmonics can be obtained. In general, for an open pipe the nth harmonic or $(n - 1)$th overtone,

$$f_n = \frac{nv}{2L} = nf_1 \qquad (8.10.8)$$

where, $n = 1, 2, 3....$

Thus, all the harmonics (f_1, $2f_1$, $3f_1$....) are possible for an open pipe.

Thus, in both the types of the pipes there are normal mode of oscillation for the air columns of the pipes.

Illustration 16 : If the second overtone of a closed pipe and third overtone of an open pipe are same, find the ratio of their lengths.

Solution :

For a closed pipe second overtone means fifth harmonics. Put $n =$ 5, in following equation,

$$f = \frac{nv}{4L} = \frac{5v}{4L_1}$$

for an open pipe, third overtone means fourth harmonics. Put $n = 4$ in following equation,

$$f = \frac{nv}{2L} = \frac{4v}{2L_2}$$

Now, the frequency is same for both the pipes.

$$\frac{5v}{4L_1} = \frac{4v}{2L_2}$$

$$\therefore \frac{L_1}{L_2} = \frac{5}{8} \text{ OR } L_1 : L_2 = 5 : 8$$

Illustration 17 : Air column of a resonance tube resonates with a tuning fork of frequency 800 Hz when its lengths is 9.75 cm. If the length of the air column is increased to 31.25 cm then also it resonates with the same tuning fork. Find speed of sound in air.

Solution : In the experiment of resonance tube, the arrangement of a closed pipe is obtained by immersing one end of a pipe in the water.

When oscillations are produced in the air column with the help of a tuning fork having frequency equal to the natural frequency of air column it oscillates with large amplitude, and large intense of sound is heard. This is a phenomenon of **resonance.**

Here, $f = 800$ Hz, $L_1 = 9.75$ cm, $L_2 = 31.25$ cm

Resonance tube is a closed pipe. For a closed pipe the natural frequency is given by

$$f = (2n - 1)\frac{v}{4L}$$

Taking $n = 1$ for first resonance,

$$f = \frac{v}{4L_1}$$

$$\therefore L_1 = \frac{v}{4f}$$

Taking $n = 2$ for second resonance,

$$f = (2 \times 2 - 1)\frac{v}{4L_2} = \frac{3v}{4L_2}$$

$$\therefore L_2 = \frac{3v}{4f}$$

$$\therefore L_2 - L_1 = \frac{3v}{4f} - \frac{v}{4f} = \frac{2v}{4f} = \frac{v}{2f}$$

$\therefore$ Speed of sound $v = (L_2 - L_1)(2f)$

$= (31.25 - 9.75)(2 \times 800)$

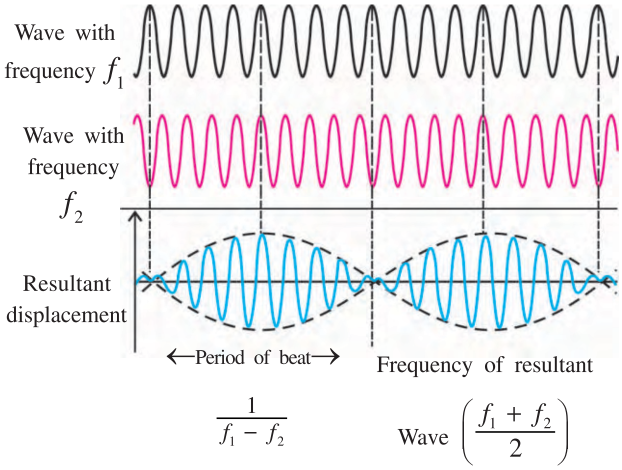
$= 34400$ cm/s

$= 344$ m/s

8.11 Beats

So far we have applied principle of superposition to two waves propagating in opposite direction with equal amplitude and equal frequency. It produces the non-progressive wave like stationary waves.

Let us now consider two waves having equal amplitudes and travelling in a medium in the same direction but having slightly different frequencies. Now we will apply principle of superposition to study the oscillation of a particle of a medium.



Beats

Figure 8.16

Suppose, two harmonic waves superpose at a particular position in the medium are,

$y_1 = A \sin \omega_1 t = A \sin 2\pi f_1 t$ and

$y_2 = A \sin \omega_2 t = A\sin 2\pi f_2 t$

Here, initial phase of both waves is zero. f_1 and f_2 are the frequency of first wave and second wave respectively.

Remember that here we are locally observing the effect of superposition of two waves on any one particle.

Suppose, at time t, the resultant displacement of a given particle is y then according to superposition principle,

$$y = y_1 + y_2$$

$$= A \sin 2\pi f_1 t + A \sin 2\pi f_2 t$$

$$\therefore\ y = [2A\cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t] \sin 2\pi\left(\frac{f_1 + f_2}{2}\right)t \qquad (8.11.1)$$

$$y = A' \sin 2\pi\left(\frac{f_1 + f_2}{2}\right)t$$

or $y = A' \sin 2\pi ft$ (8.11.2)

Above equation shows that the resultant oscillations of a given particle are the oscillation with a frequency $f = \left(\frac{f_1 + f_2}{2}\right)$. Here, f is the average of the two combining frequencies. The resultant amplitude is,

$$A' = 2A\cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t \qquad (8.11.3)$$

and it changes periodically with time. Here, amplitude is a periodic function of time. Its frequency is $\left(\frac{f_1 - f_2}{2}\right) = f'$

Therefore, the period of oscillation is,

$$T = \frac{1}{f'} = \frac{2}{f_1 - f_2} \qquad (8.11.4)$$

In time period T, the 'cosine' function attains its maximum value and zero twice. Hence, this function becomes $f_1 - f_2$ times maximum in unit time. Therefore, the amplitude of oscillations becomes $f_1 - f_2$ times maximum and $f_1 - f_2$ times zero in unit time.

If these waves are sound waves, then loudness of sound is proportional to the square of the amplitude ($I \propto A^2$), the loudness of sound also becomes $f_1 - f_2$ times maximum and $f_1 - f_2$ times zero in unit time.

Thus, phenomenon of the loudness of sound becoming maximum periodically due to superposition of two sound waves of equal amplitude and slightly different frequencies is called the 'beats'. The number of beats in unit time is $f_1 - f_2$. It is also called frequency of beat.

Note : In case of sound waves, in order to hear the beats clearly, $f_1 - f_2$ should not exceed about 6 to 7.

The phenomenon of beats can be experienced by taking two tuning forks of the same frequency and putting some wax on the prongs of one of the forks. Loading with wax decreases the frequency of a tuning fork a little. (By filling one of the prongs of a fork, its frequency will increase a little) When these two forks are vibrated and kept side by side, the listener can recognise the periodic variation of loudness of resulting sound. Musician tune their different musical instruments with the help of beat phenomenon.

Illustration 18 : When two tuning forks A and B were sounded together, 20 beats were produced in 8 seconds. After loading one of the tuning forks with a little wax, they produce 32 beats in 8 seconds. If the unloaded fork had a frequency of 512 Hz. calculate the frequency of the other.

Solution : Suppose, tuning fork B is loaded with wax. Frequency of tuning fork A,

$f_A = 512$ Hz

Frequency of tuning fork B, $f_B = ?$

Before loading wax, number of beats per second,

$$= \frac{20}{8} = 2.5 \text{ Hz}$$

$\therefore$ Frequency of B before loading wax either $512 + 2.5 = 514.5$ Hz

or $512 - 2.5 = 509.5$ Hz

Foot Note : $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

After loading wax on B,

beats per second = $\frac{32}{8}$ = 4 Hz.

Frequency of B after loading is,

either 512 + 4 = 516 Hz

or 512 − 4 = 508 Hz

Since, after loading the wax frequency of a tuning fork B is lowered. In above calculation we can see that before loading wax, frequency of B is 509.5 Hz and after loading it is 508 Hz.

Hence, original frequency of B (i.e. before loading) will be 509.5 Hz.

8.12 Doppler Effect

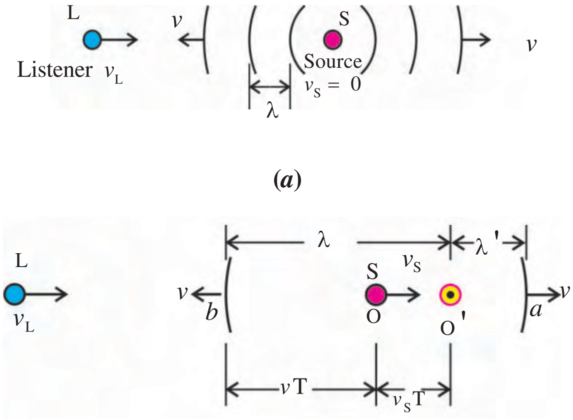
Whenever there is a relative motion between a source of a sound and a listener with respect to medium in which the waves are propagating, the frequency of sound experienced by the listener is different from that which is emitted by the source. This phenomenon is called **Doppler effect.** This effect was discovered by Austrian physicist Johann Christian Doppler (1803–1853).

The frequency of sound of a whistle of the train is found to be more than original frequency and hence its sound appears more shrill (of higher pitch) when the train is approaching you. When it is passing by you, the frequency of sound experienced is same of that of actual sound emitted, and when the train is receding from you, the frequency listens lower than actual and sound appears less shrill than the actual.

To understand Doppler effect, consider, as shown in Fig. 8.17, a listener moving with velocity v_L and a source of sound moving with a velocity v_S along straight line with respect to stationary air.

As a convention, the velocities in the direction from listener to source are considered as positive and from the source to the listener are considered as negative. The speed of sound is always considered positive. With this convention, we will obtain a general result from which other cases can be obtained easily.

Moving Listener : Suppose, a listener L moving with velocity v_L towards a stationary sources S. (See Figure 8.17a) The source emits a sound wave with frequency f_s and wavelength $\lambda = \frac{v}{f_S}$. Where, v is the speed of sound wave in air.



(*a*)

(*b*)

Doppler effect

Figure 8.17

These waves are travelling towards the listener. Hence, the speed of waves travelling towards the listener, relative to the listener will be $v + v_L$. So the frequency f_L listened by the listener will be,

$$f_L = \frac{v + v_L}{\lambda} \qquad (8.12.1)$$

Moving Source and Moving Listener : Now suppose the source is also moving with velocity v_s in the direction of L to S. (See Figure 8.17 *b*)

Let the source of sound (S) be at O at time $t = 0$ and at O'at time t = T. Where, T = $\frac{1}{f_S}$ is the periodic time of emitted sound.

Now, the distance travelled by the source in time T will be,

$$OO' = v_S T$$

The wave (crest) emitted by the source at $t = 0$ will cover the distance vT in time T. From the figure, Oa = Ob = vT

Now, at time t = T, the source is at O' and it emits successive wave (crest). The wave moving towards the listener will be in the region O'b and the wave moving away from the listener will be in region O'a.

The wavelength of the wave moving towards the listner,

λ = Distance betwen successive wave (crest) in region O'*b*

$= v_S T + vT$

$$\therefore \lambda = \frac{v_S + v}{f_S} \quad (\because \ T = \frac{1}{f_S}) \qquad (8.12.3)$$

Substituting value of λ in equation (8.12.1)

$$f_L = \frac{v + v_L}{v + v_S} \cdot f_S \qquad (8.12.3)$$

$$\text{or} \quad \frac{f_L}{v + v_L} = \frac{f_S}{v + v_S} \qquad (8.12.4)$$

From the Figure (8.17) it is clear that the waves in the front of the source (region O'*a*) are compressed, hence the wavelength is decreasing due to motion of the source, while behind the source (region O'*b*) waves are stretched out hence its wavelength is increasing. Here, waves are travelling in the same medium (air), then why their wavelengths are changing ? Think over it. The relative displacement of source and wave is responsible for that.

