ગુજરાત રાજ્યના શિક્ષણવિભાગના પત્ર–ક્રમાંક
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MATHEMATICS

Standard 10

PLEDGE

India is my country.
All Indians are my brothers and sisters.
I love my country and I am proud of its rich and varied heritage.
I shall always strive to be worthy of it.
I shall respect my parents, teachers and all my elders and treat everyone with courtesy.
I pledge my devotion to my country and its people.
My happiness lies in their well-being and prosperity.

રાજ્ય સરકારની વિનામૂલ્યે યોજના હેઠળનું પુસ્તક

Gujarat State Board of School Textbooks
'Vidyayan', Sector 10-A, Gandhinagar-382 010

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PREFACE

The Gujarat State Secondary and Higher Secondary Education Board has prepared new syllabi in accordance with the new national syllabi prepared by the N.C.E.R.T. These syllabi are sanctioned by the Government of Gujarat.

It is pleasure of the Gujarat State Board of School Textbooks, to prepare and publish this textbook of **Mathematics** for **Standard 10** based on the new syllabus.

Before publishing the textbook, its manuscript has been fully reviewed by experts and teachers teaching at this level. We have followed suggestions given by teachers and experts and made necessary changes in the manuscript.

The Board has taken special care to see that this textbook is interesting, useful and free from errors. However, we welcome the suggestions, if any to improve the quality of the textbook.

Dr. Bharat Pandit
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Date : 3-3-2015

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FUNDAMENTAL DUTIES

It shall be the duty of every citizen of India

(A) to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;

(B) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(C) to uphold and protect the sovereignty, unity and integrity of India;

(D) to defend the country and render national service when called upon to do so;

(E) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;

(F) to value and preserve the rich heritage of our composite culture;

(G) to protect and improve the natural environment including forests, lakes, rivers and wild life, and to have compassion for living creatures;

(H) to develop the scientific temper, humanism and the spirit of inquiry and reform;

(I) to safeguard public property and to abjure violence;

(J) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievement;

(K) to provide opportunities for education by the parent or the guardian, to his child or a ward between the age of 6-14 years as the case may be.

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About This Textbook...

With the intention that a student of Gujarat can consolidate his position at national level in present times, Gujarat Secondary and Higher Secondary Education Board has prepared a new syllabus equivalent to NCERT syllabus with the help of experts at school, college and university level. To prepare a textbook keeping in view NCF 2005, a committee of experts was formed. In the sequence of textbook of both semesters of standard IX, we are happy to prepare and publish a textbook for standard X.

As earlier, this textbook was first prepared in English. It was prepared according to the syllabus prescribed by Gujarat Secondary and Higher Secondary Board and Gujarat State Textbook Board. The manuscript was reviewed in the end of August by experts from various schools, college and university in a workshop. Keeping in view the suggestions, amendments were made in the manuscript. Then in the end of September the Gujarati translation of the English version was reviewed by a panel of expert school teachers and college teachers. Then the final draft was prepared. This was reviewed again in the office of Gujarat Secondary and Higher Secondary Board by a panel of experts and authors.

Chapter 1 deals with Euclid's algorithm to find *g.c.d.* of two integers, their *l.c.m.*, decimal presentation of numbers, irrational numbers according to NCERT syllabus. A small section on surd is useful in trigonometry in std. 11. In chapter 2 we discuss polynomials and their zeros. In chapter 3, we discuss solution of a pair of linear equations and its applications. We explain both algebraic and graphical methods. Chapter 4 deals with quadratic equations and their practical applications. We give some information on Arithmetic progression in chapter 5. Chapter 6 and 7 deal with geometry using the concept of the set theory as in standard 9. In chapter 8, we begin with study of elementary co-ordinate geomerty. In chapter 9 and 10, we begin with the study of trigonometry and its applications using a right angled triangle. Chapter 11 gives information on areas of a circle, a segment and a circle. Chapter 12 is about some constructions using straight-edge and compass only. Chapter 13 and 14 are about areas and volumes of solid figures. Chapter 15 and 16 are about statistics and probability.

Attractive printing in four colours, variety of illustrations and exercises and information about Indian mathematicians add to the usefulness of the book.

We have tried to give enough information so that a student can study by himself too. Variety of illustrations explains diversity of questions and train a student on how to approach a problem.

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The aim of this textbook is that a student of Gujarat can study national level syllabus interestingly and without any burden and place himself on national map.

At the end, we express our thanks to all who gave kind co-operation in preparing this textbook. We hope that students and teachers all would like the textbook. To enhance the quality of the textbook your valuable suggestions are welcome.

– Authors

•

Born	22 December, 1887 Erode, Madras Presidency
Died	26 April, 1920 (aged 32) Chetput, Madras, Madras Presidency
Nationality	India
Fields	Mathematics
Alma mater	Government Arts College Pachaiyappa's College The University of Cambridge
Academic advisors	G. H. Hardy J. E. Littlewood
Known for	Landau–Ramanujan constant Mock theta functions Ramanujan conjecture Ramanujan prime Ramanujan–Soldner constant Ramanujan theta function Ramanujan's sum Rogers–Ramanujan identities
Influences	G. H. Hardy

Shreenivas Ramanujan

Government of India has declared to celebrate the year 2012 as 'a year of Mathematics' in memory of Shree Ramanujan on his 125th birthday.

Euclid's Algorithm and Real Numbers 1

It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. We shall appreciate the grandeur of the achievement to be more when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by the antiquity.

- Piere Simon Laplace

1.1 Introduction

In standard IX, we have studied rational numbers and irrational numbers collectively forming the set of real numbers. Actually the philosophy of numbers starts from the theory of numbers (integers), their divisibility properties and consequent results.

1.2 The Division Algorithm

In mathematics and particularly in arithmetic, the usual process of division of integers gives unique integers called a quotient and a remainder. The integer division algorithm is an effective method for obtaining a quotient and a remainder. Using decimal presentation of integers (or infact any other positional notation), long division is an algorithm.

Mainly there are two versions of division algorithm in use :

Division Algorithm

(1) Given integers a and b, $b \neq 0$, there exist unique integers q and r such that $a = bq + r$, $0 \leq r < |b|$.

This applies to division process in all integers with the condition that $b \neq 0$.

a is called the dividend, b is called the divisor, q is called the quotient and r is called the remainder.

(2) Generally we use the following form of division algorithm; also called Euclid's division Lemma.

Given positive integers a and b, there exist unique non-negative integers q and r such that $a = bq + r$, $0 \leq r < b$.

Since we apply division process mainly for natural numbers form (2) is widely used.

Sometimes another version of (1) is also used. $a = bq + r \quad -\frac{1}{2}|b| < r \leq \frac{1}{2}|b|$

What is an algorithm ?

An algorithm is a series of well-defined steps leading to solving a problem. The word algorithm comes from the name of the 9th century Persian mathematician al-Khwarizmi.

Muhammad ibn Musa al-Khwarizmi
(A.D. 780 – 850)

Some conclusions from remainder theorem

(1) If $b = 2$, then from (1), $0 \le r < 2$ gives $r = 0$ or 1.

$\therefore$ $a = 2q$ or $a = 2q + 1$

Thus, **every integer is of the form $2q$ or $2q + 1$ for some integer q. If $a = 2q$, $q \in Z$, a is divisible by 2 and a is called an even integer. If $a = 2q + 1$, $q \in Z$, a is called an odd integer. Thus, even integers are of the form $2k$ ($k \in Z$) and odd integers are of the form $2k + 1$ ($k \in Z$) (writing $q = k$)**

(2) If $b = 3$, then $0 \le r < 3$ gives $r = 0$ or 1 or 2.

$\therefore$ **Every integer a is of the form $3k$ or $3k + 1$ or $3k + 2$, (taking $q = k$), ($k \in Z$)**

(3) **Every odd integer is of the form $4k + 1$ or $4k + 3$. ($k \in Z$)**

If $b = 4$, $a = 4k + r$, $0 \le r < 4$

Hence $r = 0$ or 1 or 2 or 3

If $r = 0$ or 2, then $a = 4k$ or $a = 4k + 2 = 2(2k + 1)$ is even.

But a is odd.

$\therefore$ $r = 1$ or 3

$\therefore$ $a = 4k + 1$ or $4k + 3$, ($k \in Z$)

$\therefore$ Every odd integer is of the form $4k + 1$ or $4k + 3$, ($k \in Z$)

Note : $a = 4k + 3 = 4k + 4 + 3 - 4 = 4(k + 1) - 1 = 4k' - 1$, where $k' = k + 1$

$\therefore$ Every odd integer can also be written as $4k \pm 1$

(4) Similarly in (2) $3k + 2 = 3(k + 1) - 1 = 3k' - 1$, where $k' = k + 1$

If 3 does not divide a, a is of the form $3k \pm 1$

Note : Thus, is obvious from the form

$$a = bq + r \qquad -\frac{1}{2}|b| < r \le \frac{1}{2}|b|$$

For $b = 3$, $-\frac{3}{2} < r \le \frac{3}{2}$ gives $r = -1, 0$ or 1, but $r = 0$ gives $a = 3q$ which is divisible by 3. Hence $r = \pm 1$. Hence $a = 3k \pm 1$. (writing $q = k$)

Example 1 : Prove that the square of an odd integer decreased by 1 is a multiple of 8.

Solution : Method 1 : Let $a = 4k + 1$ or $4k + 3$ **(Remark 3, above)**

$\therefore$ $a^2 = 16k^2 + 8k + 1$ or $a^2 = 16k^2 + 24k + 9$

$\therefore$ $a^2 - 1 = 8k(2k + 1)$ or $a^2 - 1 = 8(2k^2 + 3k + 1)$

$a^2 - 1$ is a multiple of 8

Thus, when 1 is subtracted from the square of an odd integer, then the integer so obtained is a multiple of 8.

Method 2 : According to the division algorithm for any integer a, $a = bq + r$, $-\frac{1}{2}|b| < r \le \frac{1}{2}|b|$,

Taking $b = 4$, $a = 4q + r$, $-2 < r \le 2$

$r = -1, 0, 1, 2$

But a is an odd, so $r \ne 0, 2$ **($a = 4q$, $a = 4q + 2$ are even)**

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∴ $a = 4k \pm 1$.

∴ $a^2 = 16k^2 \pm 8k + 1$ **(Squaring on both the sides)**

∴ $a^2 - 1 = 8k(2k \pm 1)$

∴ $a^2 - 1$ is a multiple of 8.

Example 2 : Prove that the square of an integer is of the form $9k$ or $3k + 1$.

Solution : Every integer a is of the form $3m$ or $3m + 1$ or $3m + 2$. We can take them as $3m$ or $3m \pm 1$ as $3m + 2 = 3m + 3 - 1 = 3(m + 1) - 1 = 3m' - 1$, where $m' = m + 1$.

$a^2 = 9m^2$ or $a^2 = 9m^2 \pm 6m + 1 = 3(3m^2 \pm 2m) + 1 = 3k + 1$, where $k = 3m^2 \pm 2m$

∴ Either 9 divides a^2 or $a^2 = 3k + 1$

∴ If $a = 3m$, then $a^2 = 9k$ where $k = m^2$ and if $a = 3m \pm 1$, then $a^2 = 3k + 1$ where $k = 3m^2 \pm 2m$

Example 3 : Show that the cube of any integer is of the form $9m$ or $9m \pm 1$

Solution : For every integer a and non-zero integer b, there exist unique integers q and r such that

$a = bq + r$ $\qquad -\frac{1}{2}|b| < r \leq \frac{1}{2}|b|$

Taking $b = 3$,

$-\frac{3}{2} < r \leq \frac{3}{2}$

∴ $r = -1$ or 0 or 1

∴ We can write every integer a in the form $a = 3q + r$, where $r = 0$ or 1 or -1

∴ We can write every integer a as $a = 3k$ or $3k \pm 1$, for $k \in Z$.

Now, if $a = 3k$, then $a^3 = 27k^3 = 9(3k^3) = 9m$, where $m = 3k^3$

If $a = 3k + 1$, then $a^3 = (3k + 1)^3 = 27k^3 + 1 + 9k(3k + 1)$

$= 27k^3 + 27k^2 + 9k + 1$

$= 9(3k^3 + 3k^2 + k) + 1$

$= 9m + 1$, where $m = 3k^3 + 3k^2 + k$

If $a = 3k - 1$, then $a^3 = (3k - 1)^3 = 27k^3 - 1 - 9k(3k - 1)$

$= 27k^3 - 27k^2 + 9k - 1$

$= 9(3k^3 - 3k^2 + k) - 1$

$= 9m - 1$, where $m = 3k^3 - 3k^2 + k$

∴ The cube of every integer is of form $9m$ or $9m + 1$ or $9m - 1$.

> The method shown above is not general. For $b = 4$ it does not give similar results.
>
> For this we have to take $a = 16k, 16k \pm 1, 16k \pm 2, 16k \pm 3, 16k \pm 4, 16k \pm 5, 16k \pm 6, 16k \pm 7$ or $16k + 8$.
>
> We can think of such results for a prime b.

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$\therefore \quad a^3 = 9m \pm 1$

$\therefore \quad a^3$ is always of the form $9m$ or $9m \pm 1$.

Example 4 : Prove that 6 divides $n(n + 1)(2n + 1)$ for any $n \in N$.

Solution : Now if $n = 1$ or 2, obviously $n(n + 1)(2n + 1) = 6$ or 30 is divisible by 6.

Let $n \geq 3$,

n or $n + 1$ is even, they being consecutive integers.

$\therefore \quad$ 2 divides $n(n + 1)(2n + 1)$

Also, $n = 3k$ or $3k + 1$ or $3k + 2$ for some $k \in N$

(i) If $n = 3k$, $n(n + 1)(2n + 1) = 3k(3k + 1)(6k + 1)$ is divisible by 3

(ii) $n = 3k + 1$, $n(n + 1)(2n + 1) = (3k + 1)(3k + 2)(6k + 3)$

$= 3(3k + 1)(3k + 2)(2k + 1)$ is divisible by 3

(iii) $n = 3k + 2$, $n(n + 1)(2n + 1) = (3k + 2)(3k + 3)(6k + 5)$

$= 3(3k + 2)(k + 1)(6k + 5)$ is divisible by 3

$\therefore \quad$ In any case 2 and 3 divide $n(n + 1)(2n + 1)$

$\therefore \quad n(n + 1)(2n + 1)$ is divisible by 6 as 2 and 3 have no common factor.

Note : For $a, b \in N$, if a and b have no common factor and if $a \mid n$, $b \mid n$ then $ab \mid n$; $n \in N$.

Example 5 : Prove that if n is an even positive integer, then $3^n + 1$ is divisible by 2 and not by 2^m for $m \geq 2$, $m \in N$. If n is an odd positive integer, then $3^n + 1$ is divisible by 4 and not by 2^m for $m \geq 3$, $m \in N$. (Use the fact that square of an odd integer is of form $8k + 1$.)

Solution : If n is even, say $n = 2k$, then $3^n = 3^{2k} = (3^k)^2 = 8a + 1$ **(3^k is odd)**

$\therefore \quad 3^n + 1 = 8a + 2 = 2(4a + 1)$

$\therefore \quad$ 2 divides $3^n + 1$ and $4a + 1$ is odd. Therefore 2^m, does not divide $3^n + 1$ for $m \geq 2$, $m \in N$.

If n is odd, say $n = 2k + 1$,

Then $3^n + 1 = 3^{2k+1} + 1 = 3 \cdot 3^{2k} + 1 = 3(8a + 1) + 1$ **(3^k is odd)**

$= 24a + 4$

$= 4(6a + 1)$

$\therefore \quad$ 4 divides $3^n + 1$ and 2^m, $m \in N$, $m \geq 3$ does not divide $3^n + 1$ as $6a + 1$ is odd.

EXERCISE 1.1

1. Prove that 16 divides $n^4 + 4n^2 + 11$, if n is an odd integer.
2. Prove that if n is a positive even integer, then 24 divides $n(n + 1)(n + 2)$.
3. Prove that if either of $2a + 3b$ and $9a + 5b$ is divisible by 17, so is the other. $a, b \in N$
   (**Hint :** $4(2a + 3b) + 9a + 5b = 17a + 17b$)
4. Prove that every natural number can be written in the form $5k$ or $5k \pm 1$ or $5k \pm 2$, $k \in N \cup \{0\}$.

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5. Prove that if 6 has no common factor with n, $n^2 - 1$ is divisible by 6.
6. Prove that product of four consecutive positive integers is divisible by 24.

*

1.3 Euclid's algorithm

We have studied **division algorithm** also called **Euclid's division lemma**. Its most important application is to find ***g.c.d. (Greatest Common Divisor)*** of two positive integers. As such, we know about ***g.c.d.*** and ***l.c.m. (Least Common Multiple)*** of two positive integers. A formal approach to this concept is what we are going to understand.

Before taking it up, let us understand following example from ***Brahmasphuta Siddhanta (Brahma's correct system)*** **by Brahmagupta (born 598 AD).**

An old woman goes to market and a horse steps on her basket and crashes the eggs. The rider offers to pay for the damages and asks her how many eggs she had brought. She does not remember exact figure. When she had taken them two at a time, there was one egg left. The same happened when she had taken them three, four, five or six at a time. But when she took them seven at a time they came out even. What is the smallest number of eggs she could have had ? (Congratulations for the patience of the rider !)

The problem says if n is the number of eggs she had, then n divided by 2, 3, 4, 5 or 6 leaves remainder 1 and n is divisible by 7.

Thus $n - 1$ is a multiple of 2, 3, 4, 5 and 6.

$\therefore$ $n - 1$ is a multiple of 60 **(*l.c.m.* of 2, 3, 4, 5 and 6)**

$\therefore$ n could be 61, 121, 181, 241, 301 etc.

7 divides 301 out of them.

$\therefore$ 301 is the smallest number of eggs the old woman had.

l.c.m. and *g.c.d.* are co-related concepts.

Definition : If for a positive integer a, there exists a positive integer b such that $a = bc$ for some positive integer c, then we say positive integer b is a factor of a or divisor of a. We write $b \mid a$. Since $c \geq 1$, $a \geq b$ or in other words $b \leq a$. If b is not a factor of a, we write $b \nmid a$.

If a and b are positive integers and if c divides a and c divides b, we say c is a common divisor of a and b. Since $c \leq a$ and $c \leq b$ and the number of positive integers less than a and b both is finite, there must be the largest common divisor from amongst common divisors of a and b. This positive integer is called the greatest common divisor of a and b. It is called *g.c.d.* of a and b in short. So if d is a *g.c.d.* of a and b, then

(i) $d \mid a$ and $d \mid b$ ($d \mid a$ means d divides a)

(ii) If $c \mid a$ and $c \mid b$, then $c \leq d$ (d is the greatest common divisor of a and b or it can be proved that (ii) is equivalent to (iii).

(iii) If $c \mid a$ and $c \mid b$, then $c \mid d$.

Now how to find *g.c.d.* of two positive integers a and b ? The procedure is known as Euclid's algorithm.

The algorithm runs as follows :

Let a and b be positive integers with $a > b$. (any harm if $a < b$?)

Divide a by b and continue the iterative division as shown :

$$
\begin{array}{ll}
a = bq_0 + r_0 & 0 \le r_0 < b \\
b = r_0 q_1 + r_1 & 0 \le r_1 < r_0 \\
r_0 = r_1 q_2 + r_2 & 0 \le r_2 < r_1 \\
\dots & \dots \\
\dots & \dots \\
\dots & \dots \\
r_{n-3} = r_{n-2}\, q_{n-1} + r_{n-1} & 0 \le r_{n-1} < r_{n-2}
\end{array}
$$

Since $r_0, r_1, r_2 \dots r_{n-1}$ decrease and are non-negative, at some stage say $r_{n+1} = 0$.

Then $r_{n-2} = r_{n-1}\, q_n + r_n$ and $0 \le r_n < r_{n-1}$

$$r_{n-1} = r_n \; q_{n+1}$$

The last non-zero remainder r_n is the *g.c.d.* of a and b and $r_n | r_{n-1}, r_n | r_{n-2}, \dots, r_n | b, r_n | a$.

Hence $r_n | d$, where $d = g.c.d.(a, b)$. Also $d|a$, $d|b$, $d|r_0$, $d|r_1, \dots$, $d | r_{n-2}$, $d | r_{n-1}$, $d | r_n$.

$\therefore$ $r_n = d$

$\therefore$ ***g.c.d.*** $(a, b) = r_n$ = the last non-zero remainder

If b is a factor of a, then for some positive integer c, we have $a = bc$. b is a common factor of a and b, also b is the largest factor of b.

$\therefore$ **If $b | a$, then *g.c.d.*$(a, b) = b$.**

Also substituting values of $r_{n-1}, r_{n-2} \dots$ etc. from the end we will get an equation like $ax - by = r_n = d$ where x, y are integers.

The identity $ax - by =$ **g.c.d.** (a, b) is called Be'zout's identity.

Example 6 : Find *g.c.d.* of (1) 120 and 23 (2) 38220 and 196 by Euclid's algorithm.

Solution :

(1) $120 = 23 \times 5 + 5$ $(a = 120, b = 23, q_0 = 5, r_0 = 5)$

$23 = 5 \times 4 + 3$ $(b = 23, r_0 = 5, q_1 = 4, r_1 = 3)$

$5 = 3 \times 1 + 2$ $(r_0 = 5, r_1 = 3, q_2 = 1, r_2 = 2)$

$3 = 2 \times 1 + 1$ $(r_1 = 3, r_2 = 2, q_3 = 1, r_3 = 1)$

$2 = 1 \times 2 + 0$ $(r_2 = 2, r_3 = 1, q_4 = 2, r_4 = 0)$

$\therefore$ The last non-zero remainder is 1.

$\therefore$ ***g.c.d.***$(120, 23) = 1$

Note : (For understanding purpose, not for the examination.)

Also, $1 = 3 - 2 \cdot 1 = 3 - (5 - 3 \cdot 1)$

$= 2.3 - 5$

$= 2(23 - 5 \cdot 4) - 5$

$= 2.23 - 9 \cdot 5$

$= 2.23 - 9(120 - 23 \cdot 5)$

$= 47.23 - 9.120$

$23 \cdot 47 - 120 \cdot 9 = 1$

$ax - by = 1$ with $a = 120$, $b = 23$, $x = -9$, $y = -47$

or $23 \cdot 47 - 120 \cdot 9 + 23 \cdot 120 - 23 \cdot 120 = 1$

$\therefore$ $120 \cdot 14 - 23 \cdot 73 = 1$

$\therefore$ $a = 120$, $b = 23$, $x = 14$, $y = 73$

This is Be'Zout's identity for numbers 120 and 23.

(2) $38220 = 196 \times 195 + 0$

Thus $196 \mid 38220$

$\therefore$ *g.c.d.* (196, 38220) = 196

Historical Notes : Euclid's algorithm is one of the oldest algorithm still in use. It is stated in Euclid's Elements (300 BC) in book 7 as propositions 1, 2 and in book 10 as proposition 2, 3. In book 7 it is formulated for integers and in book 10, it is formulated for lengths of line segments. The ***g.c.d.*** of two lengths a and b is the greatest length d which measures a and b or $d|a$ and $d|b$. This algorithm was not probably discovered by Euclid. Van der Waerden suggests that book 7 derives from a text book on number theory by mathematicians in the school of Pythagoras. The algorithm was known to Eudoxus (375 BC) or even earlier. Centuries later it was discovered in India and China independently. In the fifth century Aryabhata described it as 'pulverizer' because of its use in solving **Diophantine equations**.

A Diophantine equation is an equation whose solutions are worked out in integer variables. $ax + by = n$, $x^2 + y^2 = z^2$, $x^2 - 2y^2 = 1$ etc. are Diophantine equations, where $x, y \in$ N or $x, y \in$ Z. $x^2 - ny^2 = 1$ is famous Pell's equation, where n is not a perfect square.

Least Common Multiple (L.C.M.) :

Definition : **If $a|m$ and $b|m$ for natural numbers a and b and m, then we say m is a common multiple of a and b. The set of common multiples of a and b is a set of natural numbers. So this set has a least element. This number is called the least common multiple of a and b. In short, we write it as *l.c.m* (a, b).**

If m is the least common multiple of a and b, we write it as $m =$ *l.c.m.* (a, b)

We will not prove the following relation which is true for positive integers a and b.

$l.c.m.\ (a, b)\ g.c.d.(a, b) = ab$

Hence having known one of *g.c.d.* (a, b) and *l.c.m.* (a, b), we can find the other.

Let us take an ancient problem relating to *g.c.d.* and *l.c.m.*.

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For information only. Not to be asked in the examination

(By **Sun Tsu Ching in 4th century AD**) There are certain things whose number divided by 3 leaves remainder 2, divided by 5 leaves remainder 3 and divided by 7 leaves remainder 2. What is the number of things ?

Solution : By division Lemma, if the number of things is n, then $n = 3k + 2$, $n = 5k_1 + 3$ and $n = 7k_2 + 2$, where $k, k_1, k_2 \in N$. **(Why k, k_1, k_2 are non-zero ?)**

$\therefore$ $n - 2$ is a multiple of 3 and 7.

Let $n - 2 = 21m, \quad m \in N$ **(21 = *l.c.m.* (3, 7))**

$\therefore$ $n = 21m + 2 = 5k_1 + 3$

$\therefore$ $21m + 4 = 5k_1 + 5$ is a multiple of 5.

$\therefore$ $21m + 4 = 5t$ for some $t \in N$

$\therefore$ $5t - 21m = 4$ **(i)**

Now let us find ***g.c.d.*** of 21 and 5.

$\therefore$ $21 = 5 \cdot 4 + 1$

$\therefore$ $21 - 5 \cdot 4 = 1$

$\therefore$ $21 \cdot 4 - 5 \cdot 16 = 4$

$\therefore$ $m = -4$, $t = -16$ is a solution of Be'zout's identity (i).

or $21.4 - 5 \cdot 16 + 21 \cdot 5 - 21 \cdot 5 = 4$

$\therefore$ $5 \cdot 5 - 21 \cdot 1 = 4$

$\therefore$ $t = 5$ and $m = 1$

$\therefore$ $n - 2 = 21 \cdot 1$

$\therefore$ $n = 23$ or $n = 23 + 105 = 128$ or $n = 23 + 2(105) = 233$ etc. $(105 = 21 \cdot 5)$

Example 7 : A dealer of a beauty shop in a mall has 330 tooth-pastes of one company and 65 hair-creams of another company. She wants to stack them up in a showcase in such a way that each stack has the same number of similar items and the stacks occupy the least surface area of the bottom. What is the maximum number of each item in a stack presented in the showcase ?

Solution : Let d be the number of items in a stack. Then $d \mid 330$ and $d \mid 65$, because each stack carries the same number of items. Also $\frac{330}{d}$ and $\frac{65}{d}$ is the number of stacks. To occupy the least area, d must be largest. So, d = ***g.c.d.***(330, 65).

Now, $330 = 65 \cdot 5 + 5$

$65 = 5 \cdot 13$

$\therefore$ The last non-zero divisor is 5.

$\therefore$ ***g.c.d.***(330, 65) = 5

$\therefore$ There should be five tooth-pastes or five hair-creams in a stack.

Example 8 : Find ***g.c.d.***(24871, 3466).

Solution : $24871 = 3466 \cdot 7 + 609$

$3466 = 609 \cdot 5 + 421$

$609 = 421 \cdot 1 + 188$

$421 = 188 \cdot 2 + 45$

$$188 = 45 \cdot 4 + 8$$
$$45 = 8 \cdot 5 + 5$$
$$8 = 5 \cdot 1 + 3$$
$$5 = 3 \cdot 1 + 2$$
$$3 = 2 \cdot 1 + 1$$

g.c.d.(24871, 3466) = 1

[For knowledge only and not for examination : Number of steps may be frightening. How long will it continue ? Lame proved that the number of steps does not exceed 5 times number of digits in the smaller number. So these we should expect to end before 20 steps! A better approximation is number of steps $\le 4.785 \log_{10} n + 1.6723$ where n is the smaller number. The worst case occurs in case of consecutive Fibonacci numbers of sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... For example apply Euclid's algorithm to find ***g.c.d.*** of 89 and 144.

$144 = 89 \cdot 1 + 55$ $\quad 89 = 55 \cdot 1 + 34$ $\quad 55 = 34 \cdot 1 + 21$ $\quad 34 = 21 \cdot 1 + 13$

$21 = 13 \cdot 1 + 8$ $\quad 13 = 8 \cdot 1 + 5$ $\quad 8 = 5 \cdot 1 + 3$ $\quad 5 = 3 \cdot 1 + 2$

$3 = 2 \cdot 1 + 1$

Number of steps = 9

5 times digits of smaller number = 10

$4.785 \times \log_{10} 89 + 16.723 = 4.785 \times 1.9494 + 1.6723 = 10.9998$

Thus, number of steps $9 < 10$ or $9 < 10.9998$]

EXERCISE 1.2

1. Find *g.c.d.* (1) 144, 233 (2) 765, 65 (3) 10211, 2517
2. Find *g.c.d.* of 736 and 85 by using Euclid's algorithm.
3. Prove *g.c.d.*$(a - b, a + b) = 1$ or 2, if *g.c.d.*$(a, b) = 1$
4. Using the fact that *g.c.d.*(a, b) *l.c.m.*$(a, b) = ab$, find *l.c.m.*(115, 25)

*

1.4 Fundamental Theorem of Arithmetic

We begin with some definitions :

Prime : If p is a natural number greater than 1 and p has only one factor other than 1 namely itself, then p is called a prime.

Composite number : If a natural number is a product of at least two primes (not necessarily distinct), it is called a composite number. A composite number has at least three distinct factors.

For example : $4 = 2^2$ has three factors 1, 2 and 4.

All natural numbers are divided in three classes.

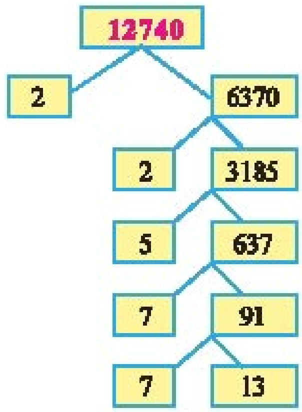
(1) 1 having only one factor 1. (1 is called a unit also.)

(2) Primes p greater than 1 and having factors 1 and p only.

(3) Composite numbers, having at least three different factors.

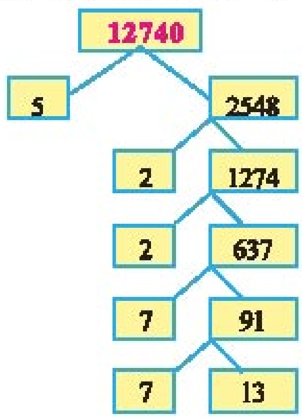
Our interest is in factorization of composite numbers.

Consider 12740.



$\therefore \quad 12740 = 2^2 \times 5 \times 7^2 \times 13$

Let us try to factorise in some other way. Let us start with the factor 5 first.



Here also we get $12740 = 5 \times 2^2 \times 7^2 \times 13$

So apart from order, all prime factorizations of 12740 are same and this applies to any natural number greater than 1.

We have the following theorem, we will not prove it.

Fundamental Theorem of Arithmetic

Theorem 1.1 : Every natural number greater than 1 can be factored uniquely as a product of primes apart from order of occurrence of primes.

An equivalent version of this theorem was given as proposition 14 in book IX in Euclid's elements, though not as 'Fundamental theorem of Arithmetic'. Its first correct proof was given by Carl Friedrich Gauss in his *Disquisitions Arithmeticae*. He completed this book in 1798 at the age of 21, though it was published in 1801. He proved that every positive number can be represented as a sum of at most three triangular numbers. He noted down in his diary.

Carl Friedrich Gauss
(1777 – 1855)

An Important Result (Not for examination) : If p is a prime and $p|ab$, then $p|a$ or $p|b$.

Solution : If $p|a$, we are done.

If $p \nmid a$, then ***g.c.d.***$(p, a) = 1$

$\therefore$ There exist integers x and y such that $ax + py = 1$

$\therefore$ $abx + pby = b$

Now $p|ab$ and $p|pby$

$\therefore$ $p|abx$ and $p|pby$

$\therefore$ $p|(abx + pby)$

$\therefore$ $p|b$

If $p|ab$, then $p|a$ or $p|b$.

Infact if $m|ab$ and ***g.c.d*** $(m, a) = 1$, then $m|b$.

Example 9 : Prove that 4^n can not end in a zero. $(n \in N)$

Solution : If 4^n ends in a zero, it has to be divisible by 10. Also $10 = 5 \cdot 2$

$\therefore$ $5|4^n$

Now, $4^n = 2^{2n}$ must have 5 in its prime factorisation.

But 4^n has only 2 as the prime in its prime factorisation and by uniqueness part of fundamental theorem of arithmetic, 5 cannot divide 4^n.

Hence 4^n is not divisible by 10 and 4^n cannot end in a zero for any $n \in N$.

An important application

From the fundamental theorem, an integer a can be factored as

$a = p_1^{a_1} p_2^{a_2} ... p_n^{a_n}$ where $p_1, p_2 ... p_n$ are prime factors of a and this expression is unique apart from the order of p_i. $a_1, a_2, ..., a_n \in N$

Let $b = q_1^{b_1} q_2^{b_2} ... q_m^{b_m}$

Some p_i may be $q_1, q_2, ...$ or q_m. Here $q_1, q_2, ... q_m$ prime factors of b. Also $b_1, b_2, ..., b_m \in N$

So we may write $a = p_1^{a_1} p_2^{a_2} ..... p_k^{a_k}$ and $b = p_1^{b_1} p_2^{b_2} ... p_k^{b_k}$

where $p_1, p_2, ... p_k$ are prime factor of a or b and $a_i \in N \cup \{0\}$, $b_i \in N \cup \{0\}$.

Consider for example, $a = 600 = 2^3 \cdot 3 \cdot 5^2$

$b = 980 = 2^2 \cdot 5 \cdot 7^2$

2, 3, 5, 7 are prime factors of a or b

$\therefore$ $a = 2^3 \cdot 3 \cdot 5^2 \cdot 7^0$

$b = 2^2 \cdot 3^0 \cdot 5 \cdot 7^2$

$\therefore$ We can write in general, as follows :

$a = p_1^{a_1} p_2^{a_2} ..... p_k^{a_k}$ and

$b = p_1^{b_1} p_2^{b_2} ..... p_k^{b_k}$ where $p_1, p_2, ... p_k$ are prime factors of at least one of a or b and $a_i (i = 1, 2, ... k)$ and $b_i (i = 1, 2, 3, ... k)$ are non-negative integers.

We can write d = g.c.d.$(a, b) = p_1^{c_1}\, p_2^{c_2}\, ...\, p_k^{c_k}$

m = l.c.m.$(a, b) = p_1^{d_1}\, p_2^{d_2}\, ...\, p_k^{d_k}$

where $c_i = \min(a_i, b_i)$ $d_i = \max(a_i, b_i)$ are respectively smaller and larger of a_i and b_i $(i = 1, 2, 3, ... k)$. Why ? Obviously $c_i \le a_i$, $c_i \le b_i$

$p_1^{c_1}\, p_2^{c_2}\, ...\, p_k^{c_k} \mid p_1^{a_1}\, p_2^{a_2}\, ...\, p_k^{a_k}$

and $p_1^{c_1}\, p_2^{c_2}\, ...\, p_k^{c_k} \mid p_1^{b_1}\, p_2^{b_2}\, ...\, p_k^{b_k}$

$\therefore$ $d|a$ and $d|b$

Moreover if $c|a$ and $c|b$, then $c|d$.

$\therefore$ d = g.c.d.(a, b) and similarly m = l.c.m.(a, b)

Also $dm = p_1^{c_1+d_1}\, p_2^{c_2+d_2}\, ...\, p_k^{c_k+d_k} = ab$

as $c_i + d_i = \min(a_i, b_i) + \max(a_i, b_i) = a_i + b_i$

$\therefore$ l.c.m.(a, b) g.c.d.$(a, b) = ab$

In the above example d = g.c.d.$(a, b) = 2^2 \cdot 3^0 \cdot 5^1 \cdot 7^0 = 20$

m = l.c.m.$(a, b) = 2^3 \cdot 3 \cdot 5^2 \cdot 7^2 = 29400$

$dm = 588000 = ab$

Example 10 : Find g.c.d. and l.c.m. of 300 and 440 using fundamental theorem of arithmetic and verify that g.c.d.(300, 440). l.c.m.(300, 440) = 300 × 440

Solution : $a = 300 = 2^2 \times 3 \times 5^2$

$b = 440 = 2^3 \times 5 \times 11$

$\therefore$ g.c.d.(300, 440) $= 2^2 \times 5^1 = 20$

l.c.m.(300, 440) $= 2^3 \times 5^2 \times 3 \times 11 = 6600$

$\therefore$ g.c.d.(300, 440). l.c.m.(300, 440) = 132000 = 300 × 440

Example 11 : Find g.c.d.(144, 610) using Euclid's algorithm and find l.c.m. (144, 610) using the relation ab = g.c.d. (a, b) × l.c.m. (a, b).

Solution : $610 = 144 \times 4 + 34$

$144 = 34 \times 4 + 8$

$34 = 8 \times 4 + 2$

$8 = 4 \times 2$

g.c.d.(144, 610) = last non-zero remainder = 2

$\therefore$ g.c.d.(144, 610) = 2

l.c.m.(144, 610) $= \frac{144 \times 610}{2} = 144 \times 305 = 43920$ $\left(\text{l.c.m.}(a, b) = \frac{ab}{\text{g.c.d.}(a, b)}\right)$

> **Note :** $144 = 2^4 \times 3^2$, $610 = 2 \times 5 \times 61$
>
> g.c.d.(144, 610) $= 2^1 = 2$, l.c.m.(144, 610) $= 2^4 \times 3^2 \times 5 \times 61 = 43920$

G.C.D. and L.C.M. of three integers.

Let $a \in N$, $b \in N$, $c \in N$. If $d \in N$, such that $d|a$, $d|b$, $d|c$ and if $d_1|a$, $d_1|b$, $d_1|c$ then $d_1|d$, then d is called the g.c.d. of a, b, c and we write d = g.c.d.(a, b, c).

Let $a = p_1^{a_1}\ p_2^{a_2}\ .....\ p_k^{a_k}$

$b = p_1^{b_1}\ p_2^{b_2}\ .....\ p_k^{b_k}$

$c = p_1^{c_1}\ p_2^{c_2}\ .....\ p_k^{c_k}$

Here each p_i divides at least one of a, b or c.

$a_i \in N \cup \{0\}$, $b_i \in N \cup \{0\}$, $c_i \in N \cup \{0\}$, $i = 1, 2, 3, ..., k$

Then **g.c.d.**$(a, b, c) = p_1^{e_1}\ p_2^{e_2}\ .....\ p_k^{e_k}$

and **l.c.m.**$(a, b, c) = p_1^{f_1}\ p_2^{f_2}\ .....\ p_k^{f_k}$

$e_i = \min(a_i, b_i, c_i)$, $f_i = \max(a_i, b_i, c_i)$

Obviously $abc = p_1^{a_1+b_1+c_1}\ p_2^{a_2+b_2+c_2}\ .....\ p_k^{a_k+b_k+c_k}$

Not necessarily $a_i + b_i + c_i = \min(a_i, b_i, c_i) + \max(a_i, b_i, c_i)$

g.c.d.(a, b, c) **l.c.m.**$(a, b, c) \neq abc$ in general.

Infact **g.c.d.(a, b, c) = g.c.d.(g.c.d.(a, b), c) = g.c.d.(b, g.c.d.(a, c)) = g.c.d.(a, g.c.d.(b, c))**

We can find integers x, y, z such that $ax + by + cz =$ **g.c.d.**(a, b, c)

Similarly, **l.c.m.(a, b, c) = l.c.m.(l.c.m.(a, b), c)** etc.

Example 12 : Find **g.c.d.**(120, 504, 882)

Solution : Method 1 : $120 = 2^3 \times 3 \times 5$

$504 = 2^3 \times 3^2 \times 7$

$882 = 2 \times 3^2 \times 7^2$

$\therefore$ **g.c.d.**$(a, b, c) =$ **g.c.d.**$(120, 504, 882) = 2 \times 3 = 6$

Method 2 : $504 = 120 \times 4 + 24$

$120 = 24 \times 5$

$\therefore$ **g.c.d.**$(504, 120) = 24$

$\therefore$ **g.c.d.**$(120, 504, 882) =$ **g.c.d.**(**g.c.d.**$(120, 504), 882)$

$=$ **g.c.d.**$(24, 882)$

Now, $882 = 24 \times 36 + 18$

$24 = 18 \times 1 + 6$

$18 = 6 \times 3$

$\therefore$ **g.c.d.** $= (882, 24) = 6$

$\therefore$ **g.c.d.** $= (120, 504, 882) =$ **g.c.d.**$(24, 882) = 6$

Example 13 : Find **g.c.d.**(28, 35, 91)

Solution : $28 = 2^2 \times 7$ $\quad 35 = 5 \times 7$ $\quad 91 = 13 \times 7$

$\therefore$ **g.c.d.**$(28, 35, 91) = 7$

Second method : Now, $91 = 35.2 + 21$

$35 = 21 \cdot 1 + 14$

$21 = 14 \cdot 1 + 7$

$14 = 7 \cdot 2$

$\therefore$ g.c.d.(91, 35) = 7

Now $28 = 7 \cdot 4$

$\therefore$ g.c.d.(28, 35, 91) = g.c.d.(28, g.c.d.(35, 91)) = g.c.d.(28, 7) = 7

Example 14 : Prove : g.c.d.$(ca, cb) = c \cdot$ g.c.d.(a, b), $c \in$ N

Solution : Euclid's algorithm for a and b is

$$\begin{array}{ll} a = bq_1 + r_1 & 0 \le r_1 < b \\ b = r_1q_2 + r_2 & 0 \le r_2 < r_1 \\ r_1 = r_2q_3 + r_3 & 0 \le r_3 < r_2 \\ \cdots & \cdots \\ \cdots & \cdots \\ \cdots & \cdots \\ r_{k-2} = r_{k-1}\, q_k + r_k & 0 \le r_k < r_{k-1} \\ r_{k-1} = r_k q_{k+1} & \end{array}$$

g.c.d.$(a, b) = r_k$

Multiply all the equations by c.

$$\begin{array}{ll} \therefore \; ca = cbq_1 + cr_1 & 0 \le cr_1 < cb \\ cb = cr_1q_2 + cr_2 & 0 \le cr_2 < cr_1 \\ cr_1 = cr_2\, q_3 + cr_3 & 0 \le cr_3 < cr_2 \\ \cdots & \cdots \\ \cdots & \cdots \\ \cdots & \cdots \\ cr_{k-2} = cr_{k-1}\, q_k + cr_k & 0 \le cr_k < cr_{k-1} \\ cr_{k-1} = cr_k\, q_{k+1} & \end{array}$$

$\therefore$ g.c.d.$(ca, cb) = cr_k = c$ g.c.d.(a, b)

Example 15 : Using ex. 14 and the fact that g.c.d.(a, b) l.c.m.$(a, b) = ab$,

prove that l.c.m.$(a, b, c) = \dfrac{abc}{g.c.d.(ab, bc, ca)}$

Solution : l.c.m.(a, b, c) = l.c.m.(l.c.m.$(a, b), c)$ **(By definition)**

But as l.c.m.$(a, b) \times$ g.c.d.$(a, b) = ab$

l.c.m.(l.c.m.$(a, b), c) \cdot$ g.c.d.(l.c.m.$(a, b), c)$ = (l.c.m.$(a, b)) \cdot c$

$$l.c.m.(a, b, c) = \frac{(l.c.m.(a,b)) \cdot c}{g.c.d.(l.c.m.(a,b), c)}$$

$$= \frac{abc}{g.c.d.(a,b)\, g.c.d.(l.c.m.(a,b), c)}$$ **(l.c.m.(a, b) g.c.d.(a, b) = ab)**

$$= \frac{abc}{g.c.d.(g.c.d.(a,b) \times l.c.m.(a,b), c \times g.c.d.(a,b))}$$

(Taking c = g.c.d.(a, b) in Ex. 14)

$$= \frac{abc}{g.c.d.(ab, g.c.d.(ac, bc))}$$

(By using g.c.d.(a, b) · l.c.m.(a, b) = ab and Ex. 14)

$$= \frac{abc}{g.c.d.(ab, bc, ca)}$$ **(By definition)**

1.5 Irrational Numbers

We studied introductory approach to irrational numbers. We know that irrational numbers and rational numbers together constitute real numbers. (as a union)

$\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ etc. are irrational numbers. Sum of an irrational number and a rational number is irrational.

Product of a non-zero rational number and an irrational number is an irrational number.

An irrational number cannot be expressed in the form $\frac{a}{b}$ ($a \in Z$, $b \in N$) of rational numbers. But we cannot prove all irrational numbers to be irrational in this way. $\sqrt{2}$ is a root of $x^2 - 2 = 0$ or $\sqrt{3}$ is a root of to be $x^2 - 3 = 0$. This type of irrational numbers are called **algebraic irrational numbers.** π (and other similar numbers) are called **transcedental irrational numbers** and they are not roots of an algebraic equation. We will try to prove that algebraic irrational numbers like $\sqrt{2}$, $\sqrt{3}$ cannot be expressed in the form $\frac{a}{b}$ using fundamental theorem of arithmetic.

We will see some important lemmas. The proof of lemmas given are not from examination point of view but useful for application.

Lemma 1 : If a prime p divides a^2 for $a \in N$, then $p \mid a$. ($a > 1$)

Proof : Let $a = p_1^{a_1} p_2^{a_2} \ldots\ldots p_k^{a_k}$ be prime factorisation of a, where p_i are distinct primes and $a_i \in N$.

$\therefore \quad a^2 = (p_1^{a_1} p_2^{a_2} \ldots\ldots p_k^{a_k})(p_1^{a_1} p_2^{a_2} \ldots\ldots p_k^{a_k})$

$\qquad = p_1^{2a_1} p_2^{2a_2} \ldots\ldots p_k^{2a_k}$

Now $p \mid a^2$.

$\therefore \quad p \mid p_1^{2a_1} p_2^{2a_2} \ldots\ldots p_k^{2a_k}$

$\therefore$ p is one of the primes in the prime factorization of a^2. By uniqueness part of fundamental theorem of arithmetic $p = p_i$ for some i.

$\therefore \quad p \mid p_1^{a_1} p_2^{a_2} \ldots\ldots p_k^{a_k}$

$\therefore \quad p \mid a$

Lemma 2 : If m is the least positive divisor of a, then $m \le \sqrt{a}$, $a \in N$.

Proof : Let $a = mn \qquad m, n \in N$

Since m is the least positive divisor and n is also a divisor of a, $m \le n$

$\therefore \quad m^2 \le mn$

$\therefore \quad m^2 \le a$

$\therefore \quad m \le \sqrt{a}$

This result is a mild test to determine whether a given number is a prime or not. Suppose we want to factorise 257. Its earliest prime factor should be less than $\sqrt{257}$ i.e. 16.

We check whether 2, 3, 5, 7, 11 or 13 divide 257 or not. Obviously none of them divides 257.

$\therefore$ 257 is a prime.

(**Note :** In fact $257 = 2^{2^3} + 1$ is a Fermat prime and regular 257 - gon be constructed using straight-edge and compass only. Guass requested that after his death heptadecagon (17 sides, $2^{2^2} + 1 = 17$) be inscribed on his tomb ! But the storeman declined saying that it would look almost like a circle.)

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Example 16 : Prove that $\sqrt{2}$ is irrational.

Solution : If possible, let $\sqrt{2}$ be rational.

Let $\sqrt{2} = \frac{a}{b}$ where *g.c.d.*$(a, b) = 1$, $a \in N$, $b \in N$

$\therefore \quad a^2 = 2b^2$ **(i)**

$\therefore \quad 2 \mid a^2$

$\therefore \quad 2 \mid a$

Let $a = 2a_1$ say, $a_1 \in N$ **(using (i))**

$\therefore \ 4{a_1}^2 = a^2 = 2b^2$

$\therefore \quad b^2 = 2{a_1}^2$

$\therefore \quad 2 \mid b^2$

$\therefore \quad 2 \mid b$

$\therefore \quad b = 2b_1$ say, $b_1 \in N$

$\therefore \quad 2 \mid a$ and $2 \mid b$

But *g.c.d.* $(a, b) = 1$

$\therefore$ We come to a contradiction

$\therefore \ \sqrt{2}$ is irrational.

Lemma 3 : If p is prime, $\sqrt{p}$ is irrational.

Proof : Let if possible $\sqrt{p} = \frac{a}{b}$, *g.c.d.* $(a, b) = 1$, $a \in N$, $b \in N$

$\therefore \quad a^2 = pb^2$ **(i)**

$\therefore \quad p \mid a^2$

$\therefore \quad p \mid a$

Let $a = pm$, $m \in N$

$\therefore \quad a^2 = p^2m^2$

$\therefore \quad p^2m^2 = pb^2$ **(using (i))**

$\therefore \quad b^2 = pm^2$

$\therefore \quad p \mid b^2$

$\therefore \quad p \mid b$

$\therefore \quad p \mid a$ and $p \mid b$

But *g.c.d.* $(a, b) = 1$

$\therefore$ We come to a contradiction

$\therefore \quad \sqrt{p}$ is irrational.

Example 17 : Prove that $5 + 2\sqrt{7}$ is irrational.

Solution : Let $m = 5 + 2\sqrt{7}$ be rational, if possible.

$\therefore \quad m - 5 = 2\sqrt{7}$ is rational.

$\therefore \quad \frac{m-5}{2} = \sqrt{7}$ is rational.

But $\sqrt{7}$ is irrational. **(lemma 3)**

$\therefore$ $5 + 2\sqrt{7}$ is irrational.

Example 18 : Prove that $\sqrt{72}$ is irrational

Solution : $\sqrt{72} = \sqrt{36 \cdot 2} = 6\sqrt{2}$

If $6\sqrt{2} = m$ is rational, so is $\frac{m}{6} = \sqrt{2}$.

But $\sqrt{2}$ is irrational

$\therefore$ $\sqrt{72}$ is irrational.

Example 19 : Prove $\sqrt{3} + \sqrt{2}$ is irrational.

Solution : Let $x = \sqrt{3} + \sqrt{2}$ be rational.

$\therefore$ $x^2 = 3 + 2 + 2\sqrt{6}$ is rational.

$\therefore$ $\frac{x^2 - 5}{2} = \sqrt{6}$ is rational.

If possible let $\sqrt{6} = \frac{a}{b}$, where *g.c.d.* $(a, b) = 1$,

$\therefore$ $a^2 = 6b^2 = 2 \cdot 3b^2$

$\therefore$ $2 \mid a^2$

$\therefore$ $2 \mid a$

$\therefore$ $a = 2a_1$, say $a_1 \in N$

$\therefore$ $4{a_1}^2 = a^2 = 6b^2$

$\therefore$ $2{a_1}^2 = 3b^2$

$\therefore$ $2 \mid 3b^2$

Also *g.c.d.* $(3, 2) = 1$

$\therefore$ $2 \mid b^2$

$\therefore$ $2 \mid b$

$\therefore$ $2 \mid a$ and $2 \mid b$

$\therefore$ A contradiction results.

$\therefore$ $\sqrt{6}$ is irrational.

$x = \sqrt{3} + \sqrt{2}$ is irrational.

Another Method : Let $x = \sqrt{3} + \sqrt{2}$ be rational.

$(x - \sqrt{2})^2 = 3$ is rational.

$\therefore$ $x^2 - 2\sqrt{2}x + 2 = 3$ is rational

$\therefore$ $\sqrt{2} = \frac{x^2 - 1}{2x}$ is rational.

But $\sqrt{2}$ is irrational.

$\therefore$ $x = \sqrt{3} + \sqrt{2}$ is irrational.

EXERCISE 1.3

1. **Express as a product of primes :**
   (1) 7007 (2) 7500 (3) 10101 (4) 15422
2. **Find *g.c.d.* and *l.c.m.* using the fundamental theorem of arithmetic :**
   (1) 250 and 336 (2) 4000 and 25 (3) 225 and 145 (4) 175 and 1001
3. **Find *g.c.d.* and *l.c.m.* :** (1) 15, 21, 35 (2) 40, 60, 80 (3) 49, 42, 91
4. **Prove that following numbers are irrational :**
   (1) $\sqrt{5}$ (2) $\sqrt{15}$ (3) $\sqrt{3} + 1$ (4) $\sqrt{5} + \sqrt{7}$ (5) $5\sqrt{2}$
5. Find *l.c.m.* (105, 91) using *g.c.d.* (a, b) *l.c.m.* $(a, b) = ab$
6. Prove $\sqrt{3} + \sqrt{2} + 1$ is irrational.
7. Using $(\sqrt{7} + \sqrt{3})(\sqrt{7} - \sqrt{3}) = 4$ and the fact that $(\sqrt{7} + \sqrt{3})$ is irrational prove that $\sqrt{7} - \sqrt{3}$ is irrational.
8. Two buses start from the same spot for the same circular root. One is a BRTS bus returning in 35 minutes. The other is a regular express bus taking 42 minutes to return. After how many minutes will they meet again at the same initial spot ?

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1.6 Presentation of Rational Numbers in decimal system

Last year, we have studied representation of rational numbers in decimal system. We know that their decimal expansion is either terminating or non-terminating recurring. Let us examine the question more closely.

$$1.05 = \frac{105}{100} = \frac{21}{20},\ 3.008 = \frac{3008}{1000} = \frac{376}{125},\ 2.003 = \frac{2003}{1000}$$

If a rational number has a terminating decimal expansion, it should be like $a.a_1 a_2 a_3 ... a_k$ where a is an integer and $a_1, a_2, ... a_k$ are digits (0, 1, 2,.....or 9) after decimal point. (a could be negative. But it does not affect the discussion). So we assume that a is positive like in 3225.47891.

$a = 3225,\ a_1 = 4,\ a_2 = 7,\ a_3 = 8,\ a_4 = 9,\ a_5 = 1$

So we could write

$$a.a_1 a_2 a_3 ... a_k = a + 0.a_1 a_2 ... a_k = a + \frac{a_1 a_2 \dots a_k}{10^k} = a + \frac{a_1 a_2 \dots a_k}{2^k 5^k}$$

After removing common factors, we can write $a.a_1 a_2 ... a_k = a + \frac{m}{n}$ where *g.c.d.*$(m, n) = 1$ and $n = 2^p 5^q$ where p and q are non-negative integers not exceeding k.

Thus we have the following theorem.

Theorem 1.2 : If a rational number a has a terminating decimal expansion, we can write $a = \frac{p}{q}$ where *g.c.d.* $(p, q) = 1$ and $q = 2^m 5^n$ for some non-negative integers m and n.

Let us consider the converse situation.

Let $a = \frac{p}{2^m 5^n}$

Let $m < n$. Then $a = \dfrac{p}{2^m \cdot 5^{n-m} \cdot 5^m} = \dfrac{p}{2^m \cdot 5^{n-m} \cdot 5^m} \times \dfrac{2^{n-m}}{2^{n-m}}$

$$= \frac{p \cdot 2^{n-m}}{(10)^m (10)^{n-m}} = \frac{p \cdot 2^{n-m}}{10^n}$$

Similarly, if $m > n$, $a = \dfrac{p \cdot 5^{m-n}}{10^m}$

If $m = n$, obviously $a = \dfrac{p}{10^m}$ or $\dfrac{p}{10^n}$

For, example, $\dfrac{1003}{2^3 5^7} = \dfrac{(1003) \cdot 2^4}{2^7 \cdot 5^7} = \dfrac{(1003) \cdot 16}{10^7}$

$$\frac{2521}{2^5 5^3} = \frac{(2521) 5^2}{10^5} = \frac{25\,(2521)}{10^5}$$

$$\frac{1433}{2^6 5^6} = \frac{1433}{10^6}$$

Hence we have the following theorem

Theorem 1.3 : If a rational number is of the form $a = \dfrac{p}{2^m 5^n}$ for non negative integers m and n, then a has a terminating decimal expansion.

Theorem 1.4 : A rational number $\dfrac{p}{q}$ has terminating decimal expansion if and only if $q = 2^m 5^n$ for non-negative integers m, n.

$\dfrac{1}{7} = 0.\overline{142857}$ is a rational number with a non-terminating recurring decimal expansion.

Theorem 1.5 : A rational number $a = \dfrac{p}{q}$ has a non-terminating recurring decimal expansion if and only if q is not of the form $2^m 5^n$ $m, n \in N \cup \{0\}$ or in other words $g.c.d.\ (p, q) = 1$ and q has at least one more prime factor other than 2 or 5.

Example 20 : Decide if the following rational numbers have terminating decimal expansion or not and if it has, find it : (1) $\dfrac{337}{125}$ (2) $\dfrac{11}{1250}$ (3) $\dfrac{12}{35}$ (4) $\dfrac{42}{35}$

Solution : (1) $\dfrac{337}{125} = \dfrac{337 \times 8}{125 \times 8} = \dfrac{2696}{1000} = 2.696$

Here $125 = 5^3 = 5^3 \cdot 2^0$ and *g.c.d.* (337, 125) = 1

$\therefore$ $\dfrac{337}{125}$ has a terminating decimal expansion.

(2) $\dfrac{11}{1250} = \dfrac{11}{5^4 \cdot 2}$

Here, $5^4 \cdot 2^1$ is of the form $2^m 5^n$ and *g.c.d.*(11, 1250) = 1.

$\therefore$ $\dfrac{11}{1250} = \dfrac{11}{5^4 \cdot 2} = \dfrac{11 \cdot 2^3}{5^4 \cdot 2^4} = \dfrac{88}{10000} = 0.0088$

$\therefore$ $\dfrac{11}{1250}$ has a terminating decimal expansion.

(3) $\dfrac{12}{35} = \dfrac{12}{5 \cdot 7}$

Here, *g.c.d.* (12, 35) = 1 and $35 = 5 \cdot 7$ has a prime factor 7 other than 2 or 5.

$\therefore$ $\dfrac{12}{35}$ does not have a terminating decimal expansion.

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(4) $\frac{42}{35} = \frac{7 \cdot 6}{7 \cdot 5} = \frac{6}{5}$

Thus equivalent fraction $\frac{6}{5}$ has *g.c.d.*$(6, 5) = 1$ and $q = 5 = 2^m 5^n$ with $m = 0$ $n = 1$

$\therefore$ $\frac{42}{35} = \frac{6}{5} = \frac{6\times2}{5\times2} = \frac{12}{10} = 1.2$ is a terminating decimal expansion.

EXERCISE 1.4

1. State whether following rational numbers have terminating decimal expansion or not and if it has terminating decimal expansion, find it

(1) $\frac{12}{625}$ (2) $\frac{17}{3125}$ (3) $\frac{13}{6250}$ (4) $\frac{14}{15625}$ (5) $\frac{47}{500}$

(6) $\frac{9}{1600}$ (7) $\frac{42}{52}$ (8) $\frac{26}{65}$ (9) $\frac{8}{343}$ (10) $\frac{5}{128}$

2. Following real numbers are expressed in decimal form. Find whether they are rational or not. If rational, express them in the form $\frac{p}{q}$. Comment on factors of q :

(1) 0.01001000100001... (2) $3.\overline{456789123}$ (3) 5.123456789 (4) 0.090909... = $0.\overline{09}$

(5) $2.3\overline{12}$ (6) $0.\overline{142857}$ (7) 0.9999...... = $0.\overline{9}$ (8) 5.781 (9) 2.312 (10) 0.12345

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1.7 Surds

Now we will study some special type of irrational numbers like $\sqrt{2}$, $\sqrt{3}$, $\sqrt{3}+\sqrt{2}$. They are called **surds**.

If a is a positive rational number, $n \in$ N, $n \neq 1$ and if $\sqrt[n]{a} \notin$ Q$^+$, then we say that $\sqrt[n]{a}$ is a surd.

So $2 \in$ Q$^+$ and $\sqrt{2} \notin$ Q$^+$. Hence $\sqrt{2}$ is a surd. $\frac{3}{4} \in$ Q$^+$. But $\sqrt{\frac{3}{4}} \notin$ Q$^+$. Thus $\sqrt{\frac{3}{4}}$ is a surd. We write $\sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$. Similarly $\sqrt{\frac{12}{47}} = \sqrt{\frac{12 \cdot 47}{(47)^2}} = \frac{1}{47}\sqrt{564}$. So we consider $\sqrt[n]{a} = m\sqrt[n]{b}$ where $m \in$ Q$^+$, $b \in$ N and $n \in$ N $- \{1\}$. Hence we will think of surds $\sqrt[n]{a}$ where a is a positive integer. Our study will be confined to surds of type $\sqrt{a}$ where a is a positive integer. This type of surds, are called **quadratic surds**. Now $\sqrt{12} = 2\sqrt{3}$, $\sqrt{48} = 4\sqrt{3}$, $\sqrt{72} = 6\sqrt{2}$. Hence we can write $\sqrt{a} = b\sqrt{c}$ where c is a square free number i.e. c has no factor p^2 where p is a prime. Surds like $\sqrt{12} = 2\sqrt{3}$, $\sqrt{48} = 4\sqrt{3}$ are called **like surds**. So if the square free part $\sqrt{c}$ is same, then $b\sqrt{c}$ and $d\sqrt{c}$ are like surds.

Surds like $\sqrt{12} = 2\sqrt{3}$, $\sqrt{72} = 6\sqrt{2}$ are called **unlike surds**. If the square free parts of $\sqrt{p}$ and $\sqrt{q}$ are different, then $\sqrt{p}$ and $\sqrt{q}$ are unlike surds.

Like surds can be added as $\sqrt{12} + \sqrt{48} = 2\sqrt{3} + 4\sqrt{3} = 6\sqrt{3}$.

$\sqrt{3}$, $\sqrt{5}$, $5\sqrt{2}$ are called **monomial surds**. $\sqrt{3}+\sqrt{2}$, $\sqrt{5}+1$, $\sqrt{3}-2$ etc. are called **binomial surds**. If product of two binomial surds is a rational number, they are called **rationalising factors** of each other. $\sqrt{3}+\sqrt{2}$ and $\sqrt{3}-\sqrt{2}$ are **'conjugate surds'**. $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$ are, in general, conjugate surds of each other. Multiplication of conjugate binomial quadratic surds results into a rational number.

Note : $2+\sqrt{3}$ and $2-\sqrt{3}$ are conjugate quadratic surds of each other as well as rationalising factor of each other. But $3+\sqrt{2}$ and $15-5\sqrt{2}$ are rationalising factors of each other, but not the conjugate surds of each other.

$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = a-b$

We will use some results about surds

Result 1 : If $a+\sqrt{b}=\sqrt{c}$ where $a \in Q$ and $\sqrt{b}$ and $\sqrt{c}$ are surds, then $a=0$ and $b=c$.

Result 2 : If $a+\sqrt{b}=c+\sqrt{d}$ where $a, c \in Q$, $\sqrt{b}$, $\sqrt{d}$ are surds, then $a=c$, $b=d$.

Note : It is necessary that $\sqrt{b}$, $\sqrt{d}$ are surds i.e. irrational.

$3+\sqrt{25} = 1+\sqrt{49}$ but $3 \neq 1$, $25 \neq 49$

Square root of a surd : $(\sqrt{3}+\sqrt{2})^2 = 3+2\sqrt{6}+2 = 5+2\sqrt{6}$

So we can write $\sqrt{5+2\sqrt{6}} = \sqrt{3}+\sqrt{2}$. Here we say that a square root of binomial surd $5+2\sqrt{6}$ is $\sqrt{3}+\sqrt{2}$.

We accept following results :

Result 3 : In order that $\sqrt{a+2\sqrt{b}} = \sqrt{x}+\sqrt{y}$

where $x, y \in Q^+$, $\sqrt{b}$ is a surd is that $a \in Q^+$ and a^2-4b is the square of a rational number.

In fact $x = \dfrac{a+\sqrt{a^2-4b}}{2}$, $y = \dfrac{a-\sqrt{a^2-4b}}{2}$ **(i)**

So that $x+y=a$ and $xy=b$

Hence there are two approaches to find $\sqrt{a+2\sqrt{b}}$. Find integers x and y such that $x+y=a$, $xy=b$ or use the formula to find x, y.

(**Note :** We will get rid of fractions and simplify to make a and b integers.)

If $\sqrt{a+2\sqrt{b}} = \sqrt{x}+\sqrt{y}$, then $\sqrt{a-2\sqrt{b}} = \sqrt{x}-\sqrt{y}$ **($x > y$)**

Example 21 : Find $\sqrt{6+2\sqrt{5}}$.

Solution : Let us try to find positive integers x, y such that $x+y=6$, $xy=5$.

Obviously $x=5$, $y=1$

$\sqrt{6+2\sqrt{5}} = \sqrt{5}+\sqrt{1} = \sqrt{5}+1$

Another method : $a=6$, $b=5$

$$x = \frac{a+\sqrt{a^2-4b}}{2} = \frac{6+\sqrt{36-20}}{2} = \frac{6+4}{2} = 5,$$

$$y = \frac{a-\sqrt{a^2-4b}}{2} = \frac{6-4}{2} = 1$$

$\therefore \quad \sqrt{6+2\sqrt{5}} = \sqrt{5}+1$

Example 22 : Find $\sqrt{\frac{7}{4}-\sqrt{3}}$.

Solution : $\sqrt{\frac{7}{4}-\sqrt{3}} = \sqrt{\frac{7-4\sqrt{3}}{4}} = \dfrac{\sqrt{7-2\sqrt{12}}}{2}$

We find $\sqrt{7-2\sqrt{12}}$.

Now, $x + y = 7$, $xy = 12$

$\therefore \quad x = 4, y = 3$ $(x > y)$

$\therefore \quad \sqrt{\frac{7}{4}-\sqrt{3}} = \frac{\sqrt{4}-\sqrt{3}}{2} = \frac{2-\sqrt{3}}{2}$

Example 23 : Find $\sqrt{2+\sqrt{3}}$.

Solution : $\sqrt{2+\sqrt{3}} = \sqrt{\frac{4+2\sqrt{3}}{2}} = \frac{\sqrt{4+2\sqrt{3}}}{\sqrt{2}}$

Now concentrate on $\sqrt{4+2\sqrt{3}}$

$x + y = 4$, $xy = 3$

$\therefore \quad x = 3, y = 1$

$\therefore \quad \sqrt{2+\sqrt{3}} = \frac{\sqrt{3}+1}{\sqrt{2}} = \frac{(\sqrt{3}+1)\sqrt{2}}{(\sqrt{2})\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{2}$

Remember :

(1) Generally we do not keep $\sqrt{m}$ in the denominator and so, we rationalise by multiplying and dividing it by $\sqrt{m}$.

(2) To find the square root, the form should be $a + 2\sqrt{b}$

If there is a factor other than 2 multiplying $\sqrt{b}$, then keep only 2 as multiplier of $\sqrt{b}$. As for example, $4\sqrt{3} = 2\sqrt{12}$, $14\sqrt{3} = 2\sqrt{147}$

(3) If there is no 2 as multiplier of $\sqrt{b}$ bring it as follows. $\sqrt{6} = \frac{1}{2}2\sqrt{6} = \frac{1}{2}(2\sqrt{6})$

Also write like $\sqrt{12} = 2\sqrt{3}$ and $\sqrt{72} = 2\sqrt{18}$.

In other words bring the binomial surd whose square root is to be found out to the form $a + 2\sqrt{b}$ where a and b are positive integers.

Example 24 : Simplify $\frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{\sqrt{4}+\sqrt{3}} +...+ \frac{1}{\sqrt{n}+\sqrt{n-1}}$

Solution : Given expression $= \frac{\sqrt{2}-1}{(\sqrt{2}+1)(\sqrt{2}-1)} + \frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})} +...+ \frac{\sqrt{n}-\sqrt{n-1}}{(\sqrt{n}+\sqrt{n-1})(\sqrt{n}-\sqrt{n-1})}$

$= \sqrt{2} - 1 + \sqrt{3} - \sqrt{2} + \sqrt{4} - \sqrt{3} + ... + \sqrt{n} - \sqrt{n-1}$

$= \sqrt{n} - 1$

Example 25 : Simplify $\frac{4}{\sqrt{6-2\sqrt{5}}} + \frac{1}{\sqrt{5+2\sqrt{6}}}$.

Solution : $\frac{4}{\sqrt{6-2\sqrt{5}}} + \frac{1}{\sqrt{5+2\sqrt{6}}} = \frac{4}{\sqrt{5}-1} + \frac{1}{\sqrt{3}+\sqrt{2}}$

$$= \frac{4(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)} + \frac{(\sqrt{3}-\sqrt{2})}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}$$

$$= \sqrt{5} + 1 + \sqrt{3} - \sqrt{2}$$

$$= 1 - \sqrt{2} + \sqrt{3} + \sqrt{5}$$

EXERCISE 1.5

1. Find the square roots of following surds :

(1) $5 + 2\sqrt{6}$ (2) $9 + 2\sqrt{14}$ (3) $2 - \sqrt{3}$ (4) $a + \sqrt{a^2 - b^2}$

(5) $7 + \sqrt{48}$ (6) $6 + 4\sqrt{2}$ (7) $5 + \sqrt{21}$ (8) $8 - 3\sqrt{7}$

2. Simplify : $\frac{1}{\sqrt{12-2\sqrt{35}}} + \frac{1}{\sqrt{8-2\sqrt{15}}} - \frac{2}{\sqrt{10-2\sqrt{21}}}$

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Miscellaneous Examples

Example 26 : Find the largest number which leaves remainders 4 and 6 respectively when it divides 220 and 186.

Solution : Given condition implies $220 = dq_1 + 4$, $186 = dq_2 + 6$ for some integers q_1 and q_2. For required divisor d, $220 - 4 = 216$ and $186 - 6 = 180$ must be divisible by the required divisor d. Since we require the largest divisor, we must find *g.c.d.* of 216 and 180.

Now, $216 = 2^3 \times 3^3$ $180 = 2^2 \times 3^2 \times 5$

$\therefore$ *g.c.d.* $(216, 180) = 2^2 \times 3^2 = 36$

$\therefore$ 36 is the largest number dividing 220 and 186 and leaving remainders 4 and 6 respectively.

Example 27 : It is required to fit square granite tiles for the flooring of a room of size $20m \times 6m$. Find the length of the each square tile and the number of square tiles required.

Solution : We want to divide the length and breadth of the room such that each is a multiple of the length of a tile. Thus we require *g.c.d.* (20, 6).

$20 = 2^2 \times 5$ and $6 = 2 \times 3$

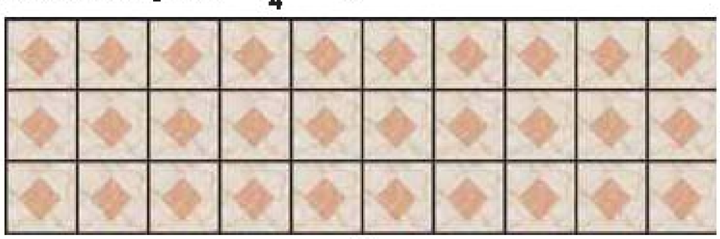
$\therefore$ *g.c.d.*$(20, 6) = 2$

So each granite tile must be a square of size $2m$. Area of room $= 120m^2$.

Area of each tile $= 4m^2$

$\therefore$ The number of tiles required $= \frac{120}{4} = 30$

6



$10 \times 3 = 30$ tiles

20

[**Note :** Infact we should arrange so that we fit $\frac{20}{2} = 10$ tiles along the length and $\frac{6}{2} = 3$ tiles along the breadth. Otherwise the tiles would have to be broken.]

Example 28 : What is the smallest number which when divided by 20, 30 and 40 leaves a remainder 5 ?

Solution : The number m must be of the form $m = 20q_1 + 5$, $m = 30q_2 + 5$, $m = 40q_3 + 5$

$\therefore$ $m - 5$ must be a multiple of 20, 30 and 40.

We require the smallest multiple i.e. *l.c.m.* of 20, 30 and 40.

$20 = 2^2 \times 5$, $30 = 2 \times 3 \times 5$, $40 = 2^3 \times 5$

$\therefore$ *l.c.m.* $(20, 30, 40) = 2^3 \times 3 \times 5 = 120$

$\therefore$ $m - 5 = 120$

The required number is 125.

Example 29 : Find the smallest number which is a multiple of all natural numbers from 2 to 10 (both inclusive).

Solution : The natural numbers from 2 to 10 are 2, 3, 4, 5, 6, 7, 8, 9, 10

$2 \mid 4$ and $4 \mid 8$ and $5 \mid 10$ and $3 \mid 6$ and $3 \mid 9$.

So may consider *l.c.m.* of 6, 7, 8, 9, 10 only.

Any multiple of 6 and 8 is a multiple of 24 (their *l.c.m.*)

we may consider *l.c.m.* of 7, 9, 10, 24.

Similarly 9 and 24 can be replaced by 72.

We require *l.c.m.* of 7, 10, 72

Now, $7 = 7^1$, $10 = 2 \cdot 5$, $72 = 2^3 \times 3^2$

l.c.m.$(7, 10, 72) = 2^3 \times 3^2 \times 5 \times 7$

The required number is 2520.

Example 30 : There is a circular path in a sports ground. Rucha takes 15 minutes to complete one round of the ring and Dev takes 20 minutes for the same. If they start running at the same initial point in the same direction at 8 A.M. when will they meet again for the first time ? How many rounds will be taken by each of them ?

Solution : Suppose Rucha takes m rounds and Dev takes n rounds before they meet. Then the time elapsed (in minutes) is

$$15m = 20n$$

$\therefore$ We want the least common multiple of 15 and 20 for meeting again for the first time.

Now $15 = 3 \cdot 5$ $\quad$ $20 = 2^2 \cdot 5$

l.c.m.$(15, 20) = 2^2 \times 3 \times 5 = 60$ minutes

They will meet again at 9 A.M.

Rucha will have completed $\frac{60}{15} = 4$ rounds and Dev would have completed $\frac{60}{20} = 3$ rounds by then.

Example 31 : Prove that $\sqrt{n+1} + \sqrt{n-1}$ is not rational for any $n \in N$.

Solution : Let $a = \sqrt{n+1} + \sqrt{n-1}$ be rational.

$\therefore$ $a - \sqrt{n-1} = \sqrt{n+1}$

$\therefore$ $a^2 - 2a\sqrt{n-1} + n - 1 = n + 1$

$\therefore$ $2a\sqrt{n-1} = a^2 - 2$

$\therefore$ $\sqrt{n-1} = \frac{a^2-2}{2a}$ is rational. **(as a is rational)**

Similarly, $\sqrt{n+1} = \frac{a^2+2}{2a}$ is rational.

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Let $\sqrt{n+1} = p$, $\sqrt{n-1} = q$. Then p and q are integers as $n \in N$ and p and q are rational.

$\therefore$ $n + 1 = p^2$ and $n - 1 = q^2$

$\therefore$ $p^2 - q^2 = 2$

$\therefore$ $(p + q)(p - q) = 2$

The only factors of 2 are 2 and 1.

$\therefore$ $p + q = 2$ and $p - q = 1$

Since $p - q \neq 0$, p and q are distinct positive integers.

Minimum value of $p + q$ is $2 + 1 = 3$ and $p + q \neq 2$.

$\therefore$ $\sqrt{n+1} + \sqrt{n-1}$ is not rational for $n \in N$.

Example 32 : Is following true ?

g.c.d. $(a, b) = 32$, *l.c.m.* $(a, b) = 48$

Solution : No, as *g.c.d.* (a, b) always divides *l.c.m.* (a, b).

But 32 does not divide 48.

$\therefore$ This is not true.

EXERCISE 1

Find *g.c.d.* and *l.c.m.* (1 to 10) :

1. 25, 35 **2.** 105, 125 **3.** 220, 132 **4.** 3125, 625 **5.** 15625, 35

6. 15, 25, 35 **7.** 18, 12, 16 **8.** 16, 24, 36 **9.** 35, 28, 63 **10.** 112, 128, 144

Prove following numbers are irrational. (11 to 20) :

11. $\sqrt{3} + \sqrt{5}$ **12.** $5\sqrt{3}$ **13.** $\frac{1}{\sqrt{5} - \sqrt{3}}$ **14.** $\sqrt{7} + \sqrt{3}$ **15.** $\sqrt{2} + 1$

16. $10\sqrt{2} + 7\sqrt{3}$ **17.** $\sqrt{5} - \sqrt{2}$ **18.** $\sqrt{12}$ **19.** $\sqrt{18}$ **20.** $\sqrt{37}$

Which of the following numbers have terminating decimal expansion and why ? (21 to 25)

21. $\frac{211}{125}$ **22.** $\frac{156}{625}$ **23.** $\frac{337}{35}$ **24.** $\frac{132}{49}$ **25.** $\frac{235}{16}$

Find the square root of the following in the form of a binomial surd (26 to 30) :

26. $12 + 2\sqrt{35}$ **27.** $8 + 2\sqrt{7}$ **28.** $2 + \frac{2}{3}\sqrt{5}$ **29.** $14 + 6\sqrt{5}$ **30.** $n + \sqrt{n^2 - 1}$

Simplify (31 to 32) :

31. $\frac{1}{\sqrt{3} + \sqrt{2}} + \frac{1}{\sqrt{4} + \sqrt{3}} + \sqrt{2}$ **32.** $\frac{6}{\sqrt{24 - 2\sqrt{135}}} - \sqrt{15}$

33. Find the largest number dividing 230 and 142 and leaving remainders 5 and 7 respectively.

34. Find the largest number dividing 110, 62, 92 and leaving remainders 5, 6 and 1 respectively.

35. The length and the breadth and the height of a room are 735 *cm*, 625 *cm* and 415 *cm*. Find the length of the largest scale measuring instrument which can measure all the three dimensions.

36. A milk man has 150 litres of milk of higher fat and 240 litres of milk of lower fat. He wants to pack the milk in tins of equal capacity. What should be the capacity of each tin ?

37. Find the smallest number which decreased by 15 is a multiple of 125 and 225

38. Find the smallest number of six digits divisible by 18, 24 and 30

39. Prove if $3 \mid (a^2 + b^2)$ then $3 \mid a$ and $3 \mid b$, $a \in N$, $b \in N$.

40. Prove $n^4 + 4$ is a composite number for $n > 1$

41. In a morning walk a man, a woman and a child step off together. Their steps measure 90 *cm*, 80 *cm* and 60 *cm*. What is the minimum distance each should walk to cover the distance in complete steps ?

42. Find the number nearest to 24001 and between 24001 and 25000 divisible by 16, 24, 40

43. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) Product of any four consecutive positive integers is divisible by .... ☐

(a) 16 (b) 48 (c) 24 (d) 32

(2) $\sqrt{4} + 3$ is ...... ☐

(a) irrational (b) rational but not integer.

(c) nonrecurring decimal (d) integer

(3) If *g.c.d.* of two numbers is 8 and their product is 384, then their *l.c.m.* is .... ☐

(a) 24 (b) 16 (c) 48 (d) 32

(4) If *l.c.m.* of two numbers (greater than 1) is the product of them, then their *g.c.d.* is .... ☐

(a) 1 (b) 2

(c) one of the numbers (d) a prime

(5) If p_1 and p_2 are distinct primes, their *g.c.d.* is .... ☐

(a) p_1 (b) p_2 (c) p_1p_2 (d) 1

(6) If p, q, r are distinct primes, their *l.c.m.* is .... ☐

(a) pqr (b) pq (c) 1 (d) $pq + qr + pr$

(7) *g.c.d.* (15, 24, 40) = ... ☐

(a) 40 (b) 1 (c) 14 (d) $15 \times 24 \times 40$

(8) *l.c.m.* (15, 24, 40) = ... ☐

(a) 1 (b) $15 \times 24 \times 40$ (c) 120 (d) 60

(9) 0.02222... is a .... ☐

(a) rational number (b) integer

(c) irrational number (d) zero

(10) $\sqrt{3+\sqrt{5}}$ = ... ☐

(a) $\sqrt{3} + \sqrt{2}$ (b) $\sqrt{5} + 1$ (c) $\frac{\sqrt{5}+1}{\sqrt{2}}$ (d) does not exist

(11) $\sqrt{9+\sqrt{141}}$ = ...

(a) does not exist as a real number (b) does not exist as a binomial surd

(c) $\sqrt{9} + \sqrt{141}$ (d) $\sqrt{141} - \sqrt{9}$

(12) g.c.d. (136, 221, 391) = ...

(a) 136 (b) 17 (c) 221 (d) 391

(13) l.c.m. (136, 221, 391) = ...

(a) 40664 (b) $136 \times 221 \times 391$

(c) g.c.d.(136, 221, 391) (d) 136×221

(14) If g.c.d. $(a, b) = 8$, l.c.m. $(a, b) = 64$ and $a > b$ then a = ...

(a) 64 (b) 8 (c) 16 (d) 32

(15) If g.c.d. $(a, b) = 1$, then g.c.d. $(a - b, a + b)$ = ...

(a) 1 or 2 (b) a or b (c) $a + b$ or $a - b$ (d) 4

(16) If $n > 1$, $n^4 + 4$ is .... $n \in$ N

(a) a prime (b) a composite integer

(c) 1 (d) infinite

(17) If g.c.d. $(a, b) = 18$, l.c.m. $(a, b) \neq$ ....

(a) 36 (b) 72 (c) 48 (d) 108

(18) $\frac{18}{5^3}$ has ...... digits after decimal point.

(a) 5 (b) 4 (c) 3 (d) 2

(19) The decimal expansion of $\frac{2517}{6250}$ will terminate after ...... digits.

(a) 4 (b) 5 (c) 3 (d) 6

(20) 5^n ($n \in$ N) ends with ....

(a) 0 (b) 5 (c) 25 (d) 10

(21) $2^m 5^n$ ($m, n \in$ N) ends with ....

(a) 0 (b) 5 (c) 25 (d) 125

(22) $\frac{317}{3125}$ represents ....

(a) a terminating decimal (b) a non recurring decimal

(c) a recurring decimal (d) an integer

(23) $(5k + 1)^2$ leaves remainder ...... on dividing by 5.

(a) 2 (b) 0 (c) −1 or 1 (d) 1

(24) On division by 6, a^2 cannot leave remainder .... ($a \in$ N)

(a) 1 (b) 4 (c) 5 (d) 3

(25) Product of three consecutive integers is divisible by ....

(a) 24 (b) 8 but not by 24 (c) 6 (d) 20

*

Summary

In this chapter we have studied following points :

1. **Division algorithm :** If a, b are positive integers, there exist unique integers q and r such that $a = bq + r$, $0 \le r < b$
2. Euclid's Algorithm to find **g.c.d.**.

   $a = bq_1 + r_1 \qquad 0 \le r_1 < b$

   $b = r_1 q_2 + r_2 \qquad 0 \le r_2 < r_1$

   $r_1 = r_2 q_3 + r_3 \qquad 0 \le r_3 < r_2$

   . . .

   $r_{n-2} = r_{n-1} q_n + r_n \qquad 0 \le r_n < r_{n-1}$

   $r_{n-1} = r_n q_{n+1}$

   g.c.d. $(a, b) = r_n$

   Also **l.c.m.** (a, b) **g.c.d.** $(a, b) = ab$
3. **Fundamental theorem of Arithmetic :** If $n > 1$ is a positive integer,

   $n = p_1^{a_1} p_2^{a_2} \ldots\ldots p_k^{a_k}$ where $p_1, p_2, \ldots p_k$ are prime divisors of n and $a_i \in$ N.

   This representation is unique.
4. Irrational numbers.
5. Decimal Expansions.
6. Surds and their square roots.

   $\sqrt{a+2\sqrt{b}} = \sqrt{x} + \sqrt{y}$, if $a \in Q^+$ and $a^2 - 4b$ is the square of a rational number. $\sqrt{b}$ is a surd.

   $$x = \frac{a+\sqrt{a^2-4b}}{2}, \; y = \frac{a-\sqrt{a^2-4b}}{2}.$$

   Also if $\sqrt{a+2\sqrt{b}} = \sqrt{x} + \sqrt{y}$, $(x > y)$, then

   $$\sqrt{a-2\sqrt{b}} = \sqrt{x} - \sqrt{y}$$

◆

- Five Pirates and a monkey are shipwrecked on an island. The pirates have collected a number of coconuts which they plan to divide the next morning. Not trusting others, one pirate wakes up during the night and divides the coconuts into five equal parts and one is left which he gives to the monkey. He hides his portion. During the night each of the five pirates does the same thing equally dividing coconuts in five equal parts, giving one left one to the monkey and hiding his share. In the morning they all wake up and divide them equally and one is left over and given to the monkey. What is the smallest number of coconuts they could have collected for their original pile ?

Let the number of coconuts be n. Then after the first pirate divides in five parts and one coconut remains, $n = 5a + 1$. Here a is the part taken by pirate 1. Now $4a$ coconuts remain. $5a + 1 - 1$ (monkey) $- a$(Pirate 1).

$\therefore$ As before $4a = 5b + 1$, where b is the part taken by pirate 2.

Similarly $4b = 5c + 1$, c is the part taken by pirate 3.

$4c = 5d + 1$, d is the part taken by pirate 4.

$4d = 5e + 1$, e is the part taken by pirate 5.

In the end $4e = 5f + 1$, where 1 coconut is given to the monkey and f is the equal part taken by each pirate.

Hence $n = 5a + 1$, $4a = 5b + 1$, $4b = 5c + 1$, $4c = 5d + 1$, $4d = 5e + 1$, $4e = 5f + 1$.

$\therefore \quad 4\left(\frac{4d-1}{5}\right) = 5f + 1$, giving us $16d = 25f + 9$. **($4d = 5e + 1$)**

$\therefore \quad 16\left(\frac{4c-1}{5}\right) = 25f + 9$, giving us $64c = 125f + 61$. **($4c = 5d + 1$)**

$\therefore \quad 64\left(\frac{4b-1}{5}\right) = 125f + 61$ **($4b = 5c + 1$)**

$\therefore \quad 256b = 625f + 369$

$\therefore \quad 256\left(\frac{4a-1}{5}\right) = 625f + 369$

Continuing finally $1024a - 3125f = 2101$ **(i)**

Now, $3125 = 1024 \cdot 3 + 53$

$1024 = 53 \cdot 19 + 17$

$53 = 17 \cdot 3 + 2$

$17 = 2 \cdot 8 + 1$

$\therefore \quad 1 = 17 - 2 \cdot 8 = 17 - 8\,(53 - 17 \cdot 3)$

$= 25 \cdot 17 - 8 \cdot 53$

$= 25(1024 - 53 \cdot 19) - 8 \cdot 53$

$= 25 \cdot 1024 - 483 \cdot 53$

$= 25 \cdot 1024 - 483(3125 - 1024 \cdot 3)$

$\therefore \quad 1 = 1474 \cdot 1024 - 483 \cdot 3125$

$\therefore \quad 1474 \cdot 2101 \cdot 1024 - 483 \cdot 2101 \cdot 3125 = 2101$

Solution of (i) is $a = 1474 \cdot 2101$, $b = 483 \cdot 2101$ or

$a = 1474 \cdot 2101 - 3125t$ $\quad t \in \mathbf{N}$

$= 3096874 - 3125t$

$t = 990$ gives the smallest $a = 3096874 - 3093750 = 3124$

$\therefore \quad a = 3124$. Hence $n = 5a + 1 = 15621$

$\therefore$ Original pile contained 15621 coconuts.

(We can see $a = 3124$, $b = 2499$, $c = 1999$, $d = 1599$, $e = 1279$, $f = 1023$ each pirate had respectively $3124 + 1023$, $2499 + 1023$, $1999 + 3123$, $1599 + 1023$, $1279 + 1023$ coconuts.

$\therefore$ They got 4147, 3522, 3022, 2622, 2302 cocontus and the monkey got 6 coconuts totallying to 15621.

POLYNOMIALS 2

Mathematicians are born, not made.

- Henry Poincare

*

Mathematicians stand on each others shoulders.

- Carl Friedrich Gauss

2.1 Introduction

We have studied about polynomials, degree and coefficients of terms in a polynomial and types of polynomials in class IX. We have also studied factor theorem, remainder theorem, division of polynomials and factorization of polynomials. We will review these by the following example.

Example 1 : Identify the type of the given polynomials and state the degree and the coefficient of each term of them. Also find their value at $x = 1$.

(1) $p(x) = 7x + 14$ **(2)** $p(x) = 3x^2 + 7x + 4$ **(3)** $p(x) = 4x^3 + 3x^2 + 2x + 1$

Solution : (1) Here $p(x) = 7x + 14$.

Degree of this polynomial is 1. So it is a linear polynomial in x.

The coefficient of x is 7 and the constant term is 14.

$p(1) = 7(1) + 14 = 7 + 14 = 21$

(2) Here $p(x) = 3x^2 + 7x + 4$

Degree of this polynomial is 2. So it is a quadratic polynomial in x.

The coefficient of x^2 is 3; that of x is 7 and the constant term is 4.

$p(1) = 3(1)^2 + 7(1) + 4 = 3 + 7 + 4 = 14$

(3) Here $p(x) = 4x^3 + 3x^2 + 2x + 1$

Degree of this polynomial is 3. So it is a cubic polynomial in x :

The coefficient of x^3 is 4, that of x^2 is 3. The coefficient of x is 2 and the constant term is 1.

$p(1) = 4(1)^3 + 3(1)^2 + 2 (1) + 1 = 4 + 3 + 2 + 1 = 10$

EXERCISE 2.1

1. Identify the type of the following polynomials : (on base of power)

(1) $p(x) = x^2 - 5x + 6$ **(2)** $p(x) = x^2 - x^3 + x + 1$

(3) $p(x) = 5x^2 + 8x + 3$ **(4)** $p(x) = x^3$

2. Obtain the degree of the following polynomials :

(1) $p(x) = 3x - x^4 + x^2 + 2x^3 + 7$ **(2)** $p(x) = x^3 - 3x - x^2 + 6$

(3) $p(x) = 3x - 9$ **(4)** $p(x) = 2x^2 - x + 1$

3. Find the coefficients of the underlined terms :

(1) $p(x) = \underline{10x^3} + 7x^2 - 3x + 5$ (2) $p(x) = 7 - \underline{5x^5} + 3x^4 + x^2 - x$

(3) $p(x) = 25 - \underline{125x}$ (4) $p(x) = x^3 - \underline{x^2} + x + 7$

4. Obtain the value of the following polynomials at the given values of x :

(1) $p(x) = 2x^3 + 3x^2 + 7x + 9$; at $x = 0, 1$

(2) $p(x) = 3x^2 + 10x + 7$; at $x = -3, 1$

(3) $p(x) = x^2 - 2x + 5$; at $x = -1, 5$

(4) $p(x) = 2x^4 - 3x^3 + 7x + 5$; at $x = -2, 2$

5. Examine the validity of the following statements :

(1) $(x + 1)$ is a factor of $p(x) = 3x^3 + 2x^2 + 7x + 8$

(2) $(x + 2)$ is a factor of $p(x) = x^3 + x^2 + x + 2$

(3) $(x - 1)$ is a factor of $p(x) = x^4 - 2x^3 + 3x - 2$

(4) $(x - 3)$ is a factor of $p(x) = x^2 - 2x - 3$

6. Factorize the following polynomials :

(1) $p(x) = x^3 - x^2 - x + 1$ (2) $p(x) = 5x^2 + 11x + 6$

(3) $p(x) = x^3 - 3x^2 + 9x - 27$ (4) $p(x) = x^3 + 2x^2 + 3x + 2$

7. Prove that $x - 2$ is a factor of $p(x) = x^3 - 2x^2$

*

2.2 Geometrical meaning of the zeros of a polynomial

We know that the graph of a linear polynomial is a straight line. Thus, the graph of $p(x) = ax + b$; $a \neq 0$ $a, b \in$ R contains at least two distinct points and it intersects X-axis at exactly one point $\left(-\frac{b}{a}, 0\right)$ where $-\frac{b}{a}$ is the zero of $p(x)$.

To understand this, we take the following example :

Example 2 : Find the zeros of the linear polynomial $p(x) = 3x - 6$ and show it on a graph.

Solution : Here $p(x) = 3x - 6$

To find the zeros of $p(x)$, consider $p(x) = 0$.

$\therefore$ $3x - 6 = 0$

$\therefore$ $3(x - 2) = 0$

$\therefore$ $x = 2$ as $3 \neq 0$.

$\therefore$ 2 is the zero of $p(x)$.

We can also find the zero of $p(x)$ by its graph. For this,

taking $x = 1$, in $p(x)$; $p(1) = 3(1) - 6 = 3 - 6 = -3$

taking $x = 3$, in $p(x)$; $p(3) = 3(3) - 6 = 9 - 6 = 3$

taking $x = 2$, in $p(x)$; $p(2) = 3(2) - 6 = 6 - 6 = 0$

We get the following table from this (Table 2.1)

Table 2.1

x	1	2	3
$p(x) = 3x - 6$	-3	0	3

Plot these points $(x, p(x))$ on a graph-paper (figure 2.1). From the graph, we can say that the graph of $p(x)$ is a straight line which intersects X-axis at (2, 0). So 2 is the zero of $p(x)$. Thus, the zero of a linear polynomial is x-coordinate of the point where it intersects X-axis.

This graph shows that the linear polynomial intersects X-axis at exactly one point i.e. (2, 0).

Now some questions arise in our mind about the graph and zeros of any quadratic polynomial $p(x) = ax^2 + bx + c$; $a \neq 0$

To answer this question, we take following examples.

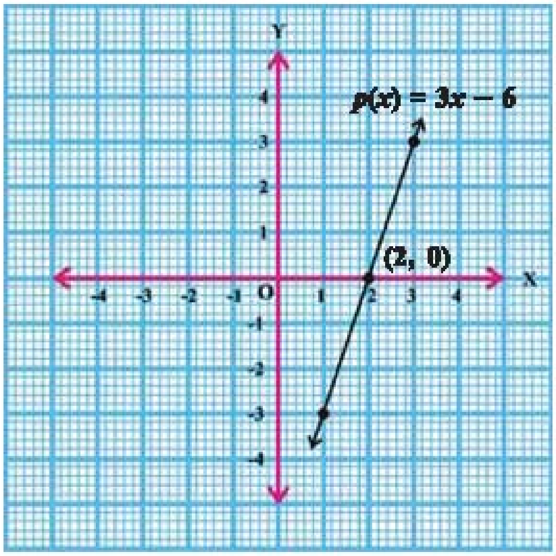


Figure 2.1

Example 3 : Find the zeros of the quadratic polynomial $p(x) = x^2 + 5x + 6$ and show them graphically.

Solution : By factorization, we can find the zeros of this quadratic polynomial as follows :

Here $p(x) = x^2 + 5x + 6$

To find the zeros of $p(x)$, let $p(x) = 0$

$\therefore \quad x^2 + 5x + 6 = 0$

$\therefore \quad x^2 + 3x + 2x + 6 = 0$

$\therefore \quad (x + 3)(x + 2) = 0$

$\therefore \quad x = -3$ or $x = -2$

$\therefore \quad -3$ and -2 are the zeros of $p(x)$.

To draw the graph of this polynomial, we take some different values of x and prepare the following table (Table 2.2)

Table 2.2

x	-4	-3	-2	-1
$p(x) = x^2 + 5x + 6$	2	0	0	2

Plotting these points on a graph-paper (figure 2.2). We can see that this graph intersects X–axis at two distinct points (–3, 0) and (–2, 0). Their x–coordinates are considered as the zeros of this polynomial. Thus, –2 and –3 are the zeros of $p(x)$. Here, we join all these points (–4, 2), (–3, 0), (–2, 0) and (–1, 2) we get the shape of the graph as shown, ⋃ i.e. open upwards.

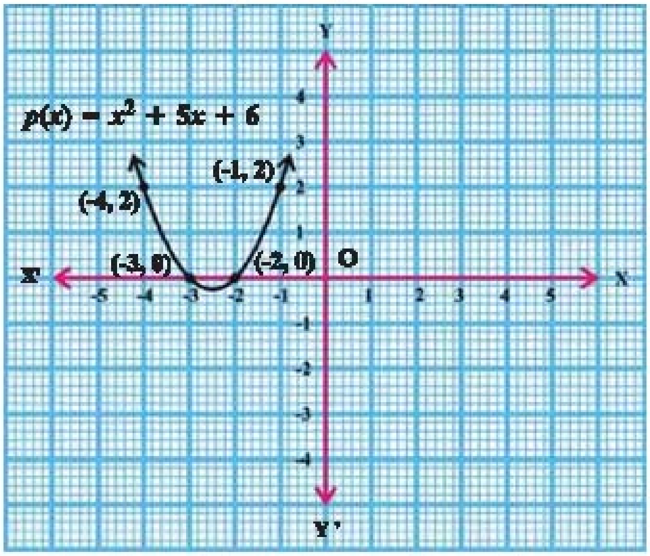


Figure 2.2

Example 4 : Find the zeros of the quadratic polynomial $p(x) = 2 + x - x^2$ and show them graphically.

Solution : Here $p(x) = 2 + x - x^2$

By taking different values of x, we get the following table (Table 2.3)

Table 2.3

x	–1	0	1	2
$p(x) = 2 + x - x^2$	0	2	2	0

From the table 2.3, we can infer that –1 and 2 are zeros of $p(x)$ because we have $p(-1) = 0 = p(2)$. We can say this from the graph in figure 2.3. The shape is ⋂ (open downwards). This curve intersects X-axis at (–1, 0) and (2, 0). Hence, their x–coordinates give the zeros of $p(x)$.

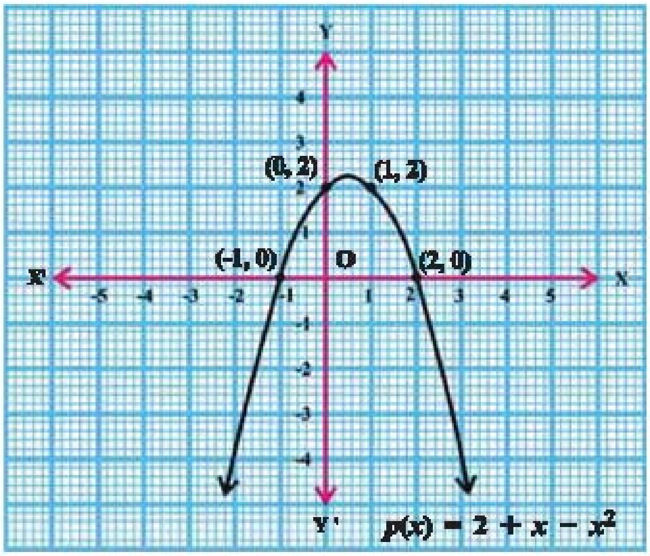


Figure 2.3

From the figure 2.2 and figure 2.3, we say that the shape of the graph of any quadratic polynomial $ax^2 + bx + c$ is either open upwards (i.e. ⋃) or open downwards (i.e. ⋂) depending upon $a > 0$ or $a < 0$ respectively.

In example 3, we can see that $a = 1 > 0$ and in example 4, we have $a = -1 < 0$ which gives the graphs as shown. These curves are called parabolas.

Thus, the zeros of any quadratic polynomial $p(x) = ax^2 + bx + c$, $a \neq 0$ $a, b, c \in R$ are the x-coordinates of the points where parabola (the graph of $p(x)$) intersects X-axis.

Example 5 : Draw the graph of the quadratic polynomial $p(x) = x^2 + 6x + 9$ and find the zeros of it.

Solution : Here $p(x) = x^2 + 6x + 9$

To find the zeros of $p(x)$, consider $p(x) = 0$

$\therefore \quad x^2 + 6x + 9 = 0$

$\therefore \quad (x + 3)^2 = 0$

$\therefore \quad x = -3$

$\therefore \quad -3$ is the zero of $p(x)$.

For the graph, we take the different values of x as follows.

Table 2.4

x	-3	-2	-1	-4
$p(x) = x^2 + 6x + 9$	0	1	4	1

Plot all these points on a graph-paper (figure 2.4). We can see that the graph intersects X–axis at the point $(-3, 0)$. Thus, -3 is the zero of $p(x)$.

In this case **we say that the graph touches X-axis.**

In the above example $a = 1 > 0$. Hence the shape of this graph is $\bigvee$ (i.e. open upwards). Now if $a < 0$ then we get the graph in the shape $\bigwedge$ (i.e. open downwards). We can observe this in the following example.

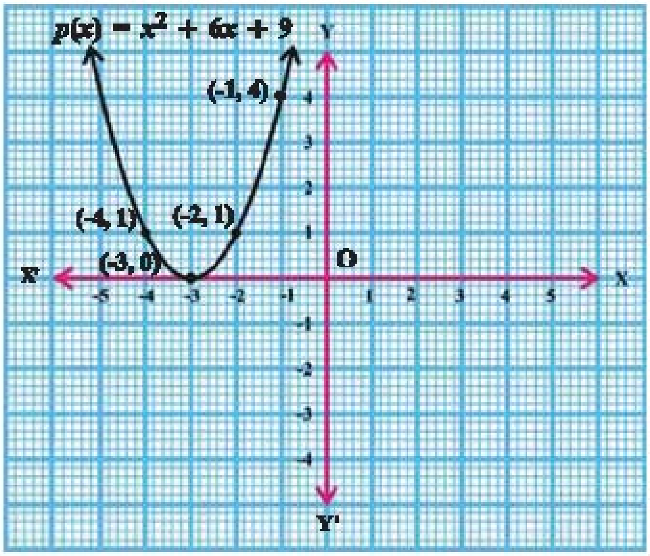


Figure 2.4

Example 6 : Find the zeros of $p(x) = -x^2 + 2x - 1$ and show them on a graph.

Solution : Here $p(x) = -x^2 + 2x - 1$

$$= -(x^2 - 2x + 1) = -(x - 1)^2$$

Now, for the zeros of $p(x)$, let $p(x) = 0$

$\therefore \quad -(x - 1)^2 = 0$

$\therefore \quad x - 1 = 0$

$\therefore \quad x = 1$

$\therefore \quad 1$ is the zero of $p(x)$.

For the graph, we take different values of x as follows :

Table 2.5

x	-1	0	1	2
$p(x) = -x^2 + 2x - 1$	-4	-1	0	-1

Plot these points on a graph-paper. We get the graph in the lower half-plane of X-axis and it touches the X-axis at (1, 0). (figure 2.5)

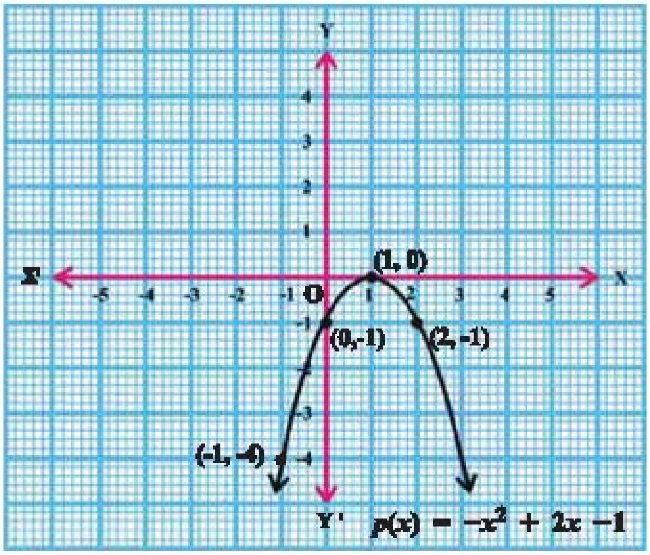


Figure 2.5

Note : If the graph lies either completely above X-axis or completely below X-axis i.e. in the upper half-plane of X-axis or in the lower half plane of X-axis, it does not intersect X-axis at any point. So the polynomial has no real zero. Let us understand this by the following example :

Example 7 : Draw the graph of $p(x) = x^2 + 4x + 5$ and from this find the zeros of $p(x)$.

Solution :

Taking $x = -4$ in $p(x)$, $p(-4) = 5$

Taking $x = -3$ in $p(x)$, $p(-3) = 2$

Taking $x = -2$ in $p(x)$, $p(-2) = 1$

Taking $x = -1$ in $p(x)$, $p(-1) = 2$

Taking $x = 0$ in $p(x)$, $p(0) = 5$

All these values are written in a tabular form as follows :

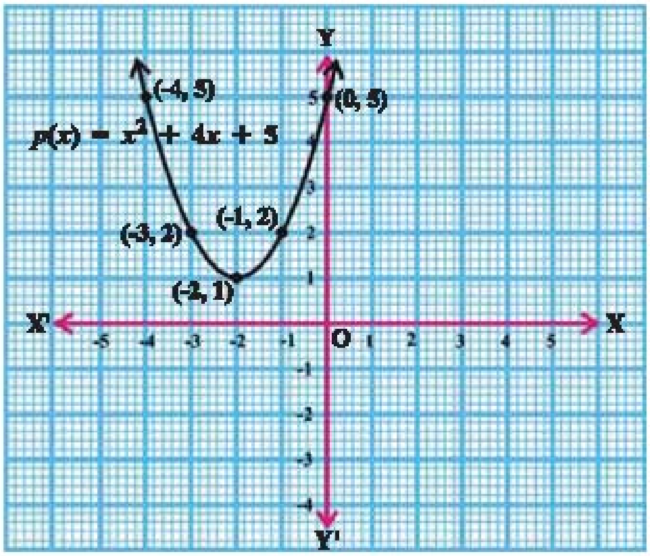


Figure 2.6

Table 2.6

x	-4	-3	-2	-1	0
$p(x) = x^2 + 4x + 5$	5	2	1	2	5

Plot all these points on graph paper as shown in figure 2.6. From this figure, we see that the graph does not intersect X-axis at any point. So $p(x)$ has no real zero. This graph is lying in the upper half plane of X-axis.

Now if we take $p(x) = -6 + x - x^2$, then how will we draw the graph ? In which part of the plane, the graph of this polynomial lies ? (Do it by yourself.)

So, we had seen that any quadratic polynomial has either two distinct zeros or exactly one zero or no zero. This means that a quadratic polynomial has at most two zeros. For the graph of any quadratic polynomial $p(x) = ax^2 + bx + c$, $a \neq 0$, $a, b, \in$ R there are three posibilities :

(1) If the graph of a quadratic polynomial intersects the X-axis in two different points A and B, then x-coodrinate of A and B are zeros of the quadratic polynomial. (Examples 3, 4).

(2) If the graph of a quadratic polynomial cuts the X-axis in one point i.e. it touches the X-axis, then the x-coordinate of that point is zero of given polynomial (See examples 5 and 6).

(3) The graph of the quadratic polynomial does not intersect the X-axis (Example 7).

Let us discuss about the geometrical meaning of the zeros of a cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$, $a, b, c, d \in$ R.

To understand this, we take the following examples.

Example 8 : Draw the graph of the cubic polynomial $p(x) = x^3 - 4x$. From this find the zeros of $p(x)$.

Solution : To draw the graph of this polynomial

Taking $x = -2$ in $p(x)$, $p(-2) = 0$

Taking $x = -1$ in $p(x)$, $p(-1) = 3$

Taking $x = 0$ in $p(x)$, $p(0) = 0$

Taking $x = 1$ in $p(x)$, $p(1) = -3$

Taking $x = 2$ in $p(x)$, $p(2) = 0$

We get the following table (Table 2.7)

Table 2.7

x	-2	-1	0	1	2
$p(x) = x^3 - 4x$	0	3	0	-3	0

Plotting all these points on a graph paper (figure 2.7), we can see that the graph intersects X–axis at the points $(-2, 0)$, $(0, 0)$ and $(2, 0)$. So, $x = -2$, 0 and 2 are the zeros of $p(x)$.

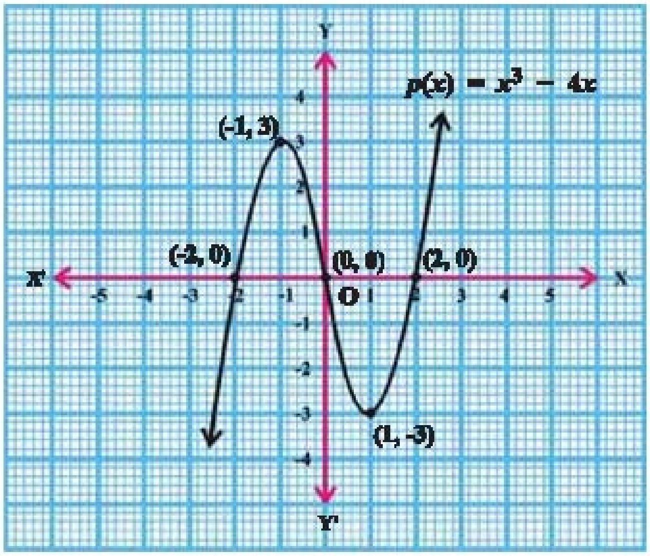


Figure 2.7

Downloaded from https:// www.studiestoday.com

Example 9 : Find the zeros of $p(x) = x^3 - 2x^2$ and show them graphically.

Solution : Here $p(x) = x^3 - 2x^2 = x^2(x - 2)$. For the zeros of $p(x)$; consider $p(x) = 0$.

$\therefore \quad x^2(x - 2) = 0$

$\therefore \quad x = 0 \quad \text{or} \quad x = 2$

$\therefore \quad 0$ and 2 are the zeros of $p(x)$.

Table 2.8

x	-1	0	1	2
$p(x) = x^3 - 2x^2$	-3	0	-1	0

Using the table 2.8. We can draw the graph of $p(x) = x^3 - 2x^2$ as given below :

Plotting all these points on a graph-paper, we can see that this graph intersects X-axis at two distinct points (0, 0) and (2, 0). Their x-coordinates are the zeros of $p(x)$. Thus, 0 and 2 are the zeros of $p(x)$.

Now, for a cubic polynomial $p(x) = x^3$, how many zeros are possible ? Can you find zeros of this polynomial ? (Do it by yourself)

From the above discussion, we can state the geometrical meaning of the zeros of a polynomial. **If the graph of any polynomial intersects the X–axis, x-coordinates of the points of intersection are the zeros of given polynomial.**

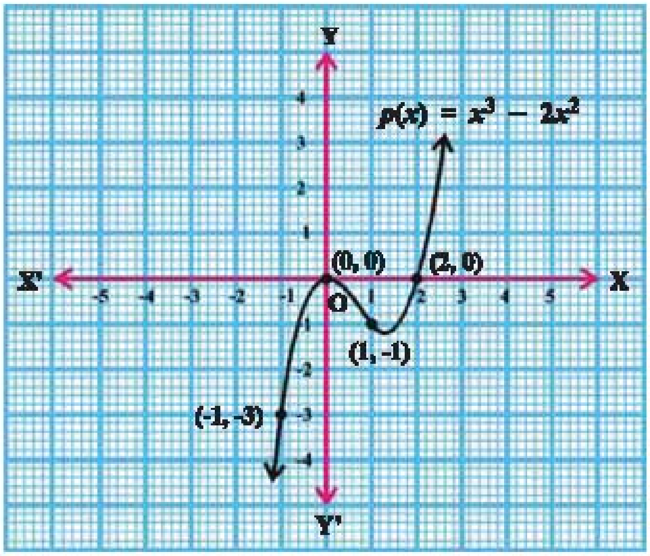
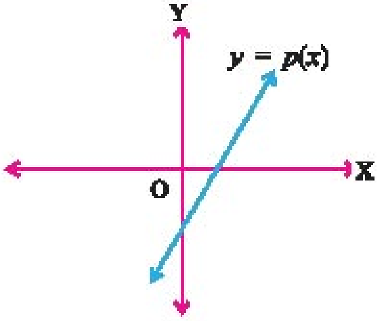


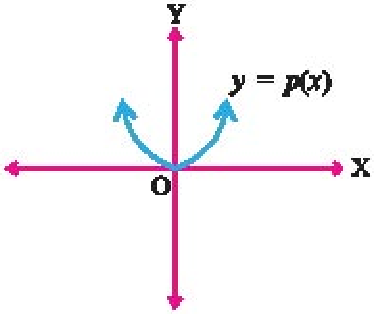
Figure 2.8

Note that a linear polynomial has exactly one zero, quadratic polynomial has at most two real zeros and a cubic polynomial has at most three real zeros. Thus, a polynomial of degree n has at most n real zeros.

Example 10 : Look at the graphs of a polynomial $y = p(x)$ in figure 2.9 and from this find the number of zeros of $p(x)$.



(i)



(ii)

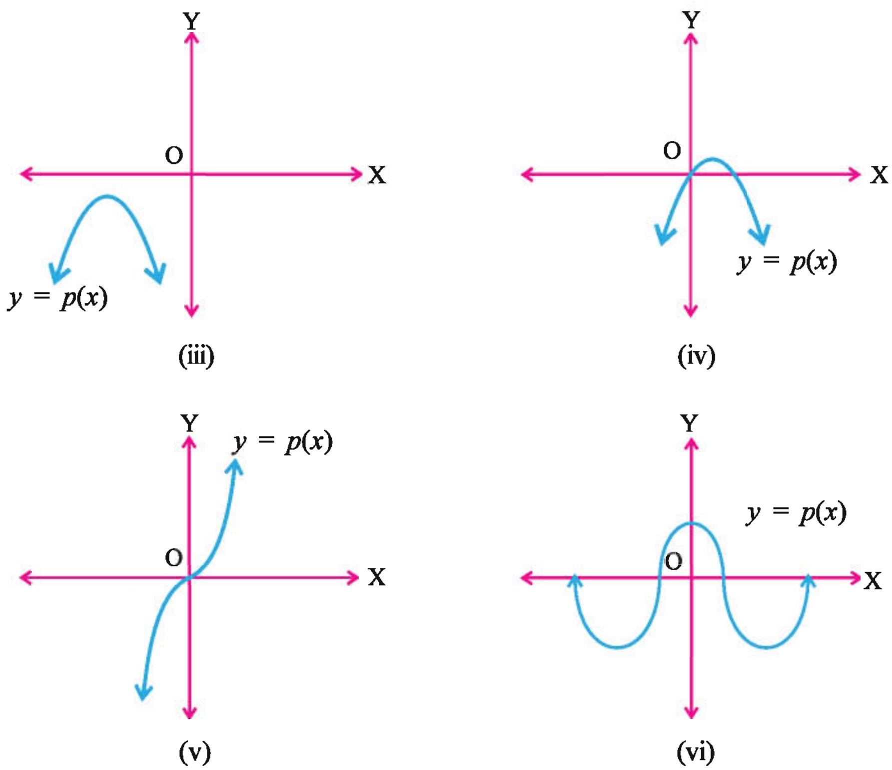


Figure 2.9

Solution : In figure 2.9

(1) The graph of $y = p(x)$ intersects X-axis at one point. So the number of real zeros of $p(x)$ is one.

(2) Here, the graph $y = p(x)$ intersects X-axis at one point. So the number of real zeros of $p(x)$ is one

(3) Here, the graph $y = p(x)$ does not intersect X-axis at any point. So the number of real zeros of $p(x)$ is zero or $p(x)$ has no real zero.

(4) Here, the graph $y = p(x)$ intersects X–axis at two distinct points. Therefore, the number of real zeros of $p(x)$ is two.

(5) The graph of $y = p(x)$ intersects X-axis at one point. So the number of real zeros of $p(x)$ is one.

(6) The graph of $y = p(x)$ intersects X-axis at four distinct points. So the number of real zeros of $p(x)$ is four.

EXERCISE 2.2

1. Find the number of zeros of the following polynomials :

(1) $p(x) = x^2 - x$ **(2)** $p(x) = x - x^2 - 1$

(3) $P(x) = 3x - 2$ **(4)** $p(x) = x^3 - x$

2. Find the number of zeros and real zeros of $p(x) = x^3 + 1$. Show them by a graph.

3. Draw the graph of $p(x) = x^2 + 1$ and find the real zeros of this polynomial.

4. From the figure 2.10 find the number of zeros of $y = p(x)$.

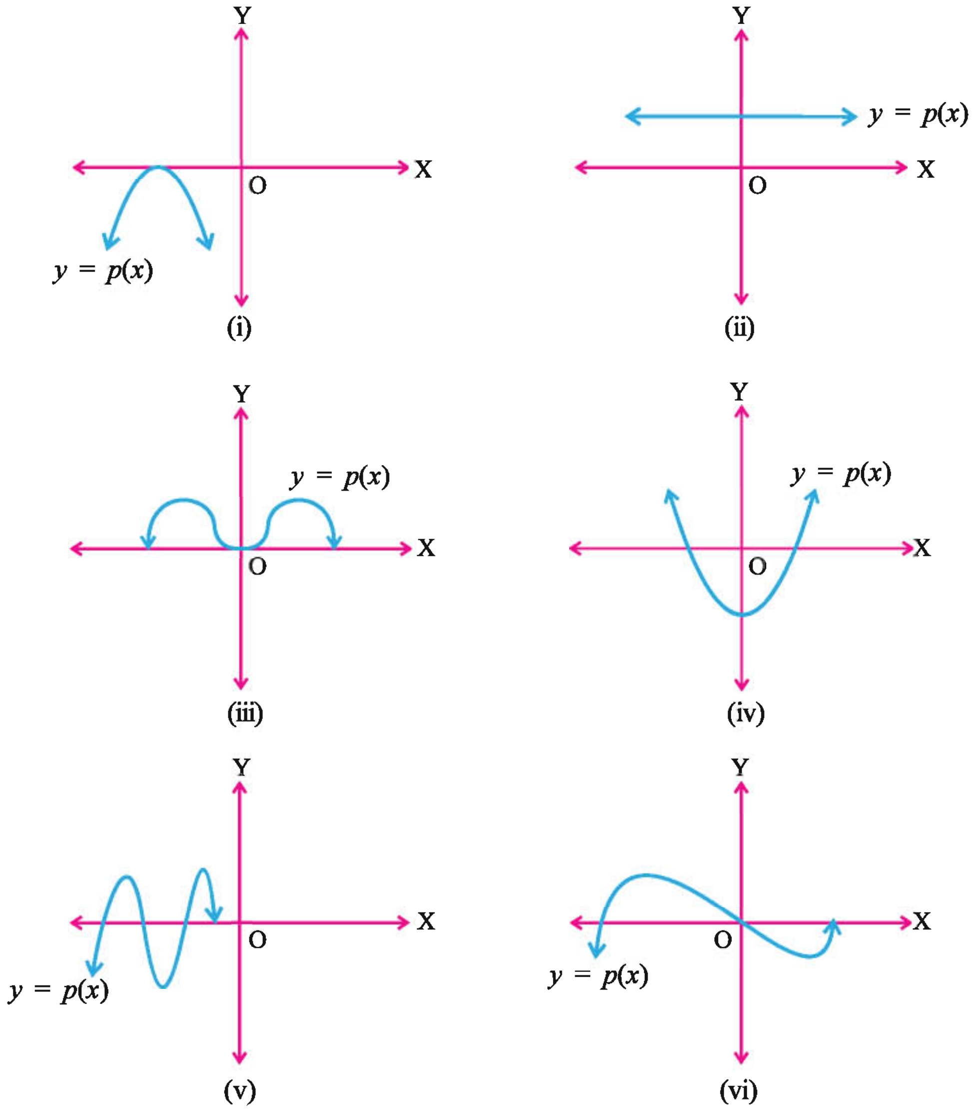


Figure 2.10

5. Find the number of zeros and zeros of $p(x) = x^2 - 4$. Represent them graphically.

*

2.3 Relationship between zeros and coefficients of a polynomial

We know that the value of $p(x)$ for $x = k$ is obtained by replacing x by k in $p(x)$ and it is denoted by $p(k)$. Also note that k is a zero of $p(x)$ if only if $p(k) = 0$.

In general, if k is a zero of a linear polynomial $p(x) = ax + b$, $a \neq 0$, then $p(k) = ak + b = 0$.

$\therefore \quad k = -\frac{b}{a}$

Thus, the zero of a linear polynomial

$$p(x) = ax + b \text{ is } -\frac{b}{a} = -\frac{\text{The constant term}}{\text{Coefficient of } x}$$

This shows the relationship between the zeros and coefficients of a linear polynomial. To understand the above relationship, we take the following example.

Example : 11 Verify that 3 is a zero of the linear polynomial $p(x) = 9x - 27$.

Solution : Here $p(x) = 9x - 27$

$a = 9,\ b = -27$

The zero of the linear polynomial is $-\frac{b}{a} = \frac{-(-27)}{9} = 3$

Thus, 3 is a zero of the linear polynomial.

Now we shall discuss about the relationship between the zeros and coefficients of a quadratic polynomial.

Suppose α and β are the zeros of a quadratic polynomial $p(x) = ax^2 + bx + c = 0,\ a \neq 0$, $a, b, c \in$ R. So $(x - \alpha)$ and $(x - \beta)$ are the factors of $p(x)$.

$$ax^2 + bx + c = k\,[(x - \alpha)(x - \beta)],\ k \in \text{R} - \{0\}$$

$$= k\,[x^2 - \alpha x - \beta x + \alpha\beta\,]$$

$$= kx^2 - k\,(\alpha + \beta)x + k\alpha\beta$$

Comparing the coefficients of x^2, x, x^0 we get $a = k,\ b = -k\,(\alpha + \beta),\ c = k\alpha\beta$

$\therefore$ The sum of its zeros $= \alpha + \beta$

$$= \frac{k(\alpha+\beta)}{k} \qquad (k \neq 0)$$

$$= \frac{-(-k(\alpha+\beta))}{k}$$

$$\alpha + \beta = -\frac{b}{a}$$

$$\therefore\ \alpha + \beta = -\frac{\text{The coefficient of } x}{\text{The coefficient of } x^2}$$

Now, the product of its zeros $= \alpha\beta$

$$= \frac{k\alpha\beta}{k} \qquad (k \neq 0)$$

$$\therefore\ \alpha\beta = \frac{c}{a}$$

$$\therefore\ \alpha\beta = \frac{\text{The constant term}}{\text{Coefficient of } x^2}$$

To understand the relationship between the zeros and the coefficients of any quadratic polynomial, we take an example.

Example 12 : Find the zeros of the quadratic polynomial $p(x) = x^2 + 3x + 2$ and hence find the sum and product of its zeros.

Solution : We have studied about the factorization of a quadratic polynomial by splitting the middle term.

Here the middle term $3x$ can be split as the sum of $2x$ and x such that their product is $2x^2$.

$\therefore$ $x^2 + 3x + 2 = x^2 + 2x + x + 2$

$$= x(x + 2) + 1\,(x + 2)$$

$$= (x + 1)\,(x + 2)$$

To find the zeros of $p(x)$, let $p(x) = 0$

$\therefore \quad (x + 1)(x + 2) = 0$

$\therefore \quad x = -1 \quad$ or $\quad x = -2$

Now the sum of the zeros $= (-1) + (-2) = -3$

The product of the zeros $= (-1)(-2) = 2$

Example 13 : Find the sum and the product of the zeros of the quadratic polynomial $p(x) = 3x^2 + 7x + 4$.

Solution : Suppose α and β are the zeros of $p(x)$. Here $a = 3$, $b = 7$, $c = 4$.

The sum of zeros $= \alpha + \beta = -\frac{b}{a} = -\frac{\text{The coefficient of } x}{\text{The coefficient of } x^2} = -\frac{7}{3}$

The product of zeros $= \alpha\beta = \frac{c}{a} = \frac{\text{The constant term}}{\text{The coefficient of } x^2} = \frac{4}{3}$

To verify the relationship between the zeros and coefficients of a given quadratic polynomial we can find the zeros of $p(x)$ by factorization. By taking sum and product of these zeros, we can verify the above results.

Example 14 : Verify the relationship between the zeros and the coefficients of $p(x) = x^2 + 9x + 14$.

Solution : Here, to verify the relationship, we have to find the zeros of $p(x)$.

$\therefore \quad x^2 + 9x + 14 = 0$

$\therefore \quad x^2 + 7x + 2x + 14 = 0$

$\therefore \quad x(x + 7) + 2(x + 7) = 0$

$\therefore \quad (x + 2)(x + 7) = 0$

$\therefore \quad x = -2 \quad$ or $\quad x = -7$

$\therefore \quad -2$ and -7 are the zeros of $p(x)$.

Now, the sum of its zeros $= (-2) + (-7) = -9 = -\frac{9}{1} = -\frac{b}{a} = -\frac{\text{The coefficient of } x}{\text{The coefficient of } x^2}$

The product of its zeros $= (-2)(-7) = 14 = \frac{14}{1} = \frac{c}{a} = \frac{\text{The constant term}}{\text{The coefficient of } x^2}$

Example 15 : Obtain a quadratic polynomial $p(x) = ax^2 + bx + c$, where sum of zeros is $\frac{8}{5}$ and product of zeros is $\frac{3}{5}$. $(a < 0)$

Solution : Suppose α and β are the zeros of the quadratic polynomial $p(x)$.

$\alpha + \beta = -\frac{b}{a} = \frac{8}{5}$. So $= \frac{b}{8} = -\frac{a}{5} = k$, say **($k > 0$ as $a < 0$)**

Thus, $b = 8k$, $a = -5k$

Now, $\alpha\beta = \frac{c}{a} = \frac{3}{5}$. Thus $c = \frac{3}{5}\ a = \frac{3}{5}(-5k) = -3k$

Again $p(x) = ax^2 + bx + c$

$\therefore \quad p(x) = -5kx^2 + 8kx - 3k$

$= k(-5x^2 + 8x - 3)$ where k is any positive real number.

Now we shall discuss about the relationship between the zeros and the coefficients of a cubic polynomial.

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Suppose α, β and γ are the zeros of a cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$; $a \neq 0$, $a, b, c, d \in$ R. So $(x - \alpha)$, $(x - \beta)$ and $(x - \gamma)$ are the factors of $p(x)$.

$\therefore \quad ax^3 + bx^2 + cx + d = k[(x-\alpha)(x-\beta)(x-\gamma)]$; $k \in$ R $-$ {0}

$$= k[(x^2 - \alpha x - \beta x + \alpha\beta)(x - \gamma)]$$

$$= k[x^3 - \gamma x^2 - \alpha x^2 + \alpha\gamma x - \beta x^2 + \beta\gamma x + \alpha\beta x - \alpha\beta\gamma]$$

$$= k[x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma]$$

Comparing the coefficients of x^3, x^2, x, x^0,

we get $a = k$, $b = -k(\alpha + \beta + \gamma)$, $c = k(\alpha\beta + \beta\gamma + \gamma\alpha)$, $d = -k(\alpha\beta\gamma)$

$\therefore$ The sum of its zeros $= \alpha + \beta + \gamma$

$$= \frac{k(\alpha+\beta+\gamma)}{k} \qquad (k \neq 0)$$

$$= -\frac{b}{a}$$

$$\therefore \alpha + \beta + \gamma = -\frac{\text{The coefficient of } x^2}{\text{The coefficient of } x^3}$$

The sum of products of two zeros taken at a time.

$$= \alpha\beta + \beta\gamma + \gamma\alpha$$

$$= \frac{k(\alpha\beta+\beta\gamma+\gamma\alpha)}{k} \qquad (k \neq 0)$$

$$= \frac{c}{a}$$

$$\therefore \alpha\beta + \beta\gamma + \gamma\alpha = \frac{\text{The coefficient of } x}{\text{The coefficient of } x^3}$$

and the product of zeros $= \alpha\beta\gamma$

$$= \frac{k(\alpha\beta\gamma)}{k} \qquad (k \neq 0)$$

$$= -\frac{d}{a}$$

$$\therefore \alpha\beta\gamma = -\frac{\text{The constant term}}{\text{The coefficient of } x^3}$$

To understand the relationship between the zeros and coefficients of any cubic polynomial, we take the following example.

Example 16 : Find the zeros of the cubic polynomial $p(x) = x^3 + 2x^2 - 3x$. Also verify the relationship between the zeros and coefficients of $p(x)$.

Solution : Here $p(x) = x^3 + 2x^2 - 3x$. Here $a = 1$, $b = 2$, $c = -3$, $d = 0$

To find the zeros of $p(x)$, let $p(x) = 0$

$\therefore \quad x^3 + 2x^2 - 3x = 0$

$\therefore \quad x(x^2 + 2x - 3) = 0$

$\therefore \quad x[x^2 + 3x - x - 3] = 0$

$\therefore \quad x[x(x + 3) - 1(x + 3)] = 0$

$\therefore$ $x(x-1)(x+3) = 0$

$\therefore$ $x = 0$ or $x = 1$ or $x = -3$

-3, 0 and 1 are the zeros of the cubic polynomial $p(x) = x^3 + 2x^2 - 3x$

Now, the sum of its zeros $= (-3) + 0 + 1 = -2 = \frac{-2}{1} = -\frac{b}{a}$

$$= -\frac{\text{The coefficient of } x^2}{\text{The coefficient of } x^3}$$

The product of zeros $= (-3)(0)(1) = 0 = \frac{0}{1} = -\frac{\text{The constant term}}{\text{The coefficient of } x^3}$

The sum of the products of zeros taken two at a time

$$= (-3)(0) + (0)(1) + (-3)(1) = -3 = \frac{c}{a} = \frac{\text{The coefficient of } x}{\text{The coefficient of } x^3}$$

EXERCISE 2.3

1. Prove that 4 and 1 are the zeros of the quadratic polynomial $p(x) = x^2 - 5x + 4$. Also verify the relationship between the zeros and the coefficients of $p(x)$.

2. Find the zeros of the following quadratic polynomials :

(1) $p(x) = x^2 + 4x - 21$ (2) $p(x) = 6x^2 - 11x + 5$ (3) $p(x) = 4x^2 + 9x + 5$

(4) $p(x) = 3x^2 + 5x - 8$ (5) $p(x) = x^2 - 81$ (6) $p(x) = x^2 - x - 6$

3. Find the zeros, the sum of the zeros and the product of the zeros of the quadratic polynomial $p(x) = 3x^2 - x - 4$

4. Obtain a quadratic polynomial with the following conditions :

(1) The sum of zeros $= 2$; the product of zeros $= -3$

(2) The sum of zeros $= -3$; the product of zeros $= -4$

(3) The sum of zeros $= \frac{1}{3}$; the product of zeros $= \frac{1}{2}$

5. Obtain the quadratic or the cubic polynomial as the case may be in the standard form with the following coefficients :

(1) $a = 6, b = 17, c = 11$ (2) $a = 1, b = -1, c = -1, d = 1$

(3) $a = 5, b = 7, c = 2$ (4) $a = 1, b = -3, c = -1, d = 3$

(5) $a = 3, b = -5, c = -11, d = -3$

*

2.4 Division algorithm for polynomials

We know that for any cubic polynomial, $p(x) = ax^3 + bx^2 + cx + d$; $a \neq 0$; $a, b, c, d \in$ R

(1) If $a + b + c + d = 0$ i.e. the sum of the coefficients of terms of the polynomials is equal to zero, $(x - 1)$ is a factor of $p(x)$ and 1 is considered to be a zero of $p(x)$.

(2) If $a + c = b + d$ i.e. the sum of the coefficients of odd power terms is equal to the sum of the coefficients of even power terms; $(x + 1)$ is a factor of $p(x)$ and -1 is considered to be a zero of $p(x)$.

We have learnt the method of division of polynomial, in class IX.

i.e. dividend polynomial = divisor polynomial × quotient polynomial + remainder polynomial

In symbols, $p(x) = s(x)\ q(x) + r(x)$

where, $p(x)$ = dividend polynomial

$s(x)$ = divisor polynomial

$q(x)$ = quotient polynomial

$r(x)$ = remainder polynomial

Also degree of $r(x)$ < degree of $s(x)$ or $r(x) = 0$. This is known as division algorithm.

We review it in the following example.

Example 17 : Divide $5x^2 + 6x + 3$ by $x + 3$.

Solution : Here dividend polynomial $p(x) = 5x^2 + 6x + 3$

divisor polynomial $s(x) = x + 3$

$$
\begin{array}{r|l}
 & 5x - 9 \\
\hline
(x+3) & 5x^2 + 6x + 3 \\
 & 5x^2 + 15x \\
 & - \quad - \\
\hline
 & -9x + 3 \\
 & -9x - 27 \\
 & + \quad + \\
\hline
 & 30
\end{array}
$$

Thus, the quotient polynomial $q(x) = 5x - 9$ and remainder polynomial $r(x) = 30$

$5x^2 + 6x + 3 = (5x - 9)(x + 3) + 30$

In the above example, we had seen that degree of $r(x)$ < degree of $s(x)$. But at the same time a question arise in our mind, what happens if $r(x) = 0$?

To answer this question, we take the following example :

Example 18 : Divide $5x^3 + 9x^2 + 8x + 20$ by $x + 2$

Solution : Here the dividend polynomial is $p(x) = 5x^3 + 9x^2 + 8x + 20$ and the divisor polynomial $s(x)$ is $x + 2$

$$
\begin{array}{r|l}
 & 5x^2 - x + 10 \\
\hline
(x+2) & 5x^3 + 9x^2 + 8x + 20 \\
 & 5x^3 + 10x^2 \\
 & - \quad - \\
\hline
 & -x^2 + 8x + 20 \\
 & -x^2 - 2x \\
 & + \quad + \\
\hline
 & 10x + 20 \\
 & 10x + 20 \\
 & - \quad - \\
\hline
 & 0
\end{array}
$$

In this example, we can see that $r(x) = 0$. So, we have

$5x^3 + 9x^2 + 8x + 30 = (x + 2)(5x^2 - x + 10)$

$(x + 2)$ and $(5x^2 - x + 10)$ are the factors of $p(x)$.

It means that if $r(x) = 0$, then the quotient polynomial $q(x)$ and the divisor polynomial $s(x)$ are the factors of dividend polynomial $p(x)$.

Let us study some more examples of the application of division algorithm.

Example 19 : Divide $-19x - 2x^2 + x^3 + 20$ by $-6x + x^2 + 5$

Solution : Here the dividend polynomial $p(x) = x^3 - 2x^2 - 19x + 20$ **(rearrange the terms)** and the divisor polynomial $s(x) = x^2 - 6x + 5$.

Now,

$$
\begin{array}{r|l}
 & x + 4 \\
\hline
(x^2 - 6x + 5) & x^3 - 2x^2 - 19x + 20 \\
 & x^3 - 6x^2 + 5x \\
 & - \quad + \quad - \\
\hline
 & \qquad 4x^2 - 24x + 20 \\
 & \qquad 4x^2 - 24x + 20 \\
 & \qquad - \quad + \quad - \\
\hline
 & \qquad\qquad 0
\end{array}
$$

$\therefore \quad x^3 - 2x^2 - 19x + 20 = (x + 4)(x^2 - 6x + 5)$

We can use an alternate method to divide any polynomial by a linear polynomial which is known as **synthetic division**. This method works as follows :

Step 1 : Arrange all the terms of the given dividend polynomial $p(x)$ in the standard form.

Step 2 : Write the coefficients of all the terms of dividend polynomial in a row.

Step 3 : Find the zero of divisor of polynomial $s(x)$ and write this value in the row of coefficients of $p(x)$ as shown on page 46.

Step 4 : Add zero to the coefficient of leading term

Step 5 : Multiply this result with the zero of $s(x)$ and place the value below the second coefficient.