Some Special Cases :

(i) Listener is stationary and source is moving towards the listener, then according to the accepted conventions, taking $v_S = -v_S$ and $v_L = 0$ in equation (8.12.3), the frequency listened by listener,

$$f_L = \frac{v}{v - v_S} f_S$$

i.e. listener will listen the frequency higher than the actual frequency ($f_L > f_S$).

(ii) Listener is stationary and source is moving away from the listener, then $v_L = 0$ and $v_S = +v_S$,

Frequency listened by listener $f_L = \frac{v}{v + v_S} f_S$

This shows that $f_L < f_S$. i.e. Listener will hear the lower frequency than actual frequency.

(iii) Both the source and listener are approaching each other, then $v_L = +v_L$ and $v_S = -v_S$ Therefore,

$$f_L = \frac{v + v_L}{v - v_S} f_S$$

In this case, $f_L > f_S$

(iv) Both the source and listener are moving aways from each other, then $v_L = -v_L$ and $v_S = +v_S$

$$\therefore f_L = \frac{v - v_L}{v + v_S} f_S$$

In this case, $f_L < f_S$

In all these cases the medium (air) is considered stationary. If wind is blowing from the source to the listener (in the direction of velocity of sound) with velocity v_w, the velocity of sound will be $v + v_w$ and if the wind is blowing in opposite direction to the motion of sound waves, the velocity of sound will be $v - v_w$.

Moreover, it is assumed that the velocities of the listener and of the source are less than the velocity of sound.

Illustration 19 : A police siren emits a wave with frequency 300 Hz. The speed of sound is 340 m/s. (*a*) Find the wavelength of the waves in the air if the police car is at rest. (*b*) If the police car is moving at 108 km/h, find the wavelength of the waves in front and behind the car.

Solution : (*a*) When the police car is at rest, $f_S = 300$ Hz, $v = 340$ m/s

Wavelength of the waves emitted from the siren.

$$\lambda = \frac{v}{f_S} = \frac{340}{300} = 1.13 \text{ m}$$

(*b*) Speed of a police car $v_S = 108$ km/h = 30 m/s

$$\text{Now, } f_L = \frac{v + v_L}{v + v_S} f_S$$

If the listener is in the region of the front of the moving car, then $v_L = 0$, and $v_S = -v_S$

$$\therefore f_{front} = \frac{v}{v - v_S} f_S$$

$$\therefore \frac{v}{\lambda_{\text{front}}} = \frac{v}{v - v_S} f_S$$

$$\therefore \lambda_{\text{front}} = \frac{v - v_S}{f_S} = \frac{340 - 30}{300} = 1.033 \text{ m}$$

For behind the police car,

$v_L = 0$ and $v_S = +v_S$

$$f_{\text{behind}} = \frac{v}{v + v_S} f_S$$

$$\therefore \lambda_{\text{behind}} = \frac{v + v_S}{f_S} = \frac{340 + 30}{300} = 1.233\text{m}$$

Illustration 20 : A SONAR system fixed in a stationary submarine in the sea operates at a frequency 40 kHz. An enemy submarine moves towards the SONAR with a speed of 360 kmh^{-1}. What is the frequency of sound reflected by the submarine ? The speed of sound in water is 1450 m s^{-1}.

Solution : $f_S = 40$ kHz, $v = 1450$ m/s

The frequency of the waves from the SONAR will undergo a change in frequency in two steps.

(i) When the waves are moving towards the enemy's submarine which is in motion, the frequency of waves will change. In this case SONAR is a source (S) and submarine is a listener (L).

Therefore, $v_S = 0$ and

$$v_L = 360 \text{ km/h} = \frac{360 \times 1000}{3600} = 100 \text{ m/s}$$

$$\text{Now, } f_{L_1} = \frac{v + v_L}{v + v_S} \times f_S$$

$$= \frac{1450 + 100}{1450 + 0} \times 40 \times 10^3$$

$$= 42.758 \text{ kHz}$$

(ii) The enemy submarine will reflect waves of frequency 42.758 kHz and will act as a source of waves, while SONAR will act as a listener (L).

$f_S = 42.758$ kHz, $v_L = 0$, $v_S = -100$ m/s

Frequency of reflected wave,

$$f_{L_2} = \frac{v + v_L}{v + v_S} \times f_S$$

$$= \frac{1450 + 0}{1450 - 100} \times 42.758 \times 10^3$$

$$= 45.92 \text{ kHz}$$

Thus the frequency of reflected waves from submarine, moving towards SONAR is 45.92 kHz.

SUMMARY

1. **Waves :** The motion of the disturbance in the medium (or in free space) is called wave pulse or generally a wave.
2. **Amplitude of a wave :** Amplitude of oscillation of particles of the medium is called the amplitude of a wave.
3. **Wavelength and frequency :** The linear distance between any two points or particles having phase difference of 2π rad is called the wavelength (λ) of the wave.

   Frequency of wave is just the frequency of oscillation of particles of the medium.

   Relation between wavelength and frequency :

   $v = f\lambda = \frac{\omega}{k}$, where, v is the speed of wave in the medium.
4. **Mechanical waves :** The waves which require elastic medium for their transmission are called mechanical waves. e.g. sound waves.
5. **Transverse and longitudinal waves :** Waves in which the oscillations are in a direction perpendicular to the direction of wave propagation are called the transverse wave.

Waves in which the oscillations of the particles of medium are along the direction of wave propagation are called longitudinal waves.

6. **Wave Equation :** The equation which describe the displacement for any particle of medium at a required time is called wave equation. Various forms of wave equations are as follows :

(i) $y = \text{A} \sin(\omega t - kx)$ (ii) $y = \text{A} \sin\left(\frac{t}{\text{T}} - \frac{x}{\lambda}\right)$

(iii) $y = \text{A} \sin 2\pi f\left(t - \frac{x}{v}\right)$ (iv) $y = \text{A} \sin \frac{2\pi}{\lambda}(vt - x)$

The above equations are for the wave travelling in the direction of increasing value of x. If the wave is travelling in the direction of decreasing value of x then put '+' instead of '–' in above equations.

7. The elasticity and inertia of the medium are necessary for the propagation of the mechanical waves.

8. The speed of the transverse waves in a medium like string kept under tension,

$$v = \sqrt{\frac{\text{T}}{\mu}}$$

where, T = Tension in the string and μ = mass per unit length of the string = $\frac{m}{\text{L}}$

9. Speed of sound waves in elastic medium, $v = \sqrt{\frac{\text{E}}{\rho}}$

where, E = Elastic constant of a medium, ρ = Density of the medium.

Speed of longitudinal waves in a fluid, $v = \sqrt{\frac{\text{B}}{\rho}} = \sqrt{\frac{\gamma \text{P}}{\rho}}$

where, B = Bulk modulus of a medium $\gamma = \frac{\text{C}_\text{P}}{\text{C}_\text{V}} = 1.41$ (for air)

Speed of longitudinal waves in a linear medium like a rod, $v = \sqrt{\frac{y}{\rho}}$

where, y = Young modulus, ρ = Density of a medium

At constant pressure and constant humidity, speed of sound waves in gas is directly proportional to the square root of its absolute temperature.

$$v = \sqrt{\frac{\gamma \text{RT}}{\text{M}}} \quad \therefore \; v \propto \sqrt{\text{T}}$$

The speed of sound in a gas does not depend on the pressure variation.

10. **Principle of Superposition :** When a particle of medium comes under the influence of two or more waves simultaneously, its net displacement is the vector sum of displacement that could occur under the influence of the individual waves.

11. **Stationary Waves :** When two waves having same amplitude and frequency and travelling in mutually opposite directions are superposed the resultant wave formed loses the property of propagation. Such a wave is called a stationary wave.

Equation of stationary wave : $y = -2 \text{ A} \sin kx \cos \omega t$

Amplitude of stationary wave : $2 \text{ A} \sin kx$

Position of nodes in stationary wave $x_n = \frac{n\lambda}{2}$

where, $n = 1, 2, 3.....$ At all these points the amplitude is zero.

Position of antinodes in stationary waves,

$x_n = (2n - 1)\frac{\lambda}{4}$ where , $n = 1, 2, 3,....$

The amplitude of all these points is 2A.

12. Frequencies corresponding to different normal modes of vibration in a stretched string of length L fixed at both the ends are given by,

$f_n = \frac{nv}{2L} = \frac{n}{2L}\sqrt{\frac{T}{\mu}}$ where $n = 1, 2, 3......$

13. In a closed pipe the values of possible wavelengths required for stationary wave pattern are given by.

$\lambda_n = \frac{4L}{(2n - 1)}$ and possible frequencies, $f_n = (2n - 1)\frac{v}{4L} = (2n - 1)f_1$

where, $n = 1, 2, 3,.....$ and L = length of pipe.

In a closed pipe only odd harmonics like f_1, $3f_1$, $5f_1$, ... are possible.

14. In an open pipe the values of possible wavelength required for stationary waves are given by,

$\lambda_n = \frac{2L}{n}$ and possible frequencies, $f_n = \frac{nv}{2L} = nf_1$ where, $n = 1, 2, 3, ...$ and L = length of pipe.

In open pipe of the harmonics like f_1, $2f_1$, $3f_1$, ... are possible.

15. **Beat :** The phenomenon of the loudness of sound becoming maximum periodically due to superposition of two sound waves of equal amplitude and slightly different frequencies is called the 'beats'.

Number of beats produced in unit time $= f_1 - f_2$

16. **Doppler Effect :** Whenever there is a relative motion between a source of sound and a listener with respect to the medium in which the waves are propagating the frequency of sound experienced by the listener is different from that which is emitted by the source. This phenomenon is called Doppler effect.

Frequency listened by the listener, $f_L = \frac{v \pm v_L}{v \pm v_S} f_S$

Where, v = velocity of sound, v_L = velocity of a listener,

v_S = velocity of a source, f_S = frequency of sound emitted by the source.

EXERCISES

Choose the correct option from the given options :

1. Mechanical waves carry ....

(A) energy (B) matter

(C) both energy and matter (D) neither energy nor matter.

2. A tuning fork makes 256 vibrations per second in air. When the velocity of sound is 330 m/s, then wavelength of the wave emitted is ....

(A) 0.56 cm (B) 0.89 m (C) 1.11 m (D) 1.29 m

3. When a sound wave of frequency 300 Hz passes through a medium, the maximum displacement of a particle of the medium is 0.1 cm. The maximum velocity of the particle is equal to ....

(A) 60π cm/s (B) 30π cm/s (C) 30 cm/s (D) 60 cm/s

4. The speed of wave of frequency 500 Hz is 360 m s^{-1}. The minimum distance between two particles on it, having phase difference of 60° is ....

(A) 0.23 m (B) 0.12 m (C) 8.33 m (D) 60 m

5. If the speed of the wave shown in the Figure is 330 m/s in the given medium, then the equation of the wave propagating in the positive x-direction will be ....