Step 6 : Add elements in the second column and multiply this result with the zero of $s(x)$. Place this result below the third coefficient.

Step 7 : Continue this process till the sum of the last coefficients is obtained.

Step 8 : The last row gives the coefficients of the quotient polynomial starting with the term having one power less than the divisor polynomial.

Step 9 : If the remainder (the last term in the last row) is zero, then the divisor exactly divides the dividend polynomial.

To understand the above steps we take some examples :

Example 20 : Divide $x^3 + 3x^2 - 7x - 9$ by $x + 1$

Solution : Here, $p(x) = x^3 + 3x^2 - 7x - 9$ is the dividend polynomial and

$s(x) = x + 1$ is the divisor polynomial

Coefficients of x^3, x^2, x, x^0 in the dividend polynomial are 1, 3, -7 and -9 respectively. Equating divisor $x + 1$ to 0, we get $x = -1$.

Thus, the above procedure can be applied as follows :

−1	1	3	−7	−9
	0	−1	−2	9
	1	2	−9	0 = remainder

Here, 1, 2 and −9 are the coefficients of x^2, x, x^0 respectively in the quotient polynomial. 0 is the remainder.

$\therefore \quad x^3 + 3x^2 - 7x - 9 = (x + 1)(x^2 + 2x - 9)$

This process is discussed stepwise as follows :

$x^3 + 3x^2 - 7x - 9$ by $x + 1$:

Step 1 : Arrange the dividend polynomial in the standard form.

$p(x) = x^3 + 3x^2 - 7x - 9$

Step 2 : Write the coefficients of the dividend polynomial in the row in order.

1 3 −7 −9

Step 3 : By taking the divisor polynomial $(x + 1)$ equal to zero, we get value of variable x.

It is shown in the row of coefficient of $p(x)$.

$x + 1 = 0$

$\therefore \quad x = -1$

Step 4 : Add zero to the leading coefficient.

$1 + 0 = 1$. Take 1 in the first column below the horizontal line.

Step 5 : Multiply 1 by (-1) (zero of the divisor) and place it in the 2nd column and add.

$(-1) \times 1 = -1$

$\therefore \quad 3 + (-1) = 2$ is obtained.

Step 6 : Similarly multiply 2 by (-1) and add it to the third column.

$(-2) + (-7) = (-9)$

Step 7 : Then multiplying (-9) by (-1) and adding to the fourth column we get remainder zero.

$(-9) \times (-1) = 9$

$\therefore \quad 9 + (-9) = 0$

$\therefore$ The remainder obtained is zero.

Step 8 : Lastly adding 9 to (−9) the procedure is completed.

Thus last row is 1, 2, −9, 0 = remainder.

They are coefficients of x^2, x, x^0 respectively and remainder is zero.

The quotient polynomial $(1 \cdot x^2 + 2 \cdot x - 9x^0) = x^2 + 2x - 9$

$\therefore \quad \mathbf{x^3 + 3x^2 - 7x - 9 = (x + 1)(x^2 + 2x - 9)}$

Example 21 : Divide $2x^4 + 5x^3 - 7x^2 - 15x - 14$ by $x - 2$

Solution : Here the dividend polynomial $p(x) = 2x^4 + 5x^3 - 7x^2 - 15x - 14$ and the divisor polynomial $s(x) = x - 2$.

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Coefficients of x^4, x^3, x^2, x and x^0 in the dividend polynomial are 2, 5, −7, −15, −14 respectively and equating divisor polynomial $x - 2$ to 0, we get, $x = 2$

2	2	5	−7	−15	−14	
	0	4	18	22	14	
	2	9	11	7	0	= remainder

Here 2, 9, 11 and 7 are the coefficients of x^3, x^2, x and x^0 respectively in the quotient polynomial.

$\therefore \quad 2x^4 + 5x^3 - 7x^2 - 15x - 14 = (x - 2)(2x^3 + 9x^2 + 11x + 7)$

Example 22 : Divide $-3x^4 - 5x^3 + 8x^2 - 7x + 15$ by $x + 3$

Solution : Here the dividend polynomial is $p(x) = -3x^4 - 5x^3 + 8x^2 - 7x + 15$ and the divisor polynomial $s(x) = x + 3$.

The coefficients of x^4, x^3, x^2, x and x^0 in the dividend polynomial are −3, −5, 8, −7, 15 respectively and equating $x + 3$ to 0; $x = -3$.

−3	−3	−5	8	−7	15	
	0	9	−12	12	−15	
	−3	4	−4	5	0	= remainder

Here −3, 4, −4 and 5 are the coefficients of x^3, x^2, x and x^0 respectively in the quotient polynomial.

$\therefore \quad -3x^4 - 5x^3 + 8x^2 - 7x + 15 = (x + 3)(-3x^3 + 4x^2 - 4x + 5)$

$= -(x + 3)(3x^3 - 4x^2 + 4x - 5)$

Now, we solve some examples of division of polynomials where the divisor polynomial is a quadratic polynomial.

Example 23 : Divide $x^4 - 3x^2 + 4x + 5$ by $-x + 1 + x^2$

Solution : Here the dividend polynomial $p(x) = x^4 - 3x^2 + 4x + 5$

and the divisor polynomial $s(x) = x^2 - x + 1$

Now,

$$
\begin{array}{r|l}
 & x^2 + x - 3 \\
\hline
(x^2 - x + 1) & x^4 + 0.x^3 - 3x^2 + 4x + 5 \\
 & x^4 - x^3 + x^2 \\
 & - \quad + \quad - \\
\hline
 & x^3 - 4x^2 + 4x + 5 \\
 & x^3 - x^2 + x \\
 & - \quad + \quad - \\
\hline
 & -3x^2 + 3x + 5 \\
 & -3x^2 + 3x - 3 \\
 & + \quad - \quad + \\
\hline
 & \boxed{8} = \text{remainder}
\end{array}
$$

$\therefore$ We get the quotient polynomial $q(x) = x^2 + x - 3$

and the remainder polynomial $r(x) = 8$

$\therefore \quad x^4 - 3x^2 + 4x + 5 = (x^2 + x - 3)(x^2 - x + 1) + 8$

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Example 24 : If $\sqrt{3}$ and $-\sqrt{3}$ are the zeros of $p(x) = x^4 + 4x^3 - 8x^2 - 12x + 15$, then find the remaining zeros of $p(x)$.

Solution : Here $\sqrt{3}$ and $-\sqrt{3}$ are the zeros of $p(x)$. So that $(x - \sqrt{3})$ and $(x + \sqrt{3})$ are the factors of $p(x)$. Thus, $(x^2 - 3)$ is also a factor of the given polynomial $p(x)$.

$\therefore$ The dividend polynomial $p(x) = x^4 + 4x^3 - 8x^3 - 12x + 15$ and the divisor polynomial $s(x) = x^2 - 3$

Now,

$$
\begin{array}{r|l}
 & x^2 + 4x - 5 \\
\hline
(x^2 - 3) & x^4 + 4x^3 - 8x^2 - 12x + 15 \\
 & x^4 + 0x^3 - 3x^2 \\
 & - \quad - \quad + \\
\hline
 & \quad 4x^3 - 5x^2 - 12x + 15 \\
 & \quad 4x^3 - 0x^2 - 12x \\
 & \quad - \quad - \quad + \\
\hline
 & \qquad -5x^2 + 15 \\
 & \qquad -5x^2 + 15 \\
 & \qquad + \quad - \\
\hline
 & \qquad\qquad 0
\end{array}
$$

$\therefore$ The quotient polynomial $q(x) = x^2 + 4x - 5$ and the remainder polynomial $r(x) = 0$

$\therefore$ We can write,

$$
\begin{aligned}
p(x) &= x^4 + 4x^3 - 8x^2 - 12x + 15 \\
&= (x^2 - 3)(x^2 + 4x - 5) \\
&= (x^2 - 3)(x + 5)(x - 1)
\end{aligned}
$$

To find the remaining zeros of $p(x)$, let $p(x) = 0$

$\therefore$ $(x^2 - 3)(x + 5)(x - 1) = 0$

$\therefore$ $x = \sqrt{3}$ or $x = -\sqrt{3}$ or $x = -5$ or $x = 1$

$\therefore$ The remaining zeros of $p(x)$ are -5 and 1.

Example 25 : The product of two polynomials is $2x^3 + 3x^2 - 1$ and one of the polynomials is $x^2 + 2x + 1$, find the other polynomial.

Solution : Here $p(x)$ = the dividend polynomial = $2x^3 + 3x^2 - 1$ and

$s(x)$ = the divisor polynomial = $x^2 + 2x + 1$

Now,

$$
\begin{array}{r|l}
 & 2x - 1 \\
\hline
(x^2 + 2x + 1) & 2x^3 + 3x^2 - 1 \\
 & 2x^3 + 4x^2 + 2x \\
 & - \quad - \quad - \\
\hline
 & \quad -x^2 - 2x - 1 \\
 & \quad -x^2 - 2x - 1 \\
 & \quad + \quad + \quad + \\
\hline
 & \qquad\qquad 0
\end{array}
$$

$\therefore$ The quotient polynomial $q(x) = 2x - 1$ and the remainder polynomial $r(x) = 0$

$\therefore$ The other polynomial is $2x - 1$

EXERCISE 2.4

1. **Divide the following polynomial $p(x)$ by the polynomial $s(x)$.**

   (1) $p(x) = 2x^3 - 13x^2 + 23x - 12$, $s(x) = 2x - 3$

   (2) $p(x) = \frac{2}{3}x^2 + 5x + 6$, $s(x) = x + 6$

   (3) $p(x) = 40x^2 + 11x - 63$, $s(x) = 8x - 9$

   (4) $p(x) = 2x^3 + 9x^2 + 13x + 6$, $s(x) = 2x^2 + 5x + 3$

   (5) $p(x) = x^4 + 4x^3 + 5x^2 - 7x - 3$, $s(x) = x^2 - 1$

2. Find the remainder polynomial when the cubic polynomial $x^3 - 3x^2 + 4x + 5$ is divided by $x - 2$.
3. 3 is a zero of $p(x) = 3x^3 - x^2 - ax - 45$. Find '$a$'.
4. The product of two polynomials is $6x^3 + 29x^2 + 44x + 21$ and one of the polynomials is $3x + 7$. Find the other polynomial.
5. If polynomial $p(x)$ is divided by $x^2 + 3x + 5$, the quotient polynomial and the remainder polynomials are $2x^2 + x + 1$ and $x - 3$ respectively. Find $p(x)$.
6. Divide $p(x) = x^3 - 4x^2 + 5x - 2$ by $x - 2$ Find $r(x)$.
7. There are $x^4 + 57x + 15$ pens to be distributed in a class of $x^2 + 4x + 2$ students. Each student should get the maximum possible number of pens. Find the number of pens received by each student and the number of pens left undistributed ($x \in$ N).
8. A trader bought $2x^2 - x + 2$ TV sets for ₹ $8x^4 + 7x - 6$. Find the price of one TV set.
9. $-\sqrt{2}$ and $\sqrt{2}$ are two of the zeros of $p(x) = 2x^4 + 7x^3 - 8x^2 - 14x + 8$. Find the remaining zeros of $p(x)$.

EXERCISE 2

1. **State whether the following statements are true or false :**

   (1) $\frac{7}{5}$ is a zero of the linear polynomial $p(x) = 5x + 7$.

   (2) $p(x) = x^2 + 2x + 1$ has two distinct zeros.

   (3) The cubic polynomial $p(x) = x^3 + x^2 - x - 1$ has two distinct zeros.

   (4) The graph of the cubic polynomial $p(x) = x^3$ meets the X–axis at only one point.

   (5) Any quadratic polynomial $p(x)$ has at least one zero, $x \in$ R

2. Find the zeros and number of zeros of $p(x) = x^2 + 9x + 18$. Show them on a graph.
3. Find the zeros, the sum and the product of zeros of $p(x) = 4x^2 + 12x + 5$.
4. -4 and 9 are the sum and product of the zeros respectively of a quadratic polynomial. Find the quadratic polynomial.
5. Find q(x) and $r(x)$, for the quadratic polynomial $p(x) = 11x - 21 + 2x^2$ when divided by $1 + 2x$
6. Divide $2x^3 + 3x^2 - 11x - 6$ by $x^2 + x - 6$
7. 4 is a zero of the cubic polynomial $p(x) = x^3 - 3x^2 - 6x + 8$. Find the remaining zeros of $p(x)$.
8. The product of two polynomials is $3x^4 + 5x^3 - 21x^2 - 53x - 30$. If one of them is $x^2 - x - 6$, find the other polynomial.

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9. $2 + \sqrt{3}$ and $2 - \sqrt{3}$ are the zeros of $p(x) = x^4 - 6x^3 - 26x^2 + 138x - 35$. Find the remaining zeros of $p(x)$.

10. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) The linear polynomial $p(x) = 7x - 3$ has the zero .... ☐

(a) $\frac{7}{3}$ (b) $\frac{3}{7}$ (c) $-\frac{7}{3}$ (d) $-\frac{3}{7}$

(2) The cubic polynomial $p(x) = x^3 - x$ has ...... zeros. ☐

(a) 0 (b) 1 (c) 2 (d) 3

(3) The graph of $p(x) = 3x - 2 - x^2$ intersects the X-axis in ...... points. ☐

(a) 0 (b) 1 (c) 2 (d) 3

(4) The sum of the zeros of $3x^2 + 5x - 2$ is .... ☐

(a) $\frac{3}{5}$ (b) $-\frac{3}{5}$ (c) $\frac{5}{3}$ (d) $-\frac{5}{3}$

(5) The graph of $p(x) = 3x + 5$ represents .... ☐

(a) a straight line (b) parabola open upwards

(c) parabola open downwards (d) a ray

(6) A quadratic polynomial has no zero. Its graph .... ☐

(a) touches X-axis at any point (b) intersects X-axis at two distinct points

(d) does not intersect X-axis at two distinct points (d) is in any one half plane of X-axis

(7) For the graph in figure 2.11 $y = p(x)$ has .... zeros. ☐

(a) 1 (b) 2

(c) 3 (d) 4

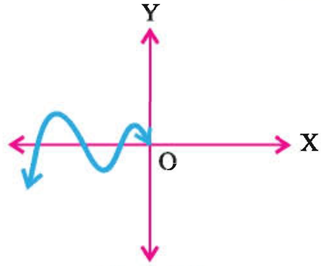


Figure 2.11

(8) The product of the zeros of $x^2 - 4x + 3$ is .... ☐

(a) 1 (b) 3 (c) 4 (d) -4

(9) $a = 3$, $b = 5$, $c = 7$, $d = 11$ in the standard notation gives the cubic polynomial .... ☐

(a) $3x^3 + 5x^2 - 7x - 11$ (b) $3x^3 - 5x^2 + 7x - 11$

(c) $3x^3 + 5x^2 - 7x + 11$ (d) $3x^3 + 5x^2 + 7x + 11$

*

Summary

In this chapter we have studied following points :

1. A linear polynomial $p(x) = ax + b, a \neq 0$, quadratic polynomial $p(x) = ax^2 + bx + c, a \neq 0$, and a cubic polynomial $p(x) = ax^3 + bx^2 + cx + d, a \neq 0$ have the degrees 1, 2 and 3 respectively, $a, b, c, d \in R$
2. The zeros of a polynomial $p(x)$ are precisely the x–coordinates of the point, where the graph of $p(x)$ intersects the X-axis.
3. A polynomial of degree n has at most n zeros.
4. Suppose α and β are the zeros of a quadratic polynomial $p(x) = ax^2 + bx + c, \; a \neq 0$.

   The sum of its zeros $= \alpha + \beta = -\frac{b}{a} = -\frac{\text{The coefficient of } x}{\text{The coefficient of } x^2}$

   and product of its zeros $= \alpha\beta = \frac{c}{a} = \frac{\text{The constant term}}{\text{The coefficient of } x^2}$
5. Suppose α, β, γ are the zeros of a cubic polynomial $p(x) = ax^3 + bx^2 + cx + d, a \neq 0$.

   The sum of its zeros $= \alpha + \beta + \gamma = -\frac{b}{a} = -\frac{\text{The coefficient of } x^2}{\text{The coefficient of } x^3}$

   Sum of zeros taken two at a time $= \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = \frac{\text{The coefficient of } x}{\text{The coefficient of } x^3}$

   and product of its zeros $= \alpha\beta\gamma = -\frac{d}{a} = -\frac{\text{The constant term}}{\text{The coefficient of } x^3}$
6. **Division algorithm :**

   $p(x) = s(x)\, q(x) + r(x)$ where $p(x)$ = the dividend polynomial, $s(x)$ = the divisor polynomial $q(x)$ = the quotient polynomial, and degree of $r(x) <$ degree of $s(x)$ or $r(x) = 0$. If $r(x) = 0$, then $s(x)$ and $q(x)$ are factors of $p(x)$.

◆

Sridhara (c. 870, India – c. 930 India) was an Indian mathematician.

Works : He was known for two treatises : Trisatika (sometimes called the Patiganitasara) and the Patiganita. His major work Patiganitasara was named Trisatika because it was witten in three hundred slokas. The book discusses counting of numbers, measures, natural number, multiplication, division, zero, squares, cubes, fraction, rule of three, interest-calculation, joint business or partnership and mensuration.

Of all the Hindu Acharyas the exposition of Sridhar Acharya on zero is the most explicit. He has written, "If 0(zero) is added to any number, the sum is the same number; If 0(zero) is subtracted from any number, the number remains unchanged; If 0(zero) is multiplied by any number, the product is 0(zero)". He has said nothing about division of any number by 0(zero).

In the case of dividing a fraction he has found out the method of mutiplying the fraction by the reciprocal of the divisor.

He wrote on practical applications of algebra separated algebra from arithmetic

He was one of the first to give a formula for solving quadratic equations.

Pair of Linear Equations in Two Variables 3

Education is what remains when one has forgotten everything learnt in school.

- Albert Einstein

*

Perfect numbers like perfect men are rare.

- Rene Des Cartes

3.1 Introduction

We have studied a linear equation in one variable and in two variables in previous standard. An equation $ax = b (a, b \in R, a \neq 0)$ is a linear equation in one variable and the equation $ax + by = c$ ($a, b, c \in R$, a and b are not simultaneously zero; i.e. $a^2 + b^2 \neq 0$) is a linear equation in two variables. We have studied the graph of a linear equation in two variables. We recall that the linear equation in one variable $ax + b = 0$ has unique solution $-\frac{b}{a}$ but a linear equation in two variables $ax + by + c = 0$ has infinite solutions. This we have studied in std. IX.

In this chapter, we shall discuss the solution of a pair of linear equations in two variables by graphical method, by algebraic methods like method of substitution, method of elimination, cross-multiplication method etc. We shall also discuss the solution of such pairs of equations which are not linear but can be reduced to linear form by some suitable substitutions.

3.2 Pair of linear equations in two variables

In day to day life we come across many problems which can be transformed in the form of linear equation in two variables.

The sum of the costs of a table and a chair is ₹ 150. Also the sum of the costs of 6 chairs and 5 tables is ₹ 800. How to put this data in a mathematical equation form ?

Let the cost of a chair be ₹ x and the cost of a table be ₹ y.

$\therefore \quad x + y = 150$ **(i)**

Now, the cost of 6 chairs is ₹ $6x$ and the cost of 5 tables is ₹ $5x$

$\therefore \quad 6x + 5y = 800$ **(ii)**

Here (i) and (ii) represent a pair of linear equations in two variables.

The general form of a pair of linear equations in two variables is

$$a_i x + b_i y + c_i = 0;\ a_i^2 + b_i^2 \neq 0;\ i = 1, 2$$

Let us take some examples to form a pair of linear equations in two variables.

Example 1 : Rakesh purchases 15 pens and 20 pencils. He has to pay ₹ 190 for it. The sum of the costs of a pen and pencil is ₹ 11. Obtain a pair of linear equations in two variables from the given data.

Solution : Suppose the cost of a pen is ₹ x and the cost of a pencil is ₹ y.

Now, sum of the costs of these two items is ₹ 11.

$\therefore \quad x + y = 11$ **(i)**

Now, Rakesh has to pay ₹ 190 for 15 pens and 20 pencils. Here the costs of 15 pens will be ₹ $15x$ and the cost of 20 pencils will be ₹ $20y$.

Thus, $15x + 20y = 190$ **(ii)**

(i) and (ii) represent a pair of linear equations in two variables.

Example 2 : Obtain a pair of linear equations for the following data.

A shop-keeper sales 5 pants and 8 shirts for ₹ 3100. The cost of a pair of a pants and a shirt is ₹ 500.

Solution : Suppose the cost of a pant is ₹ x and the cost of a shirt is ₹ y.

Now, the cost of a pair of a pant and a shirt is ₹ 500.

$\therefore \quad x + y = 500$ **(i)**

Shop-keeper sales 5 pants and 8 shirts. Thus the costs of 5 pants and 8 shirts will be ₹ $5x$ and ₹ $8y$ respectively.

He receives payment ₹ 3100 by selling them.

$5x + 8y = 3100$ **(ii)**

(i) and (ii) represent a pair of linear equations in two variables.

EXERCISE 3.1

Obtain a pair of linear equations in two variables from the following information :

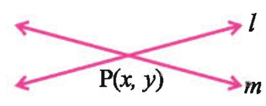
1. Father tells his son, "Five years ago, I was seven times as old as you were. After five years, I will be three times as old as you will be".
2. The sum of the cost of 1kg apple and 1kg pine-apple is ₹ 150 and the cost of 1 kg apple is twice the cost of 1kg pine-apple.
3. Nilesh got twice the marks as obtained by Ilesh, in the annual examination of mathematics of standard 10. The sum of the marks as obtained by them is 135.
4. Length of a rectangle is four less than the thrice of its breadth. The perimeter of the rectangle is 110.
5. The sum of the weights of a father and a son is 85 kg. The weight of the son is $\frac{1}{4}$ of the weight of his father.
6. In a cricket match, Sachin Tendulkar makes his score thrice the Sehwag's score. Both of them together make a total score of 200 runs.
7. In tossing a balanced coin, the probability of getting head on its face is twice to the probability of getting tail on its face. The sum of both probabilities (head and tail) is 1.

*

3.3 Graphical method of solution of a pair of linear equations

We have discussed how to form a pair of linear equations in two variables. By solving these equations we get the values of x and y. So, (x, y) is a solution of a pair of linear equations in two variables. Note that the graph of a linear equation represents a straight line. Thus, for a pair of linear equations, we get two straight lines, which represent any one of the following three possibilities.

(i) Both the straight lines intersect at a common point.
i.e. there is a unique solution.



(ii) They do not intersect at any point.
i.e. both the lines are parallel.
$\therefore$ The solution set is $\emptyset$.

(iii) Both the lines are identical or coincide
i.e. There are infinitely many common points.

By observing graphs we can see whether the given pair of equations has a unique solution, no solution or infinitely many solutions.

Let us take some examples to understand the graphical method for the solution of a pair of linear equations in two variables.

Example 3 : Solve the following pair of linear equations using graphs

$x + 2y = 5$ and $3x + 5y = 13$

Solution : Generally, we substitute $x = 0$ or $y = 0$ in the given linear equations to get corresponding y or x. So we get two points on the straight line. To find more points on the line, take different values of x related to it, we get different values for y from given equation.

We get the following tables for the given linear equations.

For $x + 2y = 5$

$y = \frac{5-x}{2}$

x	5	1	3
y	0	2	1

For $3x + 5y = 13$

$y = \frac{13-3x}{5}$

x	1	6
y	2	–1

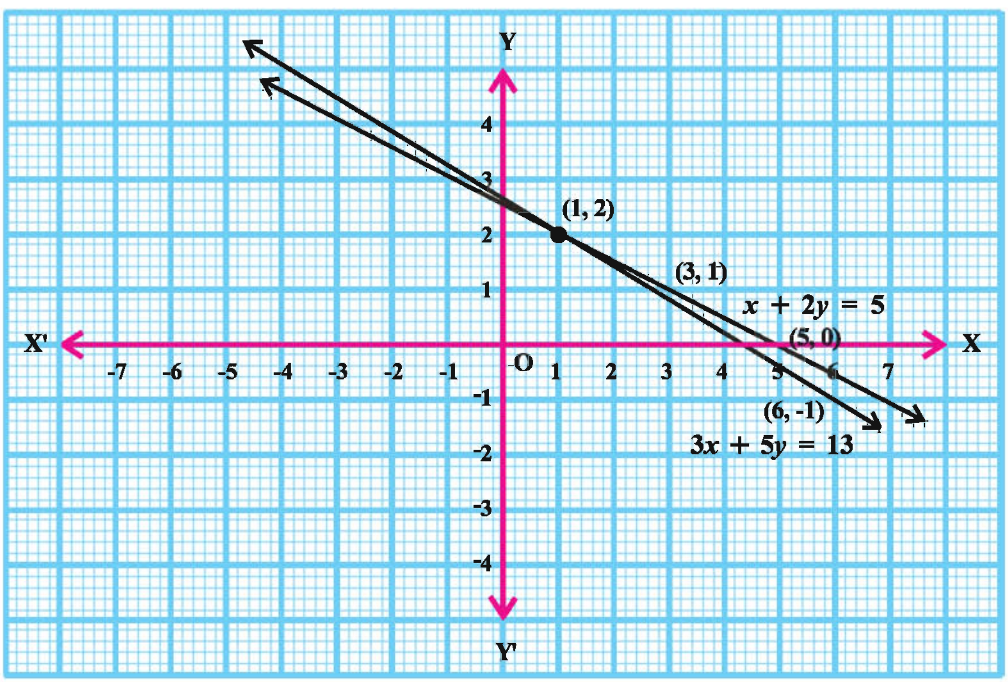


Figure 3.1

Plotting all these points on a graph paper and by joining them, we get two straight lines. From the graph we can say that both the straight lines intersect at the point (1, 2).

Thus, the solution of the pair of linear equations is (1, 2).

Example 4 : Find the solution set of the following pair of linear equations by graph.

$3x + 4y = 10$ and $3x + 4y = 15$

Solution : For $3x + 4y = 10$

$y = \frac{10 - 3x}{4}$

x	2	−2
y	1	4

For $3x + 4y = 15$

$y = \frac{15 - 3x}{4}$

x	5	1
y	0	3

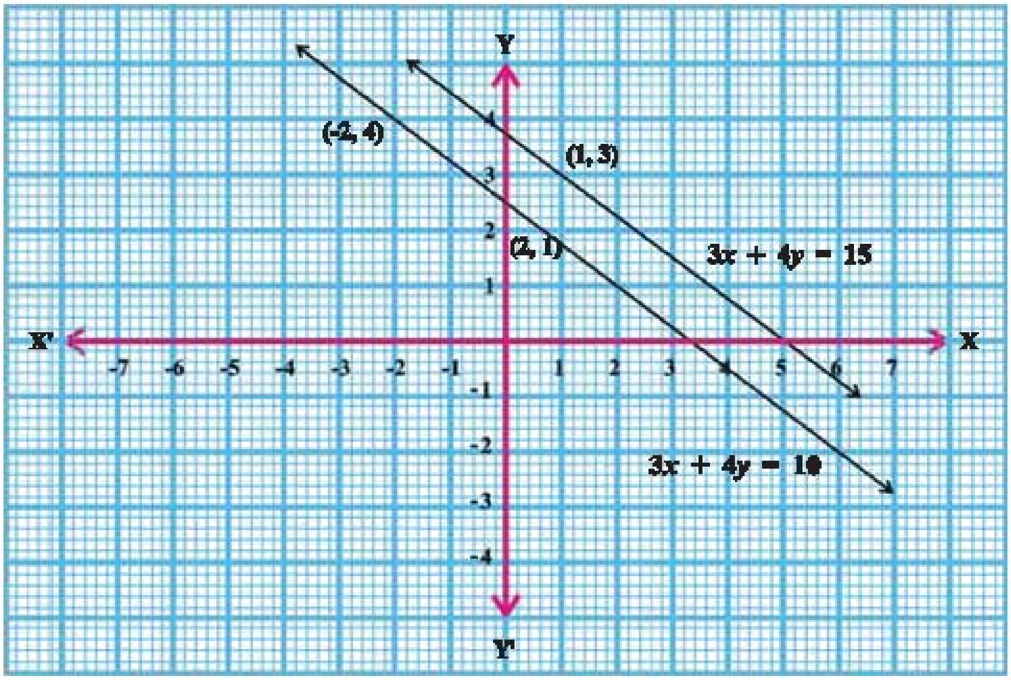


Figure 3.2

Plot all the above points i.e. (−2, 4), (2, 1) and (5, 0), (1, 3) on a graph paper. By joining corresponding points on a line, we get two straight lines. From the graph, we can observe that both these lines do not intersect at any common point. Both these lines are infact parallel. Thus, the solution of the pair of these linear equations is ϕ.

Example 5 : By graphical method, solve the following pair of linear equations :

$x + y = 7$ and $3x + 3y = 21$

Solution : Here dividing each term of $3x + 3y = 21$ we get the equation $x + y = 7$.

Thus both the equations are identical.

Hence, we say that both the lines are same. So, they are coincident. It means that there are infinitely many solutions.

To draw the graph, we make the following table :

For $x + y = 7$, $y = 7 - x$

x	0	7	2
y	7	0	5

For $3x + 3y = 21$, $y = \frac{21 - 3x}{3} = \frac{3(7 - x)}{3}$

$\therefore\ y = 7 - x$

Thus, both tables are same.

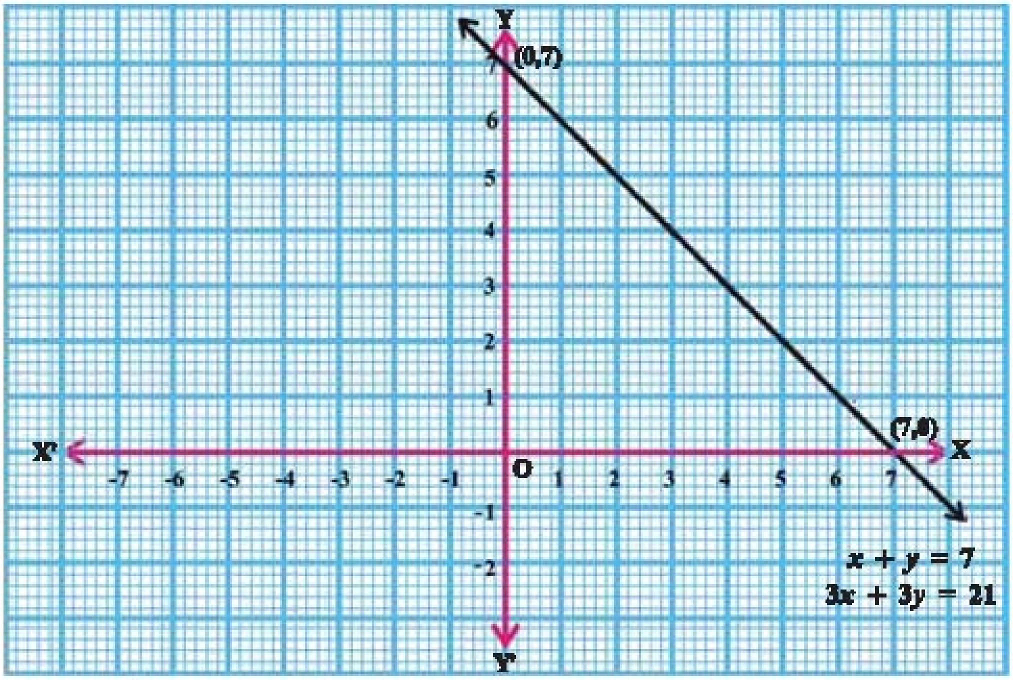


Figure 3.3

Plot all the above points (0, 7), (7, 0) and (2, 5) on a graph paper. By joining these points we get one straight line.

From this graph, we can say that there are infinitely many points on this line.

Thus, the solution of the pair of linear equation can be written as $\{(x, y) \mid x + y = 7, x, y \in R\}$

A pair of linear equations which has no solution, is called an **inconsistent** pair of linear equations. A pair of linear equations in two variables, which has at least one solution, is called a **consistent** pair of linear equations. A pair of linear equations which reduce to be the pair of same equations has infinitely many distinct common solutions. Such a pair is called a **linearly dependent pair of linear equations in two variables**. Note that a dependent pair of linear equations is always consistent as it has infinite number of solutions.

EXERCISE 3.2

1. **Solve the following pair of linear equations in two variables (by graph) :**

   (1) $2x + y = 8, x + 6y = 15$ (2) $x + y = 1, 3x + 3y = 2$

   (3) $2x + 3y = 5, x + y = 2$ (4) $x - y = 6, 3x - 3y = 18$

   (5) $(x + 2)(y - 1) = xy, (x - 1)(y + 1) = xy$

2. Draw the graphs of the pair of linear equations $3x + 2y = 5$ and $2x - 3y = -1$. Determine the coordinates of the vertices of the triangle formed by these linear equations and the X-axis.

3. 15 students of class X took part in the examination of Indian mathematics olympiad. The number of boys participants is 5 less than the number of girls participants. Find the number of boys and girls (using a graph) who took part in the examination of Indian mathematics olympiad.

4. Examine graphically whether the pair of equations $2x + 3y = 5$ and $x + \frac{9}{6}y = \frac{5}{2}$ is consistent.

*

3.4 Algebraic Methods of solving a pair of linear equations

We have discussed the graphical method of solution of a pair of linear equations in two variables. It is not convenient to plot the points like (−1.79, 3.33) or $\left(\frac{7}{3}, \frac{13}{4}\right)$ or $(\sqrt{2}, 2\sqrt{5})$ on a graph paper with accuracy. Some other methods are required. We want to discuss some algebraic methods which are as follows.

(1) Method of substitution (2) Elimination Method (3) Cross-multiplication method

(1) Method of substitution :

We have two linear equations in a pair. From any one equation the value of any one variable is to be obtained in terms of the other variable and substituted in the other equation to solve the pair of linear equations. So this method is known as **method of substitution** or **substitution method**. How do we obtain the solution of a pair of linear equations by this method ? The method is explained below :

Step 1 : Find the value of one variable, say y in terms of the other variable x.

Step 2 : Substitute the value of y in the other equation and reduce that equation to an equation in one variable. Solve it.

Step 3 : Substitute the value of x obtained in step 2 in the equation used in step 1 to obtain the value of y. If convenient, we can find x in terms of y and substitute x to find value of y.

Let us take some examples to understand the method of substitution.

Example 6 : By the method of substitution, solve the following pair of linear equations.

$2x + 3y = 10$ **(i)**

$3x - y = 4$ **(ii)**

Solution : Step 1 : From the equation (ii), we get $y = 3x - 4$

Step 2 : Substitute $y = 3x - 4$ in the equation (i) and we get

$\therefore \quad 2x + 3(3x - 4) = 10$

$\therefore \quad 2x + 9x - 12 = 10$

$\therefore \quad 11x = 22$

$\therefore \quad x = 2$

Step 3 : Substitute $x = 2$ in the equation (i) or $y = 3x - 4$

$\therefore \quad y = 3(2) - 4 = 6 - 4 = 2$

$\therefore \quad y = 2$

Thus, $(x, y) = (2, 2)$ is the solution of the pair of linear equations in two variables.

Example 7 : The cost of a table is thrice the cost of a chair. The total cost of 4 chairs and a table is ₹ 2100. Find the cost of a table and the cost of a chair.

Solution : Suppose the cost of a table is ₹ x and the cost of a chair is ₹ y.

Now, the cost of a table is thrice the cost of a chair.

$\therefore \quad x = 3y$ **(i)**

The sum of costs of 4 chairs and a table is ₹ 2100.

$\therefore \quad x + 4y = 2100$ **(ii)**

Substitute $x = 3y$ in the equation (ii) to set $3y + 4y = 2100$

$\therefore \quad 7y = 2100$

$\therefore \quad y = 300$

Now substitute $y = 300$ in equation (i), we get

$\therefore$ $x = 3(300)$

$\therefore$ $x = 900$

$\therefore$ The cost of a table is ₹ 900 and the cost of a chair is ₹ 300.

Example 8 : Solve the following pair of linear equations by the method of substitution.

$5x + 7y = 12$ **(i)**

and $10x + 14y = 20$ **(ii)**

Solution : Dividing the equation (ii) by 2, we get $5x + 7y = 10$

We also have $5x + 7y = 12$

$\therefore$ $10 = 12$ which is absurd.

Hence the pair of linear equations has no solution.

$\therefore$ The solution set is $\emptyset$.

Example 9 : Solve the following pair of linear equations :

$x + 4y = 8$ **(i)**

and $2x + 8y = 16$ **(ii)**

Solution : Dividing the equation $2x + 8y = 16$ by 2, we get $x + 4y = 8$ which is the same as equation (i).

A single linear equation has infinitely many solutions.

The solution set is $\{(x, y) \mid x = 8 - 4y, x, y \in R\}$

EXERCISE 3.3

1. Solve the following pairs of linear equations by the method of substitution :

(1) $x + y = 7, 3x - y = 1$ (2) $3x - y = 0, x - y + 6 = 0$

(3) $2x + 3y = 5, 2x + 3y = 7$ (4) $x - y = 3, 3x - 3y = 9$

(5) $\frac{3x}{2} - \frac{5y}{3} = -2, \frac{x}{3} + \frac{y}{2} = \frac{13}{6}$

2. Solve the pair of linear equations $x - y = 28$ and $x - 3y = 0$ and if the solution satisfies, $y = mx + 5$, then find m.

3. A fraction becomes $\frac{4}{5}$ if 3 is added to both the numerator and the denominator. If 5 is added to the numerator and the denominator, it becomes $\frac{5}{6}$. Find the fraction.

4. The sum of present ages of a father and his son is 50 years. After 5 years, the age of the father becomes thrice the age of his son. Find their present ages.

5. A bus traveller travelling with some of his relatives buys 5 tickets from Ahmedabad to Anand and 10 tickets from Ahmedabad to Vadodara for ₹ 1100. The total cost of one ticket from Ahmedabad to Anand and one ticket from Ahmedabad to Vadodara is ₹ 140. Find the cost of a ticket from Ahmedabad to Anand as well as the cost of a ticket from Ahmedabad to Vadodara.

*

(2) Method of Elimination of One Variable :

Let us discuss another method to solve a pair of linear equations in two variables. In this method, we eliminate any one variable and obtain a linear equation in one variable. So this method is known as method of elimination.

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The general form of a pair of linear equations is $a_i x + b_i y + c_i = 0$; $a_i^2 + b_i^2 \neq 0$, $i = 1, 2$.

$$a_1x + b_1y + c_1 = 0,\ a_1^2 + b_1^2 \neq 0 \qquad \text{(i)}$$

and $a_2x + b_2y + c_2 = 0,\ a_2^2 + b_2^2 \neq 0$ **(ii)**

Multiplying equation (i) by b_2 and (ii) by b_1, we get

$$a_1b_2x + b_1b_2y + c_1b_2 = 0, \qquad \text{(iii)}$$

and $a_2b_1x + b_2b_1y + c_2b_1 = 0$ **(iv)**

Subtracting equation (iv) from equation (iii), we get

$x\,(a_1b_2 - a_2b_1) = b_1c_2 - b_2c_1$

∴ If $a_1b_2 - a_2b_1 \neq 0$, then

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$$

Now, multiplying equation (i) by a_2 and (ii) by a_1, we get

$$a_1a_2x + b_1a_2y + a_2c_1 = 0, \qquad \text{(v)}$$

and $a_1a_2x + a_1b_2y + a_1c_2 = 0,$ **(vi)**

Subtracting equation (v) from equation (vi), we get

$$y\,(a_1b_2 - a_2b_1) = a_2c_1 - a_1c_2$$

∴ If $a_1b_2 - a_2b_1 \neq 0$, then $y = \dfrac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$

∴ The solution of these equations is

$$(x, y) = \left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}\right), \text{ where, } a_1b_2 - a_2b_1 \neq 0$$

We will accept the following without giving detailed proof. ($a_1, a_2, b_1, b_2, c_1, c_2 \neq 0$)

Thus,(1) If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then there is a unique solution. These equations are called consistent equations.

(2) If $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, but $\frac{a_1}{a_2} \neq \frac{c_1}{c_2}$ or $\frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, then the solution set is $\emptyset$.
The equations are not consistent.

(3) If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then there are infinitely many solutions.
The equations are also consistent.

From the above discussion we can write the steps of the method of elimination as follows :

(1) Multiply both the equations by a suitable non-zero constant to make the coefficients of any one variable in both the equations (either x or y) numerically equal.

(2) Eliminate the variable whose coefficients are numerically equal by addition or subtraction of the equations obtained in step 1. We get a linear equation in one variable to solve.

(3) Solve this equation in one variable and obtain the value of x or y.

(4) Substitute this value of x or y in any one of original equation, to get the value of the remaining variable.

Let us take some examples to understand the above steps of this method.

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Example 10 : Use method of elimination to find the solution of the following pair of linear equations.

$9x - 4y = 14$ **(i)**

and $7x - 3y = 11$ **(ii)**

Solution : Multiplying equation (i) by 3 and equation (ii) by 4, we get

$27x - 12y = 42$ **(iii)**

$28x - 12y = 44$ **(iv)**

Now subtracting equation (iii) from equation (iv), we get $x = 2$

Substituting $x = 2$ in equation (i), we get

$9\ (2) - 4y = 14$

$\therefore\ 18 - 4y = 14$

$\therefore\ y = \frac{18-14}{4} = 1$

$(x, y) = (2, 1)$ is the solution of the given pair of linear equations.

Note : Here x was obtained eliminating y and was substituted in any equation and y was obtained. Also the system of equations obtained by the operation of elimination in algebraic equations is equivalent to original system of equations. This means we assume that the solution of original equation and the equations obtained by elimination operations are same.

Example 11 : Solve the following pair of linear equations by method of elimination.

$x + 2y = 3$ **(i)**

and $2x + 4y = 5$ **(ii)**

Solution : From equation (i) and (ii)

$a_1 = 1,\ b_1 = 2,\ c_1 = -3$

$a_2 = 2,\ b_2 = 4,\ c_2 = -5$

$\therefore\ \frac{a_1}{a_2} = \frac{1}{2},\ \frac{b_1}{b_2} = \frac{2}{4} = \frac{1}{2},\ \frac{c_1}{c_2} = \frac{3}{5}$

$\therefore\ \frac{a_1}{a_2} = \frac{b_1}{b_2}$. Also $\frac{a_1}{a_2} \neq \frac{c_1}{c_2}$ as $\frac{1}{2} \neq \frac{3}{5}$

$\therefore$ The equations are not consistent. The solution set is $\emptyset$.

Example 12 : Solve the following pair of linear equations by the method of elimination.

$2x + y = 5$ **(i)**

and $4x + 2y = 10$ **(ii)**

Solution : We have the equation $4x + 2y = 10$. Dividing this equation by 2, we get the equation $2x + y = 5$ which is same as the equation (i).

$\therefore$ Both the equations are identical.

$\therefore$ There are infinitely many solutions, namely $\{(x, y) \mid 2x + y = 5,\ x, y \in R\}$

Example 13 : The ten's digit of a two digit number is thrice the unit digit. The new number obtained by interchanging the digits is 54 less than the original number. Find the original number.

Solution : In a two digit number, suppose the digit at ten's place is y and the digit at unit's place is x

The number is $10y + x$

As the digit at ten's place is thrice the digit at unit's place, we have $y = 3x$

$\therefore\ 3x - y = 0$ **(i)**

By interchanging their digits, the digit at ten's place becomes x and the digit at unit's place becomes y. Therefore the new number formed is $10x + y$. This number is 54 less than the original number $10y + x$.

$\therefore$ $10y + x - 54 = 10x + y$

$\therefore$ $10y + x = 10x + y + 54$

$\therefore$ $-9x + 9y = 54$

Dividing the equation by -9, we get

$x - y = -6$ **(ii)**

Now, subtracting equation (ii) from the equation (i), we get

$2x = 6$

$\therefore$ $x = 3$

Substitute $x = 3$ in the equation (i), we get

$3(3) - y = 0$

$\therefore$ $y = 9$

Therefore the original number is $10y + x = 93$.

[**Note :** Can we do this orally ? The unit digit can be 1, 2 or 3 because maximum value of 10's digit is 9. Thus numbers can be 31, 62 or 93. By interchanging the digits, number will become 13, 26, 39.

$93 - 39 = 54$.

$\therefore$ The required number is 93. Student should know such intuitive methods also.]

Example 14 : The ratio of the present ages of a mother and her daughter is 8 : 3. After 5 years, the ratio of their ages will be 9 : 4. Find their present ages.

Solution : Suppose the present age of mother is x years and the present age of her daughter is y years.

Now the ratio of their present ages is 8 : 3

$\therefore$ $\frac{x}{y} = \frac{8}{3}$

i.e. $3x = 8y$

$\therefore$ $3x - 8y = 0$ **(i)**

After 5 years, the age of the mother will be $x + 5$ years and the age of her daughter will be $y + 5$ years.

At that time the ratio of their ages will be 9 : 4

$\therefore$ $\frac{x+5}{y+5} = \frac{9}{4}$

$\therefore$ $4x + 20 = 9y + 45$

$\therefore$ $4x - 9y = 25$ **(ii)**

Multiplying equation (i) by 4 and (ii) by 3, we get

$12x - 32y = 0$ **(iii)**

and $12x - 27y = 75$ **(iv)**

Subtracting the equation (iii) from the equation (iv), we get

$5y = 75$

$\therefore$ $y = 15$

Substitute $y = 15$ in the equation (i), we get

$3x - 8(15) = 0$

$\therefore$ $3x = 120$

$\therefore$ $x = 40$

$\therefore$ The present age of the mother is 40 years and the present age of her daughter is 15 years.

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EXERCISE 3.4

1. **Solve the following pair of linear equations by elimination method :**

   (1) $\frac{x}{5} - \frac{y}{3} = \frac{4}{15}, \frac{x}{2} - \frac{y}{9} = \frac{7}{18}$ (2) $4x - 19y + 13 = 0,\ 13x - 23y = -19$

   (3) $x + y = a + b,\ ax - by = a^2 - b^2$ (4) $5ax + 6by = 28;\ 3ax + 4by = 18$

2. The sum of two numbers is 35. Four times the larger number is 5 more than 5 times the smaller number. Find these numbers.
3. There are some 25 paise coins and some 50 paise coins in a bag. The total number of coins is 140 and the amount in the bag is ₹ 50. Find the number of coins of each value in the bag.
4. The sum of the digits of two digit number is 3. The number obtained by interchanging the digits is 9 less than the original number. Find the original number.
5. The length of a rectangle is twice its breadth. The perimeter of the rectangle is 120 *cm*. Find the length and breadth of this rectangle. Also find its area.
6. An employee deposits certain amount at the rate of 8% per annum and a certain amount at the rate of 6% per annum at simple interest. He earns ₹ 500 as annual interest. If he interchanges the amount at the same rates, he earns ₹ 50 more. Find the amounts deposited by him at different rates.

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3.3 Method of Cross Multiplication

For the solution of a pair of linear equations by algebraic method, we learn one more approach. We can obtain the same results by cross multiplication method as the method of elimination.

We have the linear equations in two variables as follows :

$a_1x + b_1y + c_1 = 0,\ a_1^2 + b_1^2 \neq 0$ and $a_2x + b_2y + c_2 = 0,\ a_2^2 + b_2^2 \neq 0$

Here, a_1 and a_2 are the coefficients of x,
b_1 and b_2 are the coefficients of y and
c_1 and c_2 are the constants
x and y are the variables.

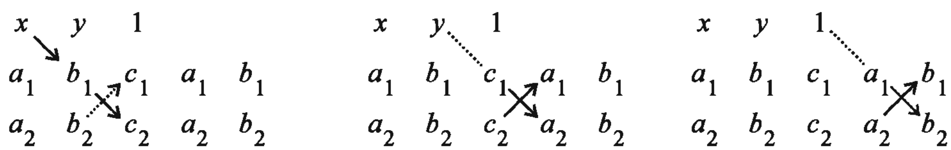
Let us write them as

$$\begin{array}{ccc} x & y & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{array}$$

Repeat the first two columns of coefficients in the same order at the end in the following manner

$$\begin{array}{ccccc} x & y & 1 & & \\ a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \end{array}$$

denominator of x **denominator of y** **denominator of 1**



The term with symbol '$\searrow$' means multiplication of numbers with + sign. The term with symbol '$\nearrow$' means multiplication of numbers with − sign (i.e. negative sign.) which indicates the difference of them in each denominator.

In the denominator of x, write the result of subtraction of the product of b_2 and c_1 from the product of b_1 and c_2. In the denominator of y, write the result of subtraction of the product of c_2 and a_1 from the product of c_1 and a_2. Similarly, in the denominator of 1, write the result of subtraction of the product of a_2 and b_1 from the product of a_1 and b_2.

Hence, the ratios are :

$$\frac{x}{b_1c_2 - b_2c_1} \quad \Big| \quad \frac{y}{a_2c_1 - a_1c_2} \quad \Big| \quad \frac{1}{a_1b_2 - a_2b_1}$$

Equating all these three ratios, we get

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{a_2c_1 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

Thus, $x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$ and $y = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$ or $y = -\frac{(a_1c_2 - a_2c_1)}{a_1b_2 - a_2b_1}$

Hence we can say that,

(i) If $a_1b_2 - a_2b_1 \neq 0$, i.e. $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then there is a unique solution.

(ii) If $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, $\frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ or $\frac{c_1}{c_2} \neq \frac{a_1}{a_2}$ i.e. $a_1b_2 - a_2b_1 = 0$ and $b_1c_2 - b_2c_1 \neq 0$ or $c_1a_2 - c_2a_1 \neq 0$; then there is no solution

$\therefore$ Solution set is ϕ

(iii) If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ i.e. $a_1b_2 - a_2b_1 = b_1c_2 - b_2c_1 = a_1c_2 - a_2c_1 = 0$, then there are infinitely many solutions.

Here, the denominator is obtained by the cross multiplication of the coefficients or constants. Hence this method is known as **cross-multiplication method.**

It can be represented in the following way

$$\frac{x}{\begin{matrix} b_1 & c_1 \\ b_2 & c_2 \end{matrix}} = \frac{y}{\begin{matrix} c_1 & a_1 \\ c_2 & a_2 \end{matrix}} = \frac{c}{\begin{matrix} a_1 & b_1 \\ a_2 & b_2 \end{matrix}}$$

How to write these coefficients and constants ? It is explained as follow :

(1) Arrange the coefficients of y and constants of the equations in their respective order in the denominator of x.

(2) Arrange the constants in the first column and coefficients of x in the second column in the denominator of y.

(3) Arrange the coefficients of x and y in their respective order in the denominator of 1.

Let us solve some examples to understand the above steps of cross multiplication method.

Example : 15 Solve the following pair of linear equations by applying the method of cross multiplication.

$2x - 5y = 4$ **(i)**

$3x - 8y = 5$ **(ii)**

Solution : From the equation $2x - 5y - 4 = 0$, we get $a_1 = 2$, $b_1 = -5$, $c_1 = -4$

and from the equation, $3x - 8y - 5 = 0$, we get $a_2 = 3$, $b_2 = -8$, $c_2 = -5$

Applying the cross-multiplication method, we get

$$\frac{x}{\begin{matrix} -5 & -4 \\ -8 & -5 \end{matrix}} = \frac{y}{\begin{matrix} -4 & 2 \\ -5 & 3 \end{matrix}} = \frac{1}{\begin{matrix} 2 & -5 \\ 3 & -8 \end{matrix}}$$

$\therefore \quad \dfrac{x}{25-32} = \dfrac{y}{-12+10} = \dfrac{1}{-16+15}$

$\therefore \quad \dfrac{x}{-7} = \dfrac{y}{-2} = \dfrac{1}{-1}$

$\therefore \quad x = \dfrac{-7}{-1} = 7 \ ; \ y = \dfrac{-2}{-1} = 2$

i.e. $x = 7$, $y = 2$

$\therefore$ The solution of the pair of linear equations is (7, 2)

Example 16 : Solve the following pair of linear equations by cross-multiplication method.

$2(2x + y + 5) + 3(x - 3y - 1) = 0,\ 2x - 3y + 1 = 0$

Solution : Simplifying the first equation of given pair, we get, $4x + 2y + 10 + 3x - 9y - 3 = 0$

$\therefore \quad 7x - 7y + 7 = 0$ i.e. $x - y + 1 = 0$

Thus, $a_1 = 1$, $b_1 = -1$, $c_1 = 1$

Now for $2x - 3y + 1 = 0$, $a_2 = 2$, $b_2 = -3$, $c_2 = 1$

Applying cross-multiplication method, we get

$$\frac{x}{\begin{matrix}-1 & 1\\ -3 & 1\end{matrix}} = \frac{y}{\begin{matrix}1 & 1\\ 1 & 2\end{matrix}} = \frac{1}{\begin{matrix}1 & -1\\ 2 & -3\end{matrix}}$$

$\therefore \quad \dfrac{x}{-1+3} = \dfrac{y}{2-1} = \dfrac{1}{-3+2}$

$\therefore \quad \dfrac{x}{2} = \dfrac{y}{1} = \dfrac{1}{-1}$

$\therefore \quad x = -2,\ y = -1$

$\therefore$ The solution of the pair of linear equations is $(-2, -1)$

Example 17 : Solve the following pair of equations by cross multiplication method.

$\dfrac{x}{a} + \dfrac{y}{b} = a + b$ and $\dfrac{x}{a^2} + \dfrac{y}{b^2} = 2,\ a \neq 0,\ b \neq 0,\ a \neq b$

Solution : $bx + ay = ab(a + b)$ and $b^2x + a^2y = 2a^2b^2$

$\therefore \quad bx + ay - (a^2b + b^2a) = 0$ **(i)**

$b^2x + a^2y - 2a^2b^2 = 0$ **(ii)**

Now, $a_1 = b \quad b_1 = a \quad c_1 = -(a^2b + b^2a)$

$a_2 = b^2 \quad b_2 = a^2 \quad c_2 = -2a^2b^2$

$$\therefore \quad \frac{x}{\begin{matrix}a & -(a^2b+b^2a)\\ a^2 & -2a^2b^2\end{matrix}} = \frac{y}{\begin{matrix}-(a^2b+b^2a) & b\\ -2a^2b^2 & b^2\end{matrix}} = \frac{1}{\begin{matrix}b & a\\ b^2 & a^2\end{matrix}}$$

$\therefore \quad \dfrac{x}{-2a^3b^2 + a^4b + a^3b^2} = \dfrac{y}{-a^2b^3 - b^4a + 2a^2b^3} = \dfrac{1}{a^2b - b^2a}$

$\therefore \quad \dfrac{x}{a^4b - a^3b^2} = \dfrac{y}{a^2b^3 - b^4a} = \dfrac{1}{ab(a-b)}$

$\therefore \quad \dfrac{x}{a^3b(a-b)} = \dfrac{y}{ab^3(a-b)} = \dfrac{1}{ab(a-b)}$ **$(a \neq 0,\ b \neq 0,\ a \neq b)$**

$\therefore$ $x = \frac{a^3b}{ab}$, $y = \frac{ab^3}{ab}$ **($a \neq 0$, $b \neq 0$)**

$\therefore$ $x = a^2$, $y = b^2$

$\therefore$ The solution set is $\{(a^2, b^2)\}$.

EXERCISE 3.5

1. Solve the following pairs of equations by cross multiplication method :

(1) $0.3x + 0.4y = 2.5$ and $0.5x - 0.3y = 0.3$

(2) $5x + 8y = 18$, $2x - 3y = 1$

(3) $\frac{x}{3} + \frac{y}{5} = 1$, $7x - 15y = 21$

(4) $3x + y = 5$, $5x + 3y = 3$

2. By cross multiplication method, find such a two digit number such that, the digit at unit's place is twice the digit at tens place and the number obtained by interchanging the digits of the number is 36 more than the original number.

3. The sum of two numbers is 70 and their difference is 6. Find these numbers by cross-multiplication method.

4. While arranging certain students of a school in rows containing equal number of students; if three rows are reduced, then three more students have to be arranged in each of the remaining rows. If three more rows are formed, then two students have to be taken off from each previously arranged rows. Find the number of students arranged.

5. In ΔABC, the measure of $\angle$B is thrice to the measure of $\angle$C and the measure of $\angle$A is $\frac{1}{2}$ the sum of the measures of $\angle$B and $\angle$C. Find the measures of all the angles of ΔABC and also state the type of this triangle.

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3.5 Equations Reducible to a pair of linear equations in two variables

We, know different methods for the solution of a pair of linear equations in two variables. Now, we shall discuss those equations which are not in linear form and how we can reduce them into linear form. Moreover, we will find the solution of the pairs of such equations.

Let us take some examples to understand the procedure of finding the solution of the pair of such equations, which are not in linear form, but can be reduced to linear form by some suitable substitutions.

Example 18 : Solve the pair of following equations.

$\frac{5}{x} + \frac{3}{y} = 8$, $\frac{3}{x} + \frac{5}{y} = 24$, $(x, y \neq 0)$

Solution : Here given equations are not in linear form. To convert them into linear form, we suppose that $\frac{1}{x} = a$ and $\frac{1}{y} = b$.

The equations will be, $5a + 3b = 8$ **(i)**

and $3a + 5b = 24$ **(ii)**

Now, we can apply any one method to solve these equations as these equations are linear.

From equation (i), we get $b = \frac{8-5a}{3}$

Substituting the value of b in the equation (ii), we get,

$$3a + 5\left(\frac{8-5a}{3}\right) = 24$$

$\therefore$ $9a + 40 - 25a = 72$

$\therefore$ $-16a = 32$

$\therefore$ $a = -2$

Substitute $a = -2$ in $b = \frac{8-5a}{3}$, we get

$b = \frac{8+10}{3} = 6$

Now, $\frac{1}{x} = a$ and $\frac{1}{y} = b$

$\therefore$ $\frac{1}{x} = -2$ and $\frac{1}{y} = 6$

$\therefore$ $x = -\frac{1}{2}$

$\therefore$ $y = \frac{1}{6}$

The solution of this pair of equations is $\left(-\frac{1}{2}, \frac{1}{6}\right)$.

Example 19 : Solve the following pair of equations $2x + 6y = 5xy$, $6x - 2y = 5xy$.

Solution : It is easy to note that $x = 0$, $y = 0$ satisfy both the equations of the pair.

$\therefore$ (0, 0) is a solution of the pair.

Suppose $x \neq 0$, $y \neq 0$

$\therefore$ $xy \neq 0$

Dividing each term of equations by xy, we get

$\frac{2}{y} + \frac{6}{x} = 5$ **(i)**

and $\frac{6}{y} - \frac{2}{x} = 5$ **(ii)**

These equations are not in linear form. To convert them into linear form, we suppose that

$\frac{1}{x} = a$ and $\frac{1}{y} = b$

$6a + 2b = 5$ **(iii)**

and $-2a + 6b = 5$ **(iv)**

Multiplying (iv) by 3 and adding to (iii), we get

$\therefore$ $20b = 20$

$\therefore$ $b = 1$

Substitute $b = 1$ in equation (iii), we get

$6a + 2(1) = 5$

$\therefore$ $6a = 3$

$\therefore$ $a = \frac{1}{2}$

Now $\frac{1}{x} = a$ and $\frac{1}{y} = b$

$\therefore$ $\frac{1}{x} = \frac{1}{2}$ and $\frac{1}{y} = 1$

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∴ $x = 2$ and $y = 1$

The solutions of pair of the equations are (0, 0) and (2, 1)

∴ The solution set is {(0, 0), (2, 1)}

Example 20 : A boat goes 25 *km* upstream and 35 *km* downstream in 10 hours. In 15 hours, it can go 40 *km* upstream and 49 *km* downstream. Determine the speed of the stream and that of the boat in still water. (Speed of boat in still water is more than the speed of the stream of river.)

Solution : Let the speed of the boat in still water be x *km/hr* and speed of the stream be y *km/hr*. It is necessary that $x > y$.

Figure 3.4

The speed of the boat in down stream = $(x + y)$ *km/hr* and the speed of the boat in upstream = $(x - y)$ *km/hr*

Also, time = $\frac{\text{distance}}{\text{speed}}$ $\left[\because \text{ Speed} = \frac{\text{distance}}{\text{time}}\right]$

In the first case, when the boat goes 25*km* upstream, let the time taken, in hours, be t_1.

Then, $t_1 = \frac{25}{x-y}$

Let t_2 be the time, in hours, taken by the boat to go 35 *km* downstream. Then $t_2 = \frac{35}{x+y}$

Total time taken, $t_1 + t_2$ is 10 hours.

We get the equation, $\frac{25}{x-y} + \frac{35}{x+y} = 10$ (i)

In the second case, in 15 hours, it can go 40 *km* upstream and 49 *km* downstream. We get the equation,

$$\frac{40}{x-y} + \frac{49}{x+y} = 15 \quad \text{(ii)}$$

Let $\frac{1}{x-y} = a$ and $\frac{1}{x+y} = b$

Then the equations will be, $25a + 35b = 10$ or $5a + 7b = 2$ (iii)

and $40a + 49b = 15$ (iv)

Now, we apply the cross-multiplication method to find 'a' and 'b'

We have $a_1 = 5, b_1 = 7, c_1 = -2$

$a_2 = 40, b_2 = 49, c_2 = -15$

$$\therefore \frac{a}{\begin{matrix}7 & -2\\ 49 & -15\end{matrix}} = \frac{b}{\begin{matrix}-2 & 5\\ -15 & 40\end{matrix}} = \frac{1}{\begin{matrix}5 & 7\\ 40 & 49\end{matrix}}$$

(Now we will not show the arrows.)

$$\therefore \frac{a}{-105+98} = \frac{b}{-80+75} = \frac{1}{245-280}$$

$$\therefore \frac{a}{-7} = \frac{b}{-5} = \frac{1}{-35}$$

$$\therefore a = \frac{1}{5} \text{ and } b = \frac{1}{7}$$

Now $\frac{1}{x-y} = a$ and $\frac{1}{x+y} = b$

$\therefore \quad \frac{1}{x-y} = \frac{1}{5}$ and $\frac{1}{x+y} = \frac{1}{7}$

$\therefore \quad x - y = 5$ **(v)**

and $x + y = 7$ **(vi)**

Adding equations (v) and (vi), we get

$2x = 12$

$\therefore \quad x = 6$ and $y = 1$

$\therefore$ The speed of the boat in still water is 6 *km/hr* and speed of the stream is 1 *km/hr*.

EXERCISE 3.6

1. Solve the following pairs of linear equations :

(1) $\frac{5}{2x} + \frac{2}{3y} = 7, \frac{3}{x} + \frac{2}{y} = 12, x \neq 0, y \neq 0$

(2) $2x + 3y = 2xy, 6x + 12y = 7xy$

(3) $\frac{4}{x-1} + \frac{5}{y-1} = 2, \frac{8}{x-1} + \frac{15}{y-1} = 3, x \neq 1, y \neq 1$

(4) $\frac{1}{3x+y} + \frac{1}{3x-y} = \frac{3}{4}, \frac{1}{2(3x+y)} - \frac{1}{2(3x-y)} = \frac{-1}{8}, 3x + y \neq 0, 3x - y \neq 0$

(5) $\frac{3}{\sqrt{x}} + \frac{4}{\sqrt{y}} = 2, \frac{5}{\sqrt{x}} + \frac{7}{\sqrt{y}} = \frac{41}{12}, x > 0, y > 0$

2. 5 women and 2 men together can finish an embroidary work in 4 days, while 6 women and 3 men can finish it in 3 days. Find the time taken by 1 woman alone to finish the work. Also find the time taken by 1 man alone to finish the work.

3. A boat goes 21 *km* upstream and 18 *km* downstream in 9 hours. In 13 hours, it can go 30 *km* upstream and 27 *km* downstream. Determine the speed of the stream and that of the boat in still water. (Speed of boat in still water is more than the speed of the stream of river.)

4. Solve the following pair of equations by cross multiplication method :

$\frac{4x+7y}{xy} = 16, \frac{10x+3y}{xy} = 11, x \neq 0, y \neq 0$

5. Mahesh travels 250 *km* to his home partly by train and partly by bus. He takes 6 hours if he travels 50 *km* by train and remaining distance by bus. If he travels 100 *km* by train and remaining distance by bus, he takes 7 hours. Find the speed of the train and the bus separately.

EXERCISE 3

1. Obtain a pair of linear equations from the following information :

"The rate of tea per *kg* is seven times the rate of sugar per *kg*. The total cost of 2 *kg* tea and 5 *kg* sugar is ₹ 570."

2. Draw the graphs of the pair of linear equations in two variables. $x + 3y = 6, 2x - y = 5$. Find its solution set.

3. Solve the following pair of equations by the method of elimination :

$\frac{4}{x} + \frac{5}{y} = 7, \frac{5}{x} + \frac{4}{y} = \frac{13}{2}$

4. Solve the following pair of linear equations by the method of cross-multiplication :

$(a + b)x + (a - b)y = a^2 + 2ab - b^2, a \neq b$

$(a - b)(x + y) = a^2 - b^2, a \neq b$

5. Solve the following pair of equations :

$\frac{4}{x+1} + \frac{7}{y+2} = 2$ and $\frac{10}{x+1} + \frac{14}{y+2} = \frac{9}{2}$, $x \neq -1, y \neq -2$

6. The difference between two natural numbers is 6. Adding 10 to the twice of the larger number, we get 2 less than 3 times of the smaller number. Find these numbers.

7. The area of a rectangle gets increased by 30 square units, if its length is reduced by 3 units and breadth is increased by 5 units. If we increase the length by 5 units and reduce the breadth by 3 units then the area of a rectangle reduces by 10 square units. Find the length and breadth of the rectangle.

8. A part of monthly hostel charges is fixed and the remaining depends on the number of days one has taken food in the mess. Yash takes food for 25 days. He has to pay ₹ 2200 as hostel charges where as Niyati takes food for 20 days. She has to pay ₹ 1800 as hostel charges. Find the fixed charges and the cost of food per day.

9. A fraction becomes $\frac{2}{5}$ when 2 is subtracted from the numerator and denominator it becomes $\frac{3}{4}$ when 5 is added to its denominator and numerator, find the fraction.

10. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) The solution set of $x - 3y = 1$ and $3x + y = 3$ is .... ☐

(a) $\{(0, 1)\}$ (b) $\{(1, 1)\}$ (c) $\{(1, 0)\}$ (d) $\left\{\left(\frac{1}{3}, 0\right)\right\}$

(2) The solution set of $2x + y = 6$ and $4x + 2y = 5$ is .... ☐

(a) $\{(x, y) \mid 2x + y = 6; x, y \in R\}$ (b) $\{(x, y) \mid 2x + y = 0; x, y \in R\}$

(c) $\emptyset$ (d) infinite set

(3) To eliminate x, from $3x + y = 7$ and $-x + 2y = 2$ second equation is multiplied by ....

(a) 1 (b) 2 (c) 3 (d) -1 ☐

(4) If $2x + 3y = 7$ and $3x + 2y = 3$, then $x - y =$ .... ☐

(a) 4 (b) -4 (c) 2 (d) -2

(5) If the pair of linear equations $ax + 2y = 7$ and $2x + 3y = 8$ has a unique solution, then $a \neq$ ....

(a) $\frac{3}{4}$ (b) $-\frac{3}{4}$ (c) $\frac{4}{3}$ (d) $-\frac{4}{3}$ ☐

(6) The pair of linear equations $2x + y - 3 = 0$ and $6x + 3y = 9$ has .... ☐

(a) a unique solution (b) two solutions (c) no solution (d) infinitely many solutions

(7) If in a two digit number, the digit at unit place is x and the digit at tens place is 5, then the number is .... ☐

(a) $50x + 5$ (b) $30x + 5$ (c) $x + 50$ (d) $5x$

(8) In a two digit number, the digit at tens place is 7 and the sum of the digits is 8 times the digit at unit place. Then the number is .... ☐

(a) 70 (b) 71 (c) 17 (d) 78

(9) The sum of two numbers is 10 and the difference of them is 2. Then the greater number of these two is .... ☐

(a) 2 (b) 4 (c) 6 (d) 8

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(10) 3 years ago, the sum of ages of a father and his son was 40 years. After 2 years the sum of ages of the father and his son will be .... ☐

(a) 40 (b) 46 (c) 50 (d) 60

(11) The solution set of $2x + 4y = 8$ and $x + 2y = 4$ is .... ☐

(a) $\{(2, 1)\}$ (b) empty set (c) infinite set (d) $\{(0, 0)\}$

(12) Equation $\frac{x}{2} - \frac{y}{3} = 1$ can be expressed in the standard form as .... ☐

(a) $2x - 3y - 6 = 0$ (b) $3x - 2y - 6 = 0$ (c) $3x - 2y = 1$ (d) $2x - 3y = 3$

*

Summary

In this chapter, we have studied the following points :

1. Two linear equations in the same two variables are called a pair of linear equations in two variables. The general form of a pair of linear equations in two variables is $a_ix + b_iy + c_i = 0$; $a_i^2 + b_i^2 \neq 0$; $i = 1, 2$
2. A pair of linear equations in two variables can be solved by (i) graphical method and (ii) algebraic method.
3. Graphical method :
   The graph of a pair of linear equations in two variables represents two straight lines.
   - If both the lines intersect at a common point, then there is a unique solution and the pair of equations is consistent.
   - If both the lines are parallel, they do not intersect at any common point and solution set is an empty set. The pair of equations is inconsistent.
   - If both the lines coincide, there are infinitely many solutions. The pair of equations is linearly dependent.
4. There are following types of algebraic methods for the solution of the pair of linear equations :
   (i) Substitution Method (ii) Elimination Method and (iii) Cross-multiplication Method
5. For the pair of linear equations $a_ix + b_iy + c_i = 0$, $a_i^2 + b_i^2 \neq 0$, $i = 1, 2$ following possibilities arise : $(a_1, a_2, b_1, b_2, c_1, c_2 \neq 0)$

   (i) If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, the pair of linear equations is consistent

   ∴ There is a unique solution

   (ii) If $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, $\frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ or $\frac{c_1}{c_2} \neq \frac{a_1}{a_2}$ the pair of linear equations is inconsistent

   ∴ The solution set is an empty set

   (iii) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ The pair of linear equations is consistent

   ∴ There are infinitely many solutions.
6. We may be able to convert a non-linear equation into linear equation by selecting proper substitutions. There are several equations which are not in linear form but they can be reduced to a pair of linear equations.

Quadratic Equations 4

Life is good for only two things discovering mathematics and teaching mathematics.

- Poisson

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Geometry is the science of correct reasoning an incorrect figures.

- George Polya

4.1 Introduction

We have already studied a linear equation in one variable. An equation of the form $ax + b = 0$, $a \neq 0$, $a, b \in R$ is called a **linear equation**. We have also solved some practical problems using a linear equation. However there are lots of problems which cannot be solved using a linear equation. Let us consider some such problems.

Suppose the length of a rectangular hall is 4 meters more than its breadth. If the area of the floor of the hall is 60 square meters, what are the length and breadth of the hall ? If we consider the breadth of the rectangular hall as x, then its length is $x + 4$.

Now the area of the rectangular floor = length $\times$ breadth = $x(x + 4)$

$\therefore$ Our problem reduces to a mathematical form $x(x + 4) = 60$.

Simplifying this equation, we get $x^2 + 4x - 60 = 0$ which is not a linear equation in x. A linear equation in x is a first degree equation in variable x, like $3x - 12 = 0$ or $ax + b = 0$, $a \neq 0$. The equation $x^2 + 4x - 60 = 0$ involves a second degree term in variable x.

A similar equation will occur when the area and the perimeter of a rectangle are given. **A polynomial equation in which the degree of the polynomial is 2 is called a quadratic equation.**

It is believed that ancient **Babylonians** were aware of such kind of problems. **Euclid** mentioned a method to find the solution of such equations. In Hindu civilization **Brahmagupta** gave the method to solve equations of this form (600 to 655 A.D.). Shridhar Acharya gave the general formula to solve quadratic equations.

Al Kwarizami (800 A.D.) visited India and learnt Hindu mathematics. He took this knowledge to what is now known as Middle East.

Let us now understand what we mean by a quadratic equation and learn different methods of solving such equations.

4.2 Quadratic Polynomial and Quadratic Equations

In the second chapter we studied polynomials in one variable. $p(x) = ax + b$, $a \neq 0$, $a, b \in R$ is a polynomial of degree one, also known as a linear polynomial. We also defined quadratic polynomial $p(x) = ax^2 + bx + c$, $a \neq 0$, $a, b, c \in R$. We have defined zeros of a polynomial.

$p(x)$ is a given polynomial. If for some $k \in R$, $p(k) = 0$, then k is called a zero of the polynomial $p(x)$. For example,

$p(x) = x^2 - 5x + 6$

$\therefore \quad p(3) = 3^2 - 5(3) + 6 = 9 - 15 + 6 = 0$

So for $x = 3$ the value of the polynomial becomes zero. We say 3 is a zero of the polynomial. In general, $p(x)$ may be any polynomial of any degree and k a real number. If $p(k) = 0$, then k is called a zero of the polynomial.

We note that some polynomials may not have any real zero. For example, $p(x) = x^2 + 9$ or $p(x) = x^2 + 2x + 5$ do not have any real zero, because

$p(x) = x^2 + 9 \geq 9$ for any real value of x, since $x^2 \geq 0$

and $p(x) = x^2 + 2x + 5 = x^2 + 2x + 1 + 4 = (x + 1)^2 + 4 \geq 4$ **$((x + 1)^2 \geq 0)$**

Thus, in both the cases mentioned above, the value of polynomial cannot be zero.

A Quadratic Equation and its Solution :

If $p(x) = ax^2 + bx + c$, $a \neq 0$, $a, b, c \in R$, then the equation $p(x) = 0$. i.e. $ax^2 + bx + c = 0$ is called a quadratic equation or a second degree equation in variable x. Zeros of the polynomial $p(x) = ax^2 + bx + c$, $a \neq 0$, $a, b, c \in R$, if they exist, are called the solutions of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, $a, b, c \in R$.

The solutions of a quadratic equation are also called the roots of the equation.

$ax^2 + bx + c = 0$, $a \neq 0$, $a, b, c \in R$ is called the standard form of a quadratic equation.

A quadratic polynomial can have at the most two zeros. Therefore a quadratic equation can have at the most two solutions (or two roots).

4.3 Solution of a Quadratic Equation by the Method of Factorization

We know that the product of two real numbers is zero if and only if at least one of the numbers is zero. In mathematical form, for $a, b \in R$, $ab = 0$ if and only if $a = 0$ or $b = 0$ (or both a and b are 0).

Now consider the quadratic equation, $x^2 - 3x + 2 = 0$ **(i)**

We know that $x^2 - 3x + 2 = x^2 - 2x - x + 2$

$= x(x - 2) - 1(x - 2)$

$= (x - 2)(x - 1)$

So, the equation (i) can be written as

$(x - 2)(x - 1) = 0$ which is in the form $a \cdot b = 0$

$\therefore \quad x - 2 = 0$ or $x - 1 = 0$ **(ii)**

Now in (ii) we have two linear equations,

$x - 2 = 0$. So, $x = 2$

or $x - 1 = 0$. So, $x = 1$

Thus $x = 1$, $x = 2$ are the solution of the equation (i).

The method in which the polynomial $ax^2 + bx + c$ is factorised to obtain the solution of $ax^2 + bx + c = 0$ is called the method of factorization.

Let us solve some examples using this method.

Example 1 : Examine whether the following equations are quadratic or not :

(1) $3x^2 + \sqrt{2}x - 7 = 0$ (2) $2x(3x - 5) + 1 = 3x(2x + 5) + 3$

(3) $x(3x + 7) = (x + 1)(x - 1)$ (4) $x^2(x + 2) = 3x(x - 1) - 5$

Solution : (1) Let us compare $3x^2 + \sqrt{2}x - 7 = 0$ with the standard form of the quadratic equation $ax^2 + bx + c = 0$.

Here, $a = 3$, $b = \sqrt{2}$, $c = -7$ a, b $c \in$ R and $a \neq 0$.

$\therefore$ The degree of the polynomial $3x^2 + \sqrt{2}x - 7 = 0$ is 2.

$\therefore$ The equation is a quadratic equation.

(2) $2x(3x - 5) + 1 = 3x(2x + 5) + 3$

$\therefore$ $6x^2 - 10x + 1 = 6x^2 + 15x + 3$

$\therefore$ $25x + 2 = 0$

The polynomial $25x + 2$ is not a quadratic polynomial. Infact it is linear.

$\therefore$ The equation is not a quadratic equation.

(3) $x(3x + 7) = (x + 1)(x - 1)$

$\therefore$ $3x^2 + 7x = x^2 - 1$

$\therefore$ $2x^2 + 7x + 1 = 0$

The polynomial $2x^2 + 7x + 1$ is a quadratic polynomial.

($a = 2$, $b = 7$, $c = 1$, $a, b, c \in$ R and $a \neq 0$)

$\therefore$ The equation is a quadratic equation.

(4) $x^2(x + 2) = 3x(x - 1) - 5$

$\therefore$ $x^3 + 2x^2 = 3x^2 - 3x - 5$

$\therefore$ $x^3 - x^2 + 3x + 5 = 0$

The polynomial $x^3 - x^2 + 3x^2 + 5$ is a cubic polynomial.

$\therefore$ The equation is not a quadratic equation.

Example 2 : Solve the following equations by the method of factorization :

(1) $2x^2 - x - 3 = 0$ (2) $x^2 + 5x + 6 = 0$

(3) $3x^2 + 4x - 4 = 0$ (4) $6x^2 + 5x - 6 = 0$

(5) $6x^2 - 13x + 6 = 0$ (6) $2x^2 + \sqrt{6}x - 6 = 0$

Solution : (1) $2x^2 - x - 3 = 0$

$\therefore$ $2x^2 - 3x + 2x - 3 = 0$

$\therefore$ $x(2x - 3) + 1(2x - 3) = 0$

$\therefore$ $(2x - 3)(x + 1) = 0$

$\therefore$ $2x - 3 = 0$ or $x + 1 = 0$

$\therefore$ $x = \frac{3}{2}$ or $x = -1$

$\therefore$ $\frac{3}{2}$ and -1 are the solutions of the quadratic equation.

Note : $\frac{3}{2}$ and -1 are the roots of the equation or we can say that $\left\{-1, \frac{3}{2}\right\}$ is the solution set of the equation.

(2) $x^2 + 5x + 6 = 0$

$\therefore$ $x^2 + 3x + 2x + 6 = 0$

$\therefore$ $(x + 3)(x + 2) = 0$

$\therefore$ $x + 3 = 0$ or $x + 2 = 0$

$\therefore$ $x = -3$ or $x = -2$

$\therefore$ -3 and -2 are the solutions of the quadratic equation.

(3) $3x^2 + 4x - 4 = 0$

$\therefore$ $3x^2 + 6x - 2x - 4 = 0$

$\therefore$ $3x(x + 2) - 2(x + 2) = 0$

$\therefore$ $(x + 2)(3x - 2) = 0$

$\therefore$ $x = -2$ or $x = \frac{2}{3}$

$\therefore$ -2 and $\frac{2}{3}$ are the solutions of the quadratic equation.

(4) $6x^2 + 5x - 6 = 0$

$\therefore$ $6x^2 + 9x - 4x - 6 = 0$

$\therefore$ $3x(2x + 3) - 2(2x + 3) = 0$

$\therefore$ $(2x + 3)(3x - 2) = 0$

$\therefore$ $x = -\frac{3}{2}$ or $x = \frac{2}{3}$

$\therefore$ $-\frac{3}{2}$ and $\frac{2}{3}$ are the solutions of the quadratic equation.

(5) $6x^2 - 13x + 6 = 0$

$\therefore$ $6x^2 - 9x - 4x + 6 = 0$

$\therefore$ $3x(2x - 3) - 2(2x - 3) = 0$

$\therefore$ $(2x - 3)(3x - 2) = 0$

$\therefore$ $x = \frac{3}{2}$ or $x = \frac{2}{3}$

$\therefore$ $\frac{3}{2}$ and $\frac{2}{3}$ are the solutions of the quadratic equation.

(6) $2x^2 + \sqrt{6}x - 6 = 0$

$\therefore$ $2x^2 + 2\sqrt{6}x - \sqrt{6}x - 6 = 0$

$\therefore$ $2x(x + \sqrt{6}) - \sqrt{6}(x + \sqrt{6}) = 0$

$\therefore$ $(x + \sqrt{6})(2x - \sqrt{6}) = 0$

$\therefore$ $x = -\sqrt{6}$ or $x = \frac{\sqrt{6}}{2}$

$\therefore$ $-\sqrt{6}$ and $\frac{\sqrt{6}}{2}$ are the solutions of the quadratic equation.

Example 3 : Find the roots of the following equations. Write the solution set of each equation :

(1) $x^2 - 30x + 221 = 0$ (2) $\sqrt{2}x^2 + \sqrt{3}(1 - 2\sqrt{2})x - 6 = 0$

(3) $12x^2 - 25x + 12 = 0$ (4) $12x^2 + 7x - 12 = 0$

Solution : (1) $x^2 - 30x + 221 = 0$

$\therefore \quad x^2 - 17x - 13x + 221 = 0$

$\therefore \quad x(x - 17) - 13(x - 17) = 0$

$\therefore \quad (x - 17)(x - 13) = 0$

$\therefore \quad x - 17 = 0$ or $x - 13 = 0$

$\therefore \quad x = 17$ or $x = 13$

$\therefore$ 13 and 17 are the roots of the quadratic equation.

The solution set of the equation is $\{13, 17\}$.

(2) $\sqrt{2}x^2 + \sqrt{3}(1 - 2\sqrt{2})x - 6 = 0$

$\therefore \quad \sqrt{2}x^2 + \sqrt{3}x - 2\sqrt{2} \cdot \sqrt{3}x - 2\sqrt{3} \cdot \sqrt{3} = 0$

$\therefore \quad x(\sqrt{2}x + \sqrt{3}) - 2\sqrt{3}(\sqrt{2}x + \sqrt{3}) = 0$

$\therefore \quad (\sqrt{2}x + \sqrt{3})(x - 2\sqrt{3}) = 0$

$\therefore \quad x = -\frac{\sqrt{3}}{\sqrt{2}}$ or $x = 2\sqrt{3}$

$\therefore \quad \left\{-\frac{\sqrt{3}}{\sqrt{2}}, 2\sqrt{3}\right\}$ is the solution set of the equation.

(3) $12x^2 - 25x + 12 = 0$

$\therefore \quad 12x^2 - 16x - 9x + 12 = 0$

$\therefore \quad 4x(3x - 4) - 3(3x - 4) = 0$

$\therefore \quad (3x - 4)(4x - 3) = 0$

$\therefore \quad x = \frac{4}{3}$ or $x = \frac{3}{4}$

$\therefore \quad \frac{4}{3}$ and $\frac{3}{4}$ are the roots of the quadratic equation.

$\therefore \quad \left\{\frac{4}{3}, \frac{3}{4}\right\}$ is the solution set of the equation.

(4) $12x^2 + 7x - 12 = 0$

$\therefore \quad 12x^2 + 16x - 9x - 12 = 0$

$\therefore \quad (3x + 4)(4x - 3) = 0$

$\therefore \quad x = -\frac{4}{3}$ or $x = \frac{3}{4}$

$\therefore \quad -\frac{4}{3}$ and $\frac{3}{4}$ are the roots of the quadratic equation.

$\therefore \quad \left\{-\frac{4}{3}, \frac{3}{4}\right\}$ is the solution set of the equation.

EXERCISE 4.1

1. Examine whether the following equations are quadratic equations or not :

(1) $x + \frac{1}{x} = 2, x \neq 0$ (2) $(x - 2)(x + 3) = 0$ (3) $2x^2 - \sqrt{5}x + 2 = 0$

(4) $\frac{1}{x+1} - \frac{1}{x-1} = 3 \ (x \neq \pm 1)$ (5) $(2x + 1)(2x - 1) = (4x + 3)(x - 5)$

(6) $\frac{x-1}{x+1} - \frac{x+1}{x-1} = \frac{2}{3} \ (x \neq \pm 1)$ (7) $(2x + 3)^2 - (3x + 2)^2 = 13$

2. Verify whether the given value of x is a solution of the quadratic equation or not :

(1) $x^2 - 3x + 2 = 0, x = 2$ (2) $x^2 + x - 2 = 0, x = 2$

(3) $\frac{1}{3x+1} - \frac{1}{2x-1} + \frac{3}{4} = 0, x = 1 \left[x \neq \frac{1}{2}, -\frac{1}{3}\right]$ (4) $(3x - 8)(2x + 5) = 0, x = -\frac{5}{2}$

3. (1) If $x = 1$ is a root of $ax^2 + bx + c = 0, a \neq 0, a, b, c \in R$, prove that $a + b + c = 0$.

(2) If $x = -1$ is a root of $x^2 - px + q = 0, p, q \in R$, prove that $p + q + 1 = 0$.

(3) Find k, if one of the roots of $x^2 - kx + 6 = 0$ is 3.

(4) Find k, if one of the roots of $x^2 + 3(k + 2)x - 9 = 0$ is -3.

4. Solve the following equations using the method of factorization :

(1) $27x^2 - 48 = 0$ (2) $(x - 7)^2 - 16 = 0$ (3) $6x^2 + 13x + 6 = 0$

(4) $15x^2 - 16x + 1 = 0$ (5) $\sqrt{5}x^2 - 4x - \sqrt{5} = 0$ (6) $x + \frac{1}{x} = 2\frac{1}{6}$

*

4.4 The Solution of a Quadratic Equation by Completing a Square

When the two linear factors of polynomial $ax^2 + bx + c$ have rational coefficients the method of factorization is easily applicable. But if the factors of polynomial $ax^2 + bx + c$ have real coefficients which are not rational, then the factorization of $ax^2 + bx + c$ becomes slightly tricky and difficult. In such a case the method of 'completing the square' is very much useful.

The polynomial $ax^2 + bx + c$ can be written as $(px + q)^2 + r$

because $(px + q)^2 + r = p^2x^2 + 2pqx + q^2 + r$

Rewrite the equation $ax^2 + bx + c = 0$ with $a > 0$.

$\therefore \quad ax^2 + bx + c = (px + q)^2 + r = p^2x^2 + 2pqx + q^2 + r$ implies

$a = p^2, b = 2pq$ and $c = (q^2 + r)$

Given a, b, c we can find p, q, r from equations $p^2 = a, 2pq = b$ and $q^2 + r = c$ respectively.

Thus a quadratic equation $ax^2 + bx + c = 0$ can be written as $(px + q)^2 + r = 0$.

In this case if $r < 0$ we have two factors of the polynomial on the left hand side of the equation. If $r > 0$ we have no real solution of the equation and if $r = 0$ we have two identical solutions of the equation. Let us understand the method by solving some of the quadratic equations using 'method of completing the square'.

How to make a perfect square

We know that $(ax + b)^2 = a^2x^2 + 2abx + b^2$

We have also learnt how to find the missing term of a perfect square when two terms of the square are given. If we take first term = a^2x^2, second term = $2abx$ and third term = b^2 then following rules, particularly the second will be useful in solving the following examples :

(1) Middle term = $\pm 2\sqrt{\text{First term} \times \text{Third term}}$ (3) First term = $\frac{(\text{middle term})^2}{4 \times \text{Third term}}$

(2) Third term = $(\text{middle term})^2 / (4 \times \text{First term})$

We will abbreviate Middle Term = M.T. and First Term = F.T.

Example 4 : Solve the following equations using the method of 'completing a square' :

(1) $x^2 - 8x + 15 = 0$ (2) $9x^2 + 6x - 35 = 0$ (3) $x^2 + 6x + 7 = 0$

(4) $x^2 - x - 1 = 0$ (5) $x^2 + 2x + 5 = 0$

Solution : (1) $x^2 - 8x + 15 = 0$

Note that the quadratic polynomial on the left hand side of the equation is easily factorizable,

$x^2 - 8x + 15 = (x - 5)(x - 3)$

Hence $x^2 - 8x + 15 = 0$

$\therefore$ $(x - 5)(x - 3) = 0$

$\therefore$ $x = 5$ or $x = 3$

Let us now, solve the example by completing the square,

$x^2 - 8x + 15 = 0$

$\therefore$ $x^2 - 8x + 16 - 1 = 0$

$\therefore$ $(x - 4)^2 - 1 = 0$

Third term $= \frac{(\text{M.T.})^2}{4 \times \text{F.T.}} = \frac{64x^2}{4 \times x^2} = 16$

$\therefore$ $(x - 4 + 1)(x - 4 - 1) = 0$

$\therefore$ $(x - 3)(x - 5) = 0$

$\therefore$ $x = 3$ or $x = 5$

$\therefore$ 3 and 5 are the solutions of the equation.

(2) $9x^2 + 6x - 35 = 0$

$\therefore$ $9x^2 + 6x + 1 - 36 = 0$

Third term $= \frac{(\text{M.T.})^2}{4 \times \text{F.T.}} = \frac{36x^2}{4 \times 9x^2} = 1$

$\therefore$ $(3x + 1)^2 - 6^2 = 0$

$\therefore$ $(3x + 1 - 6)(3x + 1 + 6) = 0$

$\therefore$ $(3x - 5)(3x + 7) = 0$

$\therefore$ $3x - 5 = 0$ or $3x + 7 = 0$

$\therefore$ $x = \frac{5}{3}$ or $x = -\frac{7}{3}$

$\frac{5}{3}$ and $-\frac{7}{3}$ are the solutions of the quadratic equation.

Remark : Note that in both the examples No (i) and (ii) the result in the third steps are

$(x - 4)^2 - 1^2 = 0$ in example **(i)**

and $(3x + 1)^2 - 6^2 = 0$ in example **(ii)**

both of these are in the form $A^2 - B^2 = 0$

Here B^2 is a perfect square (like 1^2 or 6^2), but this does not mean that B^2 is always a perfect square of a rational number. It is only necessary that B^2 must be a positive real number. Observe this in next two examples.

(3) $x^2 + 6x + 7 = 0$

$\therefore$ $x^2 + 6x + 9 - 2 = 0$

Third term $= \frac{(\text{M.T.})^2}{4 \times \text{F.T.}} = \frac{36x^2}{4x^2} = 9$

$\therefore$ $(x + 3)^2 - 2 = 0$ **(Note here that 2 is not a perfect square)**

$\therefore$ $(x + 3)^2 - (\sqrt{2})^2 = 0$

$\therefore$ $(x + 3 - \sqrt{2})(x + 3 + \sqrt{2}) = 0$

$\therefore \quad x + 3 - \sqrt{2} = 0$ or $x + 3 + \sqrt{2} = 0$

$\therefore \quad x = \sqrt{2} - 3$ or $x = -\sqrt{2} - 3$

$\therefore \quad \sqrt{2} - 3$ and $-\sqrt{2} - 3$ are the solutions of the given quadratic equation.

(4) $x^2 - x - 1 = 0$

Let us multiply the equation by 4,

$\therefore \quad 4x^2 - 4x - 4 = 0$

$\therefore \quad 4x^2 - 4x + 1 - 5 = 0$

$\therefore \quad (2x - 1)^2 - (\sqrt{5})^2 = 0$

$\therefore \quad (2x - 1 + \sqrt{5})(2x - 1 - \sqrt{5}) = 0$

$\therefore \quad 2x - 1 + \sqrt{5} = 0$ or $2x - 1 - \sqrt{5} = 0$

$\therefore \quad 2x = 1 - \sqrt{5}$ or $2x = 1 + \sqrt{5}$

$\therefore \quad x = \frac{1-\sqrt{5}}{2}$ or $x = \frac{1+\sqrt{5}}{2}$

$\therefore \quad \frac{1-\sqrt{5}}{2}$ and $\frac{1+\sqrt{5}}{2}$ are the solutions of the given quadratic equation.

> **Remark :** The value $x = \frac{1+\sqrt{5}}{2}$ is called the golden number. This number occurs in many branches of mathematics. In geometry it is known as golden ratio.
> In number theory also this number occurs in Fibonacci Sequence.

(5) $x^2 + 2x + 5 = 0$

$\therefore \quad x^2 + 2x + 1 + 4 = 0$

$\therefore \quad (x + 1)^2 + 4 = 0$

Now $(x + 1)^2$ is the square of real number and hence it is a non-negative number. 4 is also a positive number. So $(x + 1)^2 + 4 \neq 0$ for any real number x.

In another way $(x + 1)^2 = -4$ which is also not possible as the square of a real number cannot be negative.

$\therefore$ The equation has no real solution.

4.5 The Solution of $ax^2 + bx + c = 0$, $a \neq 0$, $a, b, c \in R$

The General Formula :

The method of completing the square leads us to find the formula for the solution of quadratic equation in general form.

$ax^2 + bx + c = 0$, $a \neq 0$, $a, b, c \in$ R is the general form of the quadratic equation.

Let us multiply the equation by $4a$ $(a \neq 0)$

$\therefore \quad 4a(ax^2 + bx + c) = 0$

$\therefore \quad 4a^2x^2 + 4abx + 4ac = 0$

$\therefore \quad 4a^2x^2 + 4abx + b^2 - b^2 + 4ac = 0$

Third term $= \frac{(\text{M.T.})^2}{4 \times \text{F.T.}} = \frac{16a^2b^2x^2}{4 \times 4a^2x^2} = b^2$

$\therefore \quad (2ax + b)^2 = b^2 - 4ac$ **(i)**

Left hand side of the equation (i) is the square of real number. Hence the right hand side $b^2 - 4ac$ should be a non-negative real number.

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$\therefore$ If $b^2 - 4ac \geq 0$, then $\sqrt{b^2 - 4ac}$ exists as a real number. The expression $b^2 - 4ac$ is denoted by symbol D. $D = b^2 - 4ac$ is called the **discriminant** of the quadratic equation. ($b^2 - 4ac$ is also denoted by greek capital letter Δ (Delta).

Thus, equation (i) can be written as

$(2ax + b)^2 = D = (\sqrt{D})^2$ **(D ≥ 0)**

$\therefore$ $2ax + b = \pm\sqrt{D}$ **($\because$ If $x^2 = a^2$ then $x = \pm a$)**

$\therefore$ $2ax = -b \pm \sqrt{D}$

$\therefore$ $x = \frac{-b \pm \sqrt{D}}{2a}$

$\therefore$ $\frac{-b + \sqrt{D}}{2a}$ and $\frac{-b - \sqrt{D}}{2a}$, where $D = b^2 - 4ac$, are the solutions of the general quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, $a, b, c \in R$.

> The formula $x = \frac{-b \pm \sqrt{D}}{2a}$, $D = b^2 - 4ac$ is known as **Shridhar's Formula. Shridhar Acharya** was a Hindu mathematician of mediaval India (975 to 1050 A.D.) He lived (it is believed) on the western bank of river Ganga in Bengal. His work was mainly recorded in Algebra and Mensuration. It is believed that he was the first to give the formula for the roots of quadratic equations in terms of the coefficients a, b, c in the equation $ax^2 + bx + c = 0$.

4.6 Nature of the Roots of Quadratic Equation $ax^2 + bx + c = 0$, $a \neq 0$, $a, b, c \in R$

The general formula obtained in 4.5 also gives us some information regarding the nature of the roots in real number system.

The roots of the general quadratic equation :

$ax^2 + bx + c = 0$, $a \neq 0$, $a, b, c \in R$ are $\frac{-b + \sqrt{D}}{2a}$ and $\frac{-b - \sqrt{D}}{2a}$. We call them α and β.

Without loss of generality, we will take $\alpha = \frac{-b + \sqrt{D}}{2a}$ and $\beta = \frac{-b - \sqrt{D}}{2a}$.

The square of any real number, positive or negative or zero, is always a non-negative real number. The square root of a negative number does not exist in R.

So if the roots of $ax^2 + bx + c = 0$ are real numbers, it is essential that $D \geq 0$. (so that $\sqrt{D}$ is a real number.) Now if D is the perfect square of a rational number like 4, $\frac{9}{16}$ etc. then $\sqrt{D}$ is rational. In that case if a, b, c are also rational, then the roots $\frac{-b + \sqrt{D}}{2a}$ and $\frac{-b - \sqrt{D}}{2a}$ are rational.

Again if $D = 0$, then the roots are $\frac{-b + 0}{2a}$ and $\frac{-b - 0}{2a}$. It means both the roots are equal and real, infact $\frac{-b}{2a}$. More over in this case also if $a, b \in Q$, then $\frac{-b}{2a} \in Q$. Hence if a, b are rational, the roots will be equal and rational.

We can summarise the above discussion by the way of a tree diagram.

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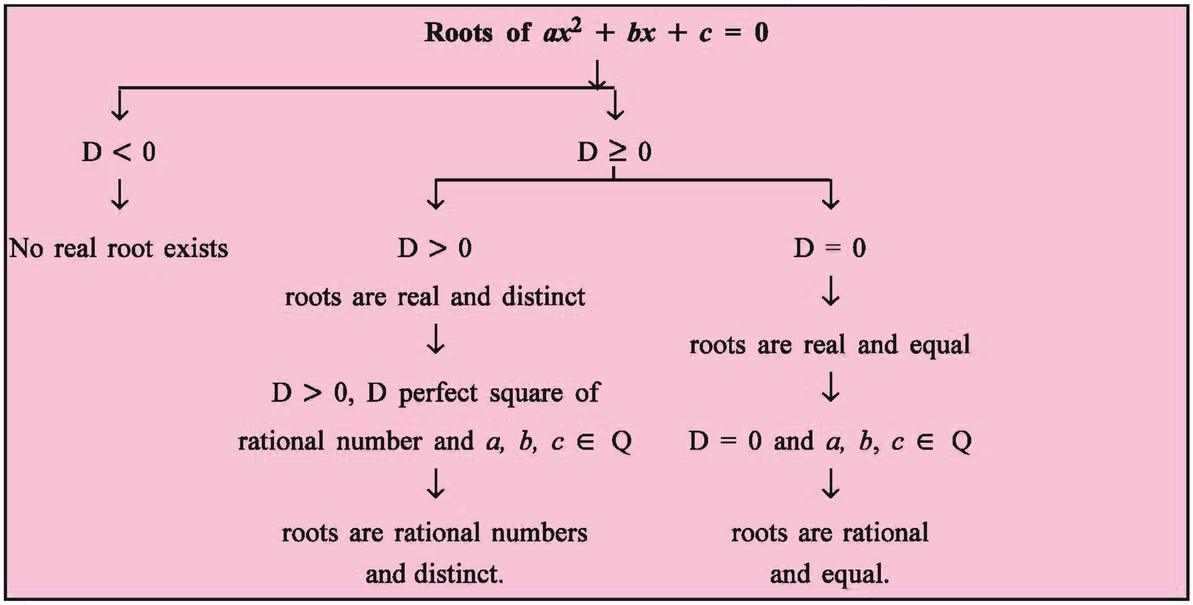

So, to find out the nature of the roots of a quadratic equation,

(1) We find the value of the discriminant D. If $D < 0$, we immediately say that equation has no solution in R.

(2) If $D \geq 0$, then we examine whether it is a perfect square of a rational number or not.

(i) If $D > 0$ and D is not a perfect square, then roots are real and distinct.

(ii) If $D > 0$ and it is a perfect square of a rational number and $a, b, c \in Q$, then the roots are real and rational. Roots are also distinct.

(3) If $D = 0$ then the roots are real and equal.

If $a, b, c \in Q$ the roots are equal rational numbers.

Now, let us solve some examples using the formula for the roots of quadratic equation.

Example 5 : Find the roots of the following quadratic equations using the general formula for the roots, if they exist :

(1) $2x^2 + x - 4 = 0$ (2) $4x^2 + 4\sqrt{3}x + 5 = 0$ (3) $9x^2 - 6x + 1 = 0$

(4) $\frac{1}{x+1} + \frac{1}{x+2} = \frac{4}{x+4}$

Solution : (1) $2x^2 + x - 4 = 0$

Let us compare the equation with $ax^2 + bx + c = 0$

$a = 2, b = 1, c = -4$

$D = b^2 - 4ac = 1^2 - 4(2)(-4) = 33$

Note that $D > 0$ and D is not a perfect square. Hence, the roots are real and distinct.

Now, $\alpha = \frac{-b+\sqrt{D}}{2a} = \frac{-1+\sqrt{33}}{2 \cdot 2} = \frac{\sqrt{33}-1}{4}$

and $\beta = \frac{-b-\sqrt{D}}{2a} = \frac{-1-\sqrt{33}}{2 \cdot 2} = -\left(\frac{\sqrt{33}+1}{4}\right)$

The roots of the equation are $\frac{\sqrt{33}-1}{4}$ and $-\left(\frac{\sqrt{33}+1}{4}\right)$

(2) $4x^2 + 4\sqrt{3}x + 5 = 0$

Here, $a = 4$, $b = 4\sqrt{3}$, $c = 5$

$\therefore$ $D = b^2 - 4ac = (4\sqrt{3})^2 - 4(4)(5) = 48 - 80 = -32 < 0$

$\therefore$ The equation has no real roots.

(3) $9x^2 - 6x + 1 = 0$

Here $a = 9$, $b = -6$, $c = 1$

$\therefore$ $D = b^2 - 4ac = (-6)^2 - 4(9)(1) = 36 - 36 = 0$

$\therefore$ Both the roots are equal.

$\therefore$ $\alpha, \beta = -\frac{b}{2a} = -\left(\frac{-6}{2 \times 9}\right) = \frac{1}{3}$

Both the roots are equal to $\frac{1}{3}$.

(4) $\frac{1}{x+1} + \frac{1}{x+2} = \frac{4}{x+4}$

Let us multiply the equation by $(x + 1)(x + 2)(x + 4)$.

$\therefore$ $\frac{(x+1)(x+2)(x+4)}{x+1} + \frac{(x+1)(x+2)(x+4)}{x+2} = \frac{4(x+1)(x+2)(x+4)}{x+4}$

$\therefore$ $(x + 2)(x + 4) + (x + 1)(x + 4) = 4(x + 1)(x + 2)$

$\therefore$ $x^2 + 6x + 8 + x^2 + 5x + 4 = 4x^2 + 12x + 8$

$\therefore$ $2x^2 + x - 4 = 0$

$\therefore$ $a = 2$, $b = 1$, $c = -4$

$\therefore$ $D = b^2 - 4ac = 1^2 - 4(2)(-4) = 33$. The roots are $\frac{-1+\sqrt{33}}{4}, \frac{-1-\sqrt{33}}{4}$. **(Refer (1))**

Example 6 : Solve the following equations using the general formula, if the equation has a solution in R :

(1) $\sqrt{3}x^2 - 5x + 2\sqrt{3} = 0$ **(2)** $x^2 + 2x + 2 = 0$

(3) $x^2 + 5x + 1 = 0$ **(4)** $(x + 4)(x + 5) = 3(x + 1)(x + 2) + 12x$

Solution : (1) $\sqrt{3}x^2 - 5x + 2\sqrt{3} = 0$

Let us compare the equation with $ax^2 + bx + c = 0$

$a = \sqrt{3}$, $b = -5$, $c = 2\sqrt{3}$

$D = b^2 - 4ac = (-5)^2 - 4(\sqrt{3})\sqrt{3}(2) = 25 - 24 = 1 > 0$

$D > 0$

$\therefore$ The equation has real distinct roots.

$\therefore$ $\alpha = \frac{-b+\sqrt{D}}{2a} = \frac{5+\sqrt{1}}{2\sqrt{3}} = \frac{6}{2\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$

$\beta = \frac{-b-\sqrt{D}}{2a} = \frac{5-\sqrt{1}}{2\sqrt{3}} = \frac{4}{2\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

$\therefore$ The solutions of the equation are $\sqrt{3}$ and $\frac{2\sqrt{3}}{3}$.

(2) $x^2 + 2x + 2 = 0$

Here $a = 1,\ b = 2,\ c = 2$

$\therefore$ $D = b^2 - 4ac = 4 - 4(1)(2) = -4 < 0$

$D < 0$

$\therefore$ There is no real solution of the equation in R.

(3) $x^2 + 5x + 1 = 0$

Comparing the equation with $ax^2 + bx + c = 0$, we have

$a = 1,\ b = 5,\ c = 1$

$\therefore$ $D = b^2 - 4ac = 25 - 4(1)(1) = 21 > 0$

$\therefore$ The equation has two real distinct roots.

$$\therefore\ \alpha = \frac{-b+\sqrt{D}}{2a} = \frac{-5+\sqrt{21}}{2} = \frac{\sqrt{21}-5}{2}$$

$$\beta = \frac{-b-\sqrt{D}}{2a} = \frac{-5-\sqrt{21}}{2} = -\frac{\sqrt{21}+5}{2}$$

$\therefore$ The solutions of the equation are $-\frac{\sqrt{21}+5}{2},\ \frac{\sqrt{21}-5}{2}$.

(4) $(x + 4)(x + 5) = 3(x + 1)(x + 2) + 12x$

Let us simplify the equation.

$\therefore$ $(x + 4)(x + 5) = 3(x + 1)(x + 2) + 12x$

$\therefore$ $x^2 + 9x + 20 = 3(x^2 + 3x + 2) + 12x$

$\therefore$ $x^2 + 9x + 20 = 3x^2 + 21x + 6$

$\therefore$ $2x^2 + 12x - 14 = 0$

$\therefore$ $x^2 + 6x - 7 = 0$

Now, comparing the equation with the general form $ax^2 + bx + c = 0$,

we have $a = 1,\ b = 6,\ c = -7$

$\therefore$ $D = b^2 - 4ac = 36 - 4(1)(-7) = 36 + 28 = 64 > 0$

$$\therefore\ \alpha = \frac{-b+\sqrt{D}}{2a} = \frac{-6+\sqrt{64}}{2\cdot 1} = \frac{-6+8}{2} = 1$$

$$\beta = \frac{-b-\sqrt{D}}{2a} = \frac{-6-\sqrt{64}}{2\cdot 1} = \frac{-6-8}{2} = -7$$

$\therefore$ 1 and -7 are the solutions of the quadratic equation.

Example 7 : Find the discriminant of the following quadratic equations and hence determine the nature of the roots of the equations. Find the roots, if they are real :

(1) $2x^2 - 4x + 5 = 0$ (2) $ax^2 - 2x + \frac{1}{a} = 0,\ a \neq 0,\ a \in R$

(3) $\sqrt{5}x^2 - 2\sqrt{2}x - 2\sqrt{5} = 0$ (4) $\sqrt{3}x^2 + 2x - \sqrt{3} = 0$

Solution : (1) $2x^2 - 4x + 5 = 0$ is in the form $ax^2 + bx + c = 0$,

where $a = 2,\ b = -4,\ c = 5$

$\therefore$ Discriminant $D = b^2 - 4ac = 16 - 4(2)(5) = -24$

Since $D < 0$, there is no real root of the equation.

(2) $ax^2 - 2x + \frac{1}{a} = 0$

$\therefore$ $D = 4 - 4(a)\left(\frac{1}{a}\right) = 0$

$D = 0$. Hence, the roots are real and equal.

Moreover if a is rational (i.e. if $a \in Q$), then roots are rational and equal.

The value of the equal roots is $\frac{-b}{2a} = \frac{2}{2a} = \frac{1}{a}$.

(3) $\sqrt{5}x^2 - 2\sqrt{2}x - 2\sqrt{5} = 0$

The equation is of the form $ax^2 + bx + c = 0$, where

$a = \sqrt{5} \neq 0$, $b = -2\sqrt{2}$, $c = -2\sqrt{5}$

$\therefore$ $D = b^2 - 4ac = (-2\sqrt{2})^2 - 4(\sqrt{5})(-2\sqrt{5}) = 8 + 40 = 48$

$\therefore$ $D = 48 > 0$

Hence the roots are real and distinct,

$$\therefore \alpha = \frac{-b+\sqrt{D}}{2a} = \frac{2\sqrt{2}+\sqrt{48}}{2\sqrt{5}} = \frac{2\sqrt{2}+4\sqrt{3}}{2\sqrt{5}} = \frac{\sqrt{2}+2\sqrt{3}}{\sqrt{5}}$$

$$\beta = \frac{-b-\sqrt{D}}{2a} = \frac{2\sqrt{2}-\sqrt{48}}{2\sqrt{5}} = \frac{2\sqrt{2}-4\sqrt{3}}{2\sqrt{5}} = \frac{\sqrt{2}-2\sqrt{3}}{\sqrt{5}}$$

Remark : Now you will have the idea how important the general formula for the roots is. Can you factorise the polynomial on the left hand side of the equation in example (3) ? Can you apply the method of completing the square to solve the equation ? Of course you can, but will it be easier than the application of the general formula for the roots to apply ?

(4) $\sqrt{3}x^2 + 2x - \sqrt{3} = 0$

The equation is of the form $ax^2 + bx + c = 0$, where

$a = \sqrt{3} \neq 0$, $b = 2$, $c = -\sqrt{3}$

$\therefore$ $D = b^2 - 4ac = 4 - 4(\sqrt{3})(-\sqrt{3}) = 4 + 12 = 16 > 0$

$\therefore$ $D > 0$

$\therefore$ The roots are real and distinct.

$$\therefore x = \frac{-b+\sqrt{D}}{2a} = \frac{-2+\sqrt{16}}{2\sqrt{3}} = \frac{-2+4}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$x = \frac{-b-\sqrt{D}}{2a} = \frac{-2-\sqrt{16}}{2\sqrt{3}} = \frac{-2-4}{2\sqrt{3}} = \frac{-3}{\sqrt{3}} = -\sqrt{3}$$

So, $\frac{1}{\sqrt{3}}$ and $-\sqrt{3}$ are the roots of the equation.

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Remark : If D > 0 and D is the perfect square of a rational number, then roots are rational. Is this statement true ? Here D > 0, D = 16 is a perfect square but roots $\frac{1}{\sqrt{3}}$ and $-\sqrt{3}$ are not rational !! Do you know what is missing in the conditional statement given above ?

Yes. If D > 0, D is a perfect square and a, b, c are rational, then the roots are rational and distinct. In example (4) $a = \sqrt{3}$, is not rational. That is why the roots are not rational even though D is a perfect square.

Example 8 : Find the value of k, if the following equations have equal real roots :

(1) $(k + 1)x^2 - 2(k + 4)x + 2k = 0,\ k \in R$

(2) $(k - 1)x^2 - 12x + 2k - 1 = 0,\ k \in R$

Solution : (1) $(k + 1)x^2 - 2(k + 4)x + 2k = 0$ is of the form $ax^2 + bx + c = 0$,

where $a = k + 1$, $b = -2(k + 4)$, $c = 2k$

The equation has equal real roots.

$\therefore$ D = 0

$\therefore$ $b^2 - 4ac = 0$

$\therefore$ $[-2(k + 4)]^2 - 4(k + 1)(2k) = 0$

$\therefore$ $4(k^2 + 8k + 16) - 4(2k^2 + 2k) = 0$

$\therefore$ $k^2 + 8k + 16 - 2k^2 - 2k = 0$

$\therefore$ $k^2 - 6k - 16 = 0$

$\therefore$ $(k - 8)(k + 2) = 0$

$\therefore$ $k = 8,\ k = -2$

$\therefore$ -2 and 8 are the values of k for which the equation has equal roots.

(2) $(k - 1)x^2 - 12x + 2k - 1 = 0,\ k \in R$

Comparing with $ax^2 + bx + c = 0$ we get,

$a = k - 1$, $b = -12$, $c = 2k - 1$

The equation has equal real roots.

$\therefore$ D = 0

$\therefore$ $b^2 - 4ac = 0$

$\therefore$ $144 - 4(k - 1)(2k - 1) = 0$

$\therefore$ $(k - 1)(2k - 1) - 36 = 0$

$\therefore$ $2k^2 - 3k - 35 = 0$

$\therefore$ $2k^2 - 10k + 7k - 35 = 0$

$\therefore$ $(2k + 7)(k - 5) = 0$

$\therefore$ $k = \frac{-7}{2}$ or $k = 5$

$\therefore$ $\frac{-7}{2}$ and 5 are the values of k for which the equation has equal roots.

EXERCISE 4.2

1. Find the discriminant of the following quadratic equations and discuss the nature of the roots :

(1) $6x^2 - 13x + 6 = 0$ (2) $\sqrt{6}x^2 - 5x + \sqrt{6} = 0$ (3) $24x^2 - 17x + 3 = 0$

(4) $x^2 + 2x + 4 = 0$ (5) $x^2 + x + 1 = 0$ (6) $x^2 - 3\sqrt{3}x - 30 = 0$

2. If $a, b, c \in R$, $a > 0$, $c < 0$, then prove that the roots of $ax^2 + bx + c = 0$ are real and distinct.

3. (1) Find k, if the roots of $x^2 - (3k - 2)x + 2k = 0$ are equal and real.

(2) If the roots of the quadratic equation $(k + 1)x^2 - 2(k - 1)x + 1 = 0$ are real and equal, find the value of k.

4. If the roots of $ax^2 + 2bx + c = 0$, $a \neq 0$, $a, b, c \in R$ are real and equal, then prove that $a : b = b : c$.

5. Solve the following equations using the general formula :

(1) $x^2 + 10x + 6 = 0$ (2) $x^2 + 5x - 1 = 0$ (3) $x^2 - 3x - 2 = 0$

(4) $x^2 - 3\sqrt{6}x + 12 = 0$ (5) $3x^2 + 5\sqrt{2}x + 2 = 0$ (6) $\frac{x^2 - 1}{x^2 + 1} = \frac{4}{5}$

*

4.7 Solutions of Problems Using Quadratic Equations

To solve practical problems we will follow the following procedure :

(1) Read the problems carefully, identify the variable and translate the problem in mathematical language. This is to make a mathematical model of the problem.

(2) Solve the equation occurring into mathematical model.

(3) Verify the solutions : We should not verify our answer by substituting the solution into mathematical equations but we should verify the answer using original problem.

Let us solve some problems.

Example 9 : Product of digits of a two-digit number is 21. If we add 36 to the number, the new number obtained is a number formed by interchange of the digits. Find the number.

Solution : Let the tens digit be x. Then the digit at unit place is $\frac{21}{x}$.

$\therefore$ The number is $10x + \frac{21}{x}$.

If we add 36 to the number, then its digits are interchanged.

$\therefore$ $10x + \frac{21}{x} + 36 = 10\left(\frac{21}{x}\right) + x$

$\therefore$ $10x^2 + 21 + 36x = 210 + x^2$ **(multiplying the equation by x)**

$\therefore$ $9x^2 + 36x - 189 = 0$

$\therefore$ $x^2 + 4x - 21 = 0$

$\therefore$ $(x + 7)(x - 3) = 0$

$\therefore$ $x = -7$ or $x = 3$

A digit of a number can not be negative (it's 0, 1, 2,..., 9)

$\therefore$ $x = 3$

$\therefore$ $\frac{21}{x} = \frac{21}{3} = 7$

$\therefore$ The number is 37.

Verification : 37 + 36 = 73. In 37 and 73 the digits are interchanged.

Note : Here product of digits is 21. Digits can be 0, 1, 2,..., 9. Also $21 = 3 \cdot 7 = 1 \cdot 21$.

$\therefore$ Digits can be 3 and 7 only. Adding 36 to the smaller number 37, we get larger number 73. The required number is 37. We can get solution intuitively in this way.

Example 10 : Area of a rectangle is 21 m^2. If the perimeter of the rectangle is 20 m, find the length and breadth of the rectangle. (Length is greater than the breadth.)

Solution : Let the length of the rectangle be x.

The area of the rectangle = length $\times$ breadth.

$\therefore$ Breadth of the rectangle = $\frac{\text{area}}{\text{length}} = \frac{21}{x}$

We will assume that length of the rectangle is greater than its breadth.

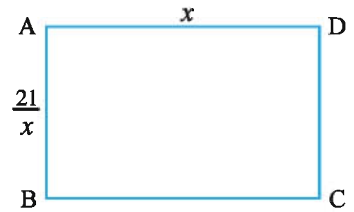

Figure 4.1

Now the perimeter of the rectangle = 2(length + breadth)

$\therefore \quad 20 = 2\left(x + \frac{21}{x}\right)$ (i)

Multiplying equation (i) by x, we get $20x = 2x^2 + 42$

$\therefore \quad 2x^2 - 20x + 42 = 0$ i.e. $x^2 - 10x + 21 = 0$

$\therefore \quad (x - 7)(x - 3) = 0$

$\therefore \quad x = 7$ or $x = 3$

Since the length x is greater than the breadth, we take, $x = 7$

$\therefore$ Breadth = $\frac{21}{x} = \frac{21}{7} = 3$

$\therefore$ Length of the rectangle = 7 m

Breadth of the rectangle = 3 m

Verification : area = length $\times$ breadth = $7 \times 3 = 21$ m^2

perimeter = 2(length + breadth) = 2(7 + 3) = 20 m

Example 11 : The sum of a non-zero number and its reciprocal is $\frac{41}{20}$. Find the number.

Solution : Let the non-zero number be x. Its reciprocal is $\frac{1}{x}$. It is given that $x + \frac{1}{x} = \frac{41}{20}$. Multiplying the equation by $20x$, we get,

$20x^2 + 20 = 41x$

$\therefore \quad 20x^2 - 41x + 20 = 0$

$\therefore \quad 20x^2 - 25x - 16x + 20 = 0$

$\therefore \quad 5x(4x - 5) - 4(4x - 5) = 0$

$\therefore \quad (4x - 5)(5x - 4) = 0$

$\therefore \quad x = \frac{5}{4}$ or $x = \frac{4}{5}$

So the required number is $\frac{5}{4}$ or $\frac{4}{5}$.

Verification : $\frac{4}{5} + \frac{5}{4} = \frac{16+25}{20} = \frac{41}{20}$

Example 12 : In a right angled triangle one of the sides forming the right angle is 4 *cm* more than twice the length of the other side. If the area of the triangle is $120(cm)^2$, find the perimeter of the triangle.

Solution : In ΔABC, let $m\angle B = 90$. If $BC = x$ then $AB = 2x + 4$

Area of $\Delta ABC = \frac{1}{2}\text{base} \times \text{altitude} = \frac{1}{2}BC \cdot AB$.

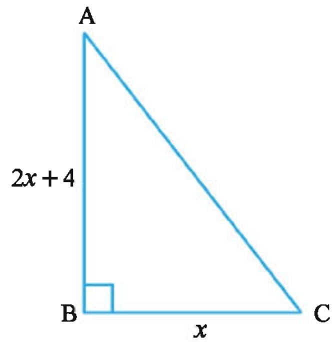

Figure 4.2

$\therefore \quad 120 = \frac{1}{2}x(2x + 4)$

$\therefore \quad 240 = 2x^2 + 4x$

$\therefore \quad x^2 + 2x - 120 = 0$

$\therefore \quad (x - 10)(x + 12) = 0$

$\therefore \quad x = 10 \quad \text{or} \quad x = -12$

The length of a side of a triangle cannot be negative.

$\therefore \quad x = 10$

$\therefore \quad BC = 10\ cm,\ AB = 2x + 4 = 24\ cm$

By Pythagoras theorem, $AC = \sqrt{AB^2 + BC^2} = \sqrt{576 + 100}$

$\therefore \quad AC = \sqrt{676} = 26\ cm$

$\therefore \quad$ Perimeter of $\Delta ABC = AB + BC + AC = 24 + 10 + 26 = 60\ cm$

Example 13 : The sum of the ages of father and son at present is 110. Ten years ago the product of their ages was 1856. What is the age of the father and the son ?

Solution : Let the present age of the son be x.

$\therefore$ The present age of the father is $110 - x$.

Ten years ago the age of the son was $x - 10$ and the age of the father was

$110 - x - 10 = 100 - x$

It is given that $(x - 10)(100 - x) = 1856$

$\therefore \quad 100x + 10x - 1000 - x^2 = 1856$

$\therefore \quad x^2 - 110x + 2856 = 0$

$\therefore \quad x^2 - 68x - 42x + 2856 = 0$

$\therefore \quad x(x - 68) - 42(x - 68) = 0$

$\therefore \quad (x - 68)(x - 42) = 0$

$\therefore \quad x - 68 = 0 \quad \text{or} \quad x - 42 = 0$

$\therefore \quad x = 68 \quad \text{or} \quad x = 42$

If the age of the son, $x = 68$, then the age of the father is $110 - x = 110 - 68 = 42$ which is impossible as son can not be older than the father.

$\therefore$ The age of the son, $x = 42$

The age of the father $110 - x = 110 - 42 = 68$.

Verification : Ten years ago their ages were 32 and 58.

The product of their ages $= 32 \times 68 = 1856$.

Example 14 : The speed of a motor boat in still water is 25 *km/hr*. In a river, it goes 60 *km* downstream and comes back the same distance upstream in 5 *hours*. Find the speed of the current. (speed of the current of the river is less than the speed of the motor boat in still water.)

Solution : Let the speed of the current of river be x *km/hr*

$\therefore$ The speed of the boat downstream is $(25 + x)$ *km/hr*

The speed of the boat upstream is $(25 - x)$ *km/hr*

$\therefore$ Time taken for going 60 km downstream $= \frac{60}{25+x}$ *hours*

Time taken for coming 60 km upstream $= \frac{60}{25-x}$ *hours*

Total time taken is 5 *hours*,

$\therefore \quad \frac{60}{25+x} + \frac{60}{25-x} = 5$

$\therefore \quad 60(25 - x) + 60(25 + x) = 5(25 + x)(25 - x)$

$\therefore \quad 12(25 - x + 25 + x) = (25 + x)(25 - x)$

$\therefore \quad 600 = 625 - x^2$

$\therefore \quad x^2 = 25$

$\therefore \quad x = 5$ **(as $x \neq -5$)**

$\therefore$ The speed of the current of the river is 5 *km/hr*.

Verification : Time taken while going downstream $= \frac{60}{25+x} = \frac{60}{30} = 2$ *hours*

Time taken while coming upstream $= \frac{60}{25-x} = \frac{60}{20} = 3$ *hours*

$\therefore$ The total time taken is $2 + 3 = 5$ *hours*

EXERCISE 4.3

1. Find two numbers whose sum is 27 and the product is 182.
2. Find two consecutive natural numbers, sum of whose squares is 365.
3. The sum of ages of two friends is 20 years. Four years ago the product of their ages was 48. Show that these statements can not be true.
4. A rectangular garden is designed such that the length of the garden is twice its breadth and the area of the garden is 800 m^2. Find the length of the garden.
5. Perimeter of a rectangular garden is 360 *m* and its area is 8000 m^2. Find the length of the garden and also find its breadth. (The length is greater than the breadth)
6. If a cyclist travels at a speed 2 *km/hr* more than his usual speed, he reaches the destination 2 *hours* earlier. If the destination is 35 *km* away, what is the usual speed of the cyclist ?
7. The diagonal of a rectangular ground is 60 meters more than the breadth of the ground. If the length of the ground is 30 meters more than the breadth, find the area of the ground.
8. The sides of a right angled triangle are consecutive positive integers. Find the area of the triangle.

*

Miscellaneous Example :

Example 15 : Form the quadratic equation (a mathematical model) in each of the following case :

(1) $\overline{AB}$ is a line segment. P ∈ $\overline{AB}$ such that A–P–B.

AP = 1. If $\frac{AP}{PB} = \frac{AB}{AP}$, find AB.

Figure 4.3

(2) In figure 4.4, □ABCD is rectangle.

AB > BC. BC = 1; □BCXY is a square.

If $\frac{AB}{BC} = \frac{XY}{AY}$, find AB.

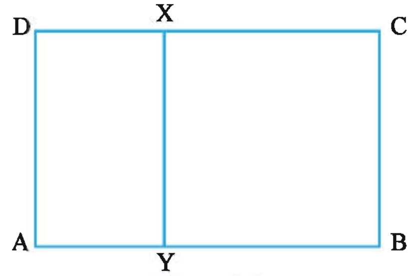

Figure 4.4

(3) If we subtract 1 from positive number, then we get the reciprocal of the number. Find the number.

Solution : (1) We have : AP = 1. Let AB = x

∴ PB = AB − AP = $x - 1$ **(A–P–B)**

Now, $\frac{AP}{PB} = \frac{AB}{AP}$

∴ $\frac{1}{x-1} = \frac{x}{1}$

∴ $x^2 - x = 1$

∴ $x^2 - x - 1 = 0$ **(i)**

(2) □ABCD is rectangle and □BCXY is a square.

BC = XY = AD = 1 **(BC = 1)**

Let AB = x. BC = BY = 1. So, AY = $x - 1$

∴ $\frac{AB}{BC} = \frac{XY}{AY}$ (given)

∴ $\frac{x}{1} = \frac{1}{x-1}$

∴ $x^2 - x = 1$

∴ $x^2 - x - 1 = 0$ **(ii)**

(3) Let the positive number be x

∴ $x - 1 = \frac{1}{x}$

∴ $x^2 - x - 1 = 0$ **(iii)**

Behold, in all the three examples the quadratic quation is the same namely $x^2 - x - 1 = 0$.

Do you remember we have solved this equation in example 4(4).

Let us solve it again.

Comparing $x^2 - x - 1 = 0$ with $ax^2 + bx + c = 0$, we have

$a = 1, b = -1, c = -1$

$D = b^2 - 4ac = 1 - 4(1)(-1) = 5$

∴ The roots are, $\frac{-b+\sqrt{D}}{2a} = \frac{1+\sqrt{5}}{2} > 0$ and $\frac{-b-\sqrt{D}}{2a} = \frac{1-\sqrt{5}}{2}$

But $\frac{1-\sqrt{5}}{2} < 0$ $(1 < \sqrt{5})$

$\therefore$ Our required length AB in (i), AB in (ii) and the positive number x in (iii), all have the same value $\frac{\sqrt{5}+1}{2}$. This is the number known as golden number !!!

EXERCISE 4

1. Solve the following quadratic equations using factorization :

(1) $x^2 - 12 = 0$ (2) $x^2 - 7x - 60 = 0$ (3) $x^2 - 15x + 56 = 0$

(4) $\frac{2x+3}{2x-3} + \frac{2x-3}{2x+3} = \frac{17}{4}, x \neq \pm\frac{3}{2}$ (5) $\frac{1}{x+5} + \frac{3}{4(3x+1)} = \frac{1}{x+2}, x \neq -5, x \neq -2, x \neq -\frac{1}{3}$

2. Find the roots of the following equations by the method of perfect square :

(1) $x^2 - 24x - 16 = 0$ (2) $3x^2 + 7x - 20 = 0$ (3) $x^2 - 10x + 25 = 0$

(4) $x^2 + (x + 5)^2 = 625$ (5) $(x + 2)(x + 3) = 240$

3. Divide 20 into two parts such that the sum of the square of the parts is 218.

4. A car takes 1 *hour* less to cover a distance of 200 *km* if its speed is increased by 10 *km/hr*, than its usual speed. What is the usual speed of the car ?

5. When there is a decrease of 5 *km/hr* in the usual uniform speed of a goods train, due to track repair work going on it takes 4 *hours* more than the usual time for travelling the distance of 400 *km*. Find the usual speed of the train.

6. A river flows at a speed of 1 *km/hr*. A boat takes 15 *hours* to travel 112 *km* downstream and coming back the same distance upstream. Find the speed of the boat in still water. (Speed of the river flow is less than the speed of the boat in still water)

7. Find a number greater than 1 such that the sum of the number and its reciprocal is $2\frac{4}{15}$.

8. The difference of the speed of a faster car and a slower car is 20 *km/hr*. If the slower car takes 1 *hour* more than the faster car to travel a distance of 400 *km*, find speed of both the cars.

9. Product of the ages of Virat 7 years ago and 7 years later is 480. Find his present age.

10. If the age of Sachin 8 year ago is multiplied by his age two years later, the result is 1200. Find the age of Sachin at present.

11. Sunita's age at present is 2 years less than 6 times the age of her daughter Anita. The product of their ages 5 years later will be 330. What was the age of Sunita when her daughter Anita was born ?

12. The formula of the sum of first n natural numbers is $S = \frac{n(n+1)}{2}$. If the sum of first n natural number is 325, find n.

13. Hypotenuse of a right angled triangle is 2 less than 3 times its shortest side. If the remaining side is 2 more than twice the shortest side, find the area of the triangle.

14. The sum of the squares of two consecutive odd positive integers is 290. Find the numbers.

15. The product of two consecutive even natural numbers is 224. Find the numbers.

16. The product of digits of a two-digit number is 8 and the sum of the squares of the digits is 20. If the number is less than 25. Find the number.

17. If price of sugar decreases by ₹ 5, one can buy 1 *kg* more sugar in ₹ 150, what is the price of the sugar ?

18. If the price of petrol is increased by ₹ 5 per litre. One gets 2 litres less petrol spending ₹ 1320. What is the increased price of the petrol ?

19. A vendor gets a profit in percentage equal to the cost price of a flower pot when he sells it for ₹ 96. Find the cost of the flower pot and the percentage of profit.

20. While selling a pen for ₹ 24 the loss in percentage is equal to its cost price. Find the cost price of pen. The cost price of pen is less than ₹ 50.

21. The difference of lengths of sides forming right angle in right angled triangle is 3 *cm*. If the perimeter of the triangle is 36 *cm*. Find the area of the triangle.

22. The sides of a right angled triangle are x, $x + 3$, $x + 6$, x being a positive integer. Find the perimeter of the triangle.

23. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) ...... is a solution of quadratic equation $x^2 - 3x + 2 = 0$ ☐

(a) −3 (b) 1 (c) 3 (d) −2

(2) Discriminant D = ... for the quadratic equation $5x^2 - 6x + 1 = 0$ ☐

(a) 16 (b) $\sqrt{56}$ (c) 4 (d) 56

(3) If $x = 2$ is a root of the equation $x^2 - 4x + a = 0$, then a = ... ☐

(a) −2 (b) 2 (c) −4 (d) 4

(4) A quadratic equation has two equal roots, if ...... ☐

(a) $D < 0$ (b) $D > 0$

(c) $D = 0$ (d) D is non-zero perfect square

(5) The quadratic equation ...... has 3 as one of its roots. ☐

(a) $x^2 - x - 6 = 0$ (b) $x^2 + x - 6 = 0$

(c) $x^2 - x + 6 = 0$ (d) $x^2 + x + 6 = 0$

(6) If 4 is a root of quadratic equation $x^2 + ax - 8 = 0$, then a = ... ☐

(a) 2 (b) 4 (c) −2 (d) −4

(7) If one of the roots of $kx^2 - 7x + 3 = 0$ is 3, then k = ... ☐

(a) −2 (b) 3 (c) −3 (d) 2

(8) The discriminant of $x^2 - 3x - k = 0$ is 1. A value of x is .... ☐

(a) −4 (b) −2 (c) 2 (d) 4

*

Summary

We studied following points in this chapter :

1. A quadratic equation and its solution.
2. Method of Factorization
3. Method of completing a square
4. General Formula for solution
5. Nature of roots based on discriminant.
6. Problems leading to a quadratic equation.

ARITHMETIC PROGRESSION 5

Mathematics consists of proving the most obvious things in the least obvious way.
– George Polya. (1887-1985)

5.1 Introduction

In day to day life, we encounter the word 'sequence' through print media, visual media, conversation etc. We read in news papers 'sequence of events that lead to...' for some breaking news. We say that the sequence of batsmen in a one-day cricket match was not proper. In mathematics a sequence is an ordered list of numbers. 1, 2, 3, 4,... is a sequence. We say 1 is its first term. 3 is its third term etc. Generally we use symbol t_n or a_n or T_n or u_n or v_n for the nth term of a sequence. For the above sequence n is the nth term.

Now let us have a look at some examples :

Figure 5.1

(1) Arun applied for a job and started the job with a monthly salary of ₹ 25000. Yearly he got an increment of ₹ 1000 per month. His salary for different years is

₹ 25000 per month in the first year.
₹ 26000 per month in the second year.
₹ 27000 per month in the third year.....

So we get a sequence 25,000, 26,000, 27,000,.... . This is a 'finite' sequence as eventually, he will get superannuation or may be promoted to a different grade. Here $T_n = 25000 + (n - 1)1000$.

Figure 5.2

(2) Sudha got ₹ 3000 on her birthday and put it in a money-box. From the next month, she deposited ₹ 100 at the end of every month in the money-box as savings from pocket expenses.

She will have ₹ 3000, ₹ 3100, ₹ 3200, ₹ 3300,... at the end of successive months till she continues the habit of saving and depositing. This sequence is definitely finite. Here $T_n = 3000 + (n - 1)100$.

Figure 5.3

(3) A series of steps lead to a temple. The first step starts from the ground obviously. Every next step is at 15 cm rise from each previous step. There are 50 more steps to reach the pundal. Then one has to climb 15, 30, 45,..., 750 cm.

The sequence is finite. $T_n = 15n$, $1 \le n \le 50$, $n \in N$

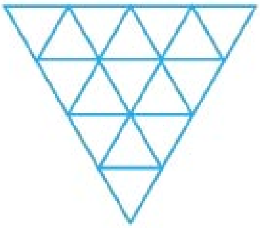

Figure 5.4

(4) A triangle is given. By joining the mid-points of the sides of the given triangle, we get 4 triangles. Again by joining the mid-points of the sides of the triangles, we get sixteen triangles. So we get a sequence of triangles 1, 4, 16, 64,... .

So here theoretically we may continue indefinitely.

Here $T_1 = 1$, $T_2 = 4 = 2^2$, $T_3 = 16 = 2^4$, $T_4 = 64 = 2^6$.

We expect $T_n = 2^{2(n-1)}$

(5) A sequence is found in nature in leaves of trees, petals of sunflower, grain on a maize cone etc.

This sequence is defined as follows :

$a_1 = 1$, $a_2 = 1$, and for $n \geq 3$, $a_n = a_{n-1} + a_{n-2}$.

So $a_3 = a_2 + a_1 = 2$, $a_4 = a_3 + a_2 = 3$ etc.

So the sequence is 1, 1, 2, 3, 5, 8, 13, 21, 34,...

It is called ***Fibonacci*** Sequence.

Leonardo Pisano Bigollo also known as *Leanardo of Pisano, Leonardo Bonacci, Leonardo Fibonacci* was an Italian mathematician. In the 13th century, he published *Liber Abaci*. He recognised that arithmetic with Hindu- Arabic numerals is simpler than Roman numerals. Original problem about how fast rabbits could breed in ideal circumstances generates *Fibonacci* sequence.

Fibonacci

Born : C 1170, **Died :** C 1250 (aged around 80),

Nationality : Italian, **Fields :** Mathematics

Known for : Fibonacci number, Fibonacci prime, Brahmagupta-Fibonacci identity, Fibonacci polynomials, Fibonacci pseudoprime, Fibonacci word, Reciprocal Fibonacci constant, Introduction of digital notation to Europe, Pisano period, Practical number.

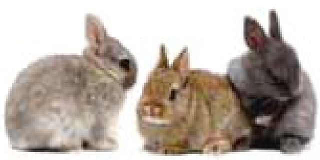

Figure 5.5

Suppose a newly-born pair of rabbits, one male, one female, are put in a field. Rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits. Suppose that rabbits **never die** and that the female **always** produces one new pair (one male, one female) **every month** from the second month on.

The puzzle that Fibonacci posed was how many pairs will there be in one year ?