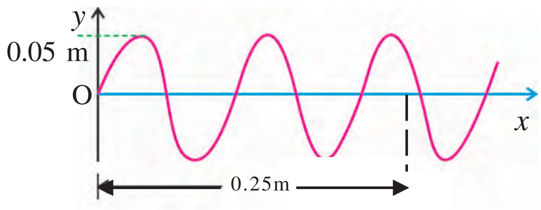


Figure 8.18

(A) $y = 0.05 \sin 2\pi (4000\, t - 12.5x)$ m

(B) $y = 0.05 \sin 2\pi (4000\, t - 122.5x)$ m

(C) $y = 0.05 \sin 2\pi (3300\, t - 10x)$ m

(D) $y = 0.05 \sin 2\pi\ (3300\, t - 10t)$ m

6. The equation $y = A \sin^2 (kx - \omega t)$ represents a wave with amplitude ...... and frequency ....

(A) A, $\omega/2\pi$ (B) $\frac{A}{2}, \frac{\omega}{\pi}$ (C) 2A, $\frac{\omega}{4\pi}$ (D) $\sqrt{A}, \frac{\omega}{2\pi}$

7. Two pulse travels in mutually opposite directions in a string with a speed of 2.5 cm/s as shown in figure. Initially (at $t = 0$) the pulses are 10 cm apart. What will be the state of string after two seconds ?

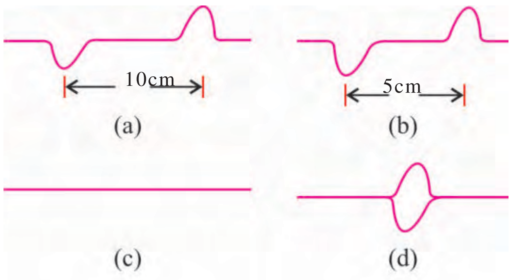


Figure 8.19

8. The speed of the component waves of a stationary wave represented by $y = 10 \sin (100\, t) \cos (0.01x)$ is ....
Where, x and y are in metre and t is in second.

(A) 1 m s^{-1} (B) 10^2 m s^{-1} (C) 10^3 m s^{-1} (D) 10^4 m s^{-1}

9. The mass of 7 m long string is 0.035 kg. If the tension in the string is 60.5 N, then the speed of wave on string will be ....

(A) 77 m s^{-1} (B) 102 m s^{-1} (C) 110 m s^{-1} (D) 165 m s^{-1}

10. If the maximum intensity of the beat produced by the superposition of two waves is x times the intensity of superposing wave then x = ...

(A) 1 (B) $\sqrt{2}$ (C) 2 (D) 4

11. Two waves of wavelengths 2.00 m and 2.02 m superpose with each other to produce beats in 1 s. If the speed of both waves is the same, their same speed is ....

(A) 400 m/s (B) 402 m/s (C) 404 m/s (D) 406 m/s

12. The speed of the component waves of a stationary wave is 1200 m/s. If the distance between consecutive antinode and node is 1 m, then frequency of standing wave will be ....

(A) 300 Hz (B) 400 Hz (C) 600 Hz (D) 1200 Hz

13. Suppose the listener and sound source both are approaching each other with speed of 50 m/s on a straight path. If the v frequency listened by listener is 440 Hz, what is the frequency of the wave produced by the source ? (speed of wave in air is 340 m/s)

(A) 327 s^{-1} (B) 367 s^{-1} (C) 390 s^{-1} (D) 591 s^{-1}

14. The fundamental frequency of the air column in a closed pipe is 512 Hz. If the pipe is open from both the ends, the fundamental frequency will be .... Hz.

(A) 1024 (B) 512 (C) 256 (D) 128

15. The air column in a closed pipe experiences first resonance with a tuning fork of frequency 264 Hz. If the length of the air column in the closed pipe is ..... cm. The speed of the sound in air is 330 m/s.

(A) 31.25 (B) 62.50 (C) 93.75 (D) 125

16. When the temperature of an ideal gas is increased by 600 K, the velocity of sound in the gas becomes $\sqrt{3}$ times the initial velocity in it. The initial temperature of the gas is ....

(A) -73 °C (B) 27 °C (C) 127 °C (D) 327 °C

17. Beats are produced by two waves given by y_1 = A sin $(2000\pi)t$ (m) and y_2 = A sin (2008π)t (m). The number of beats heard per second is ....

(A) 0 (B) 1 (C) 4 (D) 8

18. A source of sound is moving towards a stationary listener with 1/10 of the speed of sound. The ratio of apparent to real frequency is ....

(A) 10/9 (B) 11/10 (C) $(11/10)^2$ (D) $(9/10)^2$

19. A transverse wave is described by the equation $y = \text{A} \sin 2\pi \left(\frac{t}{\text{T}} - \frac{x}{\lambda}\right)$. For which wavelength of a wave, maximum particle velocity is two times the wave velocity ?

(A) $\lambda = \frac{\pi \text{A}}{4}$ (B) $\lambda = \frac{\pi \text{A}}{2}$ (C) $\lambda = \pi \text{A}$ (D) $\lambda = 2\pi \text{A}$

20. The temperature at which speed of sound in air becomes double of its value at 0°C is ....

(A) 273 K (B) 546 K (C) 1092 K (D) 0 K

ANSWERS

1. (A)	**2.** (D)	**3.** (A)	**4.** (B)	**5.** (C)	**6.** (B)
7. (C)	**8.** (D)	**9.** (C)	**10.** (D)	**11.** (C)	**12.** (A)
13. (A)	**14.** (A)	**15.** (A)	**16.** (B)	**17.** (C)	**18.** (A)
19. (C)	**20.** (C)				

Answer the following questions in short :

1. Give the definition of wave intensity and give its SI unit.

2. What is angular wave number of a wave ?

3. What is the distance travelled by the progressive wave if its wavelength is λ and frequency is f ?

4. Which characteristics of a medium are required for the propagation of mechanical wave ?

5. What is pressure wave ?

6. How the wave speed is changing with change in the temperature of a medium ?

7. What will be change in the speed of a wave in wire if the tension in wire increased four times ?

8. What will be the effect on the speed of a wave if the pressure of the medium will change ?

9. The wave equation of a wave is $y = 5 \sin (0.01x - 2t)$. Where x and y are in cm. What is the speed of a wave ?

10. What will be the change in the phase of a wave, if the wave on the string is reflected from the rigid support ?

11. What is the amplitude of node and antinode in a stationary wave ?

12. What is the distance between consecutive antinode in a stationary wave if the distance between consecutive node and antinode is 5 cm ?

13. In a closed pipe fundamental frequency is 300 Hz. What will be the frequency of second overtone ?

14. Frequency of the source of sound is 440 Hz. If the relative velocity of source and listener is zero then which frequency will be listened by a listener ?

15. What is a beat ?

Answer the following questions :

1. Explain the classification of the waves. Give the example of each wave.

2. Explain wavelength, wave number and frequency of a wave.

3. With the help of dimensional analysis, obtain the expression for the wave speed propagating in the string kept under tension.

4. Explain the propagation of sound waves in the air.

5. Write the Newton's formula for speed of a wave in air. Expain Laplace correction in Newton's formula.

6. Obtain the one dimension wave equation $y = A \sin(\omega t - kx)$ for the wave propagating in the direction of increasing value of x.
7. Write the superposition principle for the waves and explain it.
8. What are stationary waves ? Obtain the expression for the stationary wave in case of string fixed at its two ends.
9. Show that in a closed pipe the harmonics are possible only for odd multiples of fundamental frequency.
10. What is Doppler effect ? When the source of sound is stationary and listener is moving towards the source, obtain the expression for the wavelength of wave travelling towards the listner.

Solve the following problems :

1. In case of the progressive harmonic waves, prove that the ratio of the instantaneous velocity of any particle of the medium to the wave speed is equal to the negative of the slope of the waveform at that point at that instant.
2. Two types of waves, transverse (S) and longitudinal (P) are produced in the earth during an earthquake. The speed of the S wave is approximately 4.0 km/s and that of the P wave is 8.0 km/s. In a seismograph, recording the earthquake the P wave is recorded 4 min earlier than the S wave. Assuming that both types of waves travel on straight line, find the distance of the origin of the quake from the seismograph. [**Ans. :** Approximately 1920 km]
3. The amplitude of the progressive harmonic wave is 10 m. During the wave propagation, the displacement of a particle which is at a distance of 2 m from the origin is 5 m after 2 s. Another particle which is at 16 m from origin has displacement of $5\sqrt{3}$ m in 8 s. Find the angular frequency and wave vector of a wave. [**Ans. :** $\omega = \pi/8$ rad/s, $k = \pi/24$ rad/m]
4. The equation for a wave travelling in x–direction on a string is, $y = 3 \sin[(3.14x - (314)t]$. Where x is in cm and t is in second.

   (i) Find the maximum velocity of a particle of the string.

   (ii) Find the acceleration of a particle at $x = 6.0$ cm at time $= t = 0.11$ s.

   [**Ans. :** Maximum velocity = 9.4 m/s, $a = 0$]
5. At 0 °C temperature, a source of sound of frequency 250 Hz emits sound waves of wavelength 1.32 m. What will be the increase in wavelength at 27 °C ? [**Ans. :** 0.06 m]
6. At what temperature the hydrogen gas will have the speed of sound waves in it will be equal to the speed of sound in oxygen at 1200 °C ? The density of oxygen is 16 times that of hydrogen. [**Ans. :**–180.9 °C]
7. The length of a sonometer wire between its fixed ends is 110 cm. Where should the two bridges S_1 and S_2 be placed in between the ends so as to divide the wire into 3 segments whose fundamental frequencies are $f_1 : f_2 : f_3 =$ 1 : 2 : 3 ? [**Ans. :** $L_1 = 60$ cm, $L_2 = 30$ cm, $L_3 = 20$ cm]

8. A wire having a linear mass density 0.05 g/cm is stretched betwen two rigid supports with a tension of 450 N. The wire resonates at a frequency of 420 Hz. The next higher frequency at which the same wire resonates is 490 Hz. Find the length of the wire. [**Ans. :** 2.1 m]

9. The length of a string is 100 cm. The frequencies of two consecutive harmonics formed on the string are 300 Hz and 400 Hz respectively. The maximum amplitude is 10 cm when the string oscillates with its fundamental frequency. Write the equation of the stationary wave in this case.

[**Ans. :** $y = -10\sin\left(\frac{\pi x}{100}\right)\cos(200\pi)t$ (cm)]

10. Find the difference of apparent frequencies of the sound of a car horn heard by a stationary listener when the car is moving towards and away from the listener with a speed 54 km/4. The frequency of sound emitted by the horn is 500 Hz and speed of sound in air is 340 m/s. [**Ans. :** 44.2 Hz]

11. The whistle of an engine, approaching a hill with a speed of 10 m/s produces sound of frequency 660 Hz. Find the frequency experienced by the driver of the sound reflected from the hill. The speed of sound in air is 340 m/s.

[**Ans. :** 700 Hz]

• • •

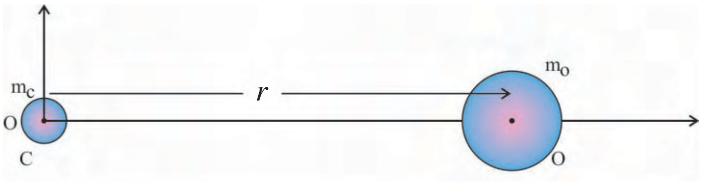
Meghnad Saha (1893-1956)

Meghnad Saha was born on October 6, 1893 in Sheoratali, a village in the District of Dacca (now in Bangladesh). In 1911, he came to Calcutta to study in Presidency College. He came to be recognised as a scientist of substance. In 1920, he went to England to prove his theory–equation of reaction–before the global scientific community. This later became Saha’s Thermo Ionization Equation. In 1927, Meghnad was elected as a fellow of London’s Royal Society. He invented an instrument to measure the weight and pressure of solar rays. The lasting memorial to him is the ‘Saha Institute of Nuclear Physics’ founded in 1943 in Calcutta. Saha passed away on February 16, 1956.

SOLUTION

CHAPTER 1

1.