1. At the **end of the first month**, they mate, but there is still only one pair.
2. At the **end of the second month** the female produces a new pair, so now there are 2 pairs of rabbits in the field.
3. At the **end of the third month**, the original female produces a second pair, making 3 pairs in all in the field.
4. At the **end of the fourth month**, the original female has produced yet another new pair, the female born two months ago produces her first pair also, making 5 pairs.

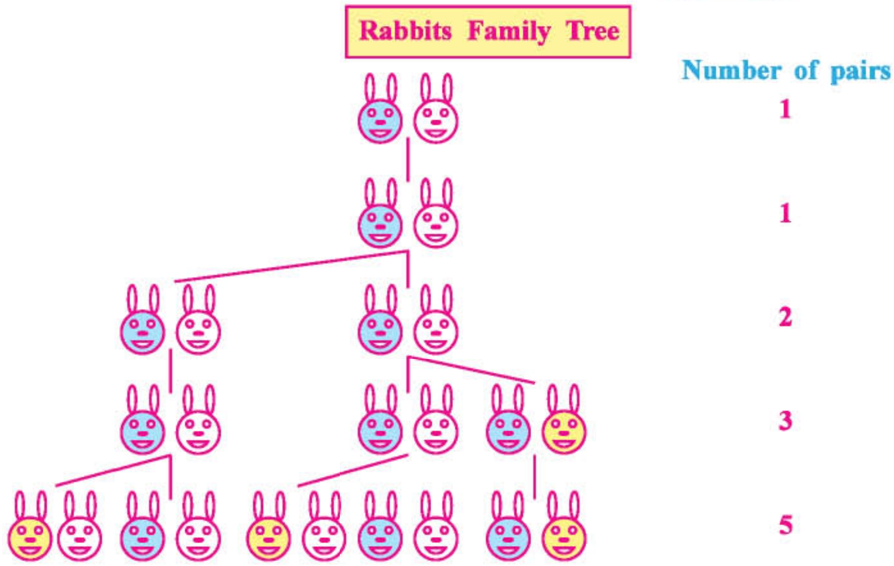

Figure 5.6

The number of rabbits in the field **at the start** of each month is 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144,...

The number of petals on many plants is a Fibonacci number.

3 petals	Lily, Iris
5 petals	Butter cup, Wild rose, Pinks
8 petals	Delphiniums
13 petals	Ragwort, Corn marigold, Cinearia, Some daisies
21 petals	Aster, Chicory
34 petals	Plantain, Pysethrum
55, 89 petals	Michael daisies

(6) Consider the sequence of primes, 2, 3, 5, 7, 11

This sequence is infinite as proved by Euclid. But there is no formula for the nth term. You do not know which prime will follow a given prime.

(7)

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Consider the sequence formed by the numbers in a diagonal of Pascal's triangle as shown. 1, 3, 6, 10,...

This is called the sequence of triangular numbers. Its nth term is given by $T_n = \frac{n(n+1)}{2}$

Look at the triangle shown in figure 5.7. Sum all the numbers upto a given row. The sum is a triangular number. (The dots form a triangle)

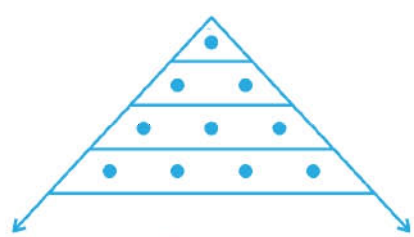

Figure 5.7

5.2 Arithmetic Progression

Consider following ordered list of numbers.

(1) 2, 5, 8, 11,...

(2) 1, 7, 13, 19,...

(3) 99, 97, 95, 93,...

(4) 10, 9.5, 9, 8.5, 8,...

(1) Here each term is 3 more than the previous term. In other words

$T_2 - T_1 = T_3 - T_2 = T_4 - T_3 = 3$

Assuming that the same pattern continues, this sequence is called an arithmetic progression. 2 is its first term and $T_{r+1} - T_r = d$ is called the **common difference** or simply the **difference**. By knowing first term called a and the difference d, we can write the progression as

2, 2 + 3 = 5, 5 + 3 = 8, 8 + 3 = 11,... etc.

This is an infinite sequence. What would be its general term ?

$T_2 = 2 + 3$, $T_3 = 5 + 3 = 2 + 3 + 3 = 2 + \underline{2}.3$ (2 is 3 − 1)

$T_4 = 8 + 3 = 5 + 3 + 3 = 2 + 3 + 3 + 3 = 2 + \underline{3}.3$ (3 is 4 − 1)

We can expect $T_n = 2 + (n - 1)3 = 3n - 1$

(2) As in (1) each term is 6 more than the pervious term. We should get,

$T_n = 1 + (n - 1)6 = 6n - 5$

(3) Here each term is 2 less i.e. −2 more than the previous term.

$T_2 = 99 - 2 = 99 + (-2)(1) = 97$, $T_3 = 97 - 2 = 99 - 2 - 2 = 99 + (-2)(2)$

$T_4 = 99 + (-2)3$

Hence $T_n = 99 + (-2)(n - 1) = -2n + 101$ or $101 - 2n$.

(4) As in case (1) $d = -0.5$

$\therefore \quad T_n = 10 - \frac{1}{2}(n - 1) = -\frac{1}{2}n + \frac{21}{2} = -\frac{1}{2}(n - 21)$

All these sequences are infinite. There is no last term. The nth term is called the general term. **The difference d is a non-zero constant.**

A sequence in which the difference between any two consecutive terms is a non-zero constant (Difference = given term − previous term) is called an arithmetic progression.

So if we are given the first term a and the common difference d, we can completely identify the sequence as,

$$a,\ a + d,\ a + 2d,\ a + 3d, ...,\ a + (n - 1)d, ...$$

If the A.P. is finite and terminates at the nth step, then $T_n = a + (n - 1)d$ is also called the last term.

> **Note :** In an example, to identify whether a sequence is an A.P. (arithmetic progression) or not just by observation only, there is some danger.

From the terms in 2, 5, 8, 11,..., one may be tempted to work like this,

$a = 2$, $d = 5 - 2 = 3$

$\therefore \quad T_n = a + (n - 1)d = 2 + 3(n - 1) = 3n - 1$

But consider $T_n = 11(n-1)(n-2)(n-3)(n-4) - \frac{(n-2)(n-3)(n-4)}{3}$

$$+ \frac{5(n-1)(n-3)(n-4)}{2} - 4(n-1)(n-2)(n-4) + \frac{n(n-1)(n-2)(n-3)}{6}$$

Also satisfies $T_1 = 2$, $T_2 = 5$, $T_3 = 8$, $T_4 = 11$

But $T_5 \neq 14$. Infact $T_5 = 278$.

So to identify a sequence as an A.P. it is to be given that $T_2 - T_1 = T_4 - T_3 = ...$ for given terms and that the sequence follows the same pattern further.

An important result :

$d = \frac{T_m - T_n}{m - n}$, $m \neq n$.

In fact $T_m - T_n = a + (m-1)d - (a + (n-1)d)$

$= (m - n)d$

$\therefore \quad d = \frac{T_m - T_n}{m - n}$ **$(m - n \neq 0)$**

Example 1 : Find the first term and the common difference for following A.P. s.

(1) 3, 6, 9, 12,...

(2) 100, 98, 96,...

(3) Natural numbers in increasing order, which are multiples of 4.

(4) Natural numbers in increasing order, ending in zero.

Solution : (1) For this A.P., $a = 3$, $d = 6 - 3 = 3$ (You can take any of $T_2 - T_1 = T_3 - T_2$ etc.)

(2) For this A.P. $a = 100$, $d = 98 - 100 = -2$

(3) 4, 8, 12,.... is the sequence of multiples of 4.

$\therefore \quad a = 4$, $d = 8 - 4 = 4$

(4) Numbers ending in zero form the sequence 10, 20, 30, 40, 50,...

$\therefore \quad a = 10$, $d = 20 - 10 = 10$

Example 2 : Write the first four terms of A.P. where,

(1) the first term is 5 and the common difference is 3.

(2) the first term is 20 and the common difference is –2.

(3) the first term is –3 and the common difference is 4.

(4) the first term is –7 and the common difference is –5.

Solution : (1) $a = 5$, $d = 3$

$\therefore \quad T_2 = 5 + 3 = 8$, $T_3 = 8 + 3 = 11$, $T_4 = 11 + 3 = 14$

$\therefore$ The A.P. is 5, 8, 11, 14,...

(2) $a = 20$, $d = -2$

$\therefore \quad T_2 = 20 + (-2) = 18$, $T_3 = 18 + (-2) = 16$, $T_4 = 16 + (-2) = 14$

$\therefore$ The A.P. is 20, 18, 16, 14,...

(3) $a = -3$, $d = 4$

$\therefore \quad T_2 = -3 + 4 = 1$, $T_3 = 1 + 4 = 5$, $T_4 = 5 + 4 = 9$

$\therefore$ The A.P. is –3, 1, 5, 9,...

(4) $a = -7$, $d = -5$

$\therefore$ $T_2 = -7 + (-5) = -12$, $T_3 = -12 + (-5) = -17$, $T_4 = -17 + (-5) = -22$

$\therefore$ The A.P. is $-7, -12, -17, -22,...$

Example 3 : Assuming that same pattern follows decide which of the following sequence is an A.P. If the sequence is A.P., then find its common difference.

(1) 11, 21, 31, 41,...

(2) 1, 0, 1, 0,...

(3) 2, 2, 2, 2,...

(4) $-8, -12, -16, -20$

Solution : (1) $T_2 - T_1 = T_3 - T_2 = T_4 - T_3 = 10$

Hence assuming that every term exceeds previous term by 10, the given sequence is an A.P. with $a = 11$, $d = 10$.

(2) Here $T_2 - T_1 = -1$, $T_3 - T_2 = 1$

Hence the sequence is not an A.P.

(3) For this sequence $T_2 - T_1 = T_3 - T_2 = ... = 0$.

The common difference should be a non-zero constant.

$\therefore$ This constant sequence is not an A.P.

(4) $T_2 - T_1 = -12 - (-8) = -4$, $T_3 - T_2 = -16 - (-12) = -4$, $T_4 - T_3 = -20 - (-16) = -4$

Thus the difference between consecutive terms is a non-zero constant. Assuming the same pattern further, the given sequence is an A.P. Here $d = -4$.

Example 4 : Find the 101st term of A.P. 5, 11, 17,...

Solution : $a = 5$, $d = 6$

$T_n = a + (n - 1)d$

$\therefore$ $T_{101} = 5 + (101 - 1)6 = 5 + 600 = 605$

$\therefore$ The 101st term is 605.

Example 5 : For a given A.P. 5, 10, 15, 20,..., 200, what is the number of terms ?

Solution : Suppose 200, the last term is the nth term.

$T_n = 200$, $a = 5$, $d = 5$

$\therefore$ $a + (n - 1)d = 200$

$\therefore$ $5 + (n - 1)5 = 200$

$\therefore$ $1 + n - 1 = 40$

$\therefore$ $n = 40$

$\therefore$ The number of terms is 40.

Example 6 : Is 0 a term of A.P. 200, 196, 192,..., -200 ? If yes, what is its order ?

Solution : $a = 200$, $d = 196 - 200 = -4$

Let $T_n = 0$, if possible.

$T_n = a + (n - 1)d$

$\therefore$ $200 - 4(n - 1) = 0$

$\therefore$ $50 - n + 1 = 0$

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∴ $n = 51$

∴ Yes, 51st term of the A.P. is zero.

Example 7 : Which term of the A.P. 5, 9, 13, 17,... is 101 ? Is any term equal to 203 ?

Solution : For the given sequence $a = 5$, $d = 9 - 5 = 4$

Let the nth term be 101.

$T_n = 101$

∴ $a + (n - 1)d = 101$

∴ $5 + (n - 1)4 = 101$

∴ $4(n - 1) = 96$

∴ $n - 1 = 24$

∴ $n = 25$

∴ 25th term is 101.

If possible, let $T_m = 203$

∴ $a + (m - 1)d = 203$

∴ $5 + (m - 1)4 = 203$

∴ $4(m - 1) = 198$

But 4 does not divide 198.

∴ m is not an integer $\left(m = \frac{99}{2} + 1 = \frac{101}{2}\right)$

∴ No term of the sequence can equal 203.

> **Note :** See that the sequence consists of natural numbers leaving remainder 1 when divided by 4. i.e. numbers of form $4n + 1$. 101 leaves remainder 1 when divided by 4. So 101 is a term of the sequence. 203 leaves remainder 3 when divided by 4. So it is not a term of the sequence.

Example 8 : If in an A.P., 7th term is 108 and 11th term is 212, find its nth term.

Solution : Here $T_7 = 108$, $T_{11} = 212$

∴ $a + 6d = 108$ and $a + 10d = 212$ $\quad$ **($T_n = a + (n - 1)d$)**

Subtraction of the equations gives $4d = 104$

∴ $d = 26$

∴ $a = 108 - 6d = 108 - 156 = -48$

∴ $T_n = a + (n - 1)d = -48 + 26(n - 1)$

∴ $T_n = 26n - 74$

Here $d = \frac{T_{11} - T_7}{11 - 7} = \frac{212 - 108}{4} = \frac{104}{4} = 26$

Example 9 : In the following finite A.P., find the 7th term from the end, 11, 17, 23, 29,..., 605.

Solution : $a = 11$, $d = 6$

If 605 is nth term, $605 = 11 + 6(n - 1)$

∴ $6(n - 1) = 594$

∴ $n = \frac{594}{6} + 1 = 99 + 1 = 100$

Now 7th term from the end is 94th term **(100 − 7 + 1)**

∴ $T_{94} = 11 + (94 - 1)6$ **$(a + (n - 1)d)$**

$= 11 + 93 \times 6$

$= 569$

Infact the kth term from the end is $(n - k + 1)$st term.

Another method : We may consider A.P., 605, 599, 593,..., going backwards.

∴ $a = 605$, $d = -6$

We want to find T_7.

∴ $T_7 = 605 + (7 - 1)(-6)$

$= 605 - 36 = 569$

For a finite A.P., you may consider it as an ordered pair, if there are two terms.

3-terms A.P. is an ordered triple.

n-terms A.P. is an ordered n-tuple.

Example 10 : In a rose garden, a trapezium is formed as a bed of roses.

The first row contains 25 roses.

The second row contains 21 roses.

The last row contains 5 roses. How many rows of roses are there ?

Solution : $a = 25$, $d = -4$.

Let the last row be the nth row.

∴ $T_n = a + (n - 1)d$

$5 = 25 + (n - 1)(-4)$

∴ $-20 = -4(n - 1)$

∴ $n = 6$

∴ The rose-bed consists of six rows.

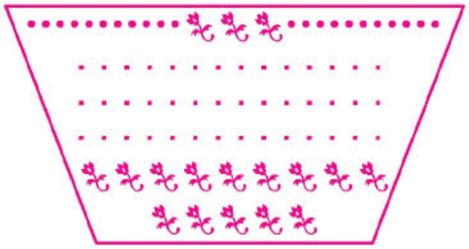

Figure 5.8

Example 11 : How many three digit multiples of 7 are there ?

Solution : Multiples of 7 having 3 digits are 105, 112,..., 994.

∴ $a = 105$, $d = 7$, $T_n = 994$

∴ $994 = 105 + 7(n - 1)$

∴ $\frac{889}{7} = n - 1$

∴ $n = 128$

∴ There are 128 three digit multiples of 7.

Note : How did we arrive at 105 and 994 ?

First three digit number is 100.

$100 = 7 \cdot 14 + 2$

∴ $105 = 7 \cdot 14 + 2 + 5 = 7 \cdot 14 + 7$

So, divide 100 by 7. Remainder is 2. Add $7 - 2$ to 100. We arrive at 105.

Similarly, the largest 3 digit number is 999.

$999 = 7 \cdot 142 + 5$

∴ Divide 999 by 7. The remainder is 5.

∴ $999 - 5 = 994$ is the largest three digit number divisible by 7.

Example 12 : Can any two terms of an A.P. be the same ?

Solution : No. If $T_m = T_n$, say $m \neq n$

$a + (m - 1)d = a + (n - 1)d$

$\therefore \quad (m - n)d = 0$

But $m \neq n$. Hence $d = 0$.

But the common difference in an A.P. is a non-zero constant.

Example 13 : Which is the first negative term of A.P. 112, 107, 102,... ?

Solution : Let the nth term of the sequence be its first negative term.

$\therefore \quad T_n < 0$

$\therefore \quad 112 + (n - 1)(-5) < 0$

$\therefore \quad 112 < 5(n - 1)$

$\therefore \quad n > \frac{112}{5} + 1$

$\therefore \quad n > 23.4$

$\therefore$ The smallest $n \in N$ greater than 23.4 is 24.

$\therefore$ 24th term is the first negative term of A.P. 112, 107, 102,...

Infact $T_{23} = 112 + (23 - 1)(-5) = 2,$

$T_{24} = 112 + (24 - 1)(-5) = -3$

Example 14 : Determine the A.P. whose 4th term is 17 and the 10th term exceeds the 7th term by 12.

Solution : We know $d = \frac{T_m - T_n}{m - n}$

Here $T_{10} = T_7 + 12$. So $T_{10} - T_7 = 12$

$\therefore \quad d = \frac{T_{10} - T_7}{10 - 7} = \frac{12}{3} = 4$

Now, $T_4 = a + 3d = 17$

$\therefore \quad a + 12 = 17$

$\therefore \quad a = 5$

$\therefore$ The A.P. is 5, 9, 13, 17, 21, 25,... $T_n = 5 + 4(n - 1) = 4n + 1$

Example 15 : For which n, some terms of 231, 228, 225,... and 3, 6, 9,... are equal ?

Solution : For the sequence 231, 228, 225,...

$T_n = 231 + (n - 1)(-3) = -3n + 234$

For the sequence 3, 6, 9,... $T_n' = 3n$

We want $T_n = T_n'$

$\therefore \quad 3n = -3n + 234$

$\therefore \quad 6n = 234$

$\therefore \quad n = 39$

The 39th terms of both A.P. s are same, namely 117.

Example 16 : Can any two terms of A.P. s 4, 7, 10, 13,... and 107, 104, 101,... be same ? Why ?

Solution : No. The terms of the first A.P. leave remainder 1 when divided by 3 and terms of the second A.P. leave remainder 2 when divided by 3. So they cannot have any term in common.

Note : $T_n = 3n + 1$ and $T'_m = -3m + 110$

If $T_n = T_m'$ for some $m, n \in N$,

$3n + 1 = -3m + 110$

$\therefore$ $3(m + n) = 109$. Impossible ! **(Why ?)**

EXERCISE 5.1

1. Given a and d for the following A.P., find the following A.P. :

(1) $a = 3, d = 2$ (2) $a = -3, d = -2$ (3) $a = 100, d = -7$

(4) $a = -100, d = 7$ (5) $a = 1000, d = -100$

2. Determine if the following sequences represent an A.P., assuming that the pattern continues. If it is an A.P., find the nth term :

(1) 5, −5, 5, −5,... (2) 2, 2, 2, 2,...

(3) 1, 11, 111, 1111,... (4) 5, 15, 25, 35, 45,...

(5) 17, 22, 27, 32,... (6) 101, 99, 97, 95,...

(7) 201, 198, 195, 192,...

(8) Natural numbers which are consecutive multiples of 5 in increasing order.

(9) Natural numbers which are multiples of 3 or 5 in increasing order.

3. Find the nth term of the following A.P.'s :

(1) 2, 7, 12, 17,... (2) 200, 195, 190, 185,...

(3) 1000, 900, 800,... (4) 50, 100, 150, 200,...

(5) $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2},...$ (6) 1.1, 2.1, 3.1, 4.1,...

(7) 1.2, 2.3, 3.4, 4.5,... (8) $\frac{5}{3}, \frac{7}{3}, 3, \frac{11}{3}, \frac{13}{3}, 5,...$

4. Find A.P. if T_n, T_m are as given below :

(1) $T_7 = 12, T_{12} = 72$ (2) $T_2 = 1, T_{12} = -9$

5. (1) In an A.P., $T_3 = 8$, $T_{10} = T_6 + 20$. Find the A.P.

(2) In an A.P. 5th term is 17 and 9th term exceeds 2nd term by 35. Find the A.P.

6. Can any term of A.P., 12, 17, 22, 27,... be zero ? Why ?

7. Can any term of A.P., 201, 197, 193,... be 5 ? Why ?

8. Which term of A.P., 8, 11, 14, 17,.. is 272 ?

9. Find the 10th term from end for A.P., 3, 6, 9, 12,... 300.

10. Find the 15th term from end for A.P., 10, 15, 20, 25, 30,..., 1000.

11. If in an A.P., $T_7 = 18$, $T_{18} = 7$, find T_{101}.

12. If in an A.P., $T_m = n$, $T_n = m$, prove $d = -1$.

*

5.3 Sum to n Terms of an Arithmetic Progression

Dia gets payment for her artwork with initial instalment ₹ 1000 and increasing it by ₹ 500 every week. How much did she get in six weeks ?

It is the sum 1000 + 1500 + 2000 + 2500 + 3000 + 3500.

i.e. we want to know about the sum of an A.P.

When famous mathematician Carl Friedrich Gauss (1777-1855) misbehaved in primary school, his teacher I. G. Buttner gave him a task to add a list of integers from 1 to 100.

Gauss's method was to realise that pairwise addition of terms from opposite ends of the list yielded identical intermediate sums :

$$1 + 100 = 2 + 99 = 3 + 98 = ... = 50 + 51 = 101$$

Here $1 + 2 + 3 + ... + 100$ = The sum of 50 sums each equal to 101.

∴ The sum is 5050.

He gave the answer within seconds to the astonishment of his teacher and his assistant Martin Bartels !

Now let us revert to the question of finding the sum of first n terms of an A.P.

Let a be the first term and d be the common difference. Then the nth term is given by

$$T_n = a + (n - 1)d$$

Now if the sum of first n terms is denoted by S_n, then

$$S_n = a + (a + d) + (a + 2d) + (a + 3d) +...+ [a + (n - 2)d] + [a + (n - 1)d]$$

$$\therefore \quad S_n = [a + (n - 1)d] + [a + (n - 2)d] + [a + (n - 3)d] + [a + (n - 4)d] +...+ (a + d) + a$$

∴ Adding the equalities,

$$2S_n = [2a + (n - 1)d] + [2a + (n - 1)d] +... n \text{ times}$$

$$\therefore \quad 2S_n = n[2a + (n - 1)d]$$

$$S_n = \frac{1}{2}n\,[2a + (n - 1)d] \qquad \text{(i)}$$

(i) can also be written as

$$S_n = \frac{1}{2}n\,[a + a + (n - 1)d]$$

$$= \frac{1}{2}n\,(a + l) \quad \text{where } l \text{ is the last term of a finite A.P. of } n \text{ terms.} \qquad \text{(ii)}$$

Hence $1 + 2 + 3 + ... + n = \frac{1}{2}n(1 + n) = \frac{n(n+1)}{2}$

To find the *n*th term, given S_n :

$$S_n = (T_1 + T_2 + T_3 +...+ T_{n-1}) + T_n$$

$$= S_{n-1} + T_n$$

$$\therefore \quad T_n = S_n - S_{n-1} \qquad n > 1 \text{ and } T_1 = S_1$$

Example 17 : Find the sum to first n terms of A.P. and sum to first 20 terms : $5 + 11 + 17 +...$

Solution : Here $a = 5$, $d = 6$

$$\therefore \quad S_n = \frac{1}{2}n\,[2a + (n - 1)d]$$

$$= \frac{1}{2}n\,[10 + (n - 1)6]$$

$$= \frac{1}{2}n\,(6n + 4)$$

$$\therefore \quad S_n = n(3n + 2)$$

∴ The sum of first 20 terms, $S_{20} = 20(60 + 2)$

$$= 20(62) = 1240$$

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Now onwards whenever we say the sum of n terms, we will mean the sum of first n terms.

Example 18 : If the sum of first 15 terms of an A.P. is 225 and the sum of first 21 terms of an A.P. is 441, find the sum to first n terms and also find the A.P.

Solution : $S_{15} = 225$ and $S_{21} = 441$

Now $S_n = \frac{1}{2}n\,[2a + (n - 1)d]$

$\therefore \quad 225 = S_{15} = \frac{1}{2}(15)\,(2a + 14d)$

$\therefore \quad 15 = a + 7d$ **(i)**

Also $441 = \frac{1}{2}(21)(2a + 20d)$

$\therefore \quad 21 = a + 10d$ **(ii)**

$\therefore$ Solving (i) and (ii) $3d = 6$ or $d = 2$

$\therefore \quad a = 21 - 10d = 21 - 20 = 1$

$\therefore$ The A.P. is 1, 3, 5, 7,... $T_n = 1 + 2(n - 1) = 2n - 1$

$S_n = \frac{1}{2}n\,(2 + 2(n - 1)) = n(1 + n - 1) = n \cdot n = n^2$

Note : Let, $a = 1$, $d = 2$

So the sequence is 1, 3, 5, 7, 9, 11,..., $(2n - 1)$

$S_1 = 1^2$, $S_2 = 1 + 3 = 4 = 2^2$,

$S_3 = 1 + 3 + 5 = 9 = 3^2$,

$S_4 = 1 + 3 + 5 + 7 = 16 = 4^2$.

We can expect $S_n = n^2$.

Example 19 : How many terms of A.P. 7, 11, 15, 19, 23,... will add up to 900 ?

Solution : We are given $S_n = 900$, $a = 7$, $d = 4$

$\therefore \quad \frac{1}{2}n\,[14 + 4(n - 1)] = 900$ $\left(S_n = \frac{1}{2}n\,[2a + (n - 1)d]\right)$

$\therefore \quad \frac{1}{2}n\,(4n + 10) = 900$

$\therefore \quad 2n^2 + 5n - 900 = 0$

$\therefore \quad (n - 20)(2n + 45) = 0$

$\therefore \quad n = 20$ as $n \in$ N and $n \neq -\frac{45}{2}$.

$\therefore$ The number of terms adding upto 900 is 20.

Example 20 : Find the sum of first 30 positive integer multiples of 6.

Solution : Multiples of 6 (natural numbers) are 6, 12, 18, 24,...

$\therefore \quad a = 6$, $d = 6$

We want $S_{30} = \frac{1}{2} \times 30(12 + 29 \cdot 6))$ $\left(S_n = \frac{1}{2}n\,[2a + (n - 1)d]\right)$

$= \frac{1}{2} \times 30(186) = 2790$

Example 21 : If $T_n = 5 - 7n$, prove that (T_n) is an A.P. and find S_n.

Solution : $T_1 = -2$, $T_2 = -9$, $T_3 = -16$

$\therefore$ The sequence is $-2, -9, -16,...$

$T_n - T_{n-1} = (5 - 7n) - [5 - 7(n - 1)]$ **$(n > 1)$**

$= 7(n - 1) - 7n$

$= -7$

$\therefore \quad (T_n)$ is an A.P.

Now $a = -2$, $d = -7$

$$S_n = \frac{1}{2}n\,[2a + (n-1)d]$$
$$= \frac{1}{2}n\,(-4 + (n-1)(-7))$$
$$= \frac{1}{2}n\,(-7n + 3)$$
$$= -\frac{7}{2}n^2 + \frac{3}{2}n$$

Example 22 : A spiral is made up of successive semicircles with centres alternatively at A and B starting with centre at A, having radii 1 *cm*, 2 *cm*, 3 *cm*,... as shown in the figure 5.9. What is the total length of such a spiral made up of 27 semicircles. (Take $\pi = \frac{22}{7}$)

Solution : Length of each semicircle is πr.

$r = 1, 2, 3, ..., 27$

$\therefore$ The A.P. is $\pi, 2\pi, 3\pi, ...$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$\therefore\ S_{27} = \frac{1}{2}(27)\,(2\pi + 26\pi)$$
$$= \frac{1}{2}(27)(28)\frac{22}{7}$$
$$= 1188\ cm$$

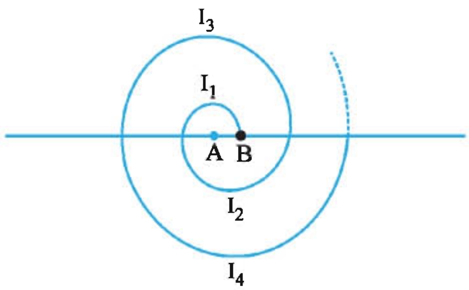

Figure 5.9

EXERCISE 5.2

1. Find the sum of the first *n* terms of the following A.P. as asked for :

(1) 2, 6, 10, 14,... upto 20 terms
(2) 5, 7, 9, 11,... upto 30 terms
(3) −10, −12, −14, −16,... upto 15 terms
(4) 1, 1.5, 2, 2.5, 3,... upto 16 terms
(5) $\frac{1}{3}, \frac{4}{3}, \frac{7}{3}, \frac{10}{3}, ...$ upto 18 terms

2. Find the sums indicated below :

(1) $3 + 6 + 9 + ... + 300$
(2) $5 + 10 + 15 + ... + 100$
(3) $7 + 12 + 17 + 22 + ... + 102$
(4) $(-100) + (-92) + (-84) + ... + 92$
(5) $25 + 21 + 17 + 13 + ... + (-51)$

3. For a given A.P. with

(1) $a = 1$, $d = 2$, find S_{10}.
(2) $a = 2$, $d = 3$, find S_{30}.
(3) $S_3 = 9$, $S_7 = 49$, find S_n and S_{10}.
(4) $T_{10} = 41$, $S_{10} = 320$, find T_n, S_n.
(5) $S_{10} = 50$, $a = 0.5$, find d.
(6) $S_{20} = 100$, $d = -2$, find a.

4. How many terms of A.P., 2, 7, 12, 17,... add upto 990 ?

5. The first term of finite A.P. is 5, the last term is 45 and the sum is 500. Find the number of terms.

6. If the first term and the last term of a finite A.P. are 5 and 95 respectively and $d = 5$, find n and S_n.

7. The sum of first n terms of an A.P. is $5n - 2n^2$. Find the A.P. i.e. a and d.

8. Find the sum of all three digit numbers divisible by 3.

9. Find the sum of all odd numbers from 5 to 205.

10. Which term of A.P. 121, 117, 113,... is its first negative term ? If it is the nth term, find S_n.

*

5.4 Miscellaneous Examples

Example 23 : If pth term of an A.P. is $\frac{1}{q}$ and the qth term is $\frac{1}{p}$, find its pqth term. $(p \neq q)$

Solution : Given $T_p = \frac{1}{q}$ and $T_q = \frac{1}{p}$

$\therefore$ $a + (p - 1)d = \frac{1}{q}$ and $a + (q - 1)d = \frac{1}{p}$

$\therefore$ Subtracting above equations,

$(p - q)d = \frac{1}{q} - \frac{1}{p} = \frac{p-q}{pq}$

$\therefore$ $d = \frac{1}{pq}$ as $p \neq q$

$\therefore$ $a = \frac{1}{q} - (p - 1)d = \frac{1}{q} - \frac{(p-1)}{pq} = \frac{p-p+1}{pq} = \frac{1}{pq}$

$\therefore$ $T_{pq} = a + (pq - 1)d = \frac{1}{pq} + \frac{pq-1}{pq} = \frac{pq}{pq} = 1$

$\therefore$ The pqth term is 1.

Example 24 : If for an A.P. $mT_m = nT_n$, prove $T_{m+n} = 0$ **$(m \neq n)$**

Solution : $m[a + (m - 1)d] = n[a + (n - 1)d]$

$\therefore$ $ma + (m^2 - m)d = na + (n^2 - n)d$

$\therefore$ $(m - n)a + (m^2 - m - n^2 + n)d = 0$

$\therefore$ $(m - n)a + [(m - n)(m + n) - 1(m - n)]d = 0$

$\therefore$ $(m - n)a + (m - n)(m + n - 1)d = 0$

$\therefore$ $a + (m + n - 1)d = 0$ **$(m \neq n)$**

$\therefore$ $T_{m+n} = 0$

Example 25 : If for an A.P. $T_l = p$, $T_m = q$, $T_n = r$, prove that
$p(m - n) + q(n - l) + r(l - m) = 0$ and $(p - q)n + (q - r)l + (r - p)m = 0$.

Solution : $a + (l - 1)d = p$ **(i)**

$a + (m - 1)d = q$ **(ii)**

$a + (n - 1)d = r$ **(iii)**

Multiply equation (i) by $m - n$, (ii) by $n - l$, (iii) by $l - m$ and add.

L.H.S. $= p(m - n) + q(n - l) + r(l - m)$

$= [a + (l - 1)d](m - n) + [a + (m - 1)d](n - l) + [a + (n - 1)d](l - m)$

$= a(m - n) + a(n - l) + a(l - m) + d[(l - 1)(m - n) + (m - 1)](n - l) + (n - 1)(l - m)]$

$= a(m - n + n - l + l - m) + d[(l - 1)(m - n) + (m - 1)(n - l) + (n - 1)(l - m)]$

$= a \cdot 0 + d\ [l(m - n) + m(n - l) + n(l - m) - (m - n + n - l + l - m)]$

$= a \cdot 0 + d \cdot 0 = 0$

Also $p - q = (l - m)d$, $q - r = (m - n)d$, $r - p = (n - l)d$

$\therefore$ $(p - q)n + (q - r)l + (r - p)m$

$= d[n(l - m) + l(m - n) + m(n - l)] = d \cdot 0 = 0$

Example 26 : Find the sum of all 3 digit positive multiples of 7.

Solution : Three digit positive numbers are 100, 101,..., 999.

$100 = 7 \cdot 14 + 2$

$\therefore$ $105 = 7 \cdot 14 + 7 = 7(14 + 1)$ **(Add 5 to remainder 2)**

$\therefore$ 105 is the smallest 3 digit multiple of 7.

$\therefore$ $999 = 7 \cdot 142 + 5$

$\therefore$ $994 = 999 - 5 = 7 \cdot 142$ **(Subtract remainder 5 from 999)**

$\therefore$ 994 is the largest 3 digit multiple of 7.

We want to find the sum :

$105 + 112 + 119 + ... + 994$

$T_n = 994 = 105 + (n - 1)7$

$\therefore$ $994 - 105 = 7(n - 1)$

$\therefore$ $n = \frac{889}{7} + 1 = 127 + 1 = 128$

$\therefore$ $S_n = \frac{1}{2}n(a + l)$

$= \frac{1}{2} \times 128(105 + 994)$

$= 64 \times 1099$

$= 64(1100 - 1)$

$= 70336$

Example 27 : For an A.P., $S_n = m$ and $S_m = n$. Prove that $S_{m+n} = -(m + n)$. **($m \neq n$)**

Solution : $S_m = n$ and $S_n = m$

$\therefore$ $\frac{1}{2}m[2a + (m - 1)d] = n$ and $\frac{1}{2}n[2a + (n - 1)d] = m$

$\therefore$ $2ma + (m^2 - m)d = 2n$ **(i)**

$\therefore$ $2na + (n^2 - n)d = 2m$ **(ii)**

Equations (i) and (ii) give

$2a(m - n) + (m^2 - n^2 - m + n)d = -2(m - n)$

$\therefore$ $2a(m - n) + (m - n)(m + n - 1)d = -2(m - n)$

$\therefore$ $2a + (m + n - 1)d = -2$ **($m \neq n$)**

Now $S_{m+n} = \frac{1}{2}(m + n)[2a + (m + n - 1)d]$

$= \frac{1}{2}(m + n)(-2)$

$= -(m + n)$

$\therefore$ $S_{m+n} = -(m + n)$

Example 28 : If $S_m = S_n$ for an A.P., prove that $S_{m+n} = 0$. **($m \neq n$)**

Solution : $S_m = S_n$

$\therefore$ $\frac{1}{2}m[2a + (m - 1)d] = \frac{1}{2}n[2a + (n - 1)d]$

$\therefore$ $2ma + (m^2 - m)d = 2na + (n^2 - n)d$

$\therefore$ $2a(m - n) + (m^2 - m - n^2 + n)d = 0$

$\therefore$ $2a(m - n) + (m - n)(m + n - 1)d = 0$

$\therefore$ $2a + (m + n - 1)d = 0$ **($m \neq n$)**

Now $S_{m+n} = \frac{1}{2}(m + n)[2a + (m + n - 1)d] = \frac{1}{2} \cdot 0 = 0$

Example 29 : If for an A.P., $S_8 = 16$ and $S_{16} = 8$, find the first negative term.

Solution : $\frac{1}{2}(8)(2a + 7d) = 16$ and $\frac{1}{2}(16)(2a + 15d) = 8$

$\therefore \quad 2a + 7d = 4$

$2a + 15d = 1$

Solving the equations given above, we get $d = -\frac{3}{8}$, $a = \frac{53}{16}$

Now, $T_n = \frac{53}{16} + (n-1)\left(-\frac{3}{8}\right) = \frac{53 - 6n + 6}{16} = \frac{59 - 6n}{16}$

If $T_n < 0$, $59 - 6n < 0$

$\therefore \quad 6n > 59$

$\therefore \quad n > \frac{59}{6} = 9.83$

$\therefore \quad n = 10$ is the least integer for which $T_n < 0$

$T_{10} = \frac{53}{16} + 9\left(\frac{-3}{8}\right) = \frac{-1}{16}$ **(What is T_9 ?)**

Example 30 : 520 logs are selected and arranged as shown in the figure 5.10. The bottom row contains 45 logs. Number of logs go on decreasing by 2 in successive upper rows. How many rows are there ? What is the number of logs in the top row ? Can you arrange 225 logs in this way ?

Solution : Here we have an arithmetic series.

$45 + 43 + 41 + ...$ so that the sum is 520.

$\therefore \quad a = 45, d = -2$

$\therefore \quad S_n = 520 = \frac{1}{2}n[90 + (n-1)(-2)]$

$= \frac{1}{2}n(92 - 2n)$

$= n(46 - n)$

$\therefore \quad n^2 - 46n + 520 = 0$

$\therefore \quad (n - 26)(n - 20) = 0$

$\therefore \quad n = 26$ or $n = 20$

If $n = 26$, the number of logs in the last row.

$a + (n-1)d = 45 - 2(25) = -5$

This is not possible.

$\therefore$ There are 20 rows.

The number of logs in the top row is

$a + (n-1)d = 45 - 2(19) = 7$

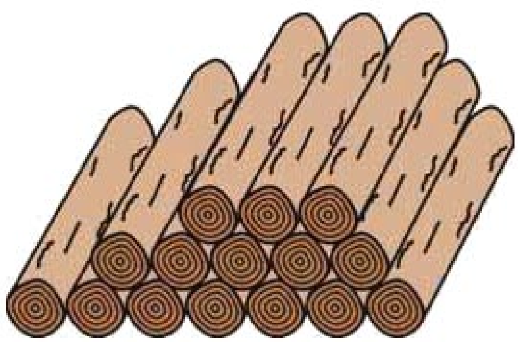

Figure 5.10

Note : The number of logs in 20th row and the rows above 20th row can be written as shown in the table given below.

Row no.	Logs
20	7
21	5
22	3
23	1
24	−1(?)

Hence if $5 + 3 + 1 = 9$ more logs are given, i.e. 529 logs are given, then we can have at most 23 rows.

No, infact if 225 logs are given we could arrange 45 + 43 + 41 + 39 + 37 = 205 logs in five rows and 20 additional logs would have to be placed in the top row. So pattern would not continue.

Example 31 : A series of steps lead to a temple. The number of steps is 20. Each step has a rise of 15 *cm* and a tread 40 *cm*. Each step is 10 *m* long. Calculate the volume of concrete required to build the stair.

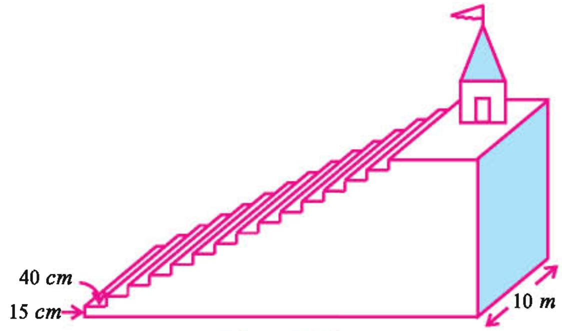

Figure 5.11

Solution : The height of the first step is 15 *cm*.

The height of the second step is is 30 *cm*. It increases by 15 *cm* at every step. i.e. 0.15 *m* at each step.

Length and width of each step are 10 *m* and 0.4 *m* respectively.

∴ The volume of concrete required for each step is

$10 \times 0.4 \times 0.15 + 10 \times 0.4 \times (2 \times 0.15) + 10 \times 0.4 \times (3 \times 0.15) + \ldots$ upto 20 terms

∴ $S_n = 10 \times 0.4 \times 0.15\ (1 + 2 + 3 + \ldots + 20)$

$= 10 \times \frac{4}{10} \times \frac{15}{100} \times \frac{20 \times 21}{2}$ $\left(1 + 2 + \ldots + n = \frac{n(n+1)}{2}\right)$

$= 126\ m^3$

Example 32 : A natural number is made up of four digits in A.P. having the sum 20. The number obtained by reversing the digits is 6174 more than the given number. Find the number.

Solution : Let the digits be $a - 3d,\ a - d,\ a + d,\ a + 3d$

(See that they are in A.P.. Difference = $2d$. For three numbers in A.P., let them be $a - d, a, a + d$. Five numbers in A.P. can be taken as $a - 2d, a - d, a, a + d, a + 2d$. This will simplify the calculations.)

∴ $a - 3d + a - d + a + d + a + 3d = 20$

∴ $4a = 20$

∴ $a = 5$

The given number is $1000(a - 3d) + 100(a - d) + 10(a + d) + a + 3d$

The number obtained by reversing the digits is

$1000(a + 3d) + 100(a + d) + 10(a - d) + a - 3d$

Their difference is $999(a + 3d) + 90(a + d) - 90(a - d) - 999(a - 3d)$

∴ $2997d + 90d + 90d + 2997d = 6174$

∴ $6174d = 6174$

∴ $d = 1$. Also $a = 5$

∴ The digits are 2, 4, 6, 8.

∴ The number is 2468.

Example 33 : There is a guest house on outskirts of a city. Kilometer stones are to be arranged on the road on both sides of the house. A man starts from the guest house G carrying 1 kilometer stone in a loading rickshaw and puts it at 1 *km* distance and comes back. Then he drops 2nd stone at 2 *km* distance and comes back. He complete the journey on both sides and leaves for home from last *km* stone. If the distance travelled is 210 *km*, find the number of stones laid.

Figure 5.12

Solution : The distance travelled on the left is (in *kms*)

$1 \cdot 2 + 2 \cdot 2 + ... + n \cdot 2$ and the distance travelled on the right is

$1 \cdot 2 + 2 \cdot 2 + ... + (n - 1)2 + n$ (he leaves now)

$\therefore$ Total distance covered is $2(1 + 2 + 3 + ... + n) + 2[1 + 2 + ... + (n - 1) + n] - n$

$\therefore$ The distance $= 4(1 + 2 + 3 + ... + n) - n$

$$= \frac{4n(n+1)}{2} - n$$

$$= 2n^2 + 2n - n$$

$$= 2n^2 + n$$

But he has travelled 210 *km*.

$\therefore$ $2n^2 + n = 210$

$\therefore$ $2n^2 + n - 210 = 0$

$(n - 10)(2n + 21) = 0$

$\therefore$ $n = 10$ as $n \neq -\frac{21}{2}$

$\therefore$ On both sides of the guest house kilometer stones are laid upto 10 *km*.

Example 34 : There are 8 bungalows in a society numbered from 1 to 8. Prove that there is a positive integer n such that sum of the house numbers preceding this house and the houses following is the same. Find n.

Solution :

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8
1	2	3	4	5	6	7	8

Let the houses be numbered 1 to 8.

Let the nth house have the required property.

$\therefore$ $1 + 2 + 3 + ... + (n - 1) = (n + 1) + (n + 2) + ... + 8$

$$= (1 + 2 + ... + 8) - (1 + 2 + 3 + ... + n)$$

$\therefore$ $\frac{(n-1)n}{2} = \frac{8 \cdot 9}{2} - \frac{n(n+1)}{2}$

$\therefore$ $\frac{n}{2}(n - 1 + n + 1) = 36$

$\therefore$ $n^2 = 36$

$\therefore$ $n = 6$

Figure 5.13

See $\underbrace{1 \quad 2 \quad 3 \quad 4 \quad 5}_{\text{sum} = 15} \quad 6 \quad \underbrace{7 \quad 8}_{\text{sum} = 15}$

$\therefore$ All houses preceding the sixth house and following it will have the same total of house numbers.

EXERCISE 5

1. If $T_n = 6n + 5$, find S_n.
2. If $S_n = n^2 + 2n$, find T_n.
3. If the sum of the first n terms of A.P.
   30, 27, 24, 21,.... is 120, find the number of terms and the last term.
4. Which term of A.P., 100, 97, 94, 91,... will be its first $-ve$ term ?
5. Find the sum of all 3 digit natural multiples of 6.
6. The ratio of the sum to m terms to sum to n terms of an A.P. is $\frac{m^2}{n^2}$. Find the ratio of its mth term to its nth term.
7. Sum to first l, m, n terms of A.P. are p, q, r. Prove that $\frac{p}{l}(m-n) + \frac{q}{m}(n-l) + \frac{r}{n}(l-m) = 0$
8. The ratio of sum to n terms of two A.P.'s is $\frac{8n+1}{7n+3}$ for every $n \in$ N. Find the ratio of their 7th terms and nth terms.
9. Three numbers in A.P. have the sum 18 and the sum of their squares is 180. Find the numbers in the increasing order.
10. In a potato race a bucket is placed at the starting point. It is 5 m away from the first potato. The rest of the potatoes are placed in a straight line each 3 m away from the other. Each competitor starts from the bucket. Picks up the nearest potato and runs back and drops it in the bucket and continues till all potatoes are placed in the bucket. What is the total distance covered if 15 potatoes are placed in the race ?

Figure 5.14

If the distance covered is 1340 m, find the number of potatoes ?

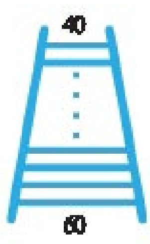

Figure 5.15

11. A ladder has rungs 25 cm apart. The rungs decrease uniformly from 60 cm at bottom to 40 cm at top. If the distance between the top rung and the bottom rung is 2.5 m, find the length of the wood required.
12. A man purchased LCD TV for ₹ 32,500. He paid ₹ 200 initially and increasing the payment by ₹ 150 every month. How many months did he take to make the complete payment ?
13. In an A.P., $T_1 = 22$, $T_n = -11$, $S_n = 66$, find n.
14. In an A.P. $a = 8$, $T_n = 33$, $S_n = 123$, find d and n.

15. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on right so that the statement becomes correct : (All the problems refer to A.P.)

(1) If $T_3 = 8$, $T_7 = 24$, then $T_{10} =$ ......

(a) -4 (b) 28 (c) 32 (d) 36

(2) If $S_n = 2n^2 + 3n$, then $d =$ ......

(a) 13 (b) 4 (c) 9 (d) -2

(3) If the sum of the three consecutive terms of A.P. is 48 and the product of the first and the last is 252, then $d =$ ......

(a) 2 (b) 3 (c) 4 (d) 16

(4) If $a = 2$ and $d = 4$, then $S_{20} =$ ......

(a) 600 (b) 800 (c) 78 (d) 80

(5) If $3 + 5 + 7 + 9 + \ldots$ upto n terms $= 288$, then $n =$ ......

(a) 12 (b) 15 (c) 16 (d) 17

(6) Four numbers are in A.P. and their sum is 72 and the largest of them is twice the smallest. Then the numbers are ......

(a) 4, 8, 12, 16 (b) 12, 16, 20, 24 (c) 10, 12, 14, 16 (d) 2, 4, 6, 8

(7) If $S_1 = 2 + 4 + \ldots + 2n$ and $S_2 = 1 + 3 + \ldots + (2n - 1)$, then $S_1 : S_2 =$ ......

(a) $\frac{n+1}{n}$ (b) $\frac{n}{n+1}$ (c) n^2 (d) $(n + 1)$

(8) For A.P., $S_n - 2S_{n-1} + S_{n-2} =$ ...... $(n > 2)$

(a) $2d$ (b) d (c) a (d) $a + d$

(9) If $S_m = n$ and $S_n = m$ then $S_{m+n} =$ ......

(a) $-(m + n)$ (b) 0 (c) $m + n$ (d) $2m - 2n$

(10) If $T_4 = 7$ and $T_7 = 4$, then $T_{10} =$ ......

(a) 9 (b) 11 (c) -11 (d) 1

(11) If $2k + 1$, 13, $5k - 3$ are three consecutive terms of A.P., then $k =$ ......

(a) 17 (b) 13 (c) 4 (d) 9

(12) $(1) + (1 + 1) + (1 + 1 + 1) + \ldots + (1 + 1 + 1 + \ldots n - 1$ times$) =$ ......

(a) $\frac{(n-1)n}{2}$ (b) $\frac{n(n+1)}{2}$ (c) n (d) n^2

(13) In the A.P., 5, 7, 9, 11, 13, 15,... the sixth term which is prime is ......

(a) 13 (b) 19 (c) 23 (d) 15

(14) For A.P. $T_{18} - T_8 =$ ......

(a) d (b) $10d$ (c) $26d$ (d) $2d$

(15) If for A.P., $T_{25} - T_{20} = 15$ then $d =$ ......

(a) 3 (b) 5 (c) 20 (d) 25

*

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Summary

In this chapter we have studied following points :

1. Introduction to a sequence.
2. Arithmetic progression.
3. nth term of an arithmetic progression $T_n = a + (n - 1)d$
4. $d = \frac{T_m - T_n}{m - n}$
5. Sum of the first n terms of A.P., $S_n = \frac{1}{2}n\,[2a + (n - 1)d] = \frac{1}{2}n\,(a + l)$

   Sum of the first n natural numbers is $\frac{n(n+1)}{2}$
6. $T_n = S_n - S_{n-1}$ $(n > 1)$ $T_1 = a = S_1$ and for $n \geq 2$ $T_n = S_n - S_{n-1}$

◆

Varahamihira (Devanagari : वराहमिहिर) (505–587), also called Varaha or Mihira, was an Indian astronomer, mathematician, and astrologer who lived in Ujjain. He is considered to be one of the nine jewels (Navaratnas) of the court of legendary ruler Vikramaditya (thought to be the Gupta emperor Chandragupta II Vikramaditya).

Pancha-Siddhantika : Varahamihira's main work is the book Pañcasiddhantika (or Pancha-Siddhantika, "[Treatise] on the Five [Astronomical] Canons) dated ca. 575 CE gives us information about older Indian texts which are now lost. The work is a treatise on mathematical astronomy and it summarises five earlier astronomical treatises, namely the Surya Siddhanta, Romaka Siddhanta, Paulisa Siddhanta, Vasishtha Siddhanta and Paitamaha Siddhantas. It is a compendium of Vedanga Jyotisha as well as Hellenistic astronomy (including Greek, Egyptian and Roman elements). He was the first one to mention in his work Pancha Siddhantika that the ayanamsa, or the shifting of the equinox is 50.32 seconds.

The 11th century Arabian scholar Alberuni also described the details of "The Five Astronomical Canons":

"They [the Indians] have 5 Siddhantas :

Surya-Siddhanta, ie. the Siddhanta of the Sun, composed by Latadeva,

Vasishtha-siddhanta, so called from one of the stars of the Great Bear, composed by Vishnucandra,

Pulisa-siddhanta, so called from Paulisa, the Greek, from the city of Saintra, which is supposed to be Alexandria, composed by Pulisa.

Romaka-siddhanta, so called from the Rum, ie. the subjects of the Roman Empire, composed by Srishena.

Brahma-siddhanta, so called from Brahman, composed by Brahmagupta, the son of Jishnu, from the town of Bhillamala between Multan and Anhilwara, 16 yojanas from the latter place."

Similarity of Triangles 6

Measure what is measurable and make measurable what is not so.

– Galilei Galileo

*

Each problem I solved became a rule which served afterwards to solve other problems.

– Rene Des Cartes

6.1 Introduction

Similarity of geometric figures is an important concept in Euclidean geometry. In a way similarity is a geometric transformation of one figure into the other figure such that the measures of all linear elements (like line segments, circumference, perimeter etc.) of one figure are in proportion to the corresponding linear elements of the other figure and all angular elements (angles formed by rays) in one figure are congruent to the corresponding elements of the other figure.

6.2 Similar Figures

A passport size photograph and its enlargement to a postcard size or cabinet size photograph or its reduction to a stamp size photograph are the examples of similar figures. A map of Gujarat state on the wall of your class and the map printed on a page of a textbook are also similar figures.

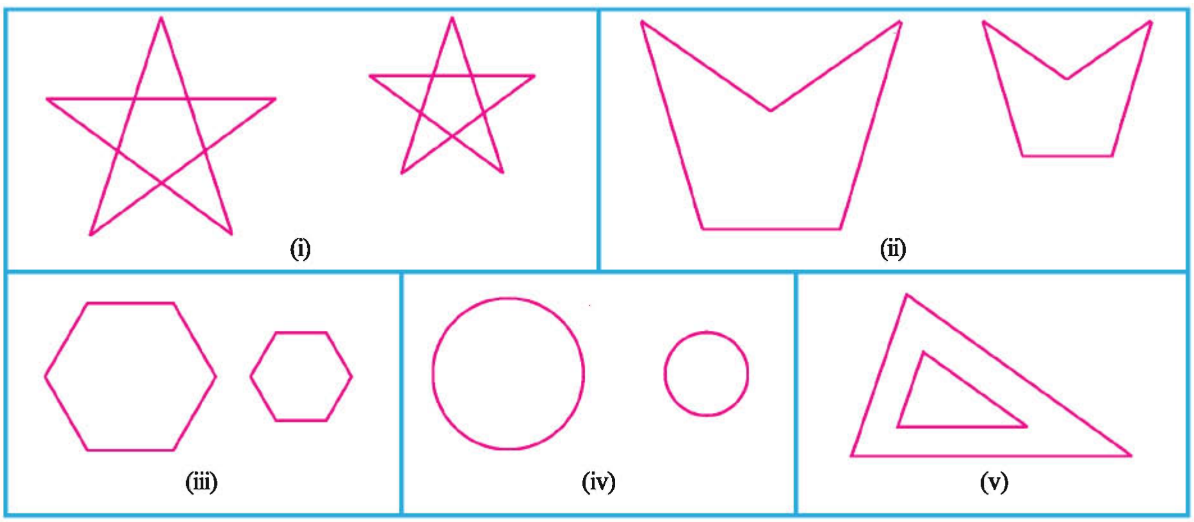

Figure 6.1

We have studied the congruence of triangles in std. IX. In congruence the shape and size of one figure is the same as those of the other figure. In similarity, the shape of the figures do not change but the size of figures may change proportionately. Two circles, two equilateral triangles, two squares, two isoceles right angled triangles have the same shape but their sizes' may differ. Figure 6.1 and 6.2 show some such similar figure.

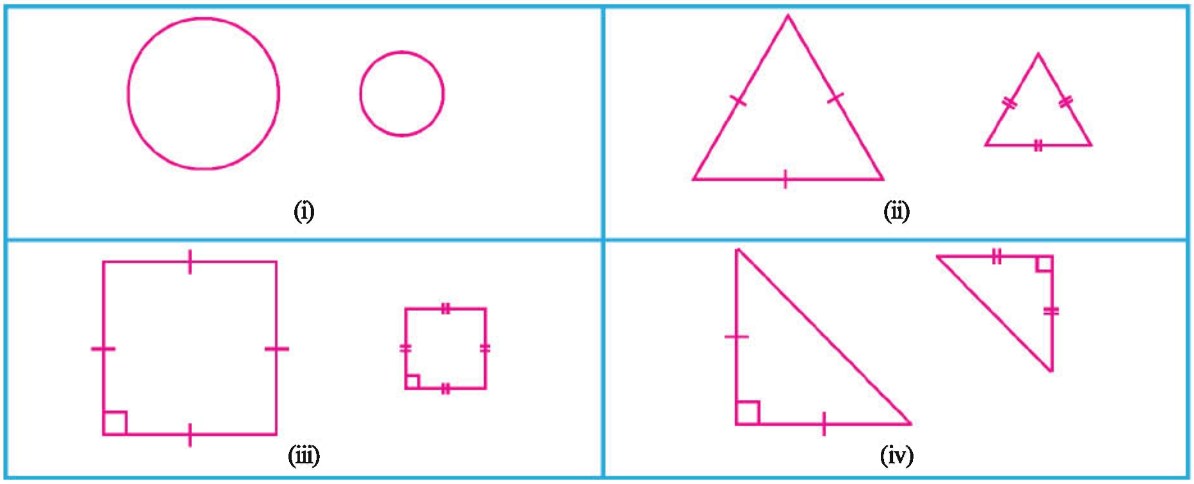

Figure 6.2

Similarity can also be viewed as a projection of one figure on the other figure. In figure 6.3 light rays are coming from a torch and incident on the transparent plane glass. A triangle is drawn, using black ink, on the glass. A white screen is placed below the glass. The plane of the glass and that of the screen are parallel.

We will be able to see the image of the triangle on the screen and it is the projection of the triangle drawn on the glass. If we are able to trace the triangle on the screen using a pencil, we can find the lengths of the sides and measure the angles of both the triangles. We will observe that,

(1) All the corresponding angles in both the figures are congruent.

(2) All the corresponding sides are having the same proportion or ratio.

These are the characteristics of the geometric transformation known as similarity of figures.

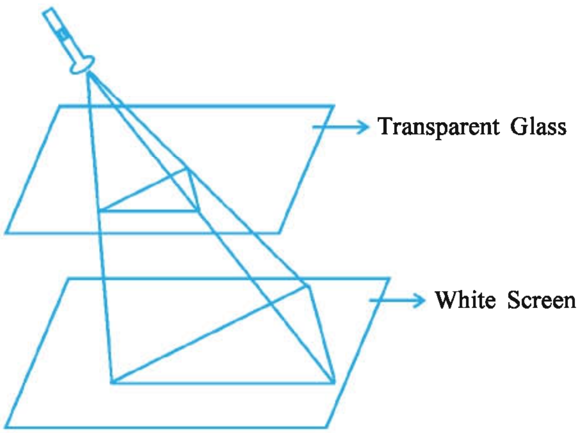

Figure 6.3

> What will happen if the plane of the glass and the plane of screen are not parallel ?
>
> The corresponding angles will not remain congruent and triangle and its projection will not be similar figures.

Now we will concentrate only on the similarity of triangles.

6.3 Similarity of Triangles

In standard IX we have studied that there are six different correspondences between the vertices of ΔABC and ΔPQR as given below :

$ABC \leftrightarrow PQR$, $ABC \leftrightarrow PRQ$

$ABC \leftrightarrow QPR$, $ABC \leftrightarrow QRP$

$ABC \leftrightarrow RPQ$ $ABC \leftrightarrow RQP$

For correspondence $ABC \leftrightarrow PQR$, the pairs of corresponding angles are $(\angle A, \angle P)$, $(\angle B, \angle Q)$, $(\angle C, \angle R)$ and the pairs of corresponding sides are $(\overline{AB}, \overline{PQ})$, $(\overline{BC}, \overline{QR})$ and $(\overline{AC}, \overline{PR})$.

If $\angle A \cong \angle P$, $\angle B \cong \angle Q$, $\angle C \cong \angle R$ and $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$, then the correspondence ABC $\leftrightarrow$ PQR is called a similarity. ΔABC and ΔPQR are called similar triangles, **if any one of the six correspondences between their vertices is a similarity.**

The statement "correspondence ABC $\leftrightarrow$ PQR is a similarity" is symbolically written as ABC $\sim$ PQR.

Definition : For a given correspondence between the vertices of two triangles, if the corresponding angles of the triangles are congruent and the lengths of the corresponding sides are in proportion, then the given correspondence is a similarity between two triangles.

If a correspondence between the vertices of two triangle is a similarity, the triangles are said to be similar and "ΔABC is similar to ΔPQR" is written as $\Delta ABC \sim \Delta PQR$.

When we say that ΔABC is similar to ΔPQR, it is the same thing as to say that ΔABC is similar to ΔQPR or ΔPRQ or any of the six different ways in which ΔPQR can be written. Which correspondence is a similarity is important. That means if the correspondence ABC $\leftrightarrow$ QRP is a similarity, we can say that ΔABC is similar to ΔPQR. So when we write $\Delta ABC \sim \Delta PQR$ we should mention the correspondence for which the triangles are similar.

Let us explain this by taking an example.

Suppose in ΔABC, $m\angle A = 30$, $m\angle B = 90$, $m\angle C = 60$, AC = 4, AB = $2\sqrt{3}$ and BC = 2. In ΔPQR $m\angle P = 90$, $m\angle Q = 30$, $m\angle R = 60$, PR = 3, QR = 6 and PQ = $3\sqrt{3}$.

From the given data, it is clear that $m\angle A = m\angle Q = 30$, $m\angle B = m\angle P = 90$, $m\angle C = m\angle R = 60$.

Also, $\frac{AB}{PQ} = \frac{2\sqrt{3}}{3\sqrt{3}} = \frac{2}{3}$, $\frac{BC}{PR} = \frac{2}{3}$, $\frac{AC}{QR} = \frac{4}{6} = \frac{2}{3}$

$\therefore$ $\angle A \cong \angle Q$, $\angle B \cong \angle P$ and $\angle C \cong \angle R$ and $\frac{AB}{PQ} = \frac{BC}{PR} = \frac{AC}{QR} = \frac{2}{3}$

So, correspondence ABC $\leftrightarrow$ QPR is a similarity and for that correspondence $\Delta ABC \sim \Delta PQR$.

The following properties of similarity are obvious :

(1) $\Delta ABC \sim \Delta ABC$ **(Reflexivity)**

(2) If $\Delta ABC \sim \Delta PQR$, then $\Delta PQR \sim \Delta ABC$ **(Symmetry)**

(3) If $\Delta ABC \sim \Delta PQR$ and $\Delta PQR \sim \Delta XYZ$, then $\Delta ABC \sim \Delta XYZ$. **(Transitivity)**

(4) Two congruent triangles are always similar.

Explanation : Let the correspondence ABC $\leftrightarrow$ PQR between the vertices of ΔABC and ΔPQR be a congruence.

$\therefore$ $\angle A \cong \angle P$, $\angle B \cong \angle Q$ and $\angle C \cong \angle R$ and $\overline{AB} \cong \overline{PQ}$, $\overline{BC} \cong \overline{QR}$ and $\overline{AC} \cong \overline{PR}$

$\therefore$ AB = PQ, BC = QR, AC = PR

$\therefore$ $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR} = 1$

Hence, whenever two triangles are congruent, the corresponding angles are congruent and the corresponding sides are in proportion, ratio of proportionality is 1.

$\therefore$ A congruence between two triangles is always a similarity.

So two congruent triangles are similar.

Can we say that converse of this statement is also true ? No, it's not true. If it is true, then all similar triangles are congruent. But it is not true. We will come across many examples in which triangles are similar but not congruent.

A student of geometry stated the definition of similarity in following words.

New definition of similarity of triangles :

For a given correspondence between the vertices of two triangles, if the measures of corresponding angles and the lengths of corresponding sides are in proportion the given correspondence is a similarity. Is this definition equivalent to the definition given by us ?

Of course, both the definitions are equivalent.

Because if for ABC $\leftrightarrow$ PQR,

$$\frac{m\angle A}{m\angle P} = \frac{m\angle B}{m\angle Q} = \frac{m\angle C}{m\angle R} = \frac{m\angle A + m\angle B + m\angle C}{m\angle P + m\angle Q + m\angle R} = \frac{180}{180} = 1$$

$\therefore$ $m\angle A = m\angle P$, $m\angle B = m\angle Q$, $m\angle C = m\angle R$

Therefore $\angle A \cong \angle P$, $\angle B \cong \angle Q$, $\angle C \cong \angle R$

We can also prove that, if the corresponding angles are congruent, then the measures of corresponding angles are in proportion, the ratio of proportionality is 1.

Some results on ratio proportion :

Here a, b, c, d, x, y etc are real non-zero numbers.

(1) If $\frac{a}{b} = \frac{c}{d}$ then $\frac{b}{a} = \frac{d}{c}$ **(invertendo)**

(2) If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a}{c} = \frac{b}{d}$ **(alternendo)**

(3) (i) If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a}{b} + 1 = \frac{c}{d} + 1$. So, $\frac{a+b}{b} = \frac{c+d}{d}$ **(componendo)**

(ii) Also $\frac{a}{b} - 1 = \frac{c}{d} - 1$. So, $\frac{a-b}{b} = \frac{c-d}{d}$ **(dividendo)**

(4) If $\frac{a}{b} = \frac{c}{d}$, then each ratio $= \frac{ax+cy}{bx+dy}$ **($bx + dy \neq 0$)**

In particular $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{a+c+e}{b+d+f}$ **($b + d + f \neq 0$)**

We are going to use these results frequently.

Example 1 : In ΔABC and ΔPQR $m\angle A = m\angle Q = 90$, AB = AC = 5, BC = $5\sqrt{2}$, PQ = QR = 2, PR = $2\sqrt{2}$. Are ΔABC and ΔPQR similar ? If yes which correspondence between them is similarity ?

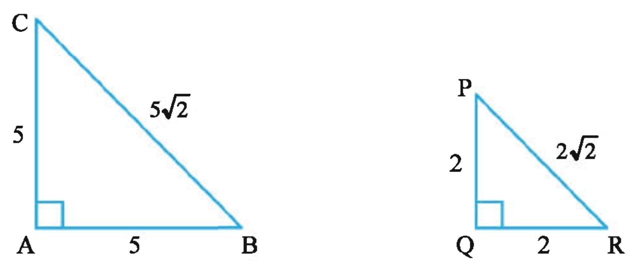

Figure 6.4

Solution : In ΔABC, AB = AC = 5

$\therefore \quad m\angle C = m\angle B.$

$\quad m\angle A = 90$

$\therefore \quad m\angle C + m\angle B = 90$

$\therefore \quad m\angle B = m\angle C = 45, \ BC = 5\sqrt{2}$

In ΔPQR, PQ = QR = 2.

$\therefore \quad m\angle R = m\angle P$

$\quad m\angle Q = 90$

$\therefore \quad m\angle P + m\angle R = 90$

$\therefore \quad m\angle P = m\angle R = 45, \ PR = 2\sqrt{2}$

Consider the correspondence ABC $\leftrightarrow$ QRP

$m\angle A = m\angle Q$ So $\angle A \cong \angle Q$

$m\angle B = m\angle R$ So $\angle B \cong \angle R$

$m\angle C = m\angle P$ So $\angle C \cong \angle P$

$$\frac{BC}{PR} = \frac{5\sqrt{2}}{2\sqrt{2}} = \frac{5}{2}, \quad \frac{CA}{PQ} = \frac{5}{2} = \frac{AB}{QR}$$

$\therefore$ For the correspondence ABC $\leftrightarrow$ QRP, corresponding angles are congruent and the lengths of the corresponding sides are in proportion.

So the correspondence is a similarity and the triangles are similar.

Note : Correspondence ABC $\leftrightarrow$ QPR is also similarity. Why is it so ?

Example 2 : Check whether the triangles given in the figure are similar or not.