Figure

Here the origin is taken on the centre of carbon (C).

r = distance of oxygen from carbon = 1.130×10^{-10} m,

m_O = mass of oxygen = 16 g mol^{-1}, m_C = mass of carbon = 12 g mol^{-1}

r_C = distance of carbon from origin = 0, r_O = distance of oxygen from origin $= r = 1.130 \times 10^{-1}$m

$$\therefore\ r_{cm} = \frac{m_C r_C + m_O r_O}{m_C + m_O}$$

2. Velocity of centre of mass $\vec{v}_{cm} = \dfrac{m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3}{m_1 + m_2 + m_3}$

3. Here for car m_1 = 1000 kg, a_1 = 4.0 m s^{-2}, initial speed v_{0_1} = 0 m s^{-1}

For truck m_2 = 2000 kg, a_2 = 0 m s^{-2}, $v_{0_2} = v_2$ = 8.0 m s^{-1}

After 3 sec, the speed of car $v_1 = v_{0_1} + a_1 t$

After 3 secs the distance travelled by car $d_1 = v_{0_1} t + \dfrac{1}{2} a_1 t^2$

The distance travelled by truck in 3 sec. $d_2 = v_2 t$ ($\because$ $a_2 = 0$)

(*a*) The distance of centre of mass of the system of car-truck is

$$d_{cm} = \frac{m_1 d_1 + m_2 d_2}{m_1 + m_2}$$

(*b*) In one dimension $\mathrm{M} v_{cm} = m_1 v_1 + m_2 v_2$

$$\therefore\ v_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \quad (\because \mathrm{M} = m_1 + m_2)$$

4. At t = 0 sec. $x_1 = -15\ m$, $x_2 = 15\ m$

m_1 = 40 kg, m_2 = 20 kg

$$\therefore\ x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

As the centre of mass is remaining stationary, x_{cm} = const. Hence find x_2 from the values of x_1 and x_{cm} for t = 2, 4 ,6 sec. At t = 0, the cat and dog are at rest.

$$\therefore\ v_1 = v_2 = 0 \Rightarrow p_1 = p_2 = 0$$

$$\Rightarrow p = p_1 + p_2 = 0$$

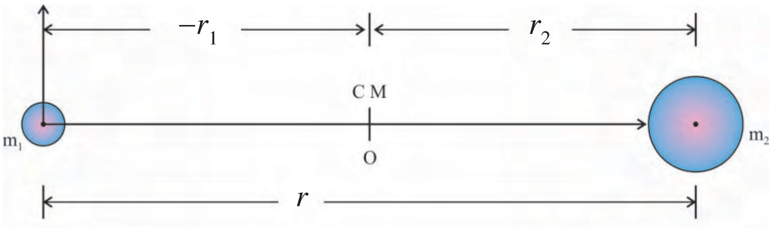
At $t = 2$ s

$$v_1 = \frac{x_1(2\text{ s}) - x_1(0\text{ s})}{2\text{ s}} = \frac{\Delta x}{\Delta t},\ v_2 = \frac{x_2(2\text{ s}) - x_2(0\text{ s})}{2\text{ s}}$$

Hence, $p_1 = m_1v_1$, $p_2 = m_2v_2$ and $p = p_1 + p_2$

Similarly, repeat calculations for $t = 4$ s and $t = 6$ s.

5.



Figure

In figure, the origin is taken at centre of mass.

$\therefore$ Position of m_1 from origin $= -r_1$, Position of m_2 from origin $= r_2$

$$\therefore r_{cm} = 0 = \frac{m_1(-r_1) + m_2r_2}{m_1 + m_2} \qquad \therefore m_1r_1 = m_2r_2$$

$$\therefore \frac{m_1}{m_2} = \frac{r_2}{r_1} \qquad (1)$$

Performing union in the denominator $\frac{m_1}{m_1 + m_2} = \frac{r_1}{r_1 + r_2} = \frac{r_2}{r}$

$$(\because r = r_1 + r_2)\ \therefore r_2 = r\left[\frac{m_1}{m_1 + m_2}\right]$$

Performing union in numerator in equation (1) $\frac{m_1 + m_2}{m_2} = \frac{r_1 + r_2}{r_1} = \frac{r}{r_1}$

$$\therefore r_1 = r\left[\frac{m_2}{m_1 + m_2}\right]$$

6. Here, the centre of mass of the system made up of three spheres is

$\vec{r}_{cm} = \dfrac{m\vec{r}_{cm1} + m\vec{r}_{cm2} + m\vec{r}_{cm3}}{m + m + m}$, Where $\vec{r}_{cm1}$ = centre of mass of sphere 1, etc.

7. Here, the density of sphere of radius R is ρ. Hence, the mass of original sphere

$$M = \rho V = \rho \times \frac{4}{3}\pi R^3 \qquad (i)$$

The mass of small sphere of radius 'a' is $m_1 = \rho \times \frac{4}{3}\pi a^3$ (ii)

Hence, the mass of the remaining sphere after removing small sphere of radius 'a' from the original sphere of radius 'R' is $m_2 = M - m_1$.

$$\therefore m_2 = \frac{4}{3}\pi\rho\ (R^3 - a^3) \qquad (iii)$$

The centre of mass of original sphere $\vec{r}_{cm} = (0, 0, 0)$

The centre of mass of small sphere of radius '*a*', $\vec{r_1} = (b, 0, 0)$

The remaining sphere has symmetry about X-axis, but no symmetry about Y and Z axes. Hence, the centre of mass of remaining sphere is, say $\vec{r_2} = (-x, 0, 0)$

The sphere of radius R is made up of small sphere of radius '*a*' and the remaining sphere (without small sphere). Hence, $M\vec{r}_{cm} = m_1\vec{r_1} + m_2\vec{r_2}$.

$\therefore$ M(0, 0, 0) = m_1(b, 0, 0) + m_2($-x$, 0, 0). Comparing x co-ordinates

$$M(0) = m_1 b - m_2 x \;\therefore\; x = \frac{m_1}{m_2} b \qquad (iv)$$

Using results (ii) and (iii), we get x.

8. From the figure, the masses of the three particles, the positions and forces acting on them during steady positions are

$m_1 = 4.0$ kg, $\vec{r_1} = (-2, 3)\ m$, $\vec{F_1} = (-6, 0)$ N

$m_2 = 8.0$ kg, $\vec{r_2} = (4, 2)\ m$, $\vec{F_2} = (12 \cos 45^o, 12\sin 45^o)$ N

$m_3 = 4.0$ kg, $\vec{r_3} = (1, -2)\ m$, $\vec{F_3} = (14, 0)$ N

$$\therefore \vec{r}_{cm} = \frac{m_1\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3}}{m_1 + m_2 + m_3}$$

According to the Newton's second law $\vec{F} = M\vec{a}_{cm}$, $M = m_1 + m_2 + m_3$

$$\therefore \vec{F_1} + \vec{F_2} + \vec{F_3} = M\vec{a}_{cm} \qquad \therefore \vec{a}_{cm} = \frac{\vec{F_1} + \vec{F_2} + \vec{F_3}}{M}$$

$$\therefore \vec{a}_{cm} = (a_{x_{cm}}, a_{y_{cm}})$$

Hence the magnitude of acceleration $|\vec{a}_{cm}| = \sqrt{(a_{xcm})^2 + (a_{ycm})^2}$

and the direction of acceleration with X-axis is $\theta = tan^{-1}\left(\frac{a_{ycm}}{a_{xcm}}\right) = ...$

9. From the figure, the centre of mass of the original plate of uniform density 'ρ' and radius 'R' is $\vec{r}_{cm} = (0, 0)$ (1)

The centre of mass of plate of radius $\frac{R}{2}$ is $\vec{r}_{cm_1} = \left(\frac{R}{2}, 0\right)$ (2)

When the plate of radius $\frac{R}{2}$ is cut from the plate of radius R, the remaining plate has symmetry about X-axis, but no symmetry about Y-axis. Hence the centre of mass of remaining plate must be away from the origin along X-axis as say $(-x)$. $\therefore\ \vec{r}_{cm_2} = (-x, 0)$ (3)

The original plate is made up plate of radius $\frac{R}{2}$ and remaining plate

$$\therefore\ \vec{r}_{cm} = \frac{M_1 \vec{r}_{cm1} + M_2 \vec{r}_{cm2}}{M_1 + M_2} \quad (4)$$

Where M_1 = Mass of plate of radius $\frac{R}{2}$ $\therefore\ M_1 = \pi\left(\frac{R}{2}\right)^2 t\rho$

$$M_2 = \text{Mass of remaining plate} = \pi R^2 t\rho - M_1 = \pi R^2 t\rho - \pi\left(\frac{R}{2}\right)^2 t\rho$$

$$M_2 = \pi t\rho\left[R^2 - \left(\frac{R}{2}\right)^2\right]$$

Where ρ = Density of plate, t = thickness of plate

Hence, from equation (4) calculate $\vec{r}_{cm2}$.

CHAPTER 2

1. Using equation $\theta = \left(\frac{\omega + \omega_0}{2}\right)t$ find ω_0. Substituting ω_0 in the equation $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ find α.

2. Substituting values $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ find α. Now from $\theta = \frac{\omega^2 + \omega_0{}^2}{2\alpha}$ find θ and represent θ in rotations. (2π rad = 1 rotation)

3. Using $\alpha = \frac{\omega - \omega_0}{t}$, find α. Now $I = m\, r^2$, using $\tau = I\alpha$, find τ from $\theta = \frac{\omega^2 + \omega_0{}^2}{2\alpha}$ find θ. Now work $= \tau \cdot \theta$

4. Use $\vec{l} = \vec{r} \times \vec{p}$, $\vec{r} = 4\hat{i} + 6\hat{j} + 12\hat{k}$ and $\vec{p} = m\vec{v} = 50\,(2\hat{i} + 3\hat{j} + 6\hat{k})$

5. Linear acceleration for a body rolling down the slope is $a = \frac{g\sin\theta}{\left[1 + \frac{K^2}{R^2}\right]}$

Substituting K = R the radius of gyration for hollow cylinder obtain a.

6. Moment of inertia of the system $I_z = I_{1z} + I_{2z}$. I_{1z} = Moment of inertia of the object of 100 kg relative to z axis I_{2z} = moment of inertia of the object of 200 kg relative to z axis. As the distances are relative to z axis, z coordinate is not taken in to calculation.

$I_z = I_x + I_y = m(x^2 + y^2)$ (1)

position vectors for the objects of 100 kg and 200 kg are (2, 4, 6) and (3, 5, 7) respectively.

$\therefore\ I_{1z} = I_{1x_1} + I_{2y_1} = 100\ (x^2_1 + y^2_1),\ I_{2z} = I_{1x_2} + I_{2y_2} = 200\ (x^2_2 + y^2_2)$

Substitute in (1)

7. Using $K = \sqrt{\frac{2}{5}}R$ for solid sphere in $v^2 = \dfrac{g\sin\theta}{\left[1+\dfrac{K^2}{R^2}\right]}$ find v.

Now using $mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$. Calculate rotational kinetic energy. $(\frac{1}{2}I\omega^2)$

8. Consider Earth as solid sphere and taking its moment of inertia $I = \frac{2}{5}MR^2$ and $\omega = \frac{2\pi}{T} = \frac{2\pi}{24\times3600}$ substitute in $L = I\omega$ and obtain L.

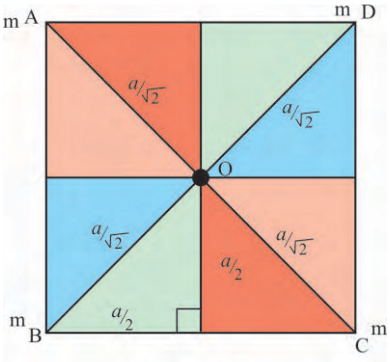
9. $I_1 = I_C + Md_1^2 \qquad \therefore\ I_C = I_1 - Md_1^2$

Now $I_2 = I_C + Md_2^2 = I_1 - Md_1^2 + Md_2^2$

$= I_1 + M(d_2^2 - d_1^2)$

10. From the figure the moment of inertia I of the system about the axis passing through O.

$$I = \frac{ma^2}{2} + \frac{ma^2}{2} + \frac{ma^2}{2} + \frac{ma^2}{2} + = 2ma^2$$



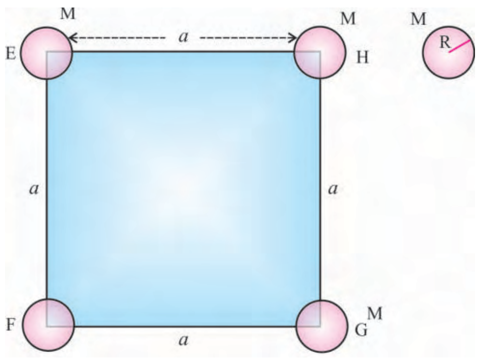
11. Moment of inertia of the sphere about the axis passing through its centre is $I_C = \frac{2}{5}MR^2$

From the figure moment of inertia of the system about the axis EF

$I = I_E + I_F + I_G + I_H$

Using $I = I_C + Md^2$

$I_E = \frac{2}{5}MR^2;\ I_F = \frac{2}{5}MR^2;$



$$I_G = \frac{2}{5}MR^2 + Ma^2; \; I_H = \frac{2}{5}MR^2 + Ma^2$$

$$\therefore \; I = \frac{2}{5}MR^2 + \frac{2}{5}MR^2 + \frac{2}{5}MR^2 + ma^2 + .\frac{2}{5}MR^2 + ma^2$$

$$= 2\left(\frac{4}{5}MR^2 + Ma^2\right)$$

12. $r_1 = 0$, $r_2 = 2m$, $r_3 = 4m$, $r_4 = 6m$, $m_1 = 1$ kg, $m_2 = 2$ kg, $m_3 = 3$ kg, $m_4 = 4$ kg

Now, $I_{AB} = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + m_4 r_4^2$

13. Total kinetic energy = Linear kinetic energy + Rotational kinetic energy

$$= \frac{1}{2}mv^2 + \frac{1}{2}\; I\omega^2$$

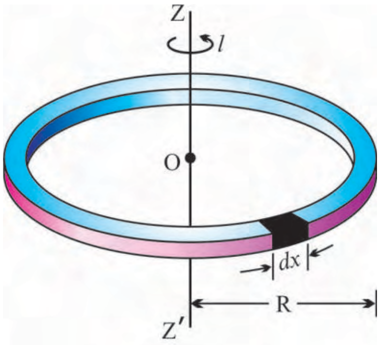
for disc $I = \frac{mr^2}{2}$ substituting $\omega = \frac{v}{r}$ $(\because \; v = r\omega)$

Total kinetic energy $= \frac{1}{2}mv^2 + \frac{1}{2}\frac{mr^2}{2}\frac{v^2}{r^2} = \frac{3}{4}mv^2$

Rotational kinetic energy $= \frac{1}{4}mv^2$

The fraction of total kinetic energy in the form of rotational kinetic energy $= \dfrac{\frac{1}{4}mv^2}{\frac{3}{4}mv^2} = \frac{1}{3}$

14. To find moment of inertia of thin circular ring or circular wire about an axis passing through its centre and perpendicular to its plane and radius of gyration, consider a thin ring with mass M and radius R as shown in the figure. Length of the ring l that is the circumference of the ring is $2\pi R$.



Mass per unit length of this ring

$$\lambda = \frac{\text{Mass of the ring}}{\text{Length of the ring}} = \frac{M}{2\pi R}$$

Mass of the element of the length dx as shown in the figure $= \lambda \cdot dx = \frac{M}{2\pi R}dx$

If dI is the moment of inertia about the axis ZZ'.

dI = (mass of the element) (perpendicular distance from ZZ' axis)

$$= \left(\frac{M}{2\pi R} \cdot dx\right)(R^2)$$

$$dI = \frac{M}{2\pi}R \cdot dx \qquad (1)$$

For the moment of inertia I of the ring as a whole about axis ZZ' integrate equation (1) in the interval from $x = 0$ to $x = 2\pi R$.

$$\therefore I = \int dI = \int_0^{2\pi R} \frac{M}{2\pi}R \cdot dx$$

$$\therefore I = \frac{M}{2\pi}R \int_0^{2\pi R} dx = \frac{M}{2\pi}R[x]_0^{2\pi R} = \frac{M}{2\pi}R[2\pi R - 0] \quad I = MR^2 \qquad (2)$$

Comparing equation (2) with $I = MK^2$, $K^2 = R^2$, Radius of gyration $K = R$

15. The vector sum of the forces acting on the light rod,

$$\vec{F} = \vec{F_1} + \vec{F_2} + \vec{F_3} + \vec{F_4} + \vec{F_5} \quad (\vec{F} \text{ is the resultant force})$$

$$\vec{F} = \vec{F_1}\,\hat{j} + \vec{F_2}\,\hat{j} + \vec{F_3}(-\hat{j}) + \vec{F_4}\,\hat{j} + \vec{F_5}(-\hat{j})$$

Now, moment of force $\vec{F}$ relative to point A = vector sum of the moments of component forces.

$\therefore F \cdot x = [F_1 \times 0] + [F_2 \times x_1] - [F_3 \times (x_1 + x_2)] + [F_4 \times (x_1 + x_2 + x_3) - [F_5 \times (x_1 + x_2 + x_3 + x_4)]$

$$\therefore x = \frac{x_1F_2 - (x_1 + x_2)F_3 + (x_1 + x_2 + x_3)F_4 - (x_1 + x_2 + x_3 + x_4)F_5}{F_1 + F_2 + F_4 - F_3 - F_5}$$

CHAPTER 3

1. If two forces become equal at distance x from the centre of the Earth,

$$\frac{GM_e m}{x^2} = \frac{GM_s m}{(r - x)^2}, \qquad M_e = \text{Mass of the Earth,}$$

M_s = Mass of the sun. r = Distance between the sun and the Earth. From this find x.

2. $M_e = \text{Volume} \times \text{Density} = \left(\frac{4}{3}\pi R_e^{\;3}\right)(\rho)$

$$\therefore g = \frac{GM_e}{R_e^{\;2}} = \frac{4}{3}\pi G\rho R_e. \text{ Hence find } g.$$

3. $$\left\{\begin{matrix}\text{The required centripetal} \\ \text{force for the Earth's} \\ \text{circular motion} \\ \dfrac{M_e v_0^{\ 2}}{r}\end{matrix}\right\} = \left\{\begin{matrix}\text{The gravitational force} \\ \text{on the Earth by} \\ \text{the Sun} \\ \dfrac{GM_s M_e}{r^2}\end{matrix}\right\}$$

$$\therefore \ M_s = \frac{r v_0^{\ 2}}{G}$$

4. For the circular motion of the satellite,

$$v_0 = \sqrt{\frac{GM_e}{r}} = \sqrt{\frac{GM_e}{2R_e}} \quad (\because r = R_e + R_e = 2R_e)$$

Find v_0 from this. Now $T^2 = \left(\dfrac{4\pi^2}{GM_e}\right) r^3$. From this find T

5. For circular motion of satellite $mv^2/r = GM_e \ m/r^2$

$\therefore$ Kinetic energy of satellite $\dfrac{1}{2}mv^2 = \dfrac{GM_e m}{2r}$

But potential energy $= \dfrac{-GM_e m}{r}$

$\therefore$ Total energy = kinetic energy + potential energy $= \dfrac{-GM_e m}{2r}$

$\therefore$ Escape energy $= \dfrac{GM_e m}{2r}$

$\therefore \ \dfrac{1}{2}mv_e^{\ 2} = \dfrac{GM_e m}{2r}$. From this find v_e.

6. For the circular motion of the satellite, $\dfrac{mv^2}{R_e} = \dfrac{GM_e m}{R_e^{\ 2}} = (g)m$ (1)

$(\because \ g = \dfrac{GM_e}{R_e^{\ 2}})$ $\therefore \ v^2 = gR_e$. But $v = \dfrac{2\pi R_e}{T}$

Put this value in equation (1) and find T.

7. For circular motion of the satellite, $\dfrac{mv_0^{\ 2}}{R_e} = \dfrac{GM_e m}{R_e^{\ 2}}$ $\therefore \ v_0 = \sqrt{\dfrac{GM_e}{R_e}}$

For the object lying on the surface of the Earth, $v_e = \sqrt{\dfrac{2GM_e}{R_e}}$. Find $\dfrac{v_0}{v_e}$

8. At the given point the total energy $= \left[-\dfrac{GM_1 m}{d/2}\right] + \left[\dfrac{-GM_2 m}{d/2}\right]$

$= \dfrac{-2G(M_1 + M_2)m}{d}$ $\therefore$ Escape energy $= \dfrac{2G(M_1 + M_2)m}{d}$

If the escape velocity is v_e, then $\dfrac{1}{2}mv_e^{\ 2} = \dfrac{2G(M_1 + M_2)m}{d}$. Hence, find v_e

9. In this special case, the circular motion is governed by,

$$\left(\text{Centripetal force } \frac{mv^2}{r}\right) = \left(\text{Gravitational force } \frac{GMm}{r^{5/2}}\right)$$

Also put $v = \frac{2\pi r}{T}$. Hence, find T^2.

CHAPTER 4

1. Here, weight of wire = tensile force = ldg and breacking stress = $\frac{\text{Tensile force}}{\text{Area}} = \text{ldg} \quad \therefore \ L = \frac{\text{Breaking stress}}{dg}$

2. If increase in lengths of AB, BC and CD wires are Δl_{AB}, Δl_{BC} and Δl_{CD}, find these increments using

$\Delta l = \frac{Fl}{AY}$, Displacement of B = Δl_{AB}, Displacement of C = $\Delta l_{AB} + \Delta l_{BC}$

and Displacement of D = $\Delta l_{AB} + \Delta l_{BC} + \Delta l_{CD}$.

3. Centripetal force necessary for circular motion is supplied by restoring force.

$Y = \frac{FL}{A\Delta l} \quad \therefore \ F = \frac{YA\Delta L}{L}$ and $F = \frac{mv^2}{L} = \frac{m\omega^2 L^2}{L}$ compose these two values of F.

4. Draw F.B.D. for both the masses and calculate tension T.

Here, Stress = $\frac{T}{A}$ and $\frac{\Delta l}{l} = \frac{\text{Stress}}{Y}$

5. First ditermine Δl using $Y = \frac{Fl}{A\Delta L}$; now use illustration 3.

Use $U = \frac{1}{2} Y \times \text{stress} \times \text{strain} \times \text{volume}$.

6. $\Delta l = l \propto \Delta t \quad \therefore \ \frac{\Delta l}{l} = \propto \Delta t$

Now use $Y = \frac{F}{A} \frac{l}{\Delta t}$; here F is tension. Now find F.

CHAPTER 5

1. Find velocity of water coming out of nozzle using $A_1 v_1 = A_2 v_2$

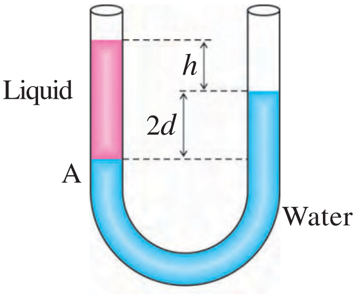
Now for vertical motion $y = \frac{1}{2} g t^2$, $y = 1$ m, for horizontal motion $x = v_2 t$

$$\therefore \ y = \frac{1}{2} g \left(\frac{x}{v_2}\right)^2 \quad \therefore \ x = \sqrt{\frac{2yv_2^2}{g}}$$

2. Pressure at A = Pressure at B

$\therefore\ (h + 2d)\rho_l g + P_a = P_a + 1(2d)g$

Now, find ρ_l.