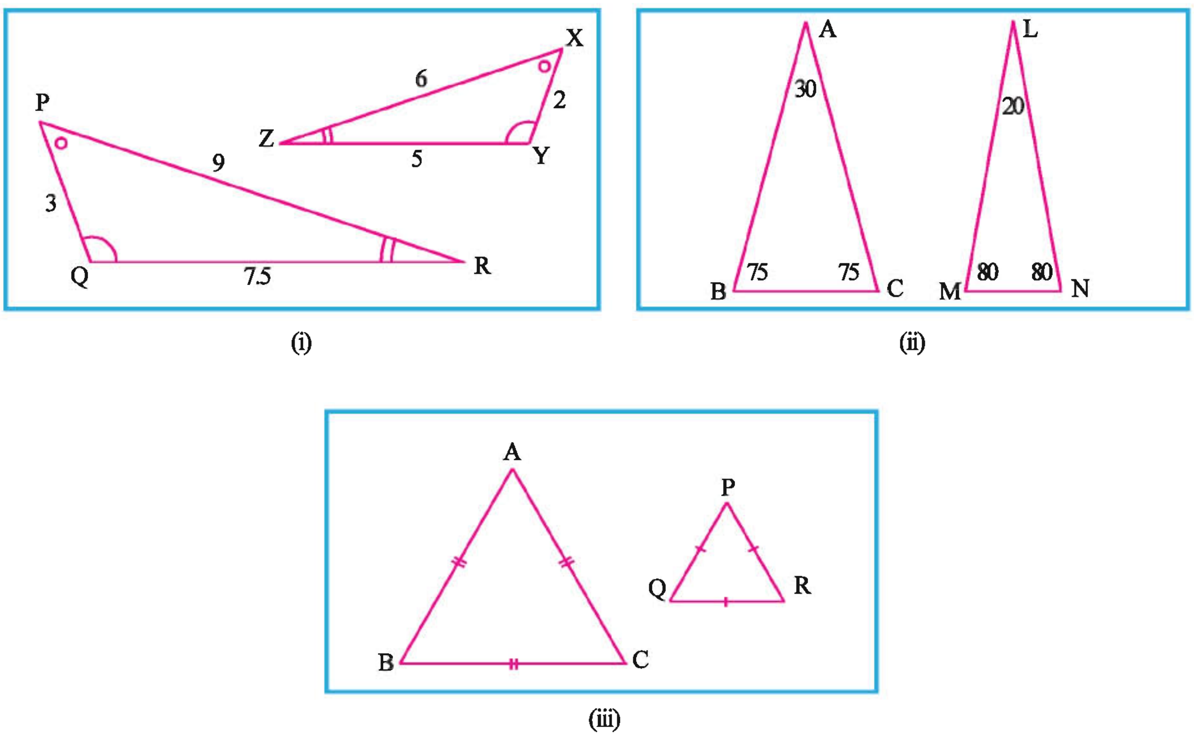

Figure 6.5

Solution : In figure 6.5(i), it is given that $\angle P \cong \angle X$, $\angle Q \cong \angle Y$, $\angle R \cong \angle Z$

Also $\frac{QR}{YZ} = \frac{7 \cdot 5}{5} = \frac{3}{2}$, $\frac{PR}{ZX} = \frac{9}{6} = \frac{3}{2}$, $\frac{PQ}{XY} = \frac{3}{2}$

$\therefore \quad \frac{QR}{YZ} = \frac{PR}{ZX} = \frac{PQ}{XY}$

Therefore, for correspondence PQR $\leftrightarrow$ XYZ, the corresponding angles are congruent and the length of the corresponding sides are in proportion.

$\therefore$ The correspondence PQR $\leftrightarrow$ XYZ is a similarity. So ΔPQR ~ ΔXYZ

In figure 6.5(ii) $\angle$A of ΔABC is not congruent to any of the angles of ΔLMN. Hence any correspondence between the given triangles cannot be a similarity.

In figure 6.5(iii), ΔABC and ΔPQR are equilateral triangles.

Hence, the measure of each of the angles of ΔABC and ΔPQR is 60.

$\therefore \quad \angle A \cong \angle P, \angle B \cong \angle Q, \angle C \cong \angle R$

Let AB = BC = AC = p and PQ = QR = PR = q

$\therefore \quad p \neq 0, q \neq 0$ and $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR} = \frac{p}{q}$, **($p$ and q are non-zero real numbers)**

Therefore for correspondence ABC $\leftrightarrow$ PQR, corresponding angles are congruent and corresponding sides are proportional.

Hence, the triangles are similar.

Note that not only ABC ~ PQR but any of the six correspondences is a similarity.

Example 3 : The correspondence XYZ $\leftrightarrow$ EFD between the vertices of ΔXYZ and ΔDEF is a similarity.

If $m\angle X : m\angle Y : m\angle Z = 2 : 3 : 5$, show that ΔDEF is a right angled triangle.

Solution : $m\angle X : m\angle Y : m\angle Z = 2 : 3 : 5$

Let $m\angle X = 2k$, $m\angle Y = 3k$, $m\angle Z = 5k$ **($k > 0$)**

In ΔXYZ, $m\angle X + m\angle Y + m\angle Z = 180$

$\therefore \quad 2k + 3k + 5k = 180$

$\therefore \quad 10k = 180$

$\therefore \quad k = 18$

$\therefore \quad m\angle Z = 5k = 5 \times 18 = 90$

Now, the correspondence XYZ $\leftrightarrow$ EFD is a similarity.

$\therefore \quad \angle Z \cong \angle D$

$\therefore \quad m\angle Z = m\angle D$

$\therefore \quad m\angle D = 90$ as $m\angle Z = 90$

Hence ΔDEF is a right angled triangle.

Example 4 : In ΔABC, the correspondence ABC $\leftrightarrow$ BAC is a similarity. Which kind of triangle is ΔABC ?

Solution : In ΔABC, ABC $\leftrightarrow$ BAC is a similarity.

$\therefore \quad \angle A \cong \angle B$

$\therefore \quad \overline{BC} \cong \overline{AC}$

ΔABC is an isosceles triangle.

Example 5 : In ΔABC, the correspondences $ABC \leftrightarrow BAC$ and $ABC \leftrightarrow ACB$ are similarities. Prove that ΔABC is an equilateral triangle.

Solution : In ΔABC, the correspondence $ABC \leftrightarrow BAC$ is a similarity.

$\therefore$ $\angle A \cong \angle B$. So $m\angle A = m\angle B$

The correspondence $ABC \leftrightarrow ACB$ is also a similarity.

$\therefore$ $\angle B \cong \angle C$. So $m\angle B = m\angle C$

Therefore $m\angle A = m\angle B = m\angle C$

$\therefore$ $\overline{BC} \cong \overline{AC}$ and $\overline{AC} \cong \overline{AB}$

$\therefore$ ΔABC is an equilateral triangle.

Example 6 : Prove that two equilateral triangles are always similar.

Solution : Let ΔABC and ΔPQR be equilateral triangles.

$\therefore$ $m\angle A = m\angle B = m\angle C = 60$ and $AB = BC = AC$

Also $m\angle P = m\angle Q = m\angle R = 60$ and $PQ = QR = PR$

$\therefore$ $\angle A \cong \angle P$, $\angle B \cong \angle Q$, $\angle C \cong \angle R$ and

$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$

$\therefore$ Correspondence $ABC \leftrightarrow PQR$ is similarity.

Hence $\Delta ABC \sim \Delta PQR$

Example 7 : $ABC \sim DEF$ is a similarity in ΔABC and ΔDEF. If $3AB = 5DE$ and $DF = 9$, find AC.

Solution : In ΔABC and ΔDEF, $ABC \sim DEF$ is a similarity.

$\therefore$ $\frac{AB}{DE} = \frac{AC}{DF}$

It is given that $3AB = 5DE$

$\therefore$ $\frac{AB}{DE} = \frac{5}{3}$

$\therefore$ $\frac{AC}{DF} = \frac{5}{3}$. But $DF = 9$

$\therefore$ $\frac{AC}{9} = \frac{5}{3}$

$\therefore$ $AC = \frac{5}{3} \times 9$

$\therefore$ $AC = 15$

Example 8 : For ΔXYZ and ΔPQR, the correspondence $XYZ \leftrightarrow QPR$ is a similarity. If $m\angle X + m\angle P = 130$ and $ZX = ZY$ find the measures of angles of ΔPQR.

Solution : For ΔXYZ and ΔPQR, the correspondence $XYZ \leftrightarrow QPR$ is a similarity.

$\therefore$ $\angle Y \cong \angle P$

$\therefore$ $m\angle Y = m\angle P$ **(i)**

$m\angle X + m\angle P = 130$

$m\angle X + m\angle Y = 130$ **(using (i))** but $m\angle X + m\angle Y + m\angle Z = 180$

$\therefore$ $m\angle Z = 180 - 130 = 50$

Also ZX = ZY

$\therefore$ $m\angle Y = m\angle X$

and $m\angle X + m\angle Y = 130$

$\therefore$ $m\angle X = m\angle Y = \frac{130}{2} = 65$

Since the correspondence XYZ $\leftrightarrow$ QPR is a similarity.

$\angle Q \cong \angle X, \angle P \cong \angle Y, \angle R \cong \angle Z$

$\therefore$ $m\angle Q = m\angle X = 65$

$m\angle P = m\angle Y = 65$

$m\angle R = m\angle Z = 50$

EXERCISE 6.1

1. According to the definition of similarity of triangles, which are the conditions for correspondence DEF $\leftrightarrow$ ZXY between ΔDEF and ΔXYZ to be a similarity ?
2. For ΔPQR and ΔXYZ, the correspondence PQR $\leftrightarrow$ YZX is a similarity. $m\angle P = 2m\angle Q$ and $m\angle X = 120$. Find $m\angle Y$.
3. The correspondence ABC $\leftrightarrow$ PQR between ΔABC and ΔPQR is a similarity. AB : PQ = 4 : 5. If AC = 6, then find PR. If QR = 15, then find BC.
4. ΔPQR ~ ΔDEF for the correspondence PQR $\leftrightarrow$ EDF. If PQ + QR = 15, DE + DF = 10 and PR = 6, find EF.
5. In ΔABC and ΔPQR, ABC $\leftrightarrow$ QPR is a similarity. The perimeter of ΔABC is 15 and perimeter of ΔPQR is 27. If BC = 7 and QR = 9, find PR and AC.
6. In ΔXYZ ~ ΔDEF consider the correspondence XYZ $\leftrightarrow$ EDF.
   If $\frac{\text{perimeter of } \Delta XYZ}{\text{perimeter of } \Delta DEF} = \frac{3}{4}$, find $\frac{XY}{ED}$ and $\frac{XZ + YZ}{EF + DF}$.
7. Using the definition of similarity prove that all the isosceles right angled triangles are similar.
8. For ΔABC and ΔXYZ, ABC $\leftrightarrow$ XYZ is a similarity. If $\frac{AB}{4} = \frac{BC}{6} = \frac{AC}{3}$, AC = 3 and XY = 5, find YZ and XZ.
9. **State whether the following statements are true or false. Give reasons for your answer :**
   (1) If ΔPQR and ΔABC are similar and none of them is equilateral, then all the six correspondences between ΔPQR and ΔABC are similarities.
   (2) All congruent triangles are similar.
   (3) All similar triangles are congruent.
   (4) If the correspondences ABC $\leftrightarrow$ BAC is similarity, then ΔABC is an isosceles triangle.
   (5) The correspondence PQR $\leftrightarrow$ YZX between ΔPQR and ΔYZX is a similarity. If $m\angle P = 60$, $m\angle R = 40$, then $m\angle Z = 80$.

10. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) $\Delta ABC \sim \Delta PQR$ for the correspondence $ABC \leftrightarrow QRP$. If $m\angle A = 50$, $m\angle C = 30$, then $m\angle R = ...$ ☐

(a) 80 (b) 50 (c) 30 (d) 100

(2) $\Delta LMN \sim \Delta XYZ$ for the correspondence $LMN \leftrightarrow ZYX$. If $m\angle Z = 50$, $m\angle X = 40$, then $m\angle L + m\angle N = ...$ ☐

(a) 10 (b) 90 (c) 110 (d) 80

(3) If the correspondence $ABC \leftrightarrow EFD$ is a similarity in ΔABC and ΔDEF, then ...... of the following is not true. ☐

(a) $\frac{BC}{DF} = \frac{AC}{DE}$ (b) $\frac{AB}{DE} = \frac{BC}{DF}$ (c) $\frac{AB}{EF} = \frac{AC}{DE}$ (d) $\frac{BC}{DF} = \frac{AB}{EF}$

(4) $ABC \sim PQR$ is a similarity in ΔABC and ΔPQR. If the perimeter of ΔABC is 12 and perimeter of a ΔPQR is 20, then $AB : PQ = ...$ ☐

(a) 3 : 5 (b) 5 : 3 (c) 4 : 3 (d) 3 : 4

*

6.4 Some Geometric Results on Proportionality : Fundamental Theorem of Proportionality

In figure 6.6, a line l parallel to side $\overline{BC}$ of ΔABC intersects the sides $\overline{AB}$ and $\overline{AC}$ in points M and N respectively.

$\angle BAC = \angle MAN$ $(\overrightarrow{AM} = \overrightarrow{AB}$ and $\overrightarrow{AN} = \overrightarrow{AC})$

$\overline{MN} \parallel \overline{BC}$ and $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ are their transversals.

$\therefore$ $\angle ABC \cong \angle AMN$ and $\angle ACB \cong \angle ANM$

What would be the ratios $\frac{AM}{MB}, \frac{AN}{NC}, \frac{AM}{AB}, \frac{AN}{AC}$?

To answer the questions, let us perform an activity.

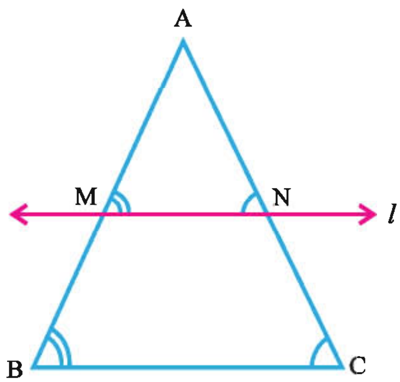

Figure 6.6

Activity : Construct ΔABC such that $AB = 8$, $AC = 12$ and $m\angle A = 70$.

(This is not a construction with straight-edge and compass only. You can use protractor to construct an angle of measure 70.)

Select M on $\overline{AB}$ such that $AM = 2$. Draw a line through M parallel to $\overline{BC}$ and let it intersect $\overline{AC}$ in N.

Now find out the ratios $\frac{AM}{AB}, \frac{AN}{AC}, \frac{MN}{BC}$

What do you notice ? You will observe that these ratios are equal.

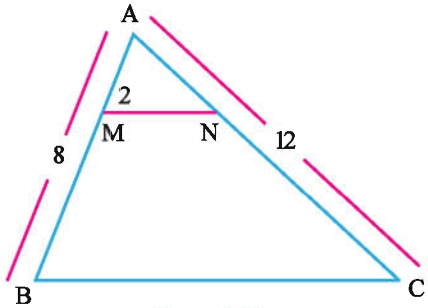

Figure 6.7

If you measure MB and NC and find out the ratios $\frac{AM}{MB}$, $\frac{AN}{NC}$ you will observe that $\frac{AM}{MB} = \frac{AN}{NC}$.

This is the fundamental theorem of proportionality. This theorem was proved by the great geometrician **Thales**. Thales was a Greek mathematician. He is regarded as the father of Geometry.

Let us prove this result as Theorem 6.1.

Theorem 6.1 : Fundamental Theorem of Proportionality

If a line parallel to one of the sides of a triangle intersects the other two sides in distinct points, then the segments of the other two sides in one halfplane are proportional to the segments in the other halfplane.

Given : In the plane of ΔABC, a line $l \parallel \overline{BC}$ and l intersects $\overline{AB}$ and $\overline{AC}$ at points P and Q respectively.

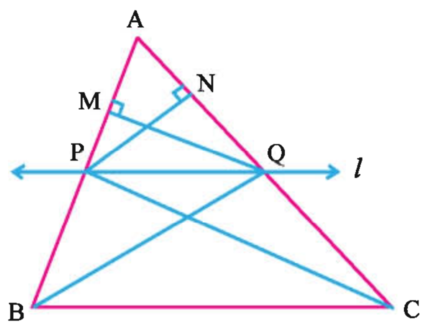

Figure 6.8(a)

To prove : $\frac{AP}{PB} = \frac{AQ}{QC}$

Proof : Let $\overline{QM} \perp \overline{AB}$, and $\overline{PN} \perp \overline{AC}$. Construct $\overline{BQ}$ and $\overline{CP}$.

Area of a triangle $= \frac{1}{2} \times$ base $\times$ altitude

$\therefore$ Area of $\Delta APQ = \frac{1}{2}AP \times QM$

Area of $\Delta PBQ = \frac{1}{2}PB \times QM$

$$\therefore \frac{\text{Area of } \Delta APQ}{\text{Area of } \Delta PBQ} = \frac{\frac{1}{2}AP \times QM}{\frac{1}{2}PB \times QM} = \frac{AP}{PB} \quad \text{(i)}$$

Also Area of $\Delta APQ = \frac{1}{2}AQ \times PN$

Area of $\Delta CPQ = \frac{1}{2}QC \times PN$

$$\therefore \frac{\text{Area of } \Delta APQ}{\text{Area of } \Delta APCQ} = \frac{\frac{1}{2}AQ \times PN}{\frac{1}{2}QC \times PN} = \frac{AQ}{QC} \quad \text{(ii)}$$

ΔPBQ and ΔPCQ are having common base $\overline{PQ}$ and they are lying between two parallel lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{BC}$.

Area of ΔPBQ = Area of ΔPCQ **(iii)**

From (i), (ii) and (iii) $\frac{AP}{PB} = \frac{AQ}{QC}$.

[**Note** :In the figure 6.8, M $\in \overline{AP}$ and N $\in \overline{AQ}$. But the proof is valid in any case. See figure 6.8(a)]

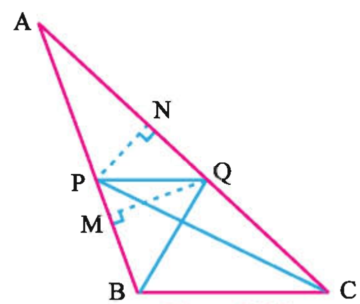

Figure 6.8(b)

Using the fundamental theorem on proportionality, we are going to obtain some results which are very useful in solving examples. We will accept these results without proof.

Some useful results :

(1) If a line parallel to one side of a triangle intersects the other two sides of the triangle in distinct points, the segments of the other sides of the triangle in the same halfplane of the line are proportional to the corresponding sides of the triangle.

In ΔABC, line $l \parallel \overline{BC}$ and l intersects $\overline{AB}$ and $\overline{AC}$ in P and Q respectively. Then we have already proved,

$$\frac{AP}{PB} = \frac{AQ}{QC}$$

$$\therefore \frac{AP + PB}{PB} = \frac{AQ + QC}{QC},$$

We have A–P–B and A–Q–C.

$$\therefore \frac{AB}{PB} = \frac{AC}{QC}$$

$$\therefore \frac{PB}{AB} = \frac{QC}{AC}$$

Again $\frac{AP}{PB} = \frac{AQ}{QC}$ and $\frac{PB}{AB} = \frac{QC}{AC}$.

Multiplying respective sides of these equalities,

$$\therefore \frac{AP}{AB} = \frac{AQ}{AC}$$

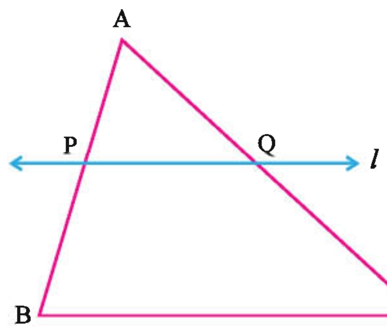

Figure 6.9

(2) If three (or more than three) parallel lines are intercepted by two transversals, the segments cut off on the transversals between the same parallel lines are proportional.

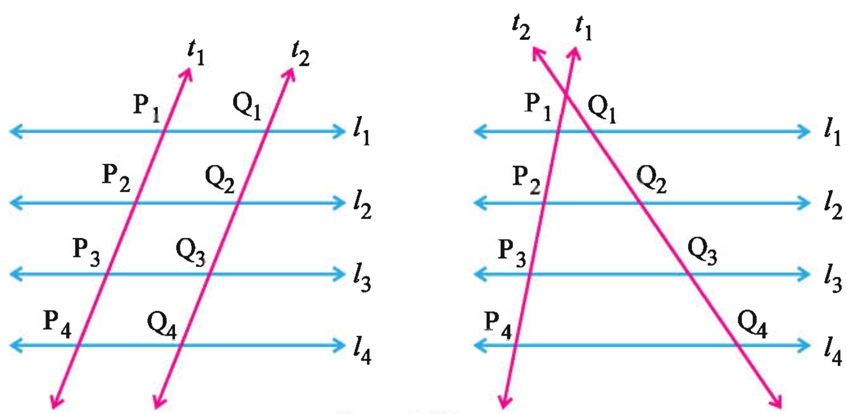

Figure 6.10

In figure 6.10, $l_1 \parallel l_2 \parallel l_3 \parallel l_4$ and transversals t_1 and t_2 intersect each of l_1 l_2 l_3 l_4 at the points shown in the figure 6.10. We have to prove.

$$\frac{P_1P_2}{Q_1Q_2} = \frac{P_2P_3}{Q_2Q_3} = \frac{P_3P_4}{Q_3Q_4}$$

If $t_1 \parallel t_2$ then the result can be proved using properties of parallelograms.

If t_1 and t_2 intersect at a point; the result can be proved using fundamental theorem of proportionality.

In ΔABC, the bisector of $\angle A$ intersects $\overline{BC}$ in D. $\overline{BD}$ and $\overline{DC}$ are the segments corresponding to sides $\overline{AB}$ and $\overline{AC}$.

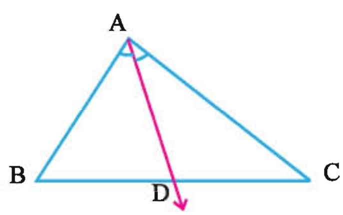

Figure 6.11

(3) In a triangle the bisector of an angle divides the side opposite to the angle in the segments whose lengths are in the ratio of their corresponding sides.

Bisector of $\angle A$ in figure 6.11 intersects $\overline{BC}$ in D. we have to prove that $\frac{BD}{DC} = \frac{AB}{AC}$.

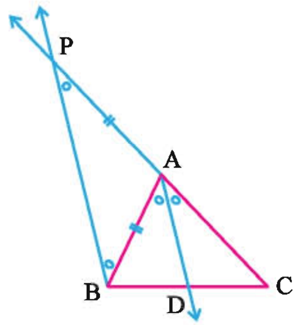

Figure 6.12

Let a line parallel to $\overleftrightarrow{AD}$ and passing through B intersects $\overrightarrow{CA}$ in P.

We have $\frac{BD}{DC} = \frac{PA}{AC}$ **(Why ?) (i)**

$\angle PBA \cong \angle BAD$ **($\overleftrightarrow{AD} \parallel \overleftrightarrow{PB}$ and $\overleftrightarrow{AB}$ is a transversal) (ii)**

$\angle BPA \cong \angle DAC$ **($\overleftrightarrow{AD} \parallel \overleftrightarrow{PB}$ and $\overleftrightarrow{AP}$ is a transversal) (iii)**

But $\angle BAD \cong \angle DAC$ **($\overrightarrow{AD}$ is the bisector of $\angle A$)**

$\therefore$ $\angle PBA \cong \angle BPA$ **(using (ii) and (iii))**

$\therefore$ $\overline{PA} \cong \overline{AB}$

$\therefore$ PA = AB

$\therefore$ from (i) $\frac{BD}{DC} = \frac{AB}{AC}$.

Example 9 : In figure 6.13, $l \parallel \overleftrightarrow{AB}$ and intersects $\overline{AC}$ in P and $\overline{BC}$ in Q. If CP = 3, PA = 4, QB = 6, find CQ and BC.

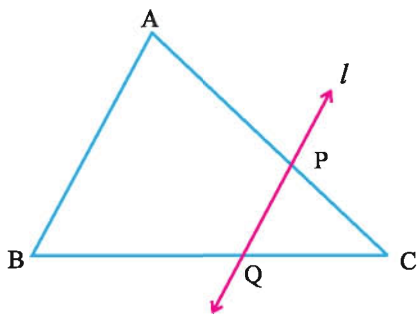

Figure 6.13

Solution : As $l \parallel \overleftrightarrow{AB}$ and l intersects $\overline{CA}$ in P and $\overline{CB}$ in Q.

$\frac{CP}{PA} = \frac{CQ}{QB}$

$\therefore$ $\frac{3}{4} = \frac{CQ}{6}$.

$\therefore$ $CQ = \frac{6 \times 3}{4} = \frac{9}{2} = 4.5$

CB = CQ + QB = 4.5 + 6 = 10.5 **(Q $\in \overline{BC}$)**

Example 10 : In ΔABC, the bisector of $\angle B$ intersects $\overline{AC}$ in D. If $\frac{AD}{DC} = \frac{3}{4}$ and AB = 7.5, find BC.

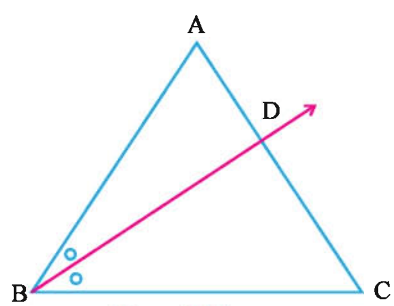

Figure 6.14

Solution : The bisector of $\angle B$ meets $\overline{AC}$ in D.

$\therefore$ $\frac{AD}{DC} = \frac{AB}{BC}$

$\therefore$ $\frac{AB}{BC} = \frac{3}{4}$

$\therefore$ $\frac{7.5}{BC} = \frac{3}{4}$

$\therefore$ $BC = \frac{7.5 \times 4}{3} = 10$

Example 11 : In ΔPQR, the bisector of $\angle P$ intersects $\overline{QR}$ in S and PQ : PR = 5 : 4. If SR = 5.6 *cm* find QR.

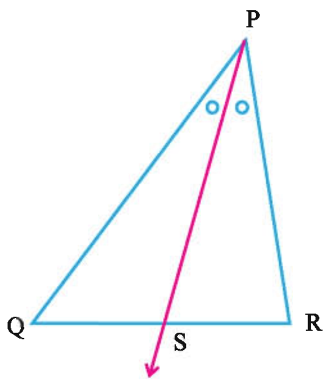

Figure 6.15

Solution : Bisector of $\angle P$ intersects $\overline{QR}$ in S.

$\therefore \frac{PQ}{PR} = \frac{QS}{RS}$

But $\frac{PQ}{PR} = \frac{5}{4}$

$\therefore \frac{QS}{SR} = \frac{5}{4}$

SR = 5.6

$\therefore \frac{QS}{5.6} = \frac{5}{4}$

$\therefore QS = \frac{5}{4} \times 5.6 = 7\ cm$

$QR = QS + SR = 7 + 5.6 = 12.6\ cm$ **($S \in \overline{QR}$)**

EXERCISE 6.2

1. In ΔABC, a line parallel to $\overline{BC}$ intersects $\overline{AB}$ and $\overline{AC}$ in D and E respectively. Fill in the blanks shown in the table :

No.	AD	DB	AB	AE	EC	AC
1.	3.6	2.4	......	......	1.8	......
2.	......	......	6.2	......	3.15	4.2
3.	......	12	......	6.4	8.0	......
4.	7.2	......	18.4	5.4	......	......
5.	......	3.4	......	......	2.55	5.10

2. In ΔABC, the bisector of $\angle C$ intersects $\overline{AB}$ in F. If 2AF = 3FB and AC = 7.2 find BC.
3. In ΔXYZ, the bisector of $\angle Y$ intersects $\overline{ZX}$ in P.
   (1) if XP : PZ = 4 : 5 and YZ = 6.5, find XY.
   (2) if XY : YZ = 2 : 3 and XP = 3.8, find PZ and ZX.
4. In ΔABC, the bisector of $\angle A$ intersects $\overline{BC}$ in D. Prove that
   $BD = \frac{BC \times AB}{AB + AC}$ and $DC = \frac{BC \times AC}{AC + AB}$
5. $\square ABCD$ is a trapezium such that $\overline{AB} \parallel \overline{CD}$. $M \in \overline{AD}$ and $N \in \overline{BC}$ such that $\overline{MN} \parallel \overline{AB}$. Prove that $\frac{AM}{MD} = \frac{BN}{NC}$.
6. In ΔABC, D and E are the mid-points of $\overline{BC}$ and $\overline{AC}$ respectively. $\overline{AD}$ and $\overline{BE}$ intersect in G. A line m passing through D and parallel to $\overleftrightarrow{BE}$ intersects $\overline{AC}$ in K. Prove that AC = 4CK.
7. In ΔPQR, $X \in \overleftrightarrow{QR}$, such that Q–X–R. A line parallel to $\overline{PR}$ and passing through X intersects $\overline{PQ}$ in Y. A line parallel to $\overline{PX}$ and passing through Y intersects $\overline{QR}$ in Z. Prove that $\frac{QZ}{ZX} = \frac{QX}{XR}$.
8. In ΔABC, $X \in \overleftrightarrow{BC}$ and B–X–C. A line passing through X and parallel to $\overline{AB}$ intersects $\overline{AC}$ in Y. A line passing through X and parallel to $\overline{BY}$ intersects $\overline{AC}$ in Z. Prove that $CY^2 = AC \cdot CZ$.
9. In ΔABC, the bisector of $\angle A$ intersects $\overline{BC}$ in D and the bisector of $\angle ADC$ intersects $\overline{AC}$ in E. Prove that AB × AD × EC = AC × BD × AE.

10. In ΔABC, D is the mid-point of $\overline{BC}$ and P is the mid-point of $\overline{AD}$. $\overrightarrow{BP}$ intersects $\overline{AC}$ in Q. Prove that (i) CQ = 2AQ (ii) BP = 3PQ

*

6.5 A Triangle and a Line Lying in the Plane

When a line is drawn in the plane of a triangle, there are three possibilities (i) The line may not intersect the triangle (ii) The line may intersect the triangle in one point (iii) The line may intersect the triangle in two points. Of course we assume that the line does not contain any of the sides of the triangle. All the three possibilities are shown in figure 6.16(i), (ii), (iii).

In figure 6.16(i) the line l does not intersect ΔABC.

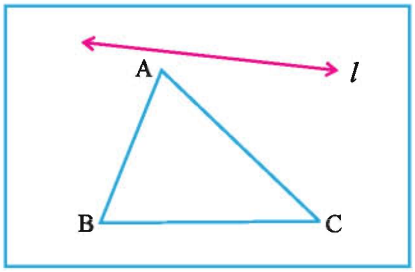

Figure 6.16(i)

In figure 6.16(ii) the line l intersects ΔABC in one point. In this case the line passes through any of the three vertices of ΔABC.

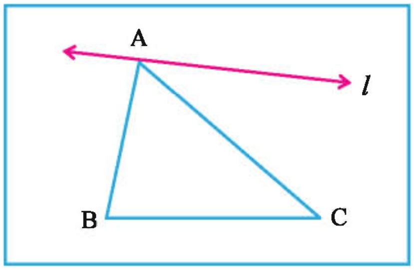

Figure 6.16(ii)

In figure 6.16(iii), line l intersects ΔABC in two distinct points. In this case the line intersects two sides of the triangle in distinct points and it will not intersect the third side.

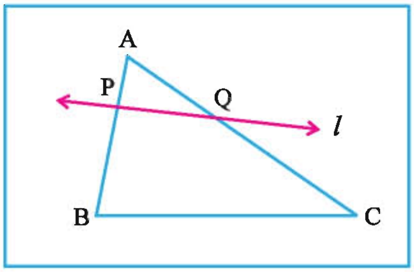

Figure 6.16(iii)

We will accept the following result as theorem 6.2 without giving a formal proof.

Theorem 6.2 : If a line lying in the plane of a triangle intersects a side of the triangle at a point other than the vertex, then it intersects one more side of the triangle but does not intersect the third side.

Now let us consider the converse of theorem 6.1.

In ΔABC, $D \in \overline{AB}$ and $E \in \overline{AC}$. D and E are the points such that $\frac{AD}{DB} = \frac{AE}{EC}$. (or $\frac{AD}{AB} = \frac{AE}{AC}$ or $\frac{DB}{AB} = \frac{EC}{AC}$) What can we say about the lines $\overleftrightarrow{DE}$ and $\overleftrightarrow{BC}$? Are they parallel ? If so why ?

We already know one such case,

In ΔABC, suppose D is the mid-point of $\overline{AB}$ and E is the mid-point of $\overline{AC}$.

Then AD = DB and AE = EC

$\therefore \frac{AD}{DB} = \frac{AE}{EC} = 1$

Do you remember, we have already proved in class IX that $\overleftrightarrow{DE}$ and $\overleftrightarrow{BC}$ are parallel in this case ?

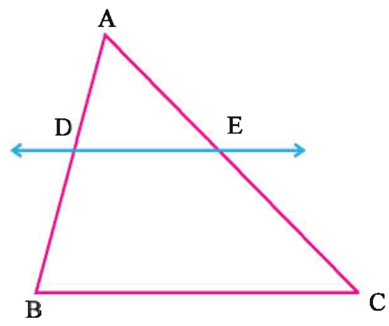

Figure 6.17

Activity : Construct ΔABC in which AB = 6, AC = 9 and BC = 5.

Now select point D on $\overline{AB}$ such that AD = 2 and a point E on $\overline{AC}$ such that AE = 3.

$\therefore$ DB = 4 and EC = 6

Measure $\angle ADE$, $\angle B$, $\angle AED$ and $\angle C$.

We will observe that $\angle ADE \cong \angle B$ and $\angle AED \cong \angle C$

From this we immediately conclude that $\overline{DE} \parallel \overline{BC}$.

Here also $\frac{AD}{DB} = \frac{2}{4} = \frac{1}{2}$, $\frac{AE}{EC} = \frac{3}{6} = \frac{1}{2}$

$\therefore \frac{AD}{DB} = \frac{AE}{EC}$.

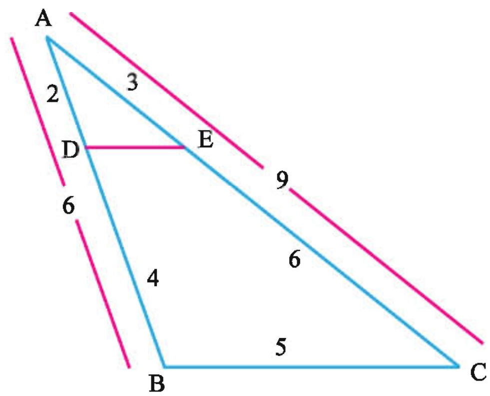

Figure 6.18

These are the particular cases of a general result which we are going to accept without proof as theorem 6.3.

Theorem 6.3 : If a line intersects two sides of a triangle such that segments of the sides in each half-plane are proportional then the line is parallel to the third side. [Converse of the fundamental theorem of proportionality.]

So, in figure 6.19, P is a point on side $\overline{AB}$ and Q is a point on side $\overline{AC}$ of ΔABC such that

$\frac{AP}{PB} = \frac{AQ}{QC}$ or $\frac{AP}{AB} = \frac{AQ}{AC}$ or $\frac{PB}{AB} = \frac{QC}{AC}$

then $\overleftrightarrow{PQ} \parallel \overleftrightarrow{BC}$.

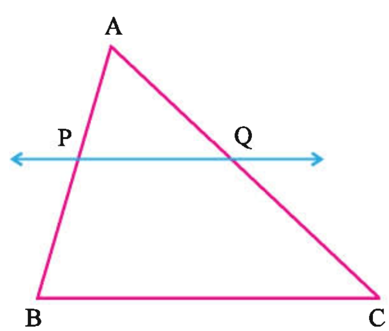

Figure 6.19

Example 12 : Prove that the mid-points of the sides of a quadrilateral are the vertices of a parallelogram using converse of the fundamental theorem of proportionality.

Solution : ABCD is a quadrilateral in which P is the mid-point of $\overline{AB}$, Q is the mid-point of $\overline{BC}$, R is the mid-point of $\overline{CD}$ and S is the mid-point of $\overline{DA}$.

Draw $\overline{BD}$.

In ΔABD, P is the mid-point of $\overline{AB}$ and S is the mid-point of $\overline{AD}$.

$\therefore$ AP = PB and AS = SD

$\therefore$ $\frac{AP}{PB} = \frac{AS}{SD} = 1$

$\therefore$ Using the converse of the theorem of proportionality we have,

$\overline{PS} \parallel \overline{BD}$ **(i)**

$\therefore$ In ΔCBD, Q is the mid-point of $\overline{CB}$ and R is the mid-point of $\overline{CD}$. As before,

$\overline{QR} \parallel \overline{BD}$ **(ii)**

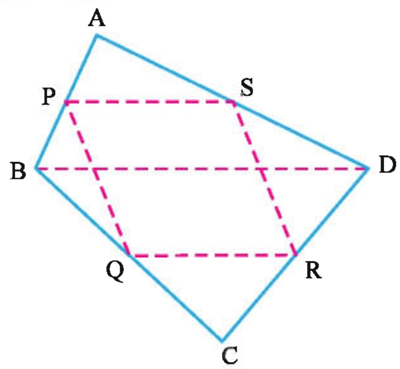

Figure 6.20

Using (i) and (ii), we have $\overline{PS} \parallel \overline{QR}$ **(iii)**

Similarly if we draw $\overline{AC}$, we can prove that $\overline{PQ} \parallel \overline{RS}$ **(as both are parallel to $\overline{AC}$) (iv)**

from (iii) and (iv) we can say that opposite sides of □PQRS are parallel.

$\therefore$ □PQRS is a parallelogram.

Example 13 : ABCD is a quadrilateral. P $\in \overline{AB}$, Q $\in \overline{BC}$, R $\in \overline{CD}$, S $\in \overline{DA}$ are the points such that $\frac{AP}{PB} = \frac{AS}{SD}$ and $\frac{CQ}{QB} = \frac{CR}{RD}$. Prove that $\overline{PS} \parallel \overline{QR}$.

Solution : Join B and D. In ΔABD, P $\in \overline{AB}$ and S $\in \overline{AD}$.

We are given $\frac{AP}{PB} = \frac{AS}{SD}$.

$\therefore$ $\overline{PS} \parallel \overline{BD}$ **(i)**

In ΔCBD, we are given $\frac{CQ}{QB} = \frac{CR}{RD}$

$\therefore$ $\overline{QR} \parallel \overline{BD}$ **(ii)**

from (i) and (ii) $\overline{PS} \parallel \overline{QR}$

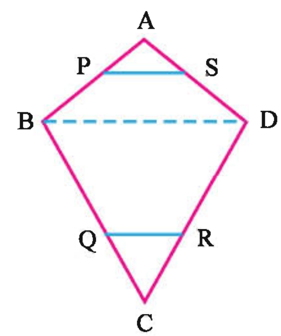

Figure 6.21

[**Note :** If in the example 13, it is given that $\frac{AP}{PB} = \frac{AS}{SD} = \frac{CQ}{QB} = \frac{CR}{RD}$, then we can prove that $\overline{PS} \parallel \overline{QR}$ and $\overline{PQ} \parallel \overline{SR}$ and in this case □PQRS will be a parallelogram.]

Example 14 : In ΔABC, D $\in \overline{BC}$. The bisectors of $\angle$ADB and $\angle$ADC intersect $\overline{AB}$ and $\overline{AC}$ respectively in P and Q. Prove that AP $\times$ AQ $\times$ BD $\times$ DC = AD^2 $\times$ PB $\times$ QC.

Hence, prove that if $\overleftrightarrow{PQ} \parallel \overleftrightarrow{BC}$ then D is the mid-point of $\overline{BC}$.

Solution : The bisector of $\angle$ADB, intersects $\overline{AB}$ in P.

$\therefore$ $\frac{AP}{PB} = \frac{AD}{BD}$ **(i)**

The bisector of $\angle$ADC, intersects $\overline{AC}$ in Q.

$\therefore$ $\frac{AQ}{QC} = \frac{AD}{DC}$ **(ii)**

from (i) and (ii), $\frac{AP}{PB} \times \frac{AQ}{QC} = \frac{AD}{BD} \times \frac{AD}{DC}$

$\therefore$ AP $\times$ AQ $\times$ BD $\times$ DC = AD^2 $\times$ PB $\times$ QC.

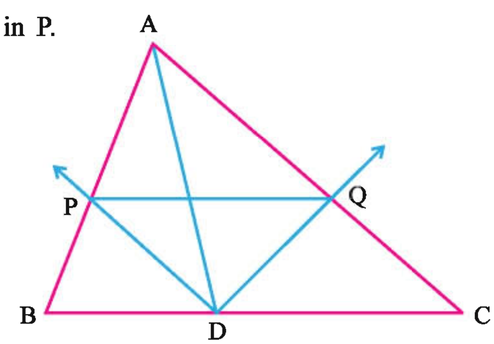

Figure 6.22

Now if $\overleftrightarrow{PQ} \parallel \overleftrightarrow{BC}$, then

$\frac{AP}{PB} = \frac{AQ}{QC}$

Therefore from (i) and (ii), $\frac{AD}{BD} = \frac{AD}{DC}$

$\therefore$ BD = DC

$\therefore$ D is the mid-point of $\overline{BC}$.

Example 15 : In ΔABC, D is the mid-point of $\overline{BC}$. G $\in$ $\overline{AD}$ such that AG = 2GD. A line l passing through G intersects $\overline{AB}$ in M and $\overline{AC}$ in N. If 3AN = 2AC prove that AM = 2MB.

Solution : The line l intersects $\overline{AB}$ in M and $\overline{AC}$ in N.

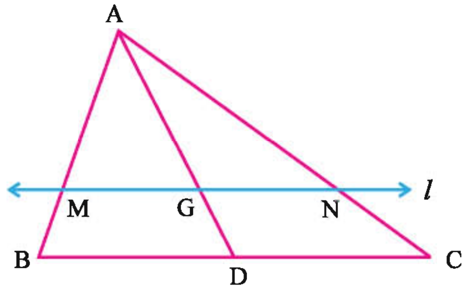

Figure 6.23

AG = 2GD

$\therefore$ $\frac{AG}{GD} = \frac{2}{1}$

$\therefore$ $\frac{AG}{AG + GD} = \frac{2}{2+1} = \frac{2}{3}$

$\therefore$ $\frac{AG}{AD} = \frac{2}{3}$ **(A–G–D) (i)**

Also, 3AN = 2AC

$\therefore$ $\frac{AN}{AC} = \frac{2}{3}$ **(ii)**

From (i) and (ii) $\frac{AG}{AD} = \frac{AN}{AC}$

$\therefore$ $\overleftrightarrow{GN} \parallel \overleftrightarrow{DC}$

$\therefore$ $l \parallel \overleftrightarrow{BC}$ **(G $\in$ l, N $\in$ l and D $\in$ $\overleftrightarrow{BC}$)**

$\therefore$ $\frac{AM}{MB} = \frac{AG}{GD}$

$\therefore$ $\frac{AM}{MB} = \frac{2}{1}$ $\left(\frac{AG}{GD} = \frac{2}{1}\right)$

$\therefore$ AM = 2MB

Example 16 : $\square$ABCD is a trapezium in which $\overline{AB} \parallel \overline{CD}$. M $\in$ $\overline{AD}$ and N $\in$ $\overline{BC}$ such that AM $\times$ NC = BN $\times$ MD. $\overline{MN}$ intersects $\overline{AC}$ at O. If $\frac{AM}{MD} = \frac{2}{3}$, find $\frac{AO}{AC}$.

Solution : In $\square$ABCD, $\overline{AB} \parallel \overline{CD}$, M and N are the points on transversals $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$,

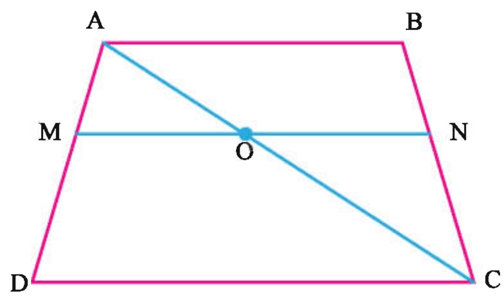

Figure 6.24

Also, AM $\times$ NC = BN $\times$ MD.

$\therefore$ $\frac{AM}{MD} = \frac{BN}{NC}$

$\therefore$ $\overleftrightarrow{MN} \parallel \overleftrightarrow{AB}$ and $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

$\overleftrightarrow{MN}$ intersects $\overline{AC}$ at O.

$\therefore$ In ΔADC, $\overleftrightarrow{MO} \parallel \overleftrightarrow{DC}$, M $\in$ $\overline{AD}$, O $\in$ $\overline{AC}$

$\therefore \frac{AM}{MD} = \frac{AO}{OC}$

but $\frac{AM}{MD} = \frac{2}{3}$

$\therefore \frac{AO}{OC} = \frac{2}{3}$

$\therefore \frac{AO}{AO + OC} = \frac{2}{2+3}$

$\therefore \frac{AO}{AC} = \frac{2}{5}$ **(A–O–C)**

Example 17 : In ΔABC, D and E are the points on $\overline{BC}$ such that B–D–E–C and BD = DE = EC. A line passing through D and parallel to $\overline{AB}$ intersects $\overline{AC}$ in P. A line passing through E and parallel to $\overline{AC}$ intersects $\overline{AB}$ in Q. Prove that $\overleftrightarrow{PQ} \parallel \overleftrightarrow{BC}$.

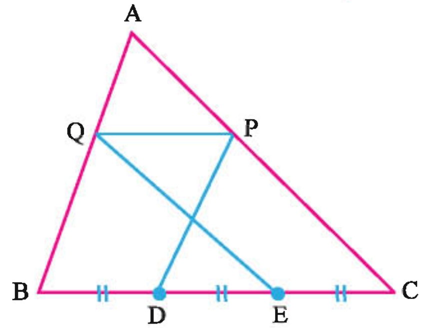

Figure 6.25

Solution : A line passing through D and parallel to $\overline{AB}$ intersects $\overline{AC}$ in P.

$\therefore \frac{AP}{PC} = \frac{BD}{DC}$ **(i)**

A line passing through E and parallel to $\overline{AC}$ intersects $\overline{AB}$ in Q.

$\therefore \frac{AQ}{QB} = \frac{CE}{EB}$ **(ii)**

Now, BD = DE = EC

$\therefore \frac{BD}{DC} = \frac{1}{2}$ and $\frac{CE}{EB} = \frac{1}{2}$ **(iii)**

From (i), (ii) and (iii), $\frac{AP}{PC} = \frac{AQ}{QB}$, $P \in \overline{AC}$ and $Q \in \overline{AB}$

$\therefore \overleftrightarrow{PQ} \parallel \overleftrightarrow{BC}$

Remark : The result is also true for the point D and E on $\overline{BC}$ such that $\frac{BD}{DC} = \frac{CE}{EB}$, instead of BD = DE = EC.

Example 18 : In ΔABC, $P \in \overline{AB}$. A line passing through P and parallel to $\overline{BC}$ intersects $\overline{AC}$ in Q and a line passing through Q parallel to $\overline{AB}$ intersects $\overline{BC}$ in R.

Prove that AP × RC = PB × BR.

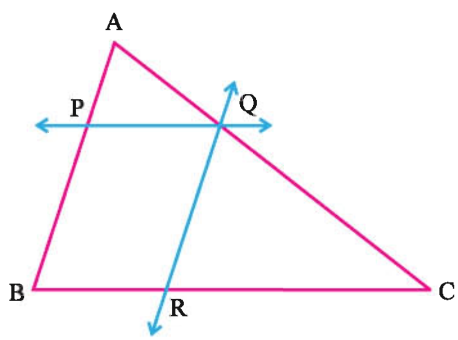

Figure 6.26

Solution :

Because of $\overleftrightarrow{PQ} \parallel \overleftrightarrow{BC}$, $\frac{AP}{PB} = \frac{AQ}{QC}$ and

Because of $\overleftrightarrow{QR} \parallel \overleftrightarrow{AB}$, $\frac{AQ}{QC} = \frac{BR}{RC}$

Hence, $\frac{AP}{PB} = \frac{AQ}{QC} = \frac{BR}{RC}$

$\therefore \frac{AP}{PB} = \frac{BR}{RC}$

$\therefore$ AP × RC = PB × BR.

EXERCISE 6.3

1. In figure 6.27, $\overline{PQ} \parallel \overline{BC}$, $\overline{PR} \parallel \overline{AC}$, $\overline{QS} \parallel \overline{AB}$. Prove that BR = CS.

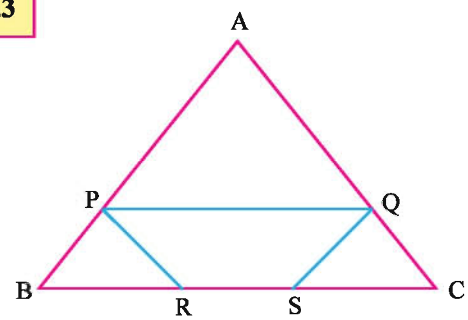

Figure 6.27

2. □ABCD is a parallelogram. Prove that ΔABD and ΔBDC are similar.

3. In □ABCD, M and N are the mid-points of $\overline{AD}$ and $\overline{BC}$. If $\overline{AB} \parallel \overline{CD}$, prove that $\overline{MN} \parallel \overline{AB}$.

4. In □ABCD, A–P–D, B–Q–C. If $\overline{AB} \parallel \overline{PQ}$ and $\overline{PQ} \parallel \overline{DC}$, prove that AP × QC = PD × BQ.

5. In ΔABC, the bisector of ∠A intersects $\overline{BC}$ in D. The bisector of ∠ADB intersects $\overline{AB}$ in F and the bisector of ∠ADC intersects $\overline{AC}$ in E. Prove that AF × AB × CE = AE × AC × BF.

6. In ΔABC, ΔPQR and ΔXYZ correspondences ABC ↔ PQR, PQR ↔ XYZ are similarity. Prove that ABC ↔ XYZ is similarity.

7. State giving reasons, whether the following statements are true or false :

In all the following questions the line does not contain a side of the triangle.

(1) A line can be drawn in the plane of a triangle not intersecting any of the sides of a triangle.

(2) A line can be drawn in the plane of a triangle which is not passing through any of the three vertices and intersecting all the three sides of the triangle.

(3) If a line drawn in the plane of a triangle intersects the triangle at only one point, the line passes through a vertex of the triangle.

(4) If a line intersects two of the three sides of a triangle in two distinct points and does not intersect the third side, then the line is parallel to the third side.

(5) In the plane of ΔABC, a line l can be drawn such that $l \cap \overline{BC} = \{P\}$, $l \cap \overline{AC} = \{Q\}$ and $l \cap \overline{AB} = \emptyset$.

*

6.6 Criteria for Similarity of Triangles

If two triangles are similar for some correspondence between the vertices of triangles,

then (1) The corresponding angles are congruent.

(2) The corresponding sides are proportional.

These conditions are sufficient for ensuring the similarity between the triangles. Are all of them necessary ? The answer is, no. If some of the conditions are satisfied, the remaining of the conditions will also be satisfied.

Let us perform some activities :

Draw ΔABC such that BC = 4 *cm*, *m*∠B = 50 and *m*∠C = 70.

Now construct $\overline{EF}$ such that EF = 8 *cm* and draw rays $\overrightarrow{EX}$ and $\overrightarrow{FY}$ such that *m*∠XEF = 50 and *m*∠YFE = 70.

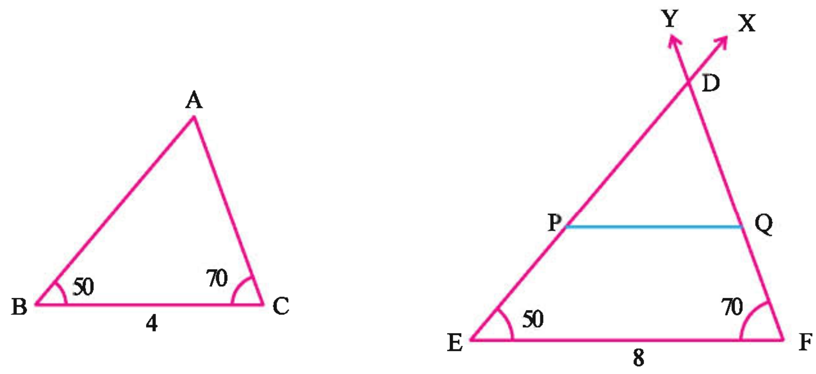

Figure 6.28

Let $\overrightarrow{EX}$ and $\overrightarrow{FY}$ intersect in D.

Let us examine correspondence ABC ↔ DEF for similarity.

$\angle B \cong \angle E$

$\angle C \cong \angle F$

$\therefore \quad \angle A \cong \angle D$

Measure AB, AC, DE, DF. Find the ratio $\frac{AB}{DE}, \frac{AC}{DF}$.

$\frac{BC}{EF} = \frac{4}{8}$ is known.

Are $\frac{AB}{DE}, \frac{AC}{DF}$ and $\frac{BC}{EF}$ equal ?

We can observe that they are equal. We can repeat our activity for different values of measures of angles and we will observe that whenever corresponding angles are congruent, the corresponding sides will be proportional and so the triangles will be similar.

We will accept the following theorem without proof.

Theorem 6.4 : (AAA Similarity Theorem) If for any correspondence between the vertices of two triangles, corresponding angles are congruent, then the correspondence is a similarity.

Whenever two pairs of corresponding angles in two triangles are congruent, the corresponding angles in the third pair are also congruent. Why ?

In ΔABC and ΔDEF, let $\angle A \cong \angle D$, $\angle B \cong \angle E$. Then $m\angle A = m\angle D$, $m\angle B = m\angle E$

Now, $m\angle A + m\angle B + m\angle C = 180 = m\angle D + m\angle E + m\angle F$

$\therefore \quad m\angle C = m\angle F$. Hence $\angle C \cong \angle F$.

Therefore we have the following corollary.

Corollary : AA criterion of similarity :

For any correspondence between the vertices of two triangles, if angles in any two pairs of corresponding angles are congruent, then the correspondence is a similarity.

Let us apply this theorem to solve some examples.

Example 19 : Line l in the plane of ΔABC intersects sides $\overline{AB}$ and $\overline{AC}$ in points P and Q respectively. $l \parallel \overleftrightarrow{BC}$. Prove that ABC ↔ APQ is similarity.

Solution : $l \parallel \overleftrightarrow{BC}$ and $\overleftrightarrow{AB}$ is a transversal of l and $\overleftrightarrow{BC}$.

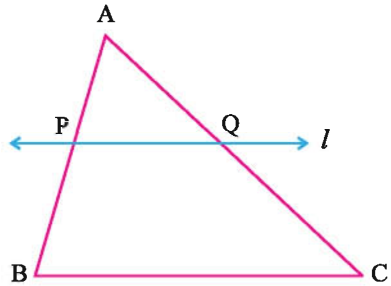

Figure 6.29

$\therefore$ $\angle APQ \cong \angle ABC$ **(corresponding angles)**

In ΔAPQ and ΔABC for the correspondence ABC $\leftrightarrow$ APQ.

$\angle PAQ \cong \angle BAC$ **(A–P–B, A–Q–C)**

(Infact $\angle PAQ = \angle BAC$)

$\angle APQ \cong \angle ABC$

$\therefore$ By AA criterion, the correspondence ABC $\leftrightarrow$ APQ is similarity.

Example 20 : In ΔXYZ and ΔABC, $m\angle A = m\angle X$ and $m\angle Y = m\angle B$. $\frac{AB}{XY} = \frac{2}{3}$. If AC = 7.2, find ZX.

Solution : In ΔXYZ and ΔABC

$m\angle A = m\angle X$ and $m\angle Y = m\angle B$

$\therefore$ $\angle A \cong \angle X$ and $\angle B \cong \angle Y$

$\therefore$ The correspondence ABC $\leftrightarrow$ XYZ is a similarity **(AA)**

$\therefore$ $\frac{AB}{XY} = \frac{AC}{ZX}$

$\therefore$ $\frac{2}{3} = \frac{7.2}{ZX}$

$\therefore$ $ZX = \frac{7.2 \times 3}{2} = 10.8$

Example 21 : $\square$ABCD is a parallelogram. A line passing through A intersects $\overline{CD}$ in M, $\overline{BD}$ in L and $\overrightarrow{BC}$ in N. Prove that $\frac{LD^2}{LB^2} = \frac{LM}{LN}$.

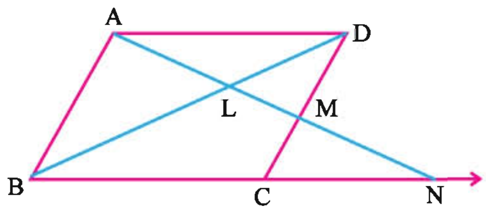

Figure 6.30

Solution : $\overleftrightarrow{AD} \parallel \overleftrightarrow{BN}$ and $\overleftrightarrow{AN}$ is the transversal.

$\therefore$ $\angle DAN \cong \angle ANB$ **(alternate angles)**

$\therefore$ $\angle DAL \cong \angle LNB$ **($\overrightarrow{AN} = \overrightarrow{AL}$, $\overrightarrow{NL} = \overrightarrow{NA}$)**

$\angle ALD \cong \angle NLB$ **(vertically opposite angles)**

$\therefore$ The correspondence ALD $\leftrightarrow$ NLB is a similarity **(AA criterion of similarity)**

$\therefore$ $\frac{LD}{LB} = \frac{AL}{LN}$ **(i)**

Similarly, the correspondence ALB $\leftrightarrow$ MLD is a similarity ($\angle ALB \cong \angle MLD$, $\angle LAB \cong \angle LMD$)

$\therefore$ $\frac{AL}{ML} = \frac{LB}{LD}$ that is $\frac{LD}{LB} = \frac{LM}{AL}$ **(ii)**

Multiply the results on corresponding sides of (i) and (ii), we get, $\frac{LD^2}{LB^2} = \frac{AL}{LN} \cdot \frac{LM}{AL} = \frac{LM}{LN}$

Let us again do an activity :

Draw ΔABC in which AB = 4, BC = 5, AC = 6

Draw ΔPQR in which PQ = 6, QR = 7.5, PR = 9

(Remember, when three sides of a triangle are given a triangle can be drawn uniquely.)

So, $\frac{AB}{PQ} = \frac{4}{6} = \frac{2}{3}$, $\frac{BC}{QR} = \frac{5}{7.5} = \frac{2}{3}$, $\frac{AC}{PR} = \frac{6}{9} = \frac{2}{3}$

Now measure all the six angles, $\angle A$, $\angle B$, $\angle C$; $\angle P$, $\angle Q$, $\angle R$.

We will observe that $m\angle A = m\angle P$, $m\angle B = m\angle Q$, $m\angle C = m\angle R$

Again draw a third ΔXYZ such that XY = 8, YZ = 10, ZX = 12.

Note that $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{ZX} = \frac{1}{2}$ and $\frac{PQ}{XY} = \frac{QR}{YZ} = \frac{PR}{YZ} = \frac{3}{4}$

If we measure, $\angle X$, $\angle Y$, $\angle Z$ we will notice that

$$m\angle X = m\angle A = m\angle P, m\angle Y = m\angle B = m\angle Q, m\angle Z = m\angle C = m\angle R$$

That induces another criterion of similarity popularly known as SSS criterion of similarity. We will accept it without proof.

Theorem 6.5 : (SSS Similarity Theorem) : If for some correspondence between the vertices of two triangles, corresponding sides are proportional, then the correspondence is a similarity.

Suppose in ΔABC and ΔDEF

$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$.

Without loss of generality we can take AB < DE.

Select $P \in \overline{DE}$ and $Q \in \overline{DF}$ such that $\overline{DP} \cong \overline{AB}$ and $\overline{DQ} \cong \overline{AC}$.

$\frac{AB}{DE} = \frac{AC}{DF}$ (given)

$\therefore \frac{DP}{DE} = \frac{DQ}{DF}$

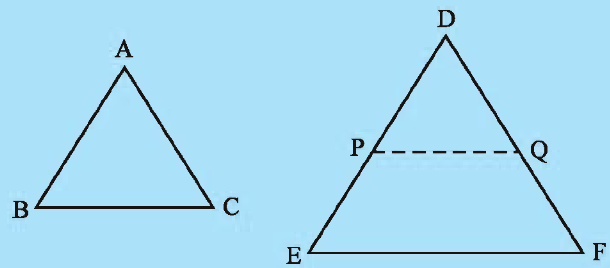

Figure 6.31

$\therefore \overleftrightarrow{PQ} \parallel \overleftrightarrow{EF}$ (The converse of fundamental theorem of proportionality) etc.

We have studied in class IX the conditions for congruence of triangles. SSS was one of them and we have seen that a similar criterion for similarity is also there. Let us now examine whether SAS can also be a criterion for similarity.

Let us draw ΔABC having AB = 4, BC = 6 and $m\angle B$ = 70 and draw another ΔPQR having PQ = 6, QR = 9 and $m\angle Q$ = 70. We will observe that,

$\frac{AB}{PQ} = \frac{4}{6} = \frac{2}{3}$, $\frac{BC}{QR} = \frac{6}{9} = \frac{2}{3}$ and $\angle B \cong \angle Q$

For this we measure the sides $\overline{AC}$ and $\overline{PR}$ and angles $\angle A$, $\angle C$, $\angle P$ and $\angle R$.

We will notice that $\frac{AC}{PR}$ is equal to $\frac{2}{3}$, $m\angle A = m\angle P$ and $m\angle C = m\angle R$.

So, by the definition of similarity the correspondence ABC $\leftrightarrow$ PQR is a similarity.

We will accept the result of this experiment as a theorem for similarity without giving formal proof.

Theorem 6.6 : (SAS Theorem) : For a correspondence between the vertices of two triangles, if two of the sides of one triangle, are proportional to the corresponding sides of the other triangle and the included angles between these sides are congruent, then the correspondence is a similarity.

6.7 Similarity and Area

We have discussed in the beginning of the chapter that the shapes of similar figures are same but their sizes may differ. Consider two equilateral triangles whose length of the sides are in the ratio 1 : 3. (e.g. consider an equivalent triangle with lengths of its sides 2 unit and another equivalent triangle with lengths of its sides 6 units.) We observe that the ratio of their areas is 1 : 9. It is because of the next theorem. Let us prove it.

Theorem 6.7 : Areas of two similar triangles are proportional to squares of corresponding sides.

Case 1 : Triangles are acute angled triangles.

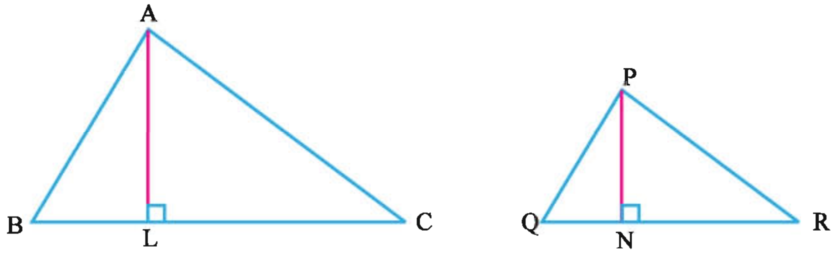

Figure 6.32

Given : Correspondence ABC $\leftrightarrow$ PQR of ΔABC and ΔPQR is a similarity.

To prove : $\frac{ABC}{PQR} = \frac{AB^2}{PQ^2} = \frac{BC^2}{QR^2} = \frac{AC^2}{PR^2}$

Proof : Draw altitudes $\overline{AL}$ and $\overline{PN}$.

The correspondence ABC $\leftrightarrow$ PQR is a similarity. **(AA)**

$\therefore \quad \angle B \cong \angle Q$

and $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$ gives $\frac{AB^2}{PQ^2} = \frac{BC^2}{QR^2} = \frac{AC^2}{PR^2}$ **(i)**

In ΔABL and ΔPQN,

$\angle B \cong \angle Q$

$\angle ALB \cong \angle PNQ$ **(right angles)**

$\therefore$ The correspondence ABL $\leftrightarrow$ PQN is a similarity.

$\therefore \quad \frac{AB}{PQ} = \frac{AL}{PN}$

$\therefore \quad \frac{AL}{PN} = \frac{AB}{PQ} = \frac{BC}{QR}$ **(ii)**

Now, area of a triangle = $\frac{1}{2}$base $\times$ altitude

$$\therefore \quad \frac{ABC}{PQR} = \frac{\frac{1}{2}BC \cdot AL}{\frac{1}{2}QR \cdot PN} = \frac{BC}{QR} \times \frac{AL}{PN}$$

$$= \frac{BC}{QR} \times \frac{BC}{QR} = \frac{BC^2}{QR^2} \quad \textbf{(using (ii))}$$

$$\therefore \quad \frac{ABC}{PQR} = \frac{AB^2}{PQ^2} = \frac{BC^2}{QR^2} = \frac{AC^2}{PR^2}$$

Case 2 : Triangles are right angled triangles.

Let $m\angle B = m\angle Q = 90$

$$\frac{ABC}{PQR} = \frac{\frac{1}{2} \times BC \times AB}{\frac{1}{2} \times QR \times PQ} = \frac{BC}{QR} \times \frac{AB}{PQ}$$

$$= \frac{BC^2}{QR^2} \qquad \left(\frac{BC}{QR} = \frac{AB}{PQ}\right)$$

$$\frac{ABC}{PQR} = \frac{AB^2}{PQ^2} = \frac{BC^2}{QR^2} = \frac{AC^2}{PR^2}$$

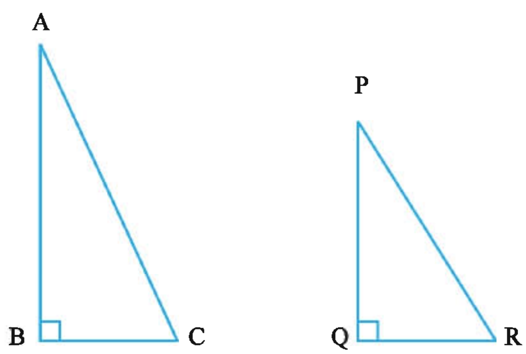

Figure 6.33

Case 3 : Triangles are obtuse angled triangles.

In ΔABC and ΔPQR, let $m\angle B = m\angle Q > 90$

Let $\overline{AL}$ and $\overline{PN}$ be altitudes on $\overleftrightarrow{BC}$ and $\overleftrightarrow{QR}$ respectively.

In this case it will be L–B–C and N–Q–R.

$\angle ABC \cong \angle PQR$ **(correspondence ABC ↔ PQR is similarity)**

$\therefore$ $\angle ABL \cong \angle PQN$ **($m\angle ABL = 180 - m\angle ABC$ and $m\angle PQN = 180 - m\angle PQR$)**

$\therefore$ $\angle ALB \cong \angle PNQ$ **(right angle)**

$\therefore$ $\Delta ALB \sim \Delta PNQ$ for correspondence ALB ↔ PNQ

$\therefore$ $\frac{AL}{PN} = \frac{AB}{PQ}$ but $\frac{AB}{PQ} = \frac{BC}{QR}$

$\therefore$ $\frac{AL}{PN} = \frac{BC}{QR}$

Now $\frac{ABC}{PQR} = \frac{\frac{1}{2} \times BC \times AL}{\frac{1}{2} \times QR \times PN} = \frac{BC}{QR} \times \frac{AL}{PN}$

$= \frac{BC}{QR} \times \frac{BC}{QR} = \frac{BC^2}{QR^2}$

$\frac{ABC}{PQR} = \frac{BC^2}{QR^2} = \frac{AB^2}{PQ^2} = \frac{AC^2}{PR^2}$

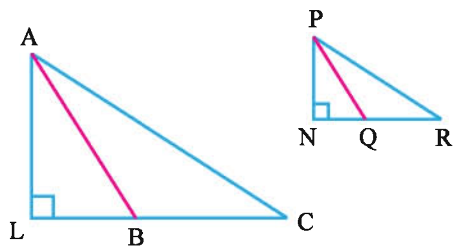

Figure 6.34

Example 22 : In ΔABC, D, E, F are the mid-points of the sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ in ΔABC. Prove that the correspondence DEF ↔ ABC is a similarity.

Solution : D, E, F are the mid-points of the sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ of ΔABC.

$\therefore$ $\overline{EF} \parallel \overline{BC}$ and $EF = \frac{1}{2}BC$ and $\frac{EF}{BC} = \frac{1}{2}$

Similarly $\frac{FD}{AC} = \frac{DE}{AB} = \frac{1}{2}$

$\therefore$ The correspondence DEF ↔ ABC is similarity. **(SSS)**

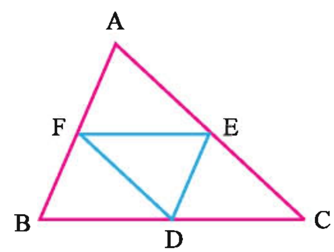

Figure 6.35

Example 23 : In ΔABC and ΔPQR, ABC ↔ PQR is similarity. $\overline{AD}$ is a median of ΔABC and $\overline{PM}$ is a median of ΔPQR. Prove that $BC \times PM = QR \times AD$.

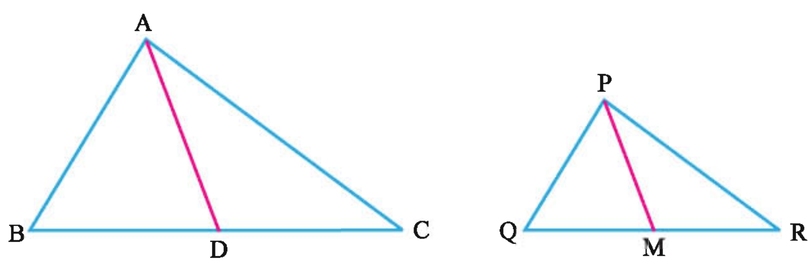

Figure 6.36

The correspondence ABC ↔ PQR is similarity.

$\therefore$ $\angle B \cong \angle Q$ **(i)**

and $\frac{AB}{PQ} = \frac{BC}{QR}$ **(ii)**

$\therefore$ $\frac{AB}{PQ} = \frac{2BD}{2QM} = \frac{BD}{QM}$ **(D is the mid-point of $\overline{BC}$ and M is the mid-point of $\overline{QR}$)**

In ΔABD and ΔPQM, $\angle B \cong \angle Q$ and $\frac{AB}{PQ} = \frac{BD}{QM}$

The correspondence ABD $\leftrightarrow$ PQM is a similarity. **(SAS)**

$\therefore \quad \frac{AB}{PQ} = \frac{AD}{PM}$ **(iii)**

From (ii) and (iii) $\frac{BC}{QR} = \frac{AD}{PM}$

$\therefore \quad BC \times PM = AD \times QR$

Example 24 : Prove that the area of the triangle formed by joining the mid-points of the sides of a triangle is $\frac{1}{4}$th of the area of the given triangle.

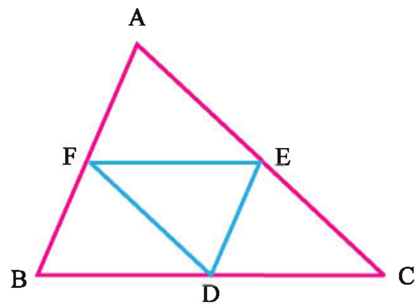

Figure 6.37

Solution : Suppose D, E, F are the mid-points of the sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ of ΔABC respectively.

As proved in example 22, $\Delta DEF \sim \Delta ABC$ for the correspondence DEF $\leftrightarrow$ ABC.

Areas of similar triangles are proportional to the squares of the corresponding sides.

$\therefore \quad \frac{DEF}{ABC} = \frac{EF^2}{BC^2}$

As E and F are the mid-points of sides $\overline{AC}$ and $\overline{AB}$ respectively $EF = \frac{1}{2}BC$.

$\therefore \quad BC = 2EF.$

$\therefore \quad \frac{DEF}{ABC} = \frac{EF^2}{(2EF)^2} = \frac{1}{4}$

Example 25 : $\square^m$ PQRS, Q–M–R and $QM = \frac{1}{4}QR$, $\overline{PM}$ and $\overline{QS}$ intersect in N. Find the ratio of area of ΔPNS to the ratio of area of ΔMNQ.

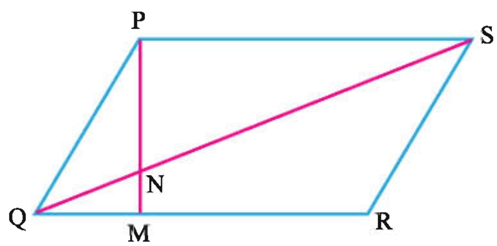

Figure 6.38

Solution : $\square$PQRS is a parallelogram.

$\therefore \quad \overline{PS} \parallel \overline{QR}$ and PS = QR

$\therefore \quad \overline{PS} \parallel \overline{QM}$ **(Q–M–R)**

$\overleftrightarrow{QS}$ is a transversal.

$\therefore \quad \angle PSQ \cong \angle SQM$

$\therefore \quad \angle PSN \cong \angle MQN$ **($\overrightarrow{SQ} = \overrightarrow{SN}$, $\overrightarrow{QN} = \overrightarrow{QS}$)**

$\angle PNS \cong \angle MNQ$ **(vertically opposite angles)**

$\therefore \quad \Delta PNS \sim \Delta MNQ$ for the correspondence PNS $\leftrightarrow$ MNQ.

$\therefore \quad \frac{PNS}{MNQ} = \frac{PS^2}{QM^2} = \frac{QR^2}{QM^2} = \left(\frac{QR}{QM}\right)^2$

But $QM = \frac{1}{4}QR$

$\therefore \quad \frac{QR}{QM} = \frac{4}{1}$

$\therefore \quad \frac{PNS}{MNQ} = \left(\frac{4}{1}\right)^2 = \frac{16}{1}$

Example 26 : In ΔABC, $m\angle B = 90$. $\overline{BD}$ is an altitude. Prove that

(1) ΔABD and ΔBCD are similar triangle.
(2) ΔABD and ΔABC are similar triangle.
(3) ΔCBD and ΔABC are similar triangle.

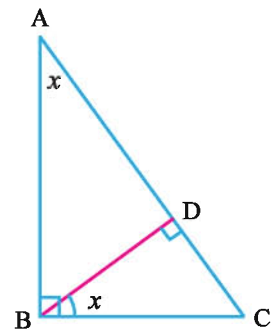

Figure 6.39

Solution : In ABC, $m\angle ABC = 90$ and $\overline{BD} \perp \overline{AC}$ and $D \in \overline{AC}$.

Suppose, $m\angle CAB = m\angle BAD = x$ **(A–D–C) (i)**

$\therefore$ $m\angle ACB = m\angle DCB = 90 - x$ **(A–D–C)**

In ΔBDC, $m\angle BDC = 90$ and $m\angle DCB = 90 - x$

$\therefore$ $m\angle CBD = 180 - 90 - (90 - x) = x$ **(ii)**

$m\angle ABC = 90$, $m\angle DBC = x$ and A–D–C

$\therefore$ $m\angle ABD = 90 - x$

(1) In ΔABD and ΔBCD, consider the correspondence $ABD \leftrightarrow BCD$.

$\angle BAD \cong \angle CBD$ **(from (i) and (ii))**

$\angle BDA \cong \angle CDB$ **(both right angles)**

$\therefore$ The correspondence $ABD \leftrightarrow BCD$ is similarity. **(AA)**

$\therefore$ ΔABD and ΔBCD are similar.

(2) In ΔABD and ΔABC, consider the correspondence $ADB \leftrightarrow ABC$.

$\angle BAD \cong \angle CAB$ **(from (i))**

$\angle ADB \cong \angle ABC$ **(both right angles)**

$\therefore$ The correspondence $ADB \leftrightarrow ABC$ is similarity.

$\therefore$ ΔADB and ΔABC are similar.

(3) In ΔCBD and ΔABC, consider the correspondence $DBC \leftrightarrow BAC$.

$\angle BDC \cong \angle ABC$ **(both right angles)**

$\angle DBC \cong \angle BAC$ **((i) and (ii))**

$\therefore$ The correspondence $DBC \leftrightarrow BAC$ is similarity.

$\therefore$ ΔCBD and ΔABC are similar.

Example 27 : In convex $\square ABCD$, diagonals intersect in O. If $OA \cdot OD = OB \cdot OC$, prove that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

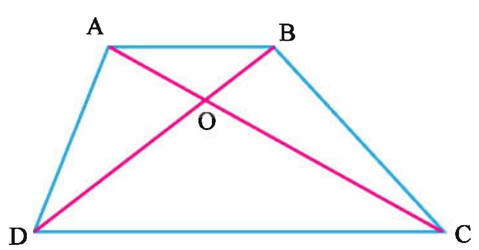

Figure 6.40

Solution : In convex $\square ABCD$, diagonals $\overline{AC}$ and $\overline{BD}$ intersect at O.

$OA \cdot OD = OB \cdot OC$

$\therefore$ $\frac{OA}{OC} = \frac{OB}{OD}$ **(i)**

In ΔAOB and ΔCOD. Consider the correspondence $AOB \leftrightarrow COD$.

$\frac{OA}{OC} = \frac{OB}{OD}$ **(from (i))**

$\angle AOB \cong \angle COD$ **(vertically opposite angles)**

$\therefore$ The correspondence $AOB \leftrightarrow COD$ is a similarity. **(SAS)**

∴ $\angle BAO \cong \angle DCO$

∴ $\angle BAC \cong \angle DCA$ $(\overrightarrow{AO} = \overrightarrow{AC}, \overrightarrow{CO} = \overrightarrow{CA})$

∴ Alternate angles of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ by transversal $\overleftrightarrow{AC}$ are congruent.

∴ $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

EXERCISE 6.4

1. $\angle B$ is a right angle in ΔABC and $\overline{BD}$ is an altitude to hypotenuse. AB = 8, BC = 6. Find the area of ΔBDC.
2. In $\square^m$ ABCD, T ∈ $\overline{BC}$ and $\overrightarrow{AT}$ intersects $\overline{BD}$ in M and $\overrightarrow{DC}$ in O. Prove that $AM^2 = MT \cdot MO$.
3. In $\square^m$ ABCD, M is the mid-point of $\overline{BC}$. $\overrightarrow{DM}$ and $\overrightarrow{AB}$ intersect in N. Prove that DN = 2MN.
4. P and Q are the mid-points of $\overline{AB}$ and $\overline{AC}$ in ΔABC. If the area of $\Delta APQ = 12\sqrt{3}$, find the area of ΔABC.
5. $\square$ABCD is a rhomus. $\overline{AC} \cap \overline{BD} = \{O\}$. Prove that the area of $\Delta OAB = \frac{1}{4}$(area of $\square$ABCD).
6. In $\square$PQRS, $\overline{PR} \cap \overline{QS} = \{T\}$, PS = QR, $\overline{PQ} \parallel \overline{RS}$. Prove that ΔTPS is similar to ΔQTR.
7. In ΔABC, a line parallel to $\overline{BC}$, passes through the mid-point of $\overline{AB}$. Prove that the line bisects $\overline{AC}$.
8. P, Q, R are the mid-points of the sides of ΔABC. X, Y, Z are the mid-points of the sides of ΔPQR. If the area of ΔXYZ is 10, find the area of ΔPQR and the area of ΔABC.
9. In ΔABC, X ∈ $\overline{BC}$, Y ∈ $\overline{CA}$, Z ∈ $\overline{AB}$ such that $\overline{XY} \parallel \overline{AB}$, $\overline{YZ} \parallel \overline{BC}$, $\overline{ZX} \parallel \overline{AC}$. Prove that X, Y, Z are the mid-points of $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ respectively.
10. Two triangles are similar. Prove that if sides in one pair of corresponding sides are congruent, then the triangles are congruent.

*

Example 28 : A line drawn in the plane of ΔABC intersects $\overline{AB}$ in F, $\overline{AC}$ in E and $\overleftrightarrow{BC}$ in D. Prove that $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$.

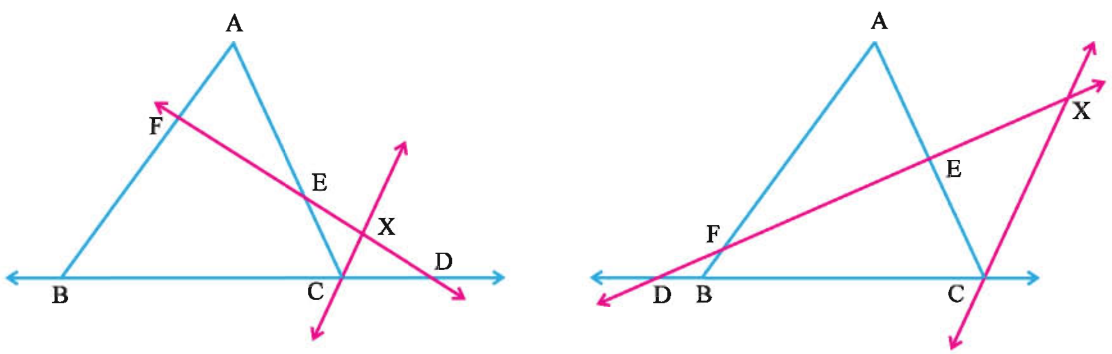

Figure 6.41

Solution : Let us draw a line passing through C and parallel to $\overline{AB}$. Let the line meet $\overleftrightarrow{EF}$ in X.

E ∈ $\overline{AC}$ and F ∈ $\overline{AB}$

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$\therefore$ $\overleftrightarrow{EF}$ does not intersect $\overline{BC}$ but it intersects $\overleftrightarrow{BC}$ in D.

There are two cases, (1) B–C–D or (2) D–B–C

Following arguments are true for both the cases.

$\frac{BD}{DC} = \frac{BF}{CX}$ ($\Delta BDF \sim \Delta CDX$ as $\overleftrightarrow{CX} \parallel \overleftrightarrow{AB}$)

$\frac{CE}{EA} = \frac{CX}{AF}$ ($\Delta CEX \sim \Delta AEF$)

Multiplying the results on the respective sides, we get

$$\frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{BF}{CX} \cdot \frac{CX}{AF}$$

$$\therefore \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{FB}{AF}$$

$$\therefore \frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$

Remarks : The result proved in example 28 is a famous theorem known as Menalaus' theorem. Here a particular case is discussed in which the line drawn in the plane of the triangle intersects two sides of the triangle and does not intersect the third side but it intersects the line containing the third side.

EXERCISE 6

1. In ΔABC, P $\in$ $\overline{AB}$ such that $\frac{AP}{PB} = \frac{m}{n}$, m, n are positive real numbers. A line passing through P and parallel to $\overline{BC}$ intersects $\overline{AC}$ in Q. Prove that $(m + n)^2$ (area of ΔAPB) = m^2(area of ΔABC)

2. D, E and F are the mid-points of $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ respectively in ΔABC.

Prove that the area of $\square$BDEF = $\frac{1}{2}$area of ΔABC.

3. Can two similar triangles have same area ? If yes, in which case they have the same area ?

4. The correspondence ABC $\leftrightarrow$ DEF is similarity in ΔABC and ΔDEF. $\overline{AM}$ is an altitude of ΔABC and $\overline{DN}$ is an altitude of ΔDEF. Prove that AB $\times$ DN = AM $\times$ DE.

5. **Explain with reasons, whether the following statements are true or false :**

(1) AAA criterion of similarity of triangles can not be the criterion for congruence of triangles.

(2) SAS criterion for congruence of triangles can not be a criterion for similarity of triangles.

(3) Two congruent triangles have the same area.

(4) Two similar triangles always have the same area.

(5) Area of similar triangles are proportional to the squares of measures of their corresponding angles.

6. **Fill in the blanks so that the following statements are true :**

(1) $\overline{AD}$ and $\overline{BE}$ are the altitudes of ΔABC. If AB = 12, AC = 9.9, AD = 8.1 and BE = 7.2, perimeter of ΔABC = ...

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(2) D, E, F are respectively the mid-points of $\overline{PQ}$, $\overline{QR}$, $\overline{PR}$ of ΔPQR. The correspondence DEF ↔ ...... is similarity.

(3) In ΔABC, $\overline{AM}$ and $\overline{CN}$ are altitudes. If AB = 12, BC = 15 and AM = 9.6, then CN = ...

(4) Areas of two similar triangles are 25 and 16. The ratio of the perimeters of the triangles is ....

(5) Area of ΔABC = 36 and area of ΔPQR = 64. The correspondence ABC ↔ PQR is a similarity. If AB = 12, then PQ = ...

(6) In ΔABC, A–M–B, A–N–C and $\overline{MN} \parallel \overline{BC}$. If AM = 8.4, AN = 6.4, CN = 19.2, then AB = ...

(7) The correspondence ABC ↔ PQR is a similarity in ΔABC and ΔPQR. AB = 16, AC = 8, PQ = 24, BC = 12, then QR + PR = ...

(8) The correspondence ABC → XYZ is a similarity in ΔABC and ΔXYZ. ABC = 72, BC = 6, YZ = 10. Then XYZ = ...

(9) □ABCD is trapezium in which $\overline{AD} \parallel \overline{BC}$. The diagonals intersect in P. If PD = 9, AP = 5, PB = 7.2, then AC = ...

(10) In ΔABC, $m\angle B$ = 90 and $\overline{BD}$ is an altitude. The correspondence BDA ↔ ...... between ΔBDA and ΔBDC is a similarity.

(11) The correspondence ABC ↔ ZXY is a similarity in ΔABC and ΔXYZ. If AB = 12, BC = 9, CA = 7.5 and ZX = 10, then YZ + XY = ...

7. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on right so that the statement becomes correct :

(1) ABC ↔ DEF is a similarity in ΔABC and ΔDEF, $m\angle A$ = 40, $m\angle E + m\angle F$ = ... ☐

(a) 40 (b) 80 (c) 140 (d) 180

(2) In ΔABC, M ∈ $\overline{AB}$, N ∈ $\overline{AC}$ such that $\overline{MN} \parallel \overline{BC}$. ...... is not true. ☐

(a) AN · MB = AM · NC (b) AM · MB = AN · NC

(c) AB · AN = AM · AC (d) AB · NC = AC · MB

(3) In ΔABC, B–M–C and A–N–C, $\overline{MN} \parallel \overline{AB}$. If NC : NA = 1 : 3 and CM = 4, then BC = ... ☐

(a) 12 (b) 16 (c) 8 (d) $\frac{1}{2}$

(4) In ΔXYZ and ΔPQR, XYZ ↔ PQR is similarity. XY = 12, YZ = 8, ZX = 16, PR = 8. So, PQ + QR = ... ☐

(a) 20 (b) 10 (c) 15 (d) 9

(5) The bisector of $\angle P$ in ΔPQR intersects $\overline{QR}$ in D. If QD : RD = 4 : 7 and PR = 14, then PQ = ...

(a) 8 (b) 4 (c) 12 (d) 16

(6) The lengths of the sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ of ΔABC are in the ratio 3 : 4 : 5. Correspondence ABC $\leftrightarrow$ PQR is similarity. If PR = 12, the perimeter of ΔPQR is ....

(a) 12 (b) 36 (c) 24 (d) 27

(7) The correspondence ABC $\leftrightarrow$ YZX in ΔABC and ΔXYZ is similarity. $m\angle B + m\angle C = 120$. So, $m\angle Y$ = ...

(a) 70 (b) 55 (c) 110 (d) 60

(8) Correspondence ABC $\leftrightarrow$ DEF of ΔABC and ΔDEF is similarity. If AB + BC = 10 and DE + EF = 12 and AC = 6, then DF = ...

(a) 6 (b) 5 (c) 7.2 (d) 16

(9) The lengths of the sides of ΔDEF are 4, 6, 8. $\Delta DEF \sim \Delta PQR$ for correspondence DEF $\leftrightarrow$ QPR. If the perimeter of ΔPQR = 36, then the length of the smallest side of ΔPQR is ....

(a) 6 (b) 2 (c) 4 (d) 8

(10) The bisector of $\angle B$ intersects $\overline{AC}$ in D. If BA = 12 and BC = 16, AD = 9, then AC = ...

(a) 15 (b) 21 (c) 18 (d) 8

(11) In ΔABC, $\overline{PQ} \parallel \overline{BC}$, P $\in \overline{AB}$, Q $\in \overline{AC}$. If 4AP = AB and AQ = 4, then AC = ...

(a) 12 (b) 4 (c) 8 (d) 16

(12) In ΔABC, the correspondence ABC $\leftrightarrow$ BAC is similarity. ...... of the following is true.

(a) $\angle B \cong \angle C$ (b) $\angle C \cong \angle A$ (c) $\angle A \cong \angle B$ (d) $\angle A \cong \angle B \cong \angle C$

(13) In ΔABC, A–P–B, A–Q–C and $\overline{PQ} \parallel \overline{BC}$. If PQ = 5, AP = 4, AB = 12, then BC = ...

(a) 9.6 (b) 20 (c) 15 (d) 5

(14) In ΔPQR, P–M–Q and P–N–R. If PQ = 18, PM = 12, PR = 9 and NR = ..., then $\overline{MN} \parallel \overline{QR}$.

(a) $\frac{27}{2}$ (b) 3 (c) 24 (d) 6

*

Summary

In this chapter we have studied the following points :

1. Two figures having the same shape but not necessarily the same size are called similar figures.
2. Congruent triangles are similar but similar triangles may or may not be congruent.
3. If two triangles are similar, for some correspondence between their vertices, then the corresponding angles are congruent and corresponding sides are proportional.
4. If a line drawn parallel to a side of a triangle intersects other two sides, then it divides the other two sides in the same ratio.
5. If a line divides any two sides of a triangle in the same ratio, then the line is parallel to the third side of the triangle.
6. If in two triangles for some correspondence between their vertices, the corresponding angles are congruent, then the triangles are similar. (AAA) In fact, if any two corresponding angles are congruent, the triangles are similar. (AA)
7. If in two triangles, for some correspondence between their vertices, two corresponding sides are proportional and the angles included between those sides are congruent, then the triangles are similar. (SAS)
8. If in two triangles, for some correspondence between their vertices, the corresponding sides are proportional, then the triangles are similar. (SSS)
9. The areas of two similar triangles are proportional to the squares of their corresponding sides.

◆

D. R. Kaprekar received his secondary school education in Thane and studied at Fergusson College in Pune. In 1927 he won the Wrangler R. P. Paranjpe Mathematical Prize for an original piece of work in mathematics.

He attended the University of Mumbai, receiving his bachelor's degree in 1929. Having never received any formal postgraduate training, for his entire career (1930–1962) he was a schoolteacher at Nashik in Maharashtra, India. He published extensively, writing about such topics as recurring decimals, magic squares, and integers with special properties.

Discoveries :

Working largely alone, Kaprekar discovered a number of results in number theory and described various properties of numbers. In addition to the Kaprekar constant and the Kaprekar numbers which were named after him, he also described self numbers or Devlali numbers, the Harshad numbers and Demlo numbers. He also constructed certain types of magic squares related to the Copernicus magic square. Initially his ideas were not taken seriously by Indian mathematicians, and his results were published largely in low-level mathematics journals or privately published, but international fame arrived when Martin Gardner wrote about Kaprekar in his March 1975 column of Mathematical Games for Scientific American. Today his name is well-known and many other mathematicians have pursued the study of the properties he discovered.

SIMILARITY AND THE THEOREM OF PYTHAGORAS 7

It is not enough to have a good mind. The main thing is to use it well.

– Rene Des Cartes

7.1 Introduction

"Which theorem of geometry is famous even among those persons who do not know geometry ?" Ask this question to any one you meet, not necessarily to a mathematics teacher or an engineer or a doctor or a science graduate but to any body including a bank employee, a lawyer or a merchant who must have completed his school studies. You will be surprised to hear the name "Pythagoras Theorem".

This theorem was invented almost three thousand years ago. It is known to the world as "Pythagoras Theorem" but it was invented independently in all ancient civilization including the civilization that flourished in the plane of rivers, Sindhu and Ganga-Yamuna. Many prominent mathematicians are fascinated by this theorem. That is why more than 370 independent proofs of this theorem are available. Do you surf internet ? Type the word "Pythagoras Theorem" in any search engine. You will get more than 100 pages related to this theorem. In this chapter we are going to study this theorem and some results associated with this theorem.

7.2 Similarity and Right Angled Triangle

We are going to use the concept of similarity to prove the "Theorem of Pythagoras". This theorem is essentially an important property of right angled triangle.

If one angle of triangle is a right angle, the remaining two angles are acute angles. If an altitude is drawn on the hypotenuse from the vertex where right angle is formed, two triangles are formed in its different semiplanes. We are going to use the relation between these two triangles and also their relation with the given triangle. We are going to accept the following theorem without proof.

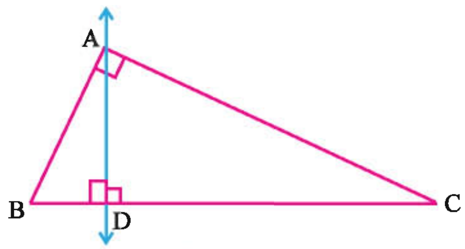

Figure 7.1

Theorem 7.1 : If an altitude is drawn to the hypotenuse of a right angled triangle, then the triangles formed in the different closed semiplanes of the altitude are similar to the given triangle and also they are similar to each other.

In other words, **If in ΔABC, $\angle A$ is right angle and $\overline{AD} \perp \overline{BC}$, $D \in \overline{BC}$, then $\Delta ADB \sim \Delta ABC$, $\Delta ADC \sim \Delta ABC$ and $\Delta ADB \sim \Delta ADC$.**

First we note that all the three triangles are right angled triangles. In a right angled triangle, sum of measures of two acute angles is 90. So if we assume that $m\angle ACD = m\angle ACB = x$, then $m\angle DAC = 90 - x$ and due to B–D–C, $m\angle BAD = 90 - (90 - x) = x$.

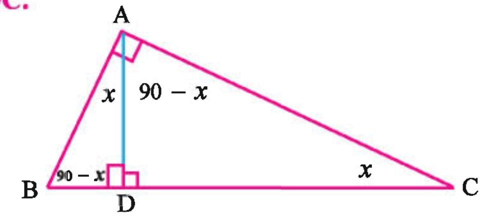

Figure 7.2

Hence, correspondence ADB $\leftrightarrow$ CAB is similarity.
correspondence ADC $\leftrightarrow$ BAC is similarity.
and correspondence ADB $\leftrightarrow$ CDA is similarity.
This is the brief out line of the proof of theorem 7.1.

Geometric Mean

If x, y, z are positive real numbers and if $\frac{x}{y} = \frac{y}{z}$ (i.e. $y^2 = zx$), then y is called the geometric mean of x and z. In other words geometric mean of two positive numbers a and b is $\sqrt{ab}$. We generally denote the geometric mean by the letter G. It can be shown that if $a < b$, then the geometric mean G of a and b satisfies the inequality $a < G < b$.

Adjacent Segments : In ΔABC, if $\overline{AD} \perp \overline{BC}$, $D \in \overline{BC}$, then $\overline{BD}$ is called the segment adjacent to $\overline{AB}$ and $\overline{CD}$ is called the segment adjacent to $\overline{AC}$.

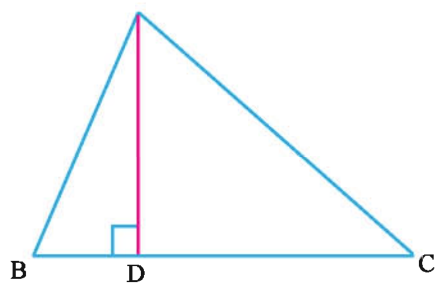

Figure 7.3

Here in ΔABC, $\angle A$ may be acute angle or right angle or obtuse angle.

As an immediate consequence of theorem 7.1 we have the following corollary, which we will accept without giving a formal proof.

Corollary 1 : If an altitude is drawn to hypotenuse of a right angled triangle, then (1) length of altitude is the geometric mean of lengths of segments of hypotenuse formed by the altitude (2) length of each side other than the hypotenuse is the geometric mean of length of hypotenuse and segment of hypotenuse adjacent to the side.

In other words,

if in ΔABC, $\angle A$ is a right angle and $\overline{AD} \perp \overline{BC}$, $D \in \overline{BC}$, then

(1) $AD^2 = BD \cdot DC$

(2) (i) $AB^2 = BD \cdot BC$

(ii) $AC^2 = CD \cdot BC$

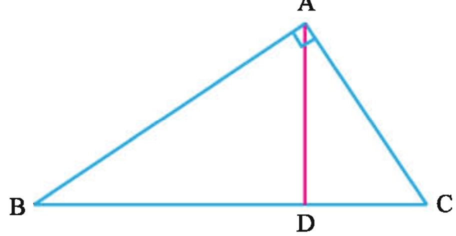

Figure 7.4

In theorem 7.1, we have seen that the correspondence ADB $\leftrightarrow$ CDA of ΔADB and ΔADC is a similarity.

$\therefore \frac{AD}{CD} = \frac{DB}{AD}$

$\therefore AD^2 = BD \cdot DC$

Also in ΔADB and ΔABC the, correspondence ADB $\leftrightarrow$ CAB is a similarity.

$\therefore \frac{AB}{BC} = \frac{BD}{AB}$

$\therefore AB^2 = BD \cdot BC$

In ΔADC and ΔABC the, correspondence ADC $\leftrightarrow$ BAC is a similarity.

$\therefore \frac{AC}{BC} = \frac{DC}{AC}$

$\therefore AC^2 = DC \cdot BC$

Now we are fully equiped to prove the famous theorem invented by Pythagoras, a greek geometrician and a student of Thales, the father of geometry. The proof given in the text books in modern days is not the proof given by Pythagoras and documented in Euclid's Elements. The proof given here is based on the concept of similarity of triangles as seen in theorem 7.1 and the corollary.

7.3 Theorem of Pythagoras

Pythagoras' theorem is famous because of its wide range of applications. We have used Pythagoras theorem to construct the line segments whose lengths are irrational numbers like $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{17}$ etc. Trigonometric ratios are defined using right angled triangle. We use Pythagoras theorem to prove the identity $sin^2\theta + cos^2\theta = 1$ (Chapter 9). In ancient Indian civilization, "Sulb Sutras" written by **Bodhayan** (800 BC) depict Pythagoras theorem. Bhaskaracharya and Brahmagupta gave different proofs of Pythagoras Theorem. Leonardo De Vinchi, the great artist, sculpturist, architect, famous for his painting "Monalisa" also gave a beautiful proof for this theorem.

Theorem 7.2 : Pythagoras Theorem : Square of the length of the hypotenuse of a right angled triangle is the sum of the squares of the lengths of other two sides.

Given : $\angle A$ is right angle in ΔABC.

To prove : $BC^2 = AB^2 + AC^2$

Proof : Let $\overline{AD} \perp \overline{BC}$, $D \in \overline{BC}$

$\angle A$ is right angle in ΔABC.

$\therefore$ $\angle B$ and $\angle C$ of ΔABC are acute angles.

$\therefore$ B–D–C

$\therefore$ $BD + DC = BC$ **(i)**

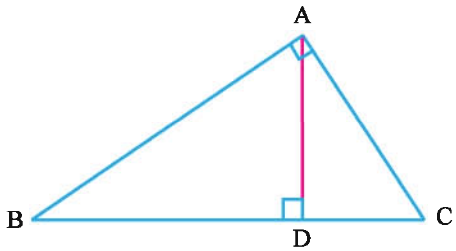

Figure 7.5

Now using corollary of theorem 7.1,

We have $AB^2 = BD \cdot BC$

and $AC^2 = DC \cdot BC$

$\therefore$ $AB^2 + AC^2 = BD \cdot BC + DC \cdot BC$

$= (BD + DC) \cdot BC$

$= BC \times BC$

$= BC^2$

Converse of Theorem of Pythagoras :

If three sides of a triangle are given, we can construct the triangle.

Let us construct a triangle in which AB = 3, BC = 4 and AC = 5. Now measure $\angle B$.

Let us construct another ΔPQR in which PQ = 5, QR = 12 and PR = 13. Measure $\angle Q$.

We will observe that in these triangles, $\angle B$ and $\angle Q$ are right angles.

Did you observe that in ΔABC, $AC^2 = AB^2 + BC^2$?

Did you note that in ΔPQR, $PR^2 = PQ^2 + QR^2$?

These activities lead us to deduce that whenever in a triangle, square of a side equals the sum of the squares of the other two sides, the triangle is a right angled triangle. That is the converse of the Pythagoras' Theorem is true.

Let us prove it.

Theorem 7.3 : Converse of Pythagoras' Theorem : In a triangle, if the square of a side is equal to the sum of the squares of other two sides, then the angle opposite to the first side is a right angle.

In other words,

If $BC^2 = AB^2 + AC^2$ in ΔABC, then $\angle A$ (opposite to $\overline{BC}$) is a right angle.

Data : $BC^2 = AB^2 + AC^2$ in ΔABC.

To prove : $\angle A$ is right angle.

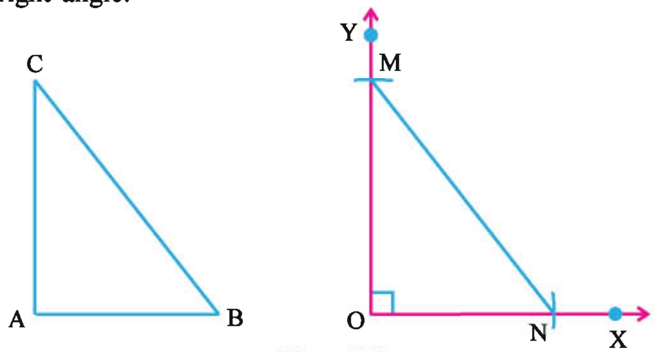

Figure 7.6

Proof : Let $\overrightarrow{OX}$ be any ray.

We can construct $\overrightarrow{OY}$ such that $\overrightarrow{OY} \perp \overrightarrow{OX}$.

Let $M \in \overrightarrow{OY}$ such that $OM = AC$.

Let $N \in \overrightarrow{OX}$ such that $ON = AB$.

Draw $\overline{MN}$.

ΔOMN is a right angled triangle, as $\overrightarrow{OM} \perp \overrightarrow{ON}$ **($M \in \overrightarrow{OY}$, $N \in \overrightarrow{OX}$)**

$\angle MON$ is right angle.

$\therefore$ $\overline{MN}$ is the hypotenuse.

$\therefore$ According to Pythagoras Theorem.

$MN^2 = OM^2 + ON^2 = AC^2 + AB^2$

But $AB^2 + AC^2 = BC^2$ **(Data)**

$\therefore$ $MN^2 = BC^2$

$\therefore$ $MN = BC$ **(i)**

$\therefore$ In ΔABC and ΔONM consider the correspondence $ABC \leftrightarrow ONM$, we have

$\overline{AB} \cong \overline{ON}$ **(ON = AB)**

$\overline{AC} \cong \overline{OM}$ **(OM = AC)**

$\overline{BC} \cong \overline{MN}$ **(BC = MN)**

$\therefore$ The correspondence $ABC \leftrightarrow ONM$ is a congruence. So, $\Delta ABC \cong \Delta ONM$ **(SSS)**

$\therefore$ $\angle A \cong \angle O$

but $\angle O$ in ΔONM is a right angle by construction.

$\angle A$ is a right angle.

Now let us solve some examples.

Example 1 : In ΔPQR, $m\angle Q = 90$ and $\overline{QM}$ is an altitude and $M \in \overline{RP}$. If $QM = 12$, $PR = 26$. Find PM and RM. If $PM < RM$, find PQ and QR.

Solution : In ΔPQR, $\overline{QM}$ is an altitude and

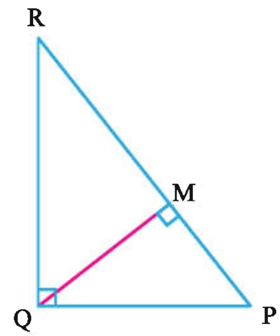

Figure 7.7

$\therefore$ $m\angle Q = 90$

$\therefore$ $M \in \overline{PR}$ and P–M–R.

Let $MP = x$.

$\therefore$ $RM = PR - MP = 26 - x$ **(PR = 26)**

Now, $QM^2 = PM \cdot RM$

$\therefore$ $12^2 = x(26 - x)$ **(QM = 12)**

$\therefore$ $x^2 - 26x + 144 = 0$

$\therefore$ $(x - 8)(x - 18) = 0$

$\therefore$ $x = 8$ or $x = 18$

$\therefore$ $PM = 8$ or $PM = 18$

Correspondingly $RM = 26 - 8 = 18$ or $RM = 26 - 18 = 8$ **(i)**

But $PM < RM$.

$\therefore$ $PM = 8$, $RM = 18$ and $PR = 26$

$PQ^2 = PM \cdot PR = 8 \times 26 = 16 \times 13$

$\therefore$ $PQ = 4\sqrt{13}$

$QR^2 = RM \cdot PR = 18 \times 26 = 36 \times 13$

$\therefore$ $QR = 6\sqrt{13}$

Example 2 : In ΔABC, $m\angle B = 90$, $\overline{BM} \perp \overline{AC}$, $M \in \overline{AC}$. If $AM = x$, $BM = y$, find AB, BC and CM in terms of x and y. $(x > 0, y > 0)$

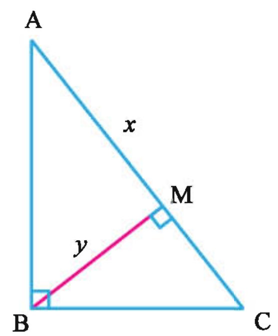

Figure 7.8

Solution : We have, $BC^2 = CM \cdot AC$ **(i)**

$AB^2 = AM \cdot AC$ **(ii)**

$AB^2 = AM^2 + BM^2$ **(iii)**

$AC^2 = AB^2 + BC^2$ **(iv)**

Here $AM = x$, $BM = y$

$AB^2 = AM^2 + BM^2 = x^2 + y^2$ **(using (iii))**

$\therefore$ $AB = \sqrt{x^2 + y^2}$ **(v)**

Using (ii), we have $AB^2 = AM \cdot AC$

$\therefore \quad x^2 + y^2 = x\,AC$

$\therefore \quad AC = \dfrac{x^2 + y^2}{x}$

$\therefore \quad CM = AC - AM = \dfrac{x^2 + y^2}{x} - x = \dfrac{y^2}{x}$ **(vi)**

$BC^2 = CM \cdot AC = \dfrac{y^2}{x}\left(\dfrac{x^2 + y^2}{x}\right) = \dfrac{y^2(x^2 + y^2)}{x^2}$ **(using (i) and (vi))**

$\therefore \quad BC = \dfrac{y}{x}\sqrt{x^2 + y^2}$ **(vii)**

Thus $AB = \sqrt{x^2 + y^2}$, $BC = \dfrac{y}{x}\sqrt{x^2 + y^2}$, $CM = \dfrac{y^2}{x}$

Example 3 : In right angled ΔPQR, $\angle P$ is a right angle and $\overline{PM}$ is the altitude on the hypotenuse. If PQ = 8, PR = 6, find PM.

Solution : $\angle P$ is a right angle in ΔPQR.

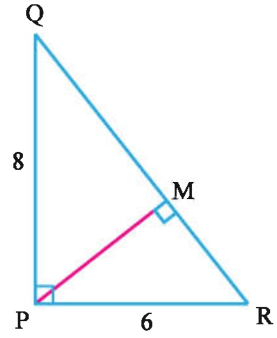

Figure 7.9

$\therefore \quad PQ^2 + PR^2 = QR^2$. Also PR = 6 and PQ = 8.

$\therefore \quad QR^2 = 6^2 + 8^2 = 100$

$\therefore \quad QR = 10$

$PQ^2 = QM \cdot QR$

$QM = \dfrac{PQ^2}{QR} = \dfrac{64}{10} = 6.4$

$\therefore \quad RM = QR - QM = 10 - 6.4 = 3.6$

$PM^2 = QM \cdot MR = 6.4 \times 3.6$

$\therefore \quad PM^2 = \dfrac{(36)(64)}{100}$

$\therefore \quad PM = \dfrac{6 \times 8}{10} = 4.8$

Example 4 : In ΔPQR, $\angle Q$ is a right angle. $PR - PQ = 9$ and $PR - QR = 18$. Find the perimeter of ΔPQR.

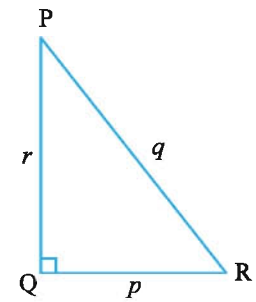

Figure 7.10

Solution : In ΔPQR, $\angle Q$ is a right angle.

Let $PQ = r$, $PR = q$, $QR = p$; $p, q, r > 0$

Now, $PR - PQ = 9$ and $PR - QR = 18$

$\therefore \quad q - r = 9$ **(i)**

$q - p = 18$ **(ii)**

Also using the theorem of Pythagoras, $PQ^2 + QR^2 = PR^2$

$\therefore \quad r^2 + p^2 = q^2$ **(iii)**

From (i) $r = q - 9$ and from (ii) $p = q - 18$

Substituting in (iii) we get,

$(q - 9)^2 + (q - 18)^2 = q^2$

$\therefore \quad q^2 - 54q + 405 = 0$

$(q - 45)(q - 9) = 0$

$q \neq 9$ **(if $q = 9$ then $r = q - 9 = 0$)**

$\therefore$ $q = 45$

$\therefore$ From (i) and (ii) $r = 36$, $p = 27$

$\therefore$ The perimeter of $\Delta PQR = PQ + QR + PR$

$= r + p + q = 36 + 27 + 45 = 108$

$\therefore$ The perimeter of ΔPQR is 108.

Example 5 : Lengths of sides $\overline{AB}$, $\overline{BC}$, $\overline{AC}$, of ΔABC are given below. In each case determine whether ΔABC is right angled triangle or not. Also state which angle is a right angle, when the triangle is a right angled triangle.

(1) AB = 25, BC = 7, AC = 24

(2) AB = 8, BC = 6, AC = 3

(3) AB = 8, BC = 6, AC = 10

(4) AB = 4, BC = 5, AC = 6

Solution : In right angled triangle hypotenuse is the greatest side.

$\therefore$ If ΔABC is right angled triangle, then the side of the greatest length is hypotenuse.

(1) AB = 25, BC = 7, AC = 24

If ΔABC is right angled triangle, then $\overline{AB}$ must be hypotenuse.

$AB^2 = 25^2 = 625$, $BC^2 + AC^2 = 7^2 + 24^2 = 49 + 576 = 625$

$\therefore$ $AB^2 = BC^2 + AC^2$

$\therefore$ ΔABC is right triangle and $\angle C$ is a right angle.

(2) AB = 8, BC = 6, AC = 3

If A, B, C form a right angled triangle, we should have $AB^2 = BC^2 + AC^2$.

$AB^2 = 8^2 = 64$, $BC^2 + AC^2 = 6^2 + 3^2 = 45$

$\therefore$ $AB^2 \neq BC^2 + AC^2$

$\therefore$ ΔABC is not a right angled triangle.

(3) AB = 8, BC = 6, AC = 10

If A, B, C form a right angled triangle, we should have $AC^2 = AB^2 + BC^2$.

$AC^2 = 10^2 = 100$, $AB^2 + BC^2 = 8^2 + 6^2 = 64 + 36 = 100$

$\therefore$ $AC^2 = AB^2 + BC^2$

$\therefore$ ΔABC is a right triangle in which $\angle B$ is right angle.

(4) AB = 4, BC = 5, AC = 6

$AC^2 = 6^2 = 36$, $AB^2 + BC^2 = 4^2 + 5^2 = 16 + 25= 41$

$\therefore$ $AC^2 \neq AB^2 + BC^2$

$\therefore$ ΔABC is not a right angled triangle.

Example 6 : In ΔABC, AC + BC = 28, AB + BC = 32 and AC + AB = 36. Determine whether ΔABC is a right angled triangle or not.

Solution : In ΔABC suppose AB = c, BC = a, AC = b

We are given AC + BC = 28 i.e. $b + a = 28$ **(i)**

AB + BC = 20 i.e. $c + a = 32$ **(ii)**

AC + AB = 24 i.e. $b + c = 36$ **(iii)**

From (i), (ii) and (iii) (by addition)

$2a + 2b + 2c = 28 + 32 + 36 = 96$

$\therefore \quad a + b + c = 48$

But $a + b = 28$

$\therefore \quad c = 48 - 28 = 20$

From (ii), we get $a = 12$ and from (iii) we get $b = 16$

$\therefore \quad a = BC = 12,\ b = AC = 16,\ c = AB = 20$

$\therefore \quad a^2 + b^2 = 12^2 + 16^2 = 144 + 256 = 400 = c^2$

$\therefore \quad a^2 + b^2 = c^2$

$\therefore \quad BC^2 + AC^2 = AB^2$

$\therefore$ In ΔABC, $BC^2 + AC^2 = AB^2$

$\therefore$ By the converse of the theorem of Pythagoras, ΔABC is a right angled triangle.

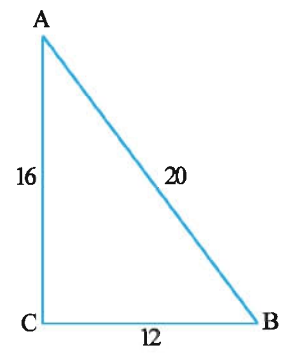

Figure 7.11

Example 7 : In ΔABC, $\overline{BM}$ is an altitude. $M \in \overline{AC}$ and $\angle B$ is a right angle. Prove that $\frac{AB^2}{BC^2} = \frac{AM}{CM}$.

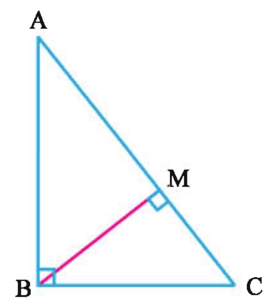

Figure 7.12

Solution : In ΔABC, $\angle B$ is a right angle and $\overline{BM} \perp \overline{AC}$, $M \in \overline{AC}$.

$\therefore$ We have $AB^2 = AM \cdot AC$ and $BC^2 = CM \cdot AC$

$$\therefore \quad \frac{AB^2}{BC^2} = \frac{AM \cdot AC}{CM \cdot AC}$$

$$\therefore \quad \frac{AB^2}{BC^2} = \frac{AM}{CM}$$

Example 8 : $m\angle B = 90$ in ΔABC and $\overline{BM} \perp \overline{AC}$, $M \in \overline{AC}$. If AM = 4MC, prove that AB = 2BC.

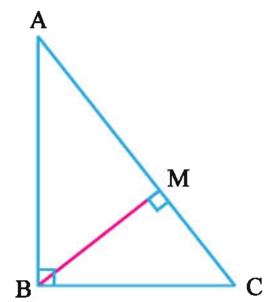

Figure 7.13

Solution : In ΔABC, $m\angle B = 90$ and $\overline{BM} \perp \overline{AC}$, $M \in \overline{AC}$.

$\therefore \quad AB^2 = AM \cdot AC$

$\quad\ \ BC^2 = CM \cdot AC$

$$\therefore \quad \frac{AB^2}{BC^2} = \frac{AM \cdot AC}{MC \cdot AC} = \frac{AM}{MC}$$

$$\therefore \quad \frac{AB^2}{BC^2} = \frac{4MC}{MC} = 4 \qquad \text{(AM = 4MC)}$$

$$\therefore \quad \frac{AB}{BC} = 2$$

$\therefore \quad AB = 2BC$

Example 9 : In ΔABC, $\angle B$ is a right angle and $\overline{BM}$ is an altitude and $M \in \overline{AC}$. If AB = 2AM, then prove that AC = 4AM.

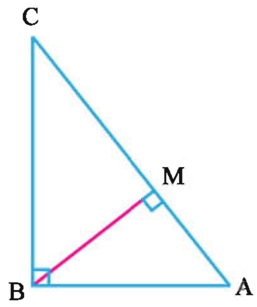

Figure 7.14

Solution : In ΔABC, $m\angle B = 90$

$\overline{BM} \perp \overline{AC}$, $M \in \overline{AC}$.

$\therefore$ $AB^2 = AM \cdot AC$

$\therefore$ $(2AM)^2 = AM \cdot AC$

$\therefore$ $4AM^2 = AM \cdot AC$

$\therefore$ $AC = 4AM$

Example 10 : In ΔABC, $m\angle A = 90$ and $\overline{AD} \perp \overline{BC}$, $D \in \overline{BC}$. Prove that $\frac{1}{AD^2} = \frac{1}{AB^2} + \frac{1}{AC^2}$.

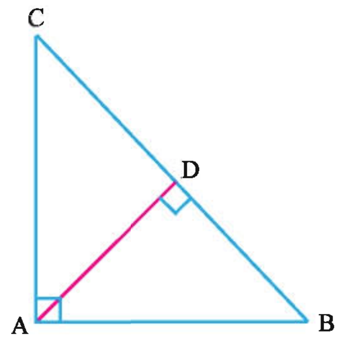

Figure 7.15

Solution : We know $AB^2 = BD \cdot BC$ **(i)**

We also know $AC^2 = DC \cdot BC$ **(ii)**

$\therefore$ $\frac{1}{AB^2} + \frac{1}{AC^2} = \frac{1}{BD \cdot BC} + \frac{1}{DC \cdot BC}$

$= \frac{DC + BD}{BD \cdot DC \cdot BC}$

$= \frac{BC}{BD \cdot DC \cdot BC} = \frac{1}{BD \cdot DC}$ **(iii)**

Also $AD^2 = BD \cdot DC$

$\frac{1}{AB^2} + \frac{1}{AC^2} = \frac{1}{AD^2}$

Example 11 : P is a point inside the rectangle ABCD, Prove that $PA^2 + PC^2 = PB^2 + PD^2$.

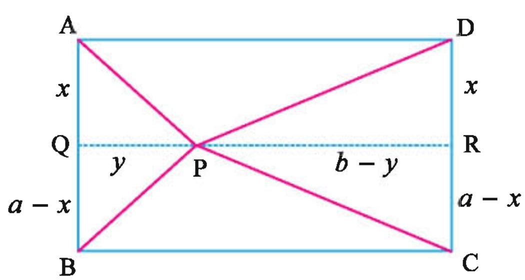

Figure 7.16

Solution : P is a point inside the rectangle ABCD.

Draw a line from P parallel to $\overline{AD}$ and let this line intersect $\overline{AB}$ and $\overline{CD}$ in Q and R respectively.

$\therefore$ A–Q–B, D–R–C.

Let AB = CD = a, AD = BC = b, AQ = DR = x.

$\therefore$ QB = CR = $a - x$ **(□ AQRD and □ QBCR are rectangles)**

Now, ΔAQP and ΔBQP are right angled triangles **($\angle Q$ is right angle)**

Similarly ΔPRD and ΔPRC are right angled triangles **($\angle R$ is right angle)**

Also let PQ = y so that PR = $b - y$ **(QR = AD = BC = b)**

$\therefore$ $PA^2 = x^2 + y^2$ and $PC^2 = (a - x)^2 + (b - y)^2$

$\therefore$ $PA^2 + PC^2 = x^2 + y^2 + (a - x)^2 + (b - y)^2$ **(i)**

$\therefore$ $PB^2 = (a - x)^2 + y^2$ and $PD^2 = (b - y)^2 + x^2$

$\therefore$ $PB^2 + PD^2 = x^2 + y^2 + (a - x)^2 + (b - y)^2$ **(ii)**

From (i) and (ii), $PA^2 + PC^2 = PB^2 + PD^2$

Example 12 : In ΔPQR, $m\angle Q = 90$, $PQ^2 - QR^2 = 260$. $\overline{QS}$ is the altitude to hypotenuse. $S \in \overline{PR}$. If $PS - SR = 10$, find PR.

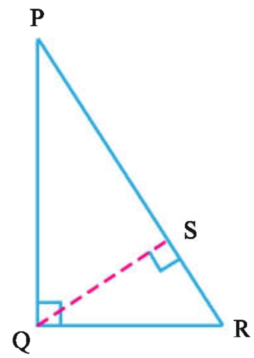

Figure 7.17

Solution : In ΔPQR, $\overline{QS}$ is the altitude on hypotenuse.

$\angle Q$ is a right angle.

We have, $PQ^2 = PS \cdot PR$

$QR^2 = SR \cdot PR$

$\therefore$ $PQ^2 - QR^2 = PR(PS - SR)$

But $PQ^2 - QR^2 = 260$ and $PS - SR = 10$

$\therefore$ $260 = PR(10)$

$\therefore$ $PR = \frac{260}{10} = 26$

EXERCISE 7.1

1. $\angle B$ is a right angle in ΔABC. $\overline{BD} \perp \overline{AC}$ and $D \in \overline{AC}$. If $AD = 4DC$, prove that $BD = 2DC$.
2. 5, 12, 13 are the lengths of the sides of a triangle. Show that the triangle is right angled. Find the length of altitude on the hypotenuse.
3. In ΔPQR, $\overline{QM}$ is the altitude to hypotenuse $\overline{PR}$. If $PM = 8$, $RM = 12$, find PQ, QR and QM.
4. In ΔABC, $m\angle B = 90$, $\overline{BM} \perp \overline{AC}$, $M \in \overline{AC}$. If $AM - MC = 7$, $AB^2 - BC^2 = 175$, find AC.
5. $\angle A$ is right angle in ΔABC. $\overline{AD}$ is an altitude of the triangle. If $AB = \sqrt{5}$, $BD = 2$, find the length of the hypotenuse of the triangle.
6. $m\angle B = 90$ in ΔABC. $\overline{BM}$ is altitude to $\overline{AC}$.
   (1) If $AM = BM = 8$, find AC.
   (2) If $BM = 15$, $AC = 34$, find AB.
   (3) If $BM = 2\sqrt{30}$, $MC = 6$, find AC.
   (4) If $AB = \sqrt{10}$, $AM = 2.5$, find MC.
7. In ΔPQR $m\angle Q = 90$, $PQ = x$, $QR = y$ and $\overline{QD} \perp \overline{PR}$. $D \in \overline{PR}$. Find PD, QD, RD in terms of x and y.
8. $\angle Q$ is a right angle in ΔPQR and $\overline{QM} \perp \overline{PR}$, $M \in \overline{PR}$. If $PQ = 4QR$, then prove that $PM = 16RM$.
9. $\square PQRS$ is a rectangle. If $PQ + QR = 7$ and $PR + QS = 10$, then find the area of $\square PQRS$.
10. The diagonals of a convex $\square ABCD$ intersect at right angles. Prove that $AB^2 + CD^2 = AD^2 + BC^2$.
11. In ΔPQR, $m\angle Q = 90$, $M \in \overline{QR}$ and $N \in \overline{PQ}$. Prove that $PM^2 + RN^2 = PR^2 + MN^2$.
12. The sides of a triangle have lengths $a^2 + b^2$, $2ab$, $a^2 - b^2$, where $a > b$ and $a, b \in R^+$. Prove that the angle opposite to the side having length $a^2 + b^2$ is a right angle.
13. In ΔABC, $m\angle B = 90$ and $\overline{BE}$ is a median. Prove that $AB^2 + BC^2 + AC^2 = 8BE^2$.
14. $AB = AC$ and $\angle A$ is right angle in ΔABC. If $BC = \sqrt{2}\,a$, then find the area of the triangle. $(a \in R, a > 0)$

*

Important result (Apolloneous Theorem) :

$\overline{AD}$ is a median of ΔABC. Prove that $AB^2 + AC^2 = 2(AD^2 + BD^2)$

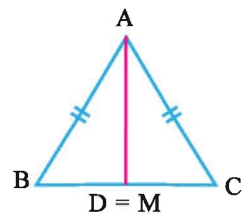

Figure 7.18(i)

Solution : Let $\overline{AM}$ be the altitude of ΔABC.

$\overline{AD}$ is a median. If $AB = AC$, then $D = M$.

Then $AD = AM$ and $\overline{AM} \perp \overline{BC}$

$AB^2 + AC^2 = 2AB^2$

$2(AD^2 + BD^2) = 2(AM^2 + BM^2) = 2AB^2$

$\therefore \quad AB^2 + AC^2 = 2(AM^2 + BM^2)$

Let $AB \neq AC$. Then any one of $\angle ADB$ and $\angle ADC$ is acute angle.

Without loss of generality we can assume $\angle ADB$ is an acute angle.

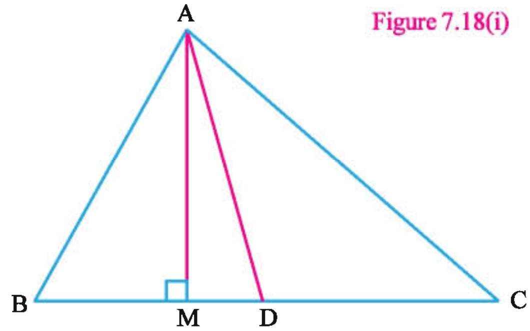

Figure 7.18(ii)

$\therefore \quad B-M-D-C$

$\therefore \quad MB = BD - DM$ and $MC = DM + DC = DM + BD$ **(BD = DC)**

ΔABM and ΔAMC are right angled triangles.

$$\begin{aligned} AB^2 + AC^2 &= (AM^2 + MB^2) + (AM^2 + MC^2) \\ &= 2AM^2 + MB^2 + MC^2 \\ &= 2AM^2 + (BD - DM)^2 + (DM + BD)^2 \quad \textbf{(BD = DC)} \\ &= 2AM^2 + 2DM^2 + 2BD^2 \\ &= 2BD^2 + 2(AM^2 + DM^2) \\ &= 2BD^2 + 2AD^2 \end{aligned}$$

$\therefore \quad AB^2 + AC^2 = 2(AD^2 + BD^2)$

Using Apolloneous theorem, let us solve some examples.

Example 13 : In ΔABC, $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ are the medians. Prove that,

$4(AD^2 + BE^2 + CF^2) = 3(AB^2 + BC^2 + AC^2)$.

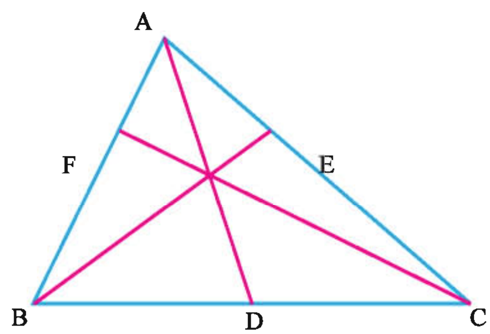

Figure 7.19

Solution : Using the theorem of Apolloneous we have proved that

$AB^2 + AC^2 = 2(AD^2 + BD^2)$ **(i)**

Let $AB = c$, $BC = a$, $CA = b$

$\overline{AD}$ is a median.

$\therefore$ D is the mid-point of $\overline{BC}$.

$\therefore \quad BD = \frac{1}{2}BC = \frac{1}{2}a$.

$c^2 + b^2 = 2\left[AD^2 + \left(\frac{a}{2}\right)^2\right] = 2AD^2 + \frac{a^2}{2}$ **(by (i))**

$\therefore \quad 2c^2 + 2b^2 = 4AD^2 + a^2$

$\therefore \quad 4AD^2 = 2c^2 + 2b^2 - a^2$ **(ii)**

Similarly we have

$$4BE^2 = 2c^2 + 2a^2 - b^2 \qquad \textbf{(iii)}$$

and $4CF^2 = 2a^2 + 2b^2 - c^2$ **(iv)**

From (ii), (iii), (iv) we have,

$$4(AD^2 + BE^2 + CF^2) = 3(a^2 + b^2 + c^2)$$
$$= 3(BC^2 + CA^2 + AB^2)$$

$\therefore \quad 4(AD^2 + BE^2 + CF^2) = 3(AB^2 + BC^2 + AC^2)$

Example 14 : ΔPQR is a right angled triangle. $m\angle P = 90$. M and N are mid-points of $\overline{PQ}$ and $\overline{PR}$ respectively. Prove that $4(RM^2 + QN^2) = 5QR^2$.

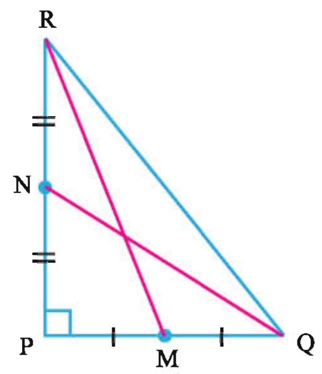

Figure 7.20

Solution : In ΔPQR, $\angle P$ is right angle.

M is the mid-point of $\overline{PQ}$.

$\therefore \quad PM = \frac{1}{2}PQ$

N is the mid-point of $\overline{PR}$.

$\therefore \quad PN = \frac{1}{2}PR$

In ΔPMR, $\angle P$ is a right angle.

$\therefore \quad RM^2 = PR^2 + PM^2$

$\therefore \quad RM^2 = PR^2 + \left(\frac{1}{2}PQ\right)^2$

$\therefore \quad RM^2 = PR^2 + \frac{1}{4}PQ^2$

$\therefore \quad 4RM^2 = 4PR^2 + PQ^2$ **(i)**

In ΔPNQ, $\angle P$ is a right angle.

$\therefore \quad QN^2 = PN^2 + PQ^2 = \left(\frac{1}{2}PR\right)^2 + PQ^2$

$\therefore \quad 4QN^2 = 4PQ^2 + PR^2$ **(ii)**

Adding the results of (i) and (ii), we get

$$4(RM^2 + QN^2) = 4(PR^2 + PQ^2) + (PQ^2 + PR^2)$$

But $PQ^2 + PR^2 = QR^2$ **(In ΔPQR, $\angle P$ is a right angle.)**

$$4(RM^2 + QN^2) = 4QR^2 + QR^2$$
$$4(RM^2 + QN^2) = 5QR^2$$

Example 15 : In ΔPQR, $m\angle Q = 90$, M, N are the points of trisection of $\overline{PR}$. Using Apolloneous theorem Prove that $QM^2 + QN^2 = 5MN^2$.

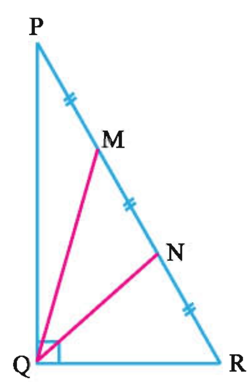

Figure 7.21

Solution : $\angle Q$ is right angle in ΔPQR.

$\therefore \quad PQ^2 + QR^2 = PR^2 = (3MN)^2 = 9MN^2$

$(PM = MN = NR = \frac{1}{3}PR)$ (i)

$\overline{QM}$ is a median in ΔPQN.

∴ By Apolloneous theorem,

$PQ^2 + QN^2 = 2QM^2 + 2MN^2$ **(ii)**

$\overline{QN}$ is a median in ΔQMR.

∴ $QM^2 + QR^2 = 2QN^2 + 2MN^2$ **(iii)**

From (ii) and (iii), by addition,

$PQ^2 + QN^2 + QM^2 + QR^2 = 2QM^2 + 2QN^2 + 4MN^2$

$9MN^2 = QM^2 + QN^2 + 4MN^2$ **(using (i))**

∴ $QM^2 + QN^2 = 5MN^2$

EXERCISE 7.2

1. In rectangle ABCD, AB + BC = 23, AC + BD = 34. Find the area of the rectangle.
2. In ΔABC $m\angle A = m\angle B + m\angle C$, AB = 7, BC = 25. Find the perimeter of ΔABC.
3. A staircase of length 6.5 meters touches a wall at height of 6 meter. Find the distance of base of the staircase from the wall.
4. In ΔABC AB = 7, AC = 5, AD = 5. Find BC, if the mid-point of $\overline{BC}$ is D.
5. In equilateral ΔABC, D ∈ $\overline{BC}$ such that BD : DC = 1 : 2. Prove that $3AD = \sqrt{7}AB$.
6. In ΔABC, AB = 17, BC = 15, AC = 8, find the length of the median on the largest side.
7. $\overline{AD}$ is a median of ΔABC. $AB^2 + AC^2 = 148$ and AD = 7. Find BC.
8. In rectangle ABCD, AC = 25 and CD = 7. Find perimeter of the rectangle.
9. In rhombus XYZW, XZ = 14 and YW = 48. Find XY.
10. In ΔPQR, $m\angle Q : m\angle R : m\angle P = 1 : 2 : 1$. If $PQ = 2\sqrt{6}$, find PR.

*

Example 16 : In ΔABC, $\overline{AB} \cong \overline{AC}$ and $\overline{AD}$ is a median. If AD = 12 and the perimeter of ΔABC is 48, then find ABC.

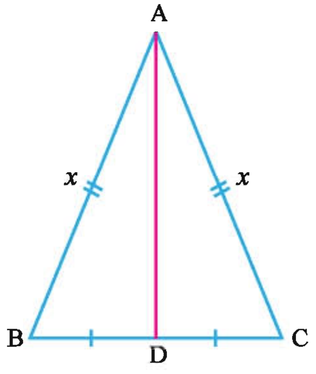

Figure 7.22

Solution : In ΔABC, $\overline{AB} \cong \overline{AC}$ and $\overline{AD}$ is a median.

∴ D is the mid-point of $\overline{BC}$.

∴ BD = DC

∴ $\overline{BD} \cong \overline{DC}$

In ΔADB and ΔADC

$\overline{AB} \cong \overline{AC}$, $\overline{BD} \cong \overline{DC}$ and $\overline{AD} \cong \overline{AD}$

∴ $\Delta ADB \cong \Delta ADC$ and $\angle ADB$ and $\angle ADC$ form a linear pair.

∴ $m\angle ADB = m\angle ADC = 90$

∴ $\overline{AD}$ is the altitude of ΔABC on $\overline{BC}$.

∴ Area of $\Delta ABC = \frac{1}{2}BC \cdot AD$. **(i)**

Let AB = AC = x. **($x > 0$)**

The perimeter of ΔABC is 48.

$\therefore$ AB + AC + BC = 48

$\therefore$ $x + x + BC = 48$

$\therefore$ $BC = 48 - 2x$. So, $BD = 24 - x$

$\therefore$ ΔADB is a right angled triangle. $\angle ADB$ is right angle.

$\therefore$ $AB^2 = BD^2 + AD^2$

$\therefore$ $x^2 = (24 - x)^2 + 12^2$

$\therefore$ $x^2 - (24 - x)^2 = 12^2$

$\therefore$ $-576 + 48x = 144$

$\therefore$ $48x = 144 + 576 = 720$

$\therefore$ $x = 15$

$\therefore$ AB = AC = 15

$\therefore$ BC = 48 − 30 = 18

$\therefore$ From (i) area of $\Delta ABC = \frac{1}{2} BC \cdot AD$

$= \frac{1}{2} \times 18 \times 12 = 108$

Example 17 : In ΔABC, $\overline{BD}$ is an altitude. AB = 2AD, CD = 3AD and A–D–C. Prove that ΔABC is a right angled triangle.

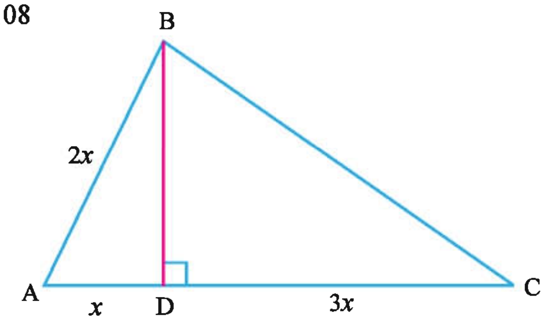

Figure 7.23

Solution : In ΔABC, AB = 2AD, CD = 3AD and A–D–C.