3. For horizontal flow

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

$$\therefore\ P_1 - P_2 = \frac{1}{2}\rho(v_2^2 - v_2^2)$$

$\therefore\ \rho_{Hg}\, g(h_1 - h_2) = \dfrac{1}{2}\rho_{water}\,(r_2^2 - r_1^2)$ insert other values to get h_2.

4. Work $= T\Delta A = T2\pi\ (r_2^2 - r_1^2)$

5. $T = \dfrac{rh\rho g}{2cos\theta}$

$\therefore\ h = \dfrac{2Tcos\theta}{r\rho g}$ Use this formula to calculate heights of water in both the arms. Then find the differences.

6. $\eta = \dfrac{2}{9}\dfrac{v^2}{vt}(\rho - \rho_0)g$

Here, constant velocity of bubble is the terminal velocity.

7. and **8.** Use equation given in hint.

9. Find P_i usng $P_i - P_o = \dfrac{4T}{R}$ $P_o = 10^5$ P*a*

Now for isothermal change $P_i V = P_i'\dfrac{V}{8}$

Find P_i', No $P_i' - P_o' = \dfrac{4T}{R'}$ take $R' = \dfrac{R}{2}$ and calculate P_o'.

CHAPTER 6

1. $m = 200$ g, $\Delta T = T_f - T_i$, $C = 0.215$ cal $g^{-1}\ C^{o-1}$, $Q = mC\Delta T$

and $H_C = \dfrac{Q}{\Delta T}$

2. (*a*) 32 g O_2 = 1 mole

$\therefore$ 10 g $O_2 = \dfrac{10}{32} = \dfrac{5}{6}$ mole

$\therefore\ \mu = \dfrac{5}{6}$ mole

$P = 3 \times 10^5\ N\ m^{-2}$, $T = 273 + 10 = 283$ K

From the ideal gas, state equation,

$PV_1 = \mu RT_1 \Rightarrow V_1 = \dfrac{\mu RT_1}{P}$

and $V_2 = 10\ L = 10^{-2}\ m^3$

Hence, the work done by the gas

$W = P(V_2 - V_1)$

(*b*) As O_2 is diatomic rigid rotator,

$$C_V = \frac{5}{2}R$$

and $PV_2 = \mu RT_2 \Rightarrow T_2 = \dfrac{PV_2}{\mu R}$

$\therefore\ \Delta E_{int} = C_V\,(T_2 - T_1)$

(*c*) $\Delta E_{int} = Q - W$

$\therefore\ Q = \Delta E_{int} + W$

3. Here $T_2 = 300$ K, $\eta = 40\ \% = 0.4$, $\eta = 1 - \dfrac{T_2}{T_1}$, Hence calculate T_1.

Keeping T_1 = constant, $\eta' = 50\% = 0.5$, then $T_2' = ?$

From, $\eta' = 1 - \dfrac{T_2'}{T_1}$, find T_2'.

4. $T_1 = 500$ K, $T_2 = 375$ K, $Q_1 = 600$ k cal

(i) Efficiency $\eta = 1 - \dfrac{T_2}{T_1}$ (ii) $\dfrac{Q_2}{Q_1} = \dfrac{T_2}{T_1} \Rightarrow Q_2 = \dfrac{T_2}{T_1} \times Q_1$

Hence network done $W = (Q_1 - Q_2) \times 4.2\dfrac{J}{cal}$

(iii) Heat gained back in heat sink is $= Q_2$

5. $T_i = 27\ ^{o}C = 27 + 273 = 300$ K

$P_i = 2$ atm, $\mu = 1$ mol, $\gamma = 1.5$, $V_f = \dfrac{1}{8}V_i$

(*a*) For adiabatic process PV^{γ} = constant

$$\therefore\ P_iV_i^{\gamma} = P_f\,V_f^{\gamma},\ \Rightarrow P_f = P_i\left(\frac{V_i}{V_f}\right)^{\gamma}$$

(*b*) According to the ideal gas state equation $P_iV_i = \mu RT_i$

$$P_fV_f = \mu RT_f\ \therefore\ \frac{P_iV_i}{P_fV_f} = \frac{T_i}{T_f} \Rightarrow T_f = T_i\,\frac{P_fV_f}{P_iV_i}$$

6. For adiabatic process $W = \dfrac{\mu R(T_i - T_f)}{\gamma - 1}$. Here the volume decreases, hence, the work done is negative.

$$\therefore\ W = \frac{-\mu R(T_i - T_f)}{\gamma - 1} = \frac{\mu R(T_f - T_i)}{\gamma - 1}$$

7. According to first law of thermodynamics, $\therefore\ \Delta E_{int} = Q - W$

For closed gas container, the volume is constant $\Rightarrow \Delta V = 0\ \therefore\ W = 0$

$$\Delta E_{int} = Q = \mu C_V \Delta T = \frac{PV}{RT}C_V\Delta T\ (\because\ PV = \mu RT,\ \therefore\ \mu = \frac{PV}{RT})$$

$$\therefore\ \Delta T = \frac{QRT}{PVC_V}\ \text{(For monoatomic gas } C_V = \frac{3}{2}R)$$

$\therefore$ Final temperature $T_f = T_i + \Delta T$. For ideal gas $P_iV_i = \mu RT_i$ $P_fV_f = \mu RT_f$

$(\because\ V_f = V_i)$

$$\therefore\ \frac{P_f}{P_i} = \frac{T_f}{T_i} \Rightarrow P_f = P_i \frac{T_f}{T_i}$$

8. Here, $\mu = 1$ mol, $\Delta T = 30\ C^o = 30$ K, $V \propto T^{\frac{2}{3}}$

$$\therefore\ V = AT^{\frac{2}{3}},\ A = \text{constant},\ \therefore\ dV = A\frac{2}{3}T^{-\frac{1}{3}}dT$$

$$\text{Hence, } W = \int_T^{T+\Delta T} PdV = \int_T^{T+\Delta T} \frac{RT}{V}dV\ (\because PV = \mu RT,\ \therefore\ PV = RT,\ \mu = 1)$$

$$= \int_T^{T+\Delta T} \frac{RT}{AT^{\frac{2}{3}}} A\frac{2}{3}T^{-\frac{1}{3}}dT = \frac{2R}{3}\int_T^{T+\Delta T} dT = \frac{2}{3}R\,[T]_T^{T+\Delta T}$$

$$= \frac{2}{3}R[T + \Delta T - T] \qquad \therefore\ W = \frac{2}{3}R\Delta T$$

9. Here, $P = 1.0$ atm $= 1.01 \times 10^5$ N m^{-2}, T = 300 K, $\mu = 2$ mol, R = 8.31 J mol^{-1} K^{-1}

For diatomic (rigid rotator) gas $\gamma = \frac{7}{5}$. According to ideal gas state equation

$PV = \mu RT$, $\therefore\ V = \frac{\mu RT}{P}$. For adiabatic process, PV^{γ} = Constant

$$\therefore\ \text{Constant} = P\left(\frac{\mu RT}{P}\right)^{\gamma}$$

10. Here $T_1 = 300$ K, $T_2 = 600$ K, $T_3 = 455$ K for monoatomic gas $f = 3$

For 1 mole gas

$E_{int,\ 1} = \frac{fRT_1}{2}$, $E_{int,\ 2} = \frac{fRT_2}{2}$ and $E_{int,\ 3}$ = Internal energy at point $3 = \frac{fRT_3}{2}$

Process 1 → 2 : Process is isobaric $\Rightarrow W_1 = 0$

$\therefore\ Q_1 = \Delta E_{int,\ 12} = E_{int,\ 2} - E_{int,\ 1}$

Process 3 → 1 : Process is isobaric

$\therefore\ \Delta E_{int,\ 31} = Q_3 - W_3,\ W_3 = PdV$

But as the volume of the gas is decreasing, W is negative

$\therefore\ W_3 = -PdV = -\mu R(T_3 - T_1)$ and $\Delta E_{int',\ 31} = E_{int',\ 1} - E_{int',\ 3}$

Hence $Q_3 = \Delta E_{int',\ 31} + W_3$

11. $\eta = 22\% = 0.22,\ Q_1 - Q_2 = 75\ J,\ \eta = \dfrac{Q_1 - Q_2}{Q_1} \Rightarrow Q_1 = \dfrac{Q_1 - Q_2}{\eta}$

and $Q_2 = Q_1 - 75$ J

12. Here $Q_1 = 10{,}000$ J, $W = 2000$ J, $L_C = 5.0 \times 10^4$ J/g

(*a*) Efficiency of engine $\eta = \dfrac{W}{Q_1}$,

(*b*) During each cycle, the heat given into heat sink is $Q_2 = Q_1 - W$,

(*c*) Let '*m*' gram gasoline is used during each cycle.

$\therefore\ Q_1 = mL_C\ \therefore\ m = \dfrac{Q_1}{L_C}$,

(*d*) Gasoline used in each cycle = *m* gram

$\therefore$ Gasoline used in 25 cycles per second is $M = 25 \times m$ gram

$\therefore$ Gasoline used in 1 hour = 60 × 60 × M g/h = ... kg/h

(*e*) Power generated by engine in 1 second = Number of cycles per second × (work done during each cycle)

CHAPTER 7

1. (*a*) $T = 3$ s, $A = 2$ cm, $\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{3},\ \phi = 60^o = \dfrac{\pi}{3}$

$$\therefore\ y = 2\sin\left(\frac{2\pi}{3}t + \frac{\pi}{3}\right)$$

(*b*) $T = 1$ min $= 60$ s, $A = 3$ cm, $\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{60},\ \phi = -90^o = -\dfrac{\pi}{2}$

$$\therefore\ y = 3\cos\left(\frac{\pi}{30}t\right)$$

2. $K = k + 2k + k = 8\ N\ m^{-1},\ T = 2\pi\sqrt{\dfrac{m}{K}} = 0.628$ s

3. Here $F = -kl = -k(l_1 + l_2)$, Also $F_1 = -k_1 l_1 = -k(l_1 + \dfrac{l_1}{n}) \therefore k_1 = (1 + \dfrac{l_1}{n})\ k$,

And $F_2 = -k_2 l_2 = -k(l_2 + l_2)\ \therefore\ k_2(n + 1)k$

4. $m = 100\ g$, $\mathrm{A}\ (t) = \dfrac{\mathrm{A}}{2}$, $t = 100 \times 2 = 200$ s, $\mathrm{A}(t) = \mathrm{A}^{-bt/2m}$

5. $v = \pm\omega\sqrt{4\mathrm{A}^2 - 3y^2}$, $v_{new} = \pm\omega\sqrt{\mathrm{A}_1{}^2 - y_1{}^2}$ As $v_{new} = 2\ v$, $2\sqrt{\mathrm{A}^2{}_{new} - y^2}$ =

$\sqrt{\mathrm{A}^2{}_{new} - y^2}$, $4(\mathrm{A}^2 - y^2) = \mathrm{A}^2{}_{new} - y^2 \quad \therefore \quad \mathrm{A}^2{}_{new}\ \mathrm{A}4^2 - 4y^2 + y,$

$\mathrm{A}_{new} = \quad \sqrt{4\mathrm{A}^2 - 3y^2}$

6. $v = \omega\sqrt{\mathrm{A}^2 - y^2}$, $a = -\omega^2 y$, $\mathrm{T} = \dfrac{2\pi}{\omega}$, $a^2\mathrm{T}^2 + 4\pi^2 v^2 = 4\pi^2\omega^2\mathrm{A}^2$ = Constant.

7. $\mathrm{T} - mg\ \cos\theta = mv^2/\mathrm{L} \ \therefore\ \mathrm{T} = mg\ \cos\theta + mv^2/\mathrm{L}$

$\mathrm{T} = \mathrm{T}_{max}$, when $\cos\theta = 1$ and v is maximum

$v^2{}_{max} = 2\ hg = 2g\ \mathrm{L}\ \dfrac{\theta_0{}^2}{2}$, $v^2{}_{max} = 2\ hg = 2g\ \mathrm{L}\ (1 - \cos\theta_1)$,

$= 2g\ \mathrm{L}\ (\sin^2\dfrac{\theta_0}{2})\ (\because\ \sin^2\theta = \dfrac{1-\cos^2\theta}{2}) = 2g\ \mathrm{L}\ \dfrac{\theta_0{}^2}{2}$

$g\mathrm{L}\left(\dfrac{\mathrm{A}}{\mathrm{L}}\right)^2 \quad \mathrm{T}_{max} = mg\left[1 + \left(\dfrac{\mathrm{A}}{\mathrm{L}}\right)^2\right]$

8. $y_1 = 10\ \sin\ (3\pi t + \dfrac{\pi}{4})$, $\mathrm{A}_1 = 10$, $\omega_1 = 3\pi \Rightarrow \mathrm{T}_1 = \dfrac{2}{3}$ s

$y_2 = 5\ (\sin 3\pi t + \sqrt{3}\cos 3\pi t) = \mathrm{A}_2\cos\phi\ \sin\ 3\pi t + \mathrm{A}_2\sin\phi\ \cos\ 3\pi t$

$y_2 = \mathrm{A}_2\ \sin\ (3\pi t + \phi)$, $\mathrm{A}_2 = \sqrt{(5)^2 + (5\sqrt{3})^2} = 10$

$\omega_2 = 3\pi$, $\mathrm{T}_2 = \dfrac{2}{3}$ s, And $\dfrac{\mathrm{A}_1}{\mathrm{A}_2} = 1$

9. PE $= \dfrac{1}{2}ky^2$. Total mechanical energy E = K + U $\therefore$ K = E − U

10. $v_1 = \omega\sqrt{\mathrm{A}^2 - y_1{}^2}$, $v_2 = \omega\sqrt{\mathrm{A}^2 - y_2{}^2}$, $v_1{}^2 - v_2{}^2 = \omega^2(y_2{}^2 - y_1{}^2)$

$\mathrm{T} = \dfrac{2\pi}{\omega}$

CHAPTER 8

1. Differentiate wave equation $y = \text{A} \sin(\omega t - kx)$ w.r.t 't', the instantaneous velocity of a particle at time 't' will be, $v_p = \frac{dy}{dt} = \text{A}\omega \cos(\omega t - kx)$.

Now, wave speed $v = \omega/k$

Slope of wave at $x = \frac{dy}{dx} = -k\text{A}\cos(\omega t - kx)$

From above all three equations, $\frac{v_\text{P}}{v} = -\frac{dy}{dx}$

2. Speed of P wave $v_\text{P} = \frac{d}{t}$, Speed of S wave $v_\text{S} = \frac{d}{t+240}$

($\because$ 4 min = 60 × 4 = 240 s)

By solving these two equations $t = 240$ s

Now, substitute value of t and v_P in equation $v_\text{P} = \frac{d}{t}$ and find out d.

3. $\text{A} = 10$ m, $x_1 = 2$ m, $t_1 = 2$ s and $y_1 = 5$ m, $x_2 = 16$ m, $t_2 = 8$ s and $y_2 = 5\sqrt{3}$ m

Now, substitute these values in equation $y_1 = a \sin(\omega t_1 - kx_1)$

$$\omega - \text{k} = \frac{\pi}{12} \qquad (1)$$

From equation $y_2 = \text{A}\sin(\omega t_2 - kx_2)$ you will get,

$$\omega - 2k = \frac{\pi}{24} \qquad (2)$$

Subtracting equation (2) from equation (1)

$k = \frac{\pi}{24}$ rad/m, substitute value of k in equation (1), $\omega = \pi/8$ rad/s

4. $y = 3 \sin((3.14)x - (314)t)$ differentiate equation w.r.t. 't'

$$v = \frac{dy}{dx} = (3)\,(314)\cos(\,(3.14)x - (314)t\,)$$

$\therefore$ Max speed of particle = (3) (314) = 9.4 m s^{-1}. Differentiate above equation w.r.t. 't'.

$$a = \frac{dv}{dt} = -(3)(314)(314)\ \sin\ (\ (3.14)x - (314)t)$$

Now put $x = 6$ cm and $t = 0.11$ s,

$$a = -\ (3)\ (314)^2\ \sin\ (6\pi - 11\pi) = (-3)\ (314)^2\ \sin\ (-5\pi) = 0.$$

5. $T_1 = 0. + 273 = 273$ K, $\lambda_1 = 1.32$ m, $T_2 = 27 + 273 = 300$ K, $\lambda_2 = ?$

$$\text{Now, } \frac{v_1}{v_2} = \sqrt{\frac{T_1}{T_2}}$$

$$\therefore\ \frac{\lambda_1}{\lambda_2} = \sqrt{\frac{T_1}{T_2}}\ (\because\ v = f\,\lambda)$$

Substitute the values in above equation, $\lambda_2 = 1.384$ m,

Increase in the wavelength $\Delta\lambda = \lambda_2 - \lambda_1 = 0.064$ m

6. $T_0 = 1200 + 273 = 1473$ K, $\rho_0 = 16\ \rho_H$, $T_H = ?$ Now, $v_0 = v_H$

$$\therefore\ \sqrt{\frac{\gamma R T_0}{\rho_0 V}} = \sqrt{\frac{\gamma R T_H}{\rho_H V}}\ \ \therefore\ T_H = T_0 \times \frac{\rho_H}{\rho_0} = 1473 \times \frac{1}{16} = 92.06\ \text{K}$$

$$\therefore\ T_H = 92.06 - 273 = -\ 180.94^\circ\text{C}$$

7. Wave speed is same in all parts of the wire as the medium (wire) is same

$$\therefore\ v = f_1\lambda_1 = f_2\lambda_2 = f_3\lambda_3$$

Each section of wire is oscillating with fundamental frequency ($f = 2L$)

$\therefore\ f_1\ (2L_1) = f_2\ (2L_2) = f_3\ (2L_3)$, Now, put $f_1 : f_2 = 1 : 2$ and $f_1 : f_3 = 1 : 3$

in above equation and determine L_1, L_2 and L_3.

8. $\mu = 0.05$ g/cm, $f_n = 420$ Hz, $f_{n+1} = 490$ Hz, T = 490 N

Suppose the wire vibrates at 420 Hz in its nth harmonic and at 490 Hz in its ($n + 1$)th harmonic. According to $f = \frac{n}{2L}\sqrt{\frac{T}{\mu}}$

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}}\quad (1)\ \text{and}\quad f_{n+1} = \frac{n+1}{2L}\sqrt{\frac{T}{\mu}}\quad (2)$$

Taking the ratio,

$$\frac{f_{n+1}}{f_n} = \frac{n+1}{n}\quad \therefore\ n = 6\quad \text{(by putting value of } f_n \text{ and } f_{n+1})$$

$$420 = \frac{6}{2L}\sqrt{\frac{450}{5 \times 10^{-3}}} = \frac{900}{L}\ \ \therefore\ L = \frac{900}{420} = 2.1\ \text{m}$$

9. L = 100 cm, f_n = 300 Hz, f_{n+1} = 400 Hz, 2A = 10 cm

Now, $f_{n+1} - f_n = (n+1) f_1 - nf_1$, $\therefore f_1 = 100$ Hz

$\pi = \frac{2L}{1} = 200$ cm, $\therefore k = \frac{2\pi}{\lambda} = \frac{\pi}{100}$ rad/cm

$\omega = 2\pi f_1 = 2\pi(100)$ rad/s

Equation of stationary wave, $y = -10 \sin(\frac{\pi}{100}x) \cos(200\pi)t$ cm

10. When the car is moving towards the listener, $f_{L_1} = \left(\frac{v+0}{v-v_S}\right) f_s$

When the car is moving away from the listener, $f_{L_2} = \left(\frac{v+0}{v+v_S}\right) f_S$

$$\therefore f_{L_1} - f_{L_2} = \left(\frac{v}{v-v_S} - \frac{v}{v+v_S}\right) f_S$$

Substitute, v = 340 m/s, v_S = 15 m/s and f_S = 500 Hz in above equation.

$f_{L_1} - f_{L_2}$ = 44.2Hz

11. f_S = 600 Hz, v = 340 m/s, v_L = 10 m s^{-1}

When the engine is moving with the speed 10 m s^{-1} towards the hill, we can consider its image moving in opposite direction. Listener is sitting in the engine and engine is moving towards the hill. Hence, direction of v_L is along L to S and direction of v_S is from S to L.

$$\therefore f_L = \frac{v+v_L}{v-v_S} \times f_S = \frac{340+10}{340-10} \times 600 = 700 \text{ Hz}$$

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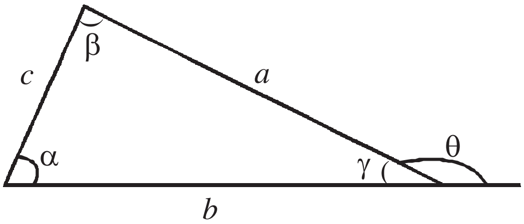
APPENDIX

SINE AND COSINE RULES

(i) $\frac{\sin\ \alpha}{a} = \frac{\sin\ \beta}{b} = \frac{\sin\ \gamma}{c}$

(ii) $c^2 = a^2 + b^2 - 2\ abcos\ \gamma$

(iii) Exterior angle, $\theta = \alpha + \beta$



TRIGONOMETRIC IDENTITIES

(i) $sin^2\theta + cos^2\theta = 1$

(ii) $1 + tan^2\theta = sec^2\theta$

(iii) $1 + cot^2\theta = cosec^2\theta$

(iv) $sec^2\theta - tan^2\theta = 1$

(v) $cosec^2\theta - cot^2\theta = 1$

(vi) $sin2\theta = 2sin\theta\ cos\theta$

(vii) $cos2\theta = cos^2\theta - sin^2\theta = 2cos^2\theta - 1 = 1 - 2sin^2\theta$

(viii) $sin(\alpha \pm \beta) = sin\alpha cos\beta \pm cos\alpha sin\beta$

(ix) $cos(\alpha \pm \beta) = cos\alpha cos\beta \mp sin\alpha sin\beta$

(x) $sin\alpha \pm sin\beta = 2sin\left(\frac{\alpha \pm \beta}{2}\right)cos\left(\frac{\alpha \mp \beta}{2}\right)$

(xi) $cos\alpha + cos\beta = 2cos\left(\frac{\alpha + \beta}{2}\right)cos\left(\frac{\alpha - \beta}{2}\right)$

(xii) $cos\alpha - cos\beta = -2sin\left(\frac{\alpha + \beta}{2}\right)sin\left(\frac{\alpha - \beta}{2}\right)$

Values of sine and cosine for special angles :

Function	**0°** **0 rad**	**30°** $\frac{\pi}{6}$ **rad**	**45°** $\frac{\pi}{4}$ **rad**	**60°** $\frac{\pi}{3}$ **rad**	**90°** $\frac{\pi}{2}$ **rad**	**180°** π **rad**	**270°** $\frac{3\pi}{2}$ **rad**	**360°** 2π **rad**
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	–1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	–1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	∞	0

Quadratic Formula :

If $ax^2 + bx + c = 0$, then, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Formulae of Log :

1. If $log\ a = x$, then $a = 10^x$

2. $log(ab) = log(a) + log(b)$

3. $log\left(\frac{a}{b}\right) = log(a) - log(b)$

4. $log(a^n) = n\ log\ a$

5. $log_a a = 1$

6. $ln\ a = log_e{}^a = 2.303\ log_{10}a$

Important Expansions :

1. Binomial Expansion $(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)x^2}{2!} + ...\ (x < 1)$

$(1 \pm x)^{-n} = 1 \mp nx + \frac{n(n+1)x^2}{2!}\ ......\ (x < 1)$

2. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^2}{3!} +.....$ when $x < < 1$, then $e^x = 1 + x$

3. $ln(1 + x) = x - \frac{x^2}{2} + \frac{x^2}{3} +.....(| x | < 1)$

when $x << 1$, then $ln(1 \pm x) = \pm x$.

4. Trigonometric Expansion (θ in radian)

(i) $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} +.....$

(ii) $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} +.....$

(iii) $\tan\theta = \theta + \frac{\theta^3}{3} + \frac{\theta^5}{15} +.....$

If θ is very small, then $sin\theta \approx \theta$; $cos\theta \approx 1$ and $tan\theta \approx \theta$ rad

y	$\frac{dy}{dx}$	y	$\frac{dy}{dx}$
x^n	nx^{n-1}	$sec\ x$	$\sec x \tan x$
$sin\ x$	$cos\ x$	$cosec\ x$	$-cosec\ x\ cot\ x$
$cos\ x$	$-sin^{-2}\ x$	lnx	$\frac{1}{x}$
$cot\ x$	$-cosec^2\ x$	$tan\ x$	$sec^2\ x$
$cos\ kx$	$-k\ sin\ x$	e^x	e^x
$sin\ kx$	$k\ cos\ x$	a^x	$a^x\ ln\ a$

Working rules of derivatives :

(1) $\frac{d}{dx}(k) = 0$ (where, k is a constant)

(2) $\frac{d}{dx}(x) = 1$

(3) $\frac{d}{dx}(ky) = k\frac{dy}{dx}$ (where k is a constant)

(4) $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

(5) If $y = u \pm v$, then $\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$

(6) If $y = uv$, then $\frac{dy}{dx} = u\frac{dv}{dx} \pm v\frac{du}{dx}$

(7) If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Integrals of Some Standard Functions :

$f(x)$	$F(x) = \int f(x)dx$	$f(x)$	$F(x) = \int f(x)dx$
x^n $(n \neq + 1)$	$\frac{x^{n+1}}{n+1} + c$	$(ax + b)^n$	$\frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c$
$\frac{1}{x}$	$ln\ x + c$	$sin\ x$	$-cos\ x + c$
e^x	$e^x + c$	$cos\ x$	$sin\ x + c$
e^{kx}	$\frac{1}{k}e^{kx} + c$	$sin\ kx$	$-\frac{1}{k}cos\ x + c$
a^x	$\frac{a^x}{lna} + c$	$cos\ kx$	$\frac{1}{k}sin\ kx + c$

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LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Difference								
											1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5566	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6345	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Difference								
											1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7769	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9843	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

AntilogarithmS

	0	1	2	3	4	5	6	7	8	9	Mean Difference								
											1	2	3	4	5	6	7	8	9
00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
.17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
.20	1585	1289	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
.26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

AntilogarithmS

	0	1	2	3	4	5	6	7	8	9	Mean Difference								
											1	2	3	4	5	6	7	8	9
50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
82	6607	6622	6637	6653	6668	6683	6699	6715	6730	6745	2	3	5	6	8	9	11	12	14
83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	12	14	16
90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

NATURAL SINES

Degree	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	Mean Deifferences				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1'	2'	3'	4'	5'
0	.0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	.0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	.0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	.0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	.0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	14
5	.0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	.1045	1063	1080	1197	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	.1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	.1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	.1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	12	14
10	.1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	11	14
11	.1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	.2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	3	6	9	11	14
13	.2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	.2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	.2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	.2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	.2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	.3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	.3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20	.3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	.3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	.3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	14
23	.3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	14
24	.4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	.4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	.4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	.4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	.4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	.4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30	.5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	.5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	.5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	.5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	.5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	.5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	9	12
36	.5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	.6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	.6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	.6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40	.6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	.6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	.6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	.6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	.6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

NATURAL SINES

Degree	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	Mean Deifferences				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1'	2'	3'	4'	5'
45	.7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	2	4	6	8	10
46	.7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2	4	6	8	10
47	.7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2	4	6	8	10
48	.7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2	4	6	8	10
49	.7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	2	4	6	8	9
50	.7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	2	4	6	7	9
51	.7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	2	4	5	7	9
52	.7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	2	4	5	7	9
53	.7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	2	3	5	7	9
54	.8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	2	3	5	7	8
55	.8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	2	3	5	7	8
56	.8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	2	3	5	6	8
57	.8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	2	3	5	6	8
58	.8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	2	3	5	6	8
59	.8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1	3	4	6	7
60	.8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	1	3	4	6	7
61	.8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1	3	4	6	7
62	.8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1	3	4	5	7
63	.8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1	3	4	5	6
64	.8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1	3	4	5	6
65	.9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	1	2	4	5	6
66	.9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	1	2	3	5	6
67	.9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1	2	3	4	6
68	.9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	1	2	3	4	5
69	.9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	1	2	3	4	5
70	.9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	1	2	3	4	5
71	.9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	1	2	3	4	5
72	.9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	1	2	3	3	4
73	.9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	1	2	2	3	4
74	.9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	1	2	2	3	4
75	.9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	1	1	2	3	4
76	.9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	1	1	2	3	3
77	.9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	1	1	2	3	3
78	.9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	1	1	2	2	3
79	.9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	1	1	2	2	3
80	.9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	0	1	1	2	2
81	.9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0	1	1	2	2
82	.9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0	1	1	2	2
83	.9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0	1	1	1	2
84	.9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0	1	1	1	2
85	.9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	0	0	1	1	1
86	.9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	0	0	1	1	1
87	.9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	0	0	0	1	1
88	.9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	0	0	0	0	0
89	.9998	9999	9999	9999	9999	1.000	1.000	1.000	1.000	1.000	0	0	0	0	0

NATURAL TANGENTS

Degree	0	6	12	18	24	30	36	42	48	54	Main Deifferences				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
0	.0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	.0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	.0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	6	9	12	15
3	.0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3	6	9	12	15
4	.0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3	6	9	12	15
5	.0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	3	6	9	12	15
6	.1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3	6	9	12	15
7	.1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3	6	9	12	15
8	.1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3	6	9	12	15
9	.1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	3	6	9	12	15
10	.1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	3	6	9	12	15
11	.1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	3	6	9	12	15
12	.2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	3	6	9	12	15
13	.2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3	6	9	12	15
14	.2493	2512	2530	2549	2568	2686	2605	2623	2642	2661	3	6	9	12	16
15	.2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	3	6	9	13	16
16	.2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3	6	9	13	16
17	.3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3	6	10	13	16
18	.3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3	6	10	13	16
19	.3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3	7	10	13	16
20	.3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3	7	10	13	17
21	.3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	3	7	10	13	17
22	.4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	3	7	10	14	17
23	.4245	4265	8286	4307	4327	4748	4369	4390	4411	4431	3	7	10	14	17
24	.4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4	7	11	14	18
25	.4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4	7	11	14	18
26	.4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	15	18
27	.5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18
28	.5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4	8	11	15	19
29	.5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4	8	12	15	19
30	.5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	4	8	12	16	20
31	.6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	4	8	12	16	20
32	.6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4	8	12	16	20
33	.6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	4	8	13	17	21
34	.6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	4	9	13	17	21
35	.7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	4	9	13	18	22
36	.7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	5	9	14	18	23
37	.7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	5	9	14	18	23
38	.7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	5	9	14	19	24
39	.8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	5	10	15	20	24
40	.8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	5	10	15	20	25
41	.8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	5	10	16	21	26
42	.9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	5	11	16	21	27
43	.9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	6	11	17	22	28
44	.9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	6	11	17	23	29

NATURAL TANGENTS

Degree	0	6	12	18	24	30	36	42	48	54	Main Deifferences				
	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	1	2	3	4	5
45	1.0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30
46	1.0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6	12	18	25	31
47	1.0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	6	13	19	25	32
48	1.1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	7	13	20	27	33
49	1.1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	7	14	21	28	34
50	1.1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7	14	22	29	36
51	1.2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	8	15	23	30	38
52	1.2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8	16	24	31	39
53	1.3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	8	16	25	33	41
54	1.3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9	17	26	34	43
55	1.4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	9	18	27	36	45
56	1.4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	10	19	29	38	48
57	1.5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	10	20	30	40	50
58	1.6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	11	21	32	43	53
59	1.6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	11	23	34	45	56
60	1.7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	12	24	36	48	60
61	1.8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	13	26	38	51	64
62	1.8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	14	27	41	55	68
63	1.9626	9711	9797	9883	9970	2.0057	2.0145	2.0233	2.0323	2.413	15	29	44	58	73
64	2.0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	16	31	47	63	78
65	2.1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	17	34	51	68	85
66	2.2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	18	37	55	73	92
67	2.3559	3673	3789	3906	4023	4142	4262	4383	4504	4627	20	40	60	79	99
68	2.4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	22	43	65	87	108
69	2.6051	6187	6325	6464	6605	6746	6889	7034	7179	7326	24	47	71	95	119
70	2.7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	26	52	78	104	131
71	2.9042	9208	9375	9544	9714	9887	3.0061	3.0237	3.0415	3.0595	29	58	87	116	145
72	3.0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	32	64	96	129	161
73	3.2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	36	72	108	144	180
74	3.4824	5105	5339	5576	5816	6059	6305	6554	6806	7062	41	81	122	163	204
75	3.7321	7583	7848	8118	8391	8667	8947	9232	9520	9812	46	93	139	186	232
76	4.0108	0408	0713	1022	1335	1653	1976	2303	2635	2972	53	107	160	213	267
77	4.3315	3662	4015	4374	4737	5107	5483	5864	6252	6646	Mean Differences no longer sufficiently accurate				
78	4.7046	7453	7867	8288	8716	9152	9594	5.0045	5.0504	5.0970					
79	5.1446	1929	2422	2924	3465	3955	4486	5026	5578	6140					
80	5.6713	7297	7894	8502	9124	9758	6.0405	6.1066	6.1742	6.2432					
81	6.3138	3859	4596	5350	6122	6912	7720	8548	9395	7.0264					
82	7.1154	2066	3002	3962	4997	5958	6996	8062	9158	8.0285					
83	8.1443	2636	3863	5126	6427	7769	9152	9.0579	9.2052	9.3572					
84	9.514	9.677	9.845	10.02	10.20	10.39	10.58	10.78	10.99	11.20					
85	11.43	11.66	11.91	12.16	12.43	12.71	13.00	13.30	13.62	13.95					
86	14.30	14.67	15.06	15.46	15.89	16.35	16.83	17.34	17.89	18.46					
87	19.08	19.74	20.45	21.20	22.02	22.90	23.86	24.90	26.03	27.27					
88	28.64	30.14	31.82	33.69	35.80	38.19	40.92	44.07	47.74	52.08					
89	57.29	63.66	71.62	81.85	95.49	114.6	143.2	191.0	286.5	573.0					