Let AD = x.

$\therefore$ $AB = 2x$, $CD = 3x$

Also A–D–C.

$\therefore$ $AC = AD + DC = x + 3x = 4x$ **(i)**

ΔADB is a right triangle.

$\therefore$ $AB^2 = AD^2 + BD^2$ **($\overline{BD} \perp \overline{AC}$)**

$\therefore$ $BD^2 = AB^2 - AD^2 = (2x)^2 - x^2 = 3x^2$ **(ii)**

ΔBDC is a right triangle. **($\overline{BD} \perp \overline{AC}$)**

$\therefore$ $BC^2 = BD^2 + CD^2 = 3x^2 + (3x)^2 = 12x^2$ **(using (ii)) (iii)**

Now $AB^2 + BC^2 = (2x)^2 + 12x^2$ **(using (iii))**

$= 16x^2$

$= AC^2$ **(using (i))**

$\therefore$ In ΔABC, $AB^2 + BC^2 = AC^2$

$\therefore$ By the converse of theorem of Pythagoras.

ΔABC is a right triangle in which $\angle B$ is right angle.

Example 18 : In ΔABC, $m\angle A + m\angle C = m\angle B$ and $AC : AB = 17 : 15$. If $BC = 12$, find the area of ΔABC.

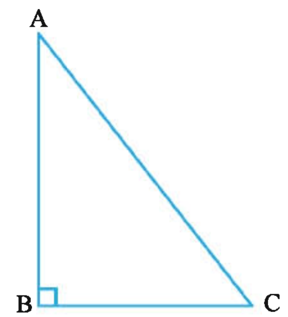

Figure 7.24

Solution : $m\angle A + m\angle C + m\angle B = 180$ in ΔABC.

But $m\angle A + m\angle C = m\angle B$

$\therefore$ $m\angle B + m\angle B = 180$

$\therefore$ $m\angle B = 90$ and $\overline{AC}$ is the hypotenuse.

$\therefore$ ΔABC is a right angled triangle in which $\angle B$ is right angle.

$AC : AB = 17 : 15$

$\therefore$ Let $AC = 17k$, $AB = 15k$ where $k > 0$

In ΔABC, $AC^2 = AB^2 + BC^2$

$\therefore$ $BC^2 = AC^2 - AB^2 = 289k^2 - 225k^2 = 64k^2$

$\therefore$ $BC = 8k$. But $BC = 12$ (given)

$\therefore$ $8k = 12$.

$\therefore$ $k = \frac{12}{8} = \frac{3}{2}$

$\therefore$ $AB = 15k = 15 \times \frac{3}{2} = \frac{45}{2}$

$BC = 12$

$\therefore$ $ABC = \frac{1}{2}BC \times AB = \frac{1}{2} \times 12 \times \frac{45}{2} = 45 \times 3 = 135$

EXERCISE 7

1. $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ are the medians of ΔABC. If $BE = 12$, $CF = 9$ and $AB^2 + BC^2 + AC^2 = 600$, $BC = 10$, find AD.
2. $\overline{AD}$ is the altitude of ΔABC such that B–D–C. If $AD^2 = BD \cdot DC$, prove that $\angle BAC$ is right angle.
[**Hint :** $\overline{AD} \perp \overline{BC}$. So, B–D–C is given. So, ΔADB and ΔADC are right angled triangles to which Pythagoras' theorem can be applied. Same method can be applied to solve Ex. 3, 4, 5.]
3. In ΔABC, $\overline{AD} \perp \overline{BC}$, B–D–C. If $AB^2 = BD \cdot BC$, prove that $\angle BAC$ is a right angle.
4. In ΔABC, $\overline{AD} \perp \overline{BC}$, B–D–C. If $AC^2 = CD \cdot BC$, prove that $\angle BAC$ is a right angle.
5. $\overline{AD}$ is a median of ΔABC. If $BD = AD$, prove that $\angle A$ is a right angle in ΔABC.
6. In figure 7.25, AC is the length of a pole standing vertical on the ground. The pole is bent at point B, so that the top of the pole touches the ground at a point 15 meters away from the base of the pole. If the length of the pole is 25, find the length of the upper part of the pole.

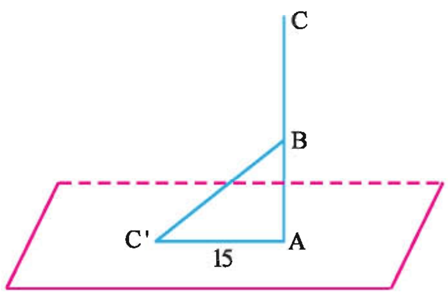

Figure 7.25

7. In ΔABC, $AB > AC$, D is the mid-point of $\overline{BC}$. $\overline{AM} \perp \overleftrightarrow{BC}$ such that B–M–C. Prove that $AB^2 - AC^2 = 2BC \cdot DM$.

8. In ΔABC, $\overline{BD} \perp \overline{AC}$, $D \in \overline{AC}$ and $\angle B$ is right angle. If $AC = 5CD$, prove that $BD = 2CD$.

9. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) In ΔPQR, if $m\angle P + m\angle Q = m\angle R$. $PR = 7$, $QR = 24$, then $PQ =$ ...... ☐

(a) 31 (b) 25 (c) 17 (d) 15

(2) In ΔABC, $\overline{AD}$ is an altitude and $\angle A$ is right angle. If $AB = \sqrt{20}$, $BD = 4$, then $CD =$ ...... ☐

(a) 5 (b) 3 (c) $\sqrt{5}$ (d) 1

(3) In ΔABC, $AB^2 + AC^2 = 50$. The length of the median $AD = 3$. So, $BC =$ ...... ☐

(a) 4 (b) 24 (c) 8 (d) 16

(4) In ΔABC, $m\angle B = 90$, $AB = BC$. Then $AB : AC =$ ...... ☐

(a) $1 : 3$ (b) $1 : 2$ (c) $1 : \sqrt{2}$ (d) $\sqrt{2} : 1$

(5) In ΔABC, $m\angle B = 90$ and $AC = 10$. The length of the median $BM =$ ...... ☐

(a) 5 (b) $5\sqrt{2}$ (c) 6 (d) 8

(6) In ΔABC, $AB = BC = \frac{AC}{\sqrt{2}}$. $m\angle B$ ...... ☐

(a) is acute (b) is obtuse (c) is right angle (d) cannot be obtained

(7) In ΔABC, if $\frac{AB}{1} = \frac{AC}{2} = \frac{BC}{\sqrt{3}}$, then $m\angle C =$ ...... ☐

(a) 90 (b) 30 (c) 60 (d) 45

(8) In ΔXYZ, $m\angle X : m\angle Y : m\angle Z = 1 : 2 : 3$. If $XY = 15$, $YZ =$ ...... ☐

(a) $\frac{15\sqrt{3}}{2}$ (b) 17 (c) 8 (d) 7.5

(9) In ΔABC, $\angle B$ is a right angle and $\overline{BD}$ is an altitude. If $AD = BD = 5$, then $DC =$ ...... ☐

(a) 1 (b) $\sqrt{5}$ (c) 5 (d) 2.5

(10) In ΔABC, $\overline{AD}$ is median. If $AB^2 + AC^2 = 130$ and $AD = 7$, then $BD =$ ...... ☐

(a) 4 (b) 8 (c) 16 (d) 32

(11) The diagonal of a square is $5\sqrt{2}$. The length of the side of the square is .... ☐

(a) 10 (b) 5 (c) $3\sqrt{2}$ (d) $2\sqrt{2}$

(12) The length of a diagonal of a rectangle is 13. If one of the side of the rectangle is 5, the perimeter of the rectangle is .... ☐

(a) 36 (b) 34 (c) 48 (d) 52

(13) The length of a median of an equilateral triangle is $\sqrt{3}$. Length of the side of the triangle is .... ☐

(a) 1 (b) $2\sqrt{3}$ (c) 2 (d) $3\sqrt{3}$

(14) The perimeter of an equilateral triangle is 6. The length of the altitude of the triangle is .... ☐

(a) $\frac{\sqrt{3}}{2}$ (b) $2\sqrt{3}$ (c) 2 (d) $\sqrt{3}$

(15) In ΔABC, $m\angle A = 90$. $\overline{AD}$ is a median. If $AD = 6$, $AB = 10$, then $AC =$ ...... ☐

(a) 8 (b) 7.5 (c) 16 (d) $2\sqrt{11}$

(16) In ΔPQR, $m\angle Q = 90$ and $PQ = QR$. $\overline{QM} \perp \overline{PR}$, $M \in \overline{PR}$. If $QM = 2$, $PQ =$ ...... ☐

(a) 4 (b) $2\sqrt{2}$ (c) 8 (d) 2

(17) In ΔABC, $m\angle A = 90$, $\overline{AD}$ is an altitude. So $AB^2 =$ ...... ☐

(a) $BD \cdot BC$ (b) $BD \cdot DC$ (c) $\frac{BD}{DC}$ (d) $BC \cdot DC$

(18) In ΔABC, $m\angle A = 90$, $\overline{AD}$ is an altitude. Therefore $BD \cdot DC =$ ...... ☐

(a) AB^2 (b) BC^2 (c) AC^2 (d) AD^2

*

Summary

In this chapter we have studied following points :

1. In ΔABC, if $m\angle B = 90$ and $\overline{BD}$ is an altitude, then the correspondence $ABC \leftrightarrow ADB$, $ABC \leftrightarrow BDC$ and $ADB \leftrightarrow BDC$ are similarities. As a consequence of these similarities following results were derived.

   (i) $AB^2 = AD \cdot AC$ (ii) $BC^2 = CD \cdot AC$ (iii) $BD^2 = AD \cdot DC$

2. **Theorem of Pythagoras :** In right angled triangle, the square of the length of the hypotenuse is sum of the squares of the lengths of the remaining two sides. In other words, if in ΔABC, $\angle A$ is a right angle, then $BC^2 = AB^2 + AC^2$.

3. **Converse of Pythagoras Theorem :** If in a triangle, square of the length of one side is the sum of squares of the lengths of the other sides, then the angle opposite to the first side is a right angle.

4. **Apolloneous Theorem :** If $\overline{AD}$ is a median of ΔABC, then $AB^2 + AC^2 = 2(AD^2 + BD^2)$.

◆

Kaprekar constant :

Kaprekar discovered the Kaprekar constant or 6174 in 1949. He showed that 6174 is reached in the limit as one repeatedly subtracts the highest and lowest numbers that can be constructed from a set of four digits that are not all identical. Thus, starting with 1234, we have

4321 − 1234 = 3087, then

8730 − 0378 = 8352, and

8532 − 2358 = 6174.

Repeating from this point onward leaves the same number (7641 - 1467 = 6174). In general, when the operation converges it does so in at most seven iterations.

A similar constant for 3 digits is 495. However, in base 10 a single such constant only exists for numbers of 3 or 4 digits.

TRIGONOMETRY 9

An engineer thinks his equations are an approximation to reality.
A physicist thinks reality is an approximations to his equations.
A mathematician does not care.

- Paul Erdos

9.1 Introduction

Trigonometry is the oldest branch of mathematics. This concept was first used by Aryabhata in *Aryabhatiyam* in 500 A.D. 'Trigonometry' is a word consisting of three Greek words : 'Tri', 'Gon' and 'Metron'. 'Tri' means three, 'Gon' means side. 'Metron' means measure. Putting it simply trigonometry is a study related to the measure of sides and angles of a triangle. We have studied about triangles and in particular, right triangles, in earlier classes. Let us take some examples from our surroundings where right triangles can be imagined to have formed. For instance,

(1) Suppose the students of a school are visiting TV transmission centre. Now, if a student is looking at the top of the TV tower, a right triangle can be imagined to have formed, as shown in the figure 9.1. Can the student find out the height of the TV tower, without actually measuring it ?

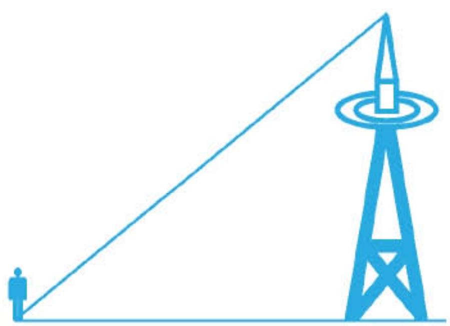

Figure 9.1

(2) Suppose a boy is sitting on the top of a light house. He is looking down and observes a ship steady at sea. A right triangle is imagined to have formed in this situation as shown in figure 9.2.

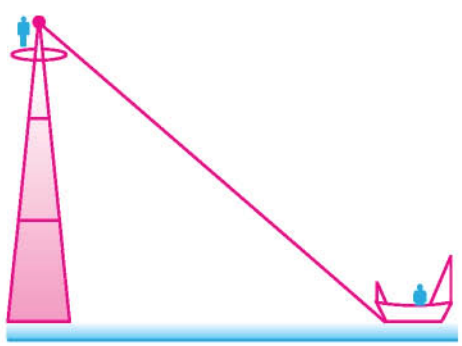

Figure 9.2

If we know the height of the light house at which the boy is sitting, can we find the distance of the ship from the light house ?

In both the situations given above, the distance of a ship from the light house and height of the TV tower can be found by using some mathematical techniques, which come under a branch of mathematics called 'trigonometry'. Trigonometry is a study related to the measure of sides and angles of a triangle. That the calculation of measures of all sides and angles of a triangle can be done using the measures of some other sides and angles of a triangle is an important feature of trigonometry. Trigonometry is used in astronomy to determine the position and the path of celestial objects. Astronomers use it to find out the distance of the stars and planets from the Earth. Captain of a ship uses it to find the direction and the distance of islands and light houses from the sea. Surveyors use to map the new lands.

9.2 Trigonometric Ratios

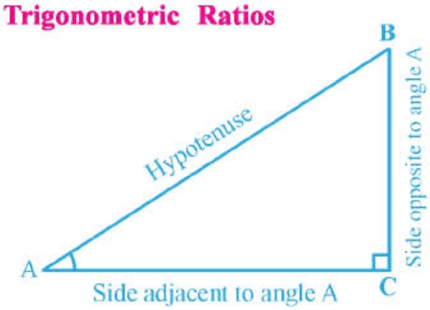

Figure 9.3

Let us take a right triangle ABC as shown in figure 9.3. Here ∠BAC (angle A) is an acute angle. side $\overline{BC}$ is called side opposite to ∠A, side $\overline{AC}$ is called side adjacent to ∠A and side $\overline{AB}$ is the hypotenuse of ΔABC. Note that these terms about sides change when we replace ∠B by ∠A.

As shown in figure 9.4 side $\overline{AC}$ is side opposite to ∠B, side $\overline{BC}$ is side adjacent to ∠B and side $\overline{AB}$ is the hypotenuse of ΔABC. (A, will also denote the measure of ∠A, if there is no ambiguity)

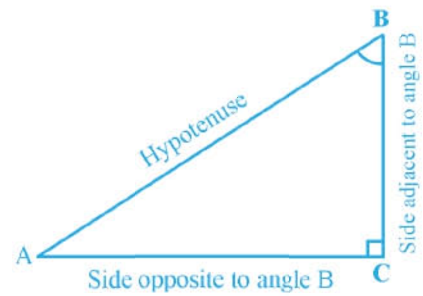

Figure 9.4

The trigonometric ratios are defined as follows. The ratio of the side opposite to ∠BAC and hypotenuse is called *sin*A (read *sine*A).

$$sinA = \frac{\text{Side opposite to angle A}}{\text{Hypotenuse}} = \frac{BC}{AB}$$

The ratio of the side adjacent to ∠A and the hypotenuse is called *cosine*A. In short we write *cosine*A as *cos*A (read *cos*A).

$$cosA = \frac{\text{Side adjacent to angle A}}{\text{Hypotenuse}} = \frac{AC}{AB}$$

The ratios, other than ones defining *sin*A and *cos*A given above, of any two of the sides from opposite side, adjacent side and hypotenuse with reference to ∠A (or ∠B) in right angled triangle ABC, have also been given special names.

- $\frac{\text{side opposite to } \angle A}{\text{side adjacent to } \angle A} = \text{tangentA}$

  tangentA is written in short, as *tan*A. So,

  $$tanA = \frac{\text{side opposite to } \angle A}{\text{side adjacent to } \angle A} = \frac{BC}{AC}$$

- $\frac{\text{adjacent side of } \angle A}{\text{opposite side of } \angle A} = \text{cotangentA}$

  CotangentA is written in short, as *cot*A. So,

  $$cotA = \frac{\text{side adjacent to } \angle A}{\text{side opposite to } \angle A} = \frac{AC}{BC}$$

- $\dfrac{\text{hypotenuse}}{\text{side adjacent to } \angle A} = \text{secant A}$

  secantA is written in short, as *sec*A. So,

  $$secA = \frac{\text{hypotenuse}}{\text{side adjacent to } \angle A} = \frac{AB}{AC}$$

- $\dfrac{\text{hypotenuse}}{\text{side opposite to } \angle A} = \text{cosecant A}$

  cosecantA is written in short, as *cosec*A. So,

  $$cosecA = \frac{\text{hypotenuse}}{\text{side opposite to } \angle A} = \frac{AB}{BC}$$

So, the trigonometric ratios of an acute angle in a right triangle express the relationship between the measure of the angle and the lengths of its sides. Each trigonometric ratio is a real number and has no unit.

Why don't you try to define the trigonometric ratios for angle B in the right triangle in figure 9.4 ?

9.3 Invariance of Trigonometric Ratio

Let $\angle XAY$ be an acute angle. Let P and Q be two points, both different from A and Y, on $\overrightarrow{AY}$. Draw $\overline{PM}$ and $\overline{QN}$ perpendiculars from P and Q respectively to $\overrightarrow{AX}$. (See figure 9.5) Two right triangles PAM and QAN are formed.

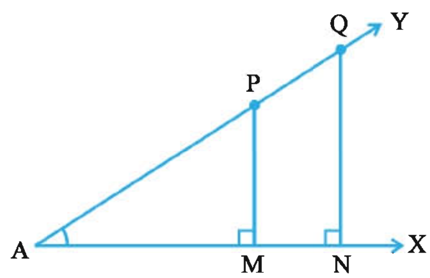

Figure 9.5

$\overline{PM}$ and $\overline{QN}$ are both perpendicular to $\overrightarrow{AX}$.

$\therefore \quad \overline{PM} \parallel \overline{QN}$

The correspondence $PAM \leftrightarrow QAN$ is a similarity relation.

$\therefore \quad \frac{AP}{AQ} = \frac{PM}{QN} = \frac{AM}{AN}$

$\therefore \quad \frac{PM}{AP} = \frac{QN}{AQ}$

In ΔPAM, we have, $sinA = \frac{PM}{AP} = \frac{QN}{AQ}$

$\therefore \quad sinA = \frac{QN}{AQ}$ as obtained from ΔQAN.

Thus trigonometric ratio *sin*A depends on the measure obtained from $\angle A$ only. Similarly other trigonometric ratios *cos*A, *tan*A, etc. depend on measure of angle A only. So, the trigonometric ratios are same for the angles having same measure. They do not vary with the length of the sides of the triangle.

Note : Any letter of the English alphabet can be used to denote an angle but in trigonometry the Greek letter θ (theta), ϕ (phi), α (alpha), β (beta) and γ (gamma) are also used to denote an angle.

Usually $sin^2\theta$ and $cos^2\theta$ is written in place of $(sin\theta)^2$ and $(cos\theta)^2$.

Example 1 : In ΔABC, $AC = 5$, $BC = 13$, $m\angle A = 90$. Find all the six trigonometric ratios of $\angle B$.

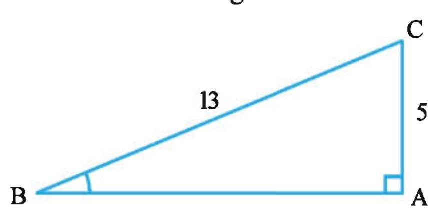

Figure 9.6

Solution : To determine trigonometric ratios for the angle B, we need to find the length of the third side $\overline{AB}$. Do you remember the Pythagoras theorem ?

Let us use it to determine the required length AB.

$$AB^2 + AC^2 = BC^2$$
$$\therefore \quad AB^2 = BC^2 - AC^2$$
$$= 169 - 25$$
$$= 144$$
$$\therefore \quad AB = \sqrt{144} = 12$$

Now, using the definitions of trigonometric ratios, we have

$$sinB = \frac{AC}{BC} = \frac{5}{13}, \quad cosB = \frac{AB}{BC} = \frac{12}{13}$$

$$tanB = \frac{AC}{AB} = \frac{5}{12}, \quad cotB = \frac{AB}{AC} = \frac{12}{5}$$

$$secB = \frac{BC}{AB} = \frac{13}{12}, \quad cosecB = \frac{BC}{AC} = \frac{13}{5}$$

Note : Since the hypotenuse is the longest side in a right triangle, the value of *sin*A or *cos*A is always less than 1.

Example 2 : In ΔABC if $m\angle C = 90$ and $tanA = \frac{1}{\sqrt{3}}$, find *sin*A and *cos*B.

Solution : Let us draw a ΔABC, right angled at C.

Now, $tanA = \frac{1}{\sqrt{3}}$

$$\therefore \quad \frac{BC}{AC} = \frac{1}{\sqrt{3}} \qquad \left(tanA = \frac{BC}{AC}\right)$$

$$\therefore \quad \frac{BC}{1} = \frac{AC}{\sqrt{3}} = k, \text{ say} \qquad (k > 0)$$

$$\therefore \quad AC = \sqrt{3}k \text{ and } BC = k$$

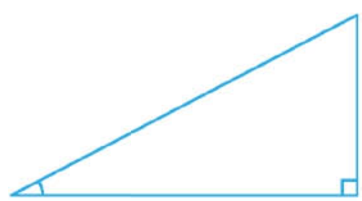

Figure 9.7

By Pythagoras theorem, we have

$$AB^2 = AC^2 + BC^2$$
$$\therefore \quad AB^2 = (\sqrt{3}k)^2 + k^2$$
$$= 3k^2 + k^2$$
$$= 4k^2$$
$$\therefore \quad AB = 2k \qquad (k > 0)$$
$$\therefore \quad sinA = \frac{BC}{AB} = \frac{k}{2k} = \frac{1}{2} \text{ and } cosB = \frac{BC}{AB} = \frac{1}{2}$$

9.4 Identities Related to Trigonometric Ratios

From the figure 9.8, $m\angle ACB = \theta$

$$sin\theta = \frac{AB}{BC} \text{ and } cos\theta = \frac{AC}{BC}$$

Now, $tan\theta = \frac{AB}{AC} = \frac{AB}{BC} \cdot \frac{BC}{AC}$

$$= sin\theta \cdot \frac{1}{cos\theta}$$

$$\therefore \quad tan\theta = \frac{sin\theta}{cos\theta}$$

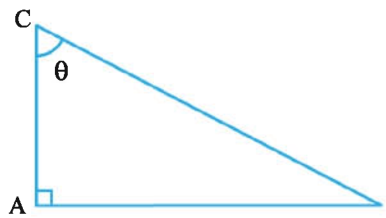

Figure 9.8

In the same way, $cot\theta = \frac{AC}{AB} = \frac{AC}{BC} \cdot \frac{BC}{AB} = cos\theta \cdot \frac{1}{sin\theta}$

$\therefore \quad cot\theta = \frac{cos\,\theta}{sin\,\theta}$

Hence, $tan\theta \cdot cot\theta = \frac{sin\,\theta}{cos\,\theta} \cdot \frac{cos\,\theta}{sin\,\theta} = 1$

$\therefore \quad tan\theta \cdot cot\theta = 1$

$sec\theta = \frac{BC}{AC} = \frac{1}{cos\,\theta}$. So, $sec\theta = \frac{1}{cos\,\theta}$.

$\therefore \quad sec\theta \cdot cos\theta = 1$

$cosec\theta = \frac{BC}{AB} = \frac{1}{sin\,\theta}$. So, $cosec\theta = \frac{1}{sin\,\theta}$.

$\therefore \quad cosec\theta \cdot sin\theta = 1$

Note : We have seen that, **the value of *sin*A and *cos*A is always less than 1**. Now, $cosecA = \frac{1}{sin\,A}$ and $secA = \frac{1}{cos\,A}$. So, **the value of *cosec*A and *sec*A is always greater than 1.** $tanA = \frac{sin\,A}{cos\,A}$ and $cotA = \frac{cos\,A}{sin\,A}$. So, the value of *tan*A and *cot*A is any real number greater than 0.

Example 3 : If $cosecA = \sqrt{10}$, find the other five trigonometric ratios.

Solution : In ΔABC, let $m\angle B = 90$.

$cosecA = \frac{AC}{BC} = \frac{\sqrt{10}}{1}$

$\therefore \quad \frac{AC}{\sqrt{10}} = \frac{BC}{1} = k$, say $\qquad (k > 0)$

$\therefore \quad AC = \sqrt{10}k,\ BC = k$

Now, $AC^2 = AB^2 + BC^2$

$\therefore \quad (\sqrt{10}k)^2 = AB^2 + k^2$

$AB^2 = 10k^2 - k^2$

$= 9k^2$

$\therefore \quad AB = 3k \qquad (k > 0)$

$\therefore \quad AC = \sqrt{10}k,\ BC = k$ and $AB = 3k$

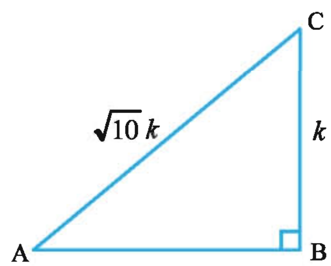

Figure 9.9

Now, $sinA \cdot cosecA = 1$.

$\therefore \quad sinA = \frac{1}{cosec\,A} = \frac{1}{\sqrt{10}}$

$cosA = \frac{AB}{AC} = \frac{3k}{\sqrt{10}k} = \frac{3}{\sqrt{10}}$

$tanA = \frac{BC}{AB} = \frac{k}{3k} = \frac{1}{3}$

$secA = \frac{AC}{AB} = \frac{1}{cos\,A} = \frac{\sqrt{10}}{3}$ and $cotA = \frac{1}{tan\,A} = 3$

Example 4 : In ΔABC, right angled at B, $BC = 7$ and $AC - AB = 1$. Determine the value of *sin*C and *cos*C.

Solution : In ΔABC, we have,

$AC - AB = 1$.

$\therefore \quad AC = AB + 1$

For right ΔABC

$AC^2 = AB^2 + BC^2$

$\therefore \quad (AB + 1)^2 = AB^2 + BC^2$

$\therefore \quad 1 + 2AB + AB^2 = AB^2 + BC^2$

$\therefore \quad 1 + 2AB = BC^2$

$\therefore \quad 2AB = 7^2 - 1$

$\therefore \quad 2AB = 48$

$\therefore \quad AB = 24$ and $AC = 1 + AB = 25$

So, $\sin C = \frac{24}{25}$ and $\cos C = \frac{7}{25}$

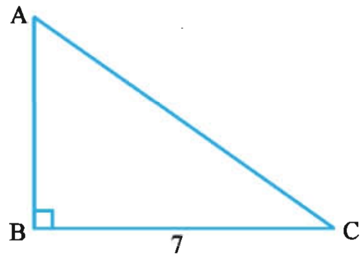

Figure 9.10

Example 5 : If $\cot\theta = \frac{a}{b}$, find the value of $\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$.

Solution : We have, $\cot\theta = \frac{a}{b}$.

$$\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = \frac{\frac{\cos\theta}{\sin\theta} - 1}{\frac{\cos\theta}{\sin\theta} + 1}$$ **(Dividing the numerator and denominator by $\sin\theta \neq 0$)**

$$= \frac{\cot\theta - 1}{\cot\theta + 1}$$

$$= \frac{\frac{a}{b} - 1}{\frac{a}{b} + 1} = \frac{a - b}{a + b}$$

Example 6 : In right triangle ABC, $m\angle B = 90$ and the ratio of BC to AC is 1 : 3. Find the value of

(1) $\sin^2 A + \cos^2 A$ (2) $\left(\frac{4\tan A - 5\cos A}{2\cos A + 4\cot A}\right)$.

Solution : We have, BC : AC = 1 : 3

$\therefore \quad \frac{BC}{AC} = \frac{1}{3}$

$\therefore \quad$ Let $\frac{BC}{1} = \frac{AC}{3} = x$ $\quad (x > 0)$

$\therefore \quad BC = x,\ AC = 3x.$

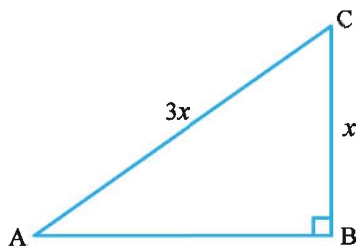

Figure 9.11

Now, $AB^2 = AC^2 - BC^2$

$= (3x)^2 - (x)^2$

$= 8x^2$

$\therefore \quad AB = 2\sqrt{2}x$ $\quad (x > 0)$

So, $\cos A = \frac{AB}{AC} = \frac{2\sqrt{2}x}{3x} = \frac{2\sqrt{2}}{3}$, $\sin A = \frac{BC}{AC} = \frac{x}{3x} = \frac{1}{3}$.

$\tan A = \frac{BC}{AB} = \frac{x}{2\sqrt{2}x} = \frac{1}{2\sqrt{2}}$, $\cot A = \frac{1}{\tan A} = 2\sqrt{2}$.

Now, (1) $\sin^2 A + \cos^2 A = \left(\frac{1}{3}\right)^2 + \left(\frac{2\sqrt{2}}{3}\right)^2$

$= \frac{1}{9} + \frac{8}{9} = \frac{9}{9} = 1$

(2) $\dfrac{4tanA - 5cosA}{2cosA + 4cotA} = \dfrac{4\left(\frac{1}{2\sqrt{2}}\right) - 5\left(\frac{2\sqrt{2}}{3}\right)}{2\left(\frac{2\sqrt{2}}{3}\right) + 4(2\sqrt{2})}$

$$= \frac{\sqrt{2} - \frac{10\sqrt{2}}{3}}{\frac{4\sqrt{2}}{3} + 8\sqrt{2}} = \frac{3\sqrt{2} - 10\sqrt{2}}{4\sqrt{2} + 24\sqrt{2}}$$

$$= \frac{-7\sqrt{2}}{28\sqrt{2}} = -\frac{1}{4}$$

Exercise 9.1

1. In ΔABC, $m\angle A = 90$. If AB = 5, AC = 12 and BC = 13, find *sin*C, *cos*C, *tan*B, *cos*B, *sin*B.

2. In ΔABC, $m\angle B = 90$. If BC = 3 and AC = 5, find all the six trigonometric ratios of $\angle A$.

3. If $cosA = \frac{4}{5}$, find *sin*A and *tan*A.

4. If $cosec\theta = \frac{13}{5}$, find $tan\theta$ and $cos\theta$.

5. If $cosB = \frac{1}{3}$, find the other five trigonometric ratios.

6. In ΔABC, $m\angle A = 90$ and if AB : BC = 1 : 2 find *sin*B, *cos*C, *tan*C.

7. If $tan\theta = \frac{4}{3}$, find the value of $\dfrac{5sin\,\theta + 2cos\,\theta}{3sin\,\theta - cos\,\theta}$.

8. If $sec\theta = \frac{13}{5}$, find the value of $\dfrac{2sin\,\theta + 3cos\,\theta}{5cos\,\theta - 4sin\,\theta}$.

9. If $sinB = \frac{1}{2}$, prove that $3cosB - 4cos^3B = 0$.

10. If $tanA = \sqrt{3}$, verify that

(1) $sin^2A + cos^2A = 1$ **(2)** $sec^2A - tan^2A = 1$ **(3)** $1 + cot^2A = cosec^2A$

11. If $cos\theta = \frac{2\sqrt{2}}{3}$, verify that $tan^2\theta - sin^2\theta = tan^2\theta \cdot sin^2\theta$.

12. In ΔABC, $m\angle B = 90$, AC + BC = 25 and AB = 5, determine the value of *sin*A, *cos*A and *tan*A.

13. In ΔABC, $m\angle C = 90$ and $m\angle A = m\angle B$, **(1)** Is *cos*A = *cos*B ? **(2)** Is *tan*A = *tan*B ? **(3)** Will the other trigonometric ratios of $\angle A$ and $\angle B$ be equal ?

14. If 3*cot*A = 4, examine whether $\dfrac{1 - tan^2A}{1 + tan^2A} = cos^2A - sin^2A$.

15. If $pcot\theta = q$, examine whether $\dfrac{psin\,\theta - qcos\,\theta}{psin\,\theta + qcos\,\theta} = \dfrac{p^2 - q^2}{p^2 + q^2}$.

16. State whether the following are true or false. Justify your answer :

(1) $sin\theta = \frac{3}{2}$, for some angle having measure θ. **(2)** $cos\theta = \frac{2}{3}$, for some angle having measure θ.

(3) $cosecA = \frac{5}{2}$, for some measure of angle A. **(4)** The value of *tan*A is always less than 1.

(5) $secB = \frac{3}{5}$ for some $\angle B$.

(6) $cos\theta = 100$ for some angle having measure θ.

*

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9.5 Value of Trigonometric Ratios for Angle of Special Measures

You are already familiar with the construction of angles having measure 30, 45 and 60. In this section, we will find the values of trigonometric ratios for these angles. If the degree measure of $\angle ABC$ is 60, we shall write $sin60^{o}$ for $sin60$ in future. This is because we shall learn another system of measure of an angle in future, in which we shall see that $sin60$ has a different meaning. At present we know only one system of measure of angle, we write $sin60$ for an angle having measure 60.

Trigonometric Ratios of 30 and 60 :

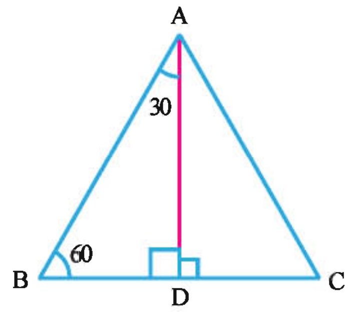

Figure 9.12

Consider an equilateral ΔABC.

$\therefore$ $m\angle A = m\angle B = m\angle C = 60$ and $AB = BC = AC$.

Draw the altitude $\overline{AD}$ from A to $\overline{BC}$, as in figure 9.12.

Now, ABD $\leftrightarrow$ ACD is a congruence (by RHS).

$\therefore$ $BD = DC$ and $m\angle BAD = m\angle CAD$ (i)

As, $m\angle BAC = m\angle BAD + m\angle CAD$ ($D \in \overline{BC}$)

$\therefore$ $60 = m\angle BAD + m\angle BAD$ ($\because$ (i))

$\therefore$ $m\angle BAD = 30$

Now, ΔABD is a right triangle with $m\angle ADB = 90$,

$m\angle BAD = 30$ and $m\angle ABD = 60$.

Let $AB = 2k$ ($k > 0$)

Then, $BD = \frac{1}{2}(BC) = k$ (i)

By Pythagoras theorem, $AD^2 = AB^2 - BD^2 = (2k)^2 - (k)^2 = 3k^2$

$\therefore$ $AD = \sqrt{3}k$ ($k > 0$)

Now, we have, $sin30 = \frac{BD}{AB} = \frac{k}{2k}$

$\therefore$ $sin30 = \frac{1}{2}$

$\therefore$ $cosec30 = 2$

$tan30 = \frac{BD}{AD} = \frac{k}{\sqrt{3}k}$

$\therefore$ $tan30 = \frac{1}{\sqrt{3}}$

$\therefore$ $cot30 = \sqrt{3}$

$cos30 = \frac{AD}{AB} = \frac{\sqrt{3}k}{2k}$

$\therefore$ $cos30 = \frac{\sqrt{3}}{2}$

$\therefore$ $sec30 = \frac{2}{\sqrt{3}}$

Similarly,

$sin60 = \frac{AD}{AB} = \frac{\sqrt{3}k}{2k}$

$\therefore$ $sin60 = \frac{\sqrt{3}}{2}$

$\therefore$ $cosec60 = \frac{2}{\sqrt{3}}$

$cos60 = \frac{BD}{AB} = \frac{k}{2k}$

$\therefore$ $cos60 = \frac{1}{2}$

$\therefore$ $sec60 = 2$

$tan60 = \frac{AD}{BD} = \frac{\sqrt{3}k}{k}$

$\therefore$ $tan60 = \sqrt{3}$

$\therefore$ $cot60 = \frac{1}{\sqrt{3}}$

[**Note :** Since $sin30 = \frac{1}{2}$, we can observe a result of geometry : 'In a right angle triangle with angles having measures 30, 60, 90 the length of hypotenuse is twice the length of the side opposite to the angle having measure 30.]

Trigonometric Ratios of 45 :

In ΔABC, $m\angle B = 90$ and $m\angle A = 45$

$\therefore$ $m\angle C = 180 - m\angle A - m\angle B$

$= 180 - 45 - 90$

$\therefore$ $m\angle C = 45$

$m\angle A = m\angle C$

$\therefore$ AB = BC

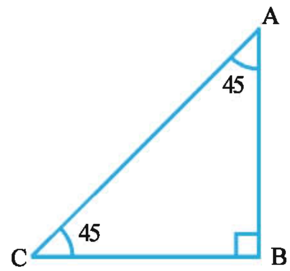

Figure 9.13

Now, suppose AB = k. Then BC = k $(k > 0)$

By Pythagoras theorem,

$AC^2 = AB^2 + BC^2 = k^2 + k^2 = 2k^2$

$\therefore$ $AC = \sqrt{2}k$ $(k > 0)$

$sin45 = \frac{k}{\sqrt{2}k}$ | $cos45 = \frac{k}{\sqrt{2}k}$ | $tan45 = \frac{k}{k}$

$\therefore$ $sin45 = \frac{1}{\sqrt{2}}$ | $\therefore$ $cos45 = \frac{1}{\sqrt{2}}$ | $\therefore$ $tan45 = 1$

$\therefore$ $cosec45 = \sqrt{2}$ | $\therefore$ $sec45 = \sqrt{2}$ | $\therefore$ $cot45 = 1$

With the help of geometrical results, we have obtained trigonometrical ratios of angles of measure 30, 45 and 60. These are special cases and it is not possible to find trigonometric ratios of angles having any measure in this way.

We have defined trigonometric ratios for the measure of an acute angle only. But we will define trigonometric ratios of numbers 0 and 90.

They are necessary for practical purpose. We define, $sin0 = 0$, $cos0 = 1$, $tan0 = 0$ and $cosec0$ and $cot0$ are not defined.

Also by definition, $sin90 = 1$, $cos90 = 0$, $cot90 = 0$, $cosec90 = 1$ and $sec90$ and $tan90$ are not defined.

Measure of an angle A	0	30	45	60	90
*sin*A	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
*cos*A	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
*tan*A	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
*cosec*A	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
*sec*A	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
*cot*A	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Note : From the table above we can observe that as $m\angle A$ increases from 0 to 90, *sin*A increases from 0 to 1 and *cos*A decreases from 1 to 0. We can also observe that value of *cosec*A and *sec*A is greater than or equal to 1.

Example 7 : Find the value of *sin*60 *sin*45 + *cos*60 *cos*45.

Solution : $sin60\ sin45 + cos60\ cos45 = \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} + \frac{1}{2} \cdot \frac{1}{\sqrt{2}}$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Example 8 : Find the value of $\frac{5sin^2 30 + sin^2 45 - 4tan^2 30}{2sin\ 30\ cos\ 30 + cot\ 45}$.

Solution : $\frac{5sin^2 30 + sin^2 45 - 4tan^2 30}{2sin\ 30\ cos\ 30 + cot\ 45} = \frac{5\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 - 4\left(\frac{1}{\sqrt{3}}\right)^2}{2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + 1}$

$$= \frac{\frac{5}{4}+\frac{1}{2}-\frac{4}{3}}{\frac{\sqrt{3}}{2}+1} = \frac{\frac{5}{12}}{\frac{\sqrt{3}+2}{2}} = \frac{5}{12} \times \frac{2}{\sqrt{3}+2}$$

$$= \frac{5}{6(2+\sqrt{3})} \times \frac{2-\sqrt{3}}{2-\sqrt{3}} = \frac{5(2-\sqrt{3})}{6}$$

Example 9 : Prove that : $\frac{sin\ 60 + cos\ 30}{1 + sin\ 30 + cos\ 60} = cos30$

Solution : L.H.S. $= \frac{sin\ 60 + cos\ 30}{1 + sin\ 30 + cos\ 60}$

$$= \frac{\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}}{1+\frac{1}{2}+\frac{1}{2}} = \frac{2\left(\frac{\sqrt{3}}{2}\right)}{2} = \frac{\sqrt{3}}{2} = cos30 = \text{R.H.S.}$$

Example 10 : If $0 < x < 90$ and $sin\ x = sin60\ cos30 - cos60\ sin30$, find x.

Solution : $sin\ x = sin60\ cos30 - cos60\ sin30$

$\therefore \quad sin\ x = \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} - \frac{1}{2} \times \frac{1}{2}$

$\therefore \quad sin\ x = \frac{3}{4} - \frac{1}{4}$

$\therefore \quad sin\ x = \frac{1}{2}$

$\therefore \quad x = 30$

Example 11 : In ΔABC, $m\angle C = 90$, $m\angle B = 60$ and AB = 15. Find the measure of remaining angles and sides.

Solution : We have $m\angle C = 90$, $m\angle B = 60$.

$m\angle A + m\angle B + m\angle C = 180$

$m\angle A + 60 + 90 = 180$

$m\angle A = 30$

Now, $sinB = \frac{AC}{AB}$

$\therefore \quad sin60 = \frac{AC}{15}$

$\therefore \quad \frac{\sqrt{3}}{2} = \frac{AC}{15}$

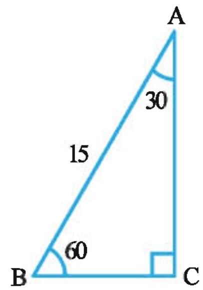

Figure 9.14

$\therefore \quad AC = \frac{\sqrt{3}}{2} \times 15$

$\therefore \quad AC = 7.5\sqrt{3}$

and $cosB = \frac{BC}{AB}$

$\therefore \quad cos60 = \frac{BC}{15}$

$\therefore \quad \frac{1}{2} = \frac{BC}{15}$

$\therefore \quad BC = \frac{15}{2} = 7.5$

Hence, $AC = 7.5\sqrt{3}$, $BC = 7.5$ and $m\angle A = 30$

Example 12 : In ΔABC, $m\angle B = 90$, $AB = 3$, $AC = 6$. Find $m\angle C$, $m\angle A$ and BC.

Solution : We have $m\angle B = 90$, $AB = 3$, $AC = 6$.

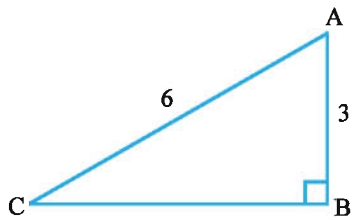

Figure 9.15

$sinC = \frac{AB}{AC} = \frac{3}{6}$

$\therefore \quad sinC = \frac{1}{2}$

$\therefore \quad m\angle C = 30$

$m\angle A + m\angle B + m\angle C = 180$

$m\angle A + 90 + 30 = 180$

$\therefore \quad m\angle A = 60$

$sinA = \frac{BC}{AC} = \frac{\sqrt{3}}{2}$ **(A = 60)**

$\therefore \quad BC = \frac{\sqrt{3}}{2} AC = \frac{\sqrt{3}}{2} \times 6$

$\therefore \quad BC = 3\sqrt{3}$

Example 13 : Given that $sin(A + B) = sinA\ cosB + cosA\ sinB$, find the value of $sin75$.

Solution : We have $sin(A + B) = sinA\ cosB + cosA\ sinB$.

Putting $A = 45$ and $B = 30$ we get

$sin(45 + 30) = sin45\ cos30 + cos45\ sin30$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Example 14 : If $\theta = 30$, verify that (1) $sin3\theta = 3sin\theta - 4sin^3\theta$ (2) $sin\theta = \sqrt{\frac{1 - cos2\theta}{2}}$.

Solution : We have $\theta = 30$

(1) $3\theta = 90$

L.H.S. $= sin3\theta = sin90 = 1$

R.H.S. $= 3sin\theta - 4sin^3\theta = 3sin30 - 4sin^3 30$

$$= 3\left(\frac{1}{2}\right) - 4\left(\frac{1}{2}\right)^3 = \frac{3}{2} - \frac{1}{2} = 1$$

(2) $\theta = 30$

$\therefore$ L.H.S. $= sin\theta = sin30 = \frac{1}{2}$

R.H.S. $= \sqrt{\frac{1 - cos2\theta}{2}}$

$= \sqrt{\frac{1 - cos\,60}{2}} = \sqrt{\frac{1 - \frac{1}{2}}{2}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

$\therefore$ L.H.S. = R.H.S.

Exercise 9.2

1. Verify :

(1) $cos60 = 1 - 2sin^2 30 = 2cos^2 30 - 1 = cos^2 30 - sin^2 30$

(2) $sin60 = 2sin30\ cos30$

(3) $sin60 = \frac{2tan\,30}{1 + tan^2 30}$

(4) $cos60 = \frac{1 - tan^2 30}{1 + tan^2 30}$

(5) $cos90 = 4cos^3 30 - 3cos30$

2. Evaluate :

(1) $\frac{sin\,30 + tan\,45 - cosec\,60}{sec\,30 + cos\,60 + cot\,45}$ (2) $\frac{5cos^2 60 + 4sec^2 30 - tan^2 45}{sin^2 30 + cos^2 30}$

(3) $2sin^2 30\ cot30 - 3cos^2 60\ sec^2 30$ (4) $3cos^2 30 + sec^2 30 + 2cos0 + 3sin90 - tan^2 60$

3. In ΔABC, $m\angle B = 90$, find the measure of the parts of the triangle other than the ones which are given below :

(1) $m\angle C = 45$, AB = 5 (2) $m\angle A = 30$, AC = 10

(3) AC = $6\sqrt{2}$, BC = $3\sqrt{6}$ (4) AB = 4, BC = 4

4. In a rectangle ABCD, AB = 20, $m\angle BAC = 60$, calculate the length of side $\overline{BC}$ and diagonals $\overline{AC}$ and $\overline{BD}$.

5. If θ is measure of an acute angle and $cos\theta = sin\theta$, find the value of $2tan^2\theta + sin^2\theta + 1$.

6. If α is measure of an acute angle and $3sin\alpha = 2cos\alpha$, prove that $\left(\frac{1 - tan^2\alpha}{1 + tan^2\alpha}\right)^2 + \left(\frac{2tan\,\alpha}{1 + tan^2\alpha}\right)^2 = 1$

7. If A = 30 and B = 60, verify that

(1) $sin(A + B) = sinA\ cosB + cosA\ sinB$, (2) $cos(A + B) = cosA\ cosB - sinA\ sinB$

8. If $sin(A - B) = sinA\ cosB - cosA\ sinB$ and $cos(A - B) = cosA\ cosB + sinA\ sinB$, find the values of $sin15$ and $cos15$.

9. State whether the following are true or false. Justify your answer :

(1) The value of $sin\theta$ increases as θ increases from 0 to 90.

(2) $sin\theta = cos\theta$ for all value of θ.

(3) $cos(A + B) = cosA + cosB$

(4) $tanA$ is not defined for A = 90.

(5) The value of $cot\theta$ increases as θ increases from 0 to 90.

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9.6 Trigonometric Ratios of Complementary Angles

We have studied that two angles are said to be complementary angles of each other, if the sum of their measures is 90. If in a right triangle measure of one angle is 90, then the measure of two acute angles can be θ and $90 - \theta$. They are always complementary angles of each other. We have studied the trigonometric ratios for $\theta = 0, 30, 45, 60$ and 90. We know that the angles with measure 30 and 60 are complementary angles of each other. We can see that $sin30 = cos60 = \frac{1}{2}$, $cos30 = sin60 = \frac{\sqrt{3}}{2}$, $tan30 = cot60 = \frac{1}{\sqrt{3}}$, $sec30 = cosec60 = \frac{2}{\sqrt{3}}$. Is this patten true for all complementary angles ? Let us verify this.

Consider a right angled ΔABC, right angled at B, as shown in figure 9.16.

Let $m\angle BAC = \theta$. Then $m\angle BCA = 90 - \theta$.

Since $m\angle BAC = \theta$,

$$\left.\begin{array}{l} sin\theta = \frac{BC}{AC},\ cos\theta = \frac{AB}{AC} \\ tan\theta = \frac{BC}{AB},\ cot\theta = \frac{AB}{BC} \\ sec\theta = \frac{AC}{AB},\ cosec\theta = \frac{AC}{BC} \end{array}\right\} \quad \text{(i)}$$

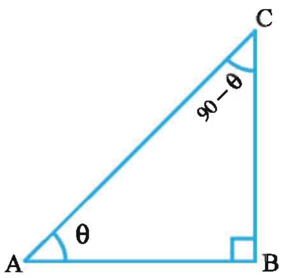

Figure 9.16

Now, $m\angle BCA = 90 - \theta$. Its opposite side is $\overline{AB}$ and adjacent side is $\overline{BC}$.

$$\therefore \left.\begin{array}{l} sin(90 - \theta) = \frac{AB}{AC},\ cos(90 - \theta) = \frac{BC}{AC},\ tan(90 - \theta) = \frac{AB}{BC} \\ cot(90 - \theta) = \frac{BC}{AB},\ sec(90 - \theta) = \frac{AC}{BC},\ cosec(90 - \theta) = \frac{AC}{AB} \end{array}\right\} \quad \text{(ii)}$$

Now, compare the ratios in (i) and (ii).

$\therefore$ $sin(90 - \theta) = \frac{AB}{AC} = cos\theta$ and $cos(90 - \theta) = \frac{BC}{AC} = sin\theta$

Also, $tan(90 - \theta) = \frac{AB}{BC} = cot\theta$, $cot(90 - \theta) = \frac{BC}{AB} = tan\theta$

Similarly, $sec(90 - \theta) = \frac{AC}{BC} = cosec\theta$, $cosec(90 - \theta) = \frac{AC}{AB} = sec\theta$

Moreover $sin0 = 0 = cos90$ and $sin90 = 1 = cos0$

Thus **for every θ, $0 \le \theta \le 90$,**

$sin(90 - \theta) = cos\theta$ and $cos(90 - \theta) = sin\theta$.

Also, $tan90$ and $sec90$ are undefined terms,

For every $\theta \in R$, $0 < \theta \le 90$, $tan(90 - \theta) = cot\theta$

For every $\theta \in R$, $0 \le \theta < 90$, $cot(90 - \theta) = tan\theta$

For every $\theta \in R$, $0 < \theta \le 90$, $sec(90 - \theta) = cosec\theta$ and

For every $\theta \in R$, $0 \le \theta < 90$, $cosec(90 - \theta) = sec\theta$

Example 15 : Show that $\frac{cos\,50}{sin\,40} + \frac{sin\,42}{cos\,48} - \frac{2tan\,18}{cot\,72} = 0$.

Solution : L.H.S. $= \frac{cos\,50}{sin\,40} + \frac{sin\,42}{cos\,48} - \frac{2tan\,18}{cot\,72}$

$$= \frac{cos\,50}{cos\,(90-40)} + \frac{sin\,42}{sin\,(90-48)} - \frac{2tan\,18}{tan\,(90-72)}$$

$$= \frac{cos\,50}{cos\,50} + \frac{sin\,42}{sin\,42} - \frac{2tan\,18}{tan\,18}$$

$$= 1 + 1 - 2(1) = 0 = \text{R.H.S.}$$

Example 16 : Prove that (1) $tan48\ tan23\ tan42\ tan67 = 1$, (2) $tan1\ tan2\ tan3 \ldots tan88\ tan89 = 1$.

Solution : (1) L.H.S. $= tan48\ tan23\ tan42\ tan67$

$= tan48\ tan23\ cot(90-42)\ cot(90-67)$

$= tan48\ tan23\ cot48\ cot23$

$= (tan48 \cdot cot48)(tan23 \cdot cot23)$

$= (1)(1) = 1 =$ R.H.S.

(2) L.H.S. $= tan1\ tan2\ tan3 \ldots tan88\ tan89$ **($tan\theta \cdot cot\theta = 1$)**

$= (tan1\ tan89)(tan2\ tan88)(tan3\ tan87) \ldots (tan44\ tan46) \cdot tan45$

$= [tan1\ cot(90-89)][tan2\ cot(90-88)][tan3\ cot(90-87)] \ldots$

$[tan44\ cot(90-46)] \cdot tan45$

$= (tan1\ cot1)(tan2\ cot2)(tan3\ cot3) \ldots (tan44\ cot44) \cdot tan45$

$= (1)(1)(1) \ldots (1)(1)$ **($tan\theta \cdot cot\theta = 1,\ tan45 = 1$)**

$= 1 =$ R.H.S.

Example 17 : Evaluate : $2\left(\frac{cos\,58}{sin\,32}\right) - \sqrt{3}\left(\frac{cos\,38\ cosec\,52}{tan\,15\ tan\,60\ tan\,75}\right)$

Solution : $2\left(\frac{cos\,58}{sin\,32}\right) - \sqrt{3}\left(\frac{cos\,38\ cosec\,52}{tan\,15\ tan\,60\ tan\,75}\right)$

$$= 2\left(\frac{sin\,(90-58)}{sin\,32}\right) - \sqrt{3}\left(\frac{cos\,38 \cdot sec\,(90-52)}{tan\,15 \cdot tan\,60 \cdot cot\,(90-75)}\right)$$

$$= 2\left(\frac{sin\,32}{sin\,32}\right) - \sqrt{3}\left(\frac{cos\,38 \cdot sec\,38}{tan\,15 \cdot cot\,15 \cdot tan\,60}\right)$$

$$= 2(1) - \sqrt{3}\left(\frac{1}{1 \times \sqrt{3}}\right) = 2 - 1 = 1$$ **($cos\theta\ sec\theta = 1,\ tan\theta\ cot\theta = 1$)**

Example 18 : If A, B, C are the measure of angles of ΔABC, prove that $tan\left(\frac{A+B}{2}\right) = cot\,\frac{C}{2}$.

Solution : For ΔABC, we have,

$A + B + C = 180$

$\therefore\ A + B = 180 - C$

$\therefore\ \frac{A+B}{2} = 90 - \frac{C}{2}$

$\therefore\ tan\left(\frac{A+B}{2}\right) = tan\left(90-\frac{C}{2}\right) = cot\,\frac{C}{2}$.

$\therefore\ tan\left(\frac{A+B}{2}\right) = cot\,\frac{C}{2}$.

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Example 19 : If $sec4A = cosec(A - 20)$, where 4A is the measure of an acute angle, find the value of A.

Solution : We have,

$sec4A = cosec(A - 20)$

$\therefore \quad sec4A = sec(90 - (A - 20))$

$\therefore \quad sec4A = sec(110 - A)$

$\therefore \quad 4A = 110 - A$

$\therefore \quad 5A = 110$

$\therefore \quad A = 22$

Example 20 : Express each of the following in terms of trigonometric ratios of angles having measure between 0 and 45 : (1) $sin70 + sec62$ (2) $cos79 + tan59$

Solution : (1) $sin70 + sec62 = cos(90 - 70) + cosec(90 - 62)$

$= cos20 + cosec28$

(2) $cos79 + tan59 = sin(90 - 79) + cot(90 - 59)$

$= sin11 + cot31$

Exercise 9.3

1. Evaluate :

(1) $\frac{cos\,18}{sin\,72}$ (2) $tan48 - cot42$ (3) $cosec32 - sec58$

(4) $\frac{cos\,70}{sin\,20} + cos59 \cdot cosec31$ (5) $sec70\,sin20 - cos20\,cosec70$

(6) $cos(40 - \theta) - sin(50 + \theta) + \frac{cos^2 40 + cos^2 50}{sin^2 40 + sin^2 50}$

(7) $\frac{cos\,70}{sin\,20} + \frac{cos\,55\,cosec\,35}{tan\,5\,tan\,25\,tan\,45\,tan\,65\,tan\,85}$

(8) $cot12 \cdot cot38 \cdot cot52 \cdot cot60 \cdot cot78$

(9) $\frac{sin\,18}{cos\,72} + \sqrt{3}(tan10\;tan30\;tan40\;tan50\;tan80)$

(10) $\frac{cos\,58}{sin\,32} + \frac{sin\,22}{cos\,68} - \frac{cos\,38\,cosec\,52}{tan\,18\,tan\,35\,tan\,60\,tan\,72\,tan\,55}$

2. Prove the following :

(1) $sin48\;sec42 + cos48\;cosec42 = 2$ (2) $\frac{sin\,70}{cos\,20} + \frac{cosec\,20}{sec\,70} - 2cos70\;cosec20 = 0$

(3) $\frac{tan(90 - A) \cdot cot\,A}{cosec^2 A} - cos^2 A = 0$ (4) $\frac{cos(90 - A) \cdot sin(90 - A)}{tan(90 - A)} = sin^2 A$

3. Express the following in terms of trigonometric ratios of angles having measure between 0 and 45 :

(1) $sin85 + cosec85$ (2) $cos89 + cosec87$ (3) $sec81 + cosec54$

4. For ΔABC, prove that (1) $tan\left(\frac{A + C}{2}\right) = cot\frac{B}{2}$, (2) $cos\left(\frac{B + C}{2}\right) = sin\frac{A}{2}$.

5. If A + B = 90, prove that $\sqrt{\dfrac{tan A \, tan B + tan A \, cot B}{sin A \, sec B}} = sec A$.

6. If 3θ is the measure of an acute angle and $sin 3\theta = cos(\theta - 26)$, then find the value of θ.

7. If $0 < \theta < 90$, θ, $sin\theta = cos 30$, then obtain the value of $2tan^2\theta - 1$.

8. If $tan A = cot B$, prove that A + B = 90, where A and B are measures of acute angles.

9. If $sec 2A = cosec(A - 42)$, where 2A is the measure of an acute angle, find the value of A.

10. If $0 < \theta < 90$ and $sec\theta = cosec 60$, find the value of $2cos^2\theta - 1$.

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9.7 Trigonometric Identities

We have studied some elementary concepts of trigonometry which include trigonometric ratios of given angle and trigonometric ratios of complementary angles. Now we shall study some fundamental identities and prove other trigonometric relations using them.

We know that, equality is one of the basic concepts and important tool of mathematics. Let us understand two types of equalities; equation and identity.

Equation : An equation may or may not be always true for any (possible) value of a variable or we can say that the equality may be true only for some definite values of a variable, while it may not be true for other possible values. Such type of equality is called an equation. For example the equation $x^2 - 7x + 12 = 0$ is true for $x = 4$ and $x = 3$ only, while it is not true for any other value of $x \in R$. Therefore, this equality is an equation.

Identity : For every value of a given variable in the equality, if all the terms of the equality are defined and make the equality true, then such an equality is called an identity. For example, $x^2 - 9 = (x + 3)(x - 3)$ is true for all real values of x. So, it is an identity. Moreover $\frac{1}{x} - \frac{1}{x+1} - \frac{1}{x(x+1)} = 0$ is true for all real values of x except $x = 0$ and $x = -1$. So, this equality is an identity. Its terms are undefined for $x = 0$ and $x = -1$. Similarly, an equality involving trigonometric ratios of an angle is called a trigonometric identity, if it is true for all values of the angle(s) involved.

Consider a right triangle ABC, right angled at B as shown in figure 9.17. Let $m\angle ACB = \theta$.

In ΔABC we have, $sin\theta = \frac{AB}{AC}$ and $cos\theta = \frac{BC}{AC}$.

By Pythagoras' theorem,

$$AB^2 + BC^2 = AC^2 \qquad \text{(i)}$$

$$\therefore \quad \frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = 1$$

$$\therefore \quad \left(\frac{AB}{AC}\right)^2 + \left(\frac{BC}{AC}\right)^2 = 1$$

$$\therefore \quad (sin\theta)^2 + (cos\theta)^2 = 1$$

$$\therefore \quad \mathbf{sin^2\theta + cos^2\theta = 1} \qquad \text{(ii)}$$

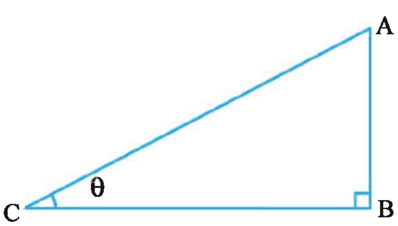

Figure 9.17

This is true for all θ, $0 < \theta < 90$. So, this is a trigonometric identity. We note that if $\theta = 0$, we have, $sin^2\theta + cos^2\theta = 0 + 1 = 1$ and if $\theta = 90$, we have, $sin^2 90 + cos^2 90 = 1 + 0 = 1$. So, (ii) is true for all $\theta \in R$ such that $0 \le \theta \le 90$.

Let us now divide (i) by BC^2. We get $\frac{AB^2}{BC^2} + \frac{BC^2}{BC^2} = \frac{AC^2}{BC^2}$.

$\therefore \quad \left(\frac{AB}{BC}\right)^2 + \left(\frac{BC}{BC}\right)^2 = \left(\frac{AC}{BC}\right)^2$

But $\frac{AB}{BC} = tan\theta$ and $\frac{AC}{BC} = sec\theta$

$\therefore \quad (tan\theta)^2 + 1 = (sec\theta)^2$

$\therefore \quad \mathbf{1 + tan^2\theta = sec^2\theta}$ **(iii)**

This is true for all θ such that $0 < \theta < 90$. We note that if $\theta = 0$, we get $1 + tan^2 0 = sec^2 0$ i.e. $1 + 0 = 1$. So this is true for $\theta = 0$. What if $\theta = 90$? Well, *tan*A and *sec*A are not defined for $\theta = 90$. So, (iii) is true for all $\theta \in R$ such that $0 \le \theta < 90$.

Let us now divide (i) by AB^2. We get

$\therefore \quad \left(\frac{AB}{AB}\right)^2 + \left(\frac{BC}{AB}\right)^2 = \left(\frac{AC}{AB}\right)^2$

But $\frac{BC}{AB} = cot\theta$ and $\frac{AC}{AB} = cosec\theta$

$\therefore \quad 1 + (cot\theta)^2 = (cosec\theta)^2$

$\therefore \quad \mathbf{1 + cot^2\theta = cosec^2\theta}$ **(iv)**

This is true for all θ such that $0 < \theta < 90$. We note that $cosec\theta$ and $cot\theta$ are not defined for $\theta = 0$. But (iv) is true for $\theta = 90$. So (iv) is true for all $\theta \in R$ such that $0 < \theta \le 90$.

From identity $sin^2\theta + cos^2\theta = 1$, we get

$\mathbf{sin^2\theta = 1 - cos^2\theta}$ and $\mathbf{cos^2\theta = 1 - sin^2\theta}$

From identity $1 + tan^2\theta = sec^2\theta$, we get

$\mathbf{sec^2\theta - tan^2\theta = 1}$ and $\mathbf{sec^2\theta - 1 = tan^2\theta}$

From identity $1 + cot^2\theta = cosec^2\theta$, we get

$\mathbf{cosec^2\theta - cot^2\theta = 1}$ and $\mathbf{cosec^2\theta - 1 = cot^2\theta}$.

Using these identities, we can express each trigonometric ratio in terms of other trigonometric ratios.

$sin^2\theta = 1 - cos^2\theta$

$\therefore \quad sin\theta = \pm\sqrt{1 - cos^2\theta}$

$\therefore \quad \mathbf{sin\theta = \sqrt{1 - cos^2\theta}}$, $\mathbf{cos\theta = \sqrt{1 - sin^2\theta}}$ **(As, $0 < \theta < 90$, $sin\theta > 0$)**

Similarly, from $sec^2\theta = 1 + tan^2\theta$, we get

$\mathbf{sec\theta = \sqrt{1 + tan^2\theta}}$ and $\mathbf{tan\theta = \sqrt{sec^2\theta - 1}}$ and from $cosec^2\theta = 1 + cot^2\theta$, we get

$\mathbf{cosec\theta = \sqrt{1 + cot^2\theta}}$ and $\mathbf{cot\theta = \sqrt{cosec^2\theta - 1}}$.

Also we can get, $sec\theta = \frac{1}{cos\theta} = \frac{1}{\sqrt{1 - sin^2\theta}}$ and $tan\theta = \frac{sin\theta}{cos\theta} = \frac{sin\theta}{\sqrt{1 - sin^2\theta}}$.

Example 21 : Express the trigonometric ratios, $sin\theta$, $cos\theta$, $sec\theta$ and $cot\theta$ in terms of $tan\theta$. **$(0 < \theta < 90)$**

Solution : We have the identity, $cosec^2\theta - cot^2\theta = 1$

$\therefore \quad cosec^2\theta = 1 + cot^2\theta$

$\therefore \quad \frac{1}{sin^2\theta} = 1 + \frac{1}{tan^2\theta}$

$\therefore \quad \frac{1}{sin^2\theta} = \frac{tan^2\theta + 1}{tan^2\theta}$

$\therefore \quad sin^2\theta = \frac{tan^2\theta}{1 + tan^2\theta}$

$\therefore \quad sin\theta = \frac{tan\theta}{\sqrt{1 + tan^2\theta}}$ **(0 < θ < 90)**

Now, we have identity, $sec^2\theta - tan^2\theta = 1$

$\therefore \quad sec^2\theta = 1 + tan^2\theta$

$\therefore \quad sec\theta = \sqrt{1 + tan^2\theta}$

Now, $cos\theta = \frac{1}{sec\theta} = \frac{1}{\sqrt{1 + tan^2\theta}}$

$cot\theta = \frac{1}{tan\theta}$

Hence, $sin\theta = \frac{tan\theta}{\sqrt{1 + tan^2\theta}}$, $cos\theta = \frac{1}{\sqrt{1 + tan^2\theta}}$, $sec\theta = \sqrt{1 + tan^2\theta}$ and $cot\theta = \frac{1}{tan\theta}$.

Example 22 : Prove the following identities :

(1) $sin^2\theta + \frac{1}{1 + tan^2\theta} = 1$

(2) $\frac{1}{1 + sin\theta} + \frac{1}{1 - sin\theta} = 2sec^2\theta$

(3) $\frac{tan\theta(1 + cot^2\theta)}{(1 + tan^2\theta)} = cot\theta$

(4) $\frac{sin\theta}{1 - cos\theta} = cosec\theta + cot\theta$

Solution : (1) L.H.S. $= sin^2\theta + \frac{1}{1 + tan^2\theta}$

$= sin^2\theta + \frac{1}{sec^2\theta}$

$= sin^2\theta + cos^2\theta = 1 =$ R.H.S.

(2) L.H.S. $= \frac{1}{1 + sin\theta} + \frac{1}{1 - sin\theta}$

$= \frac{1 - sin\theta + 1 + sin\theta}{(1 + sin\theta)(1 - sin\theta)}$

$= \frac{2}{1 - sin^2\theta} = \frac{2}{cos^2\theta} = 2sec^2\theta =$ R.H.S.

(3) L.H.S. $= \frac{tan\theta(1 + cot^2\theta)}{(1 + tan^2\theta)}$

$= \frac{tan\theta \cdot cosec^2\theta}{sec^2\theta}$

$= \frac{\frac{sin\theta}{cos\theta} \cdot \frac{1}{sin^2\theta}}{\frac{1}{cos^2\theta}}$

$= \frac{1}{cos\theta \cdot sin\theta} \times \frac{cos^2\theta}{1} = \frac{cos\theta}{sin\theta} = cot\theta =$ R.H.S.

(4) L.H.S. $= \frac{sin\,\theta}{1 - cos\theta}$

$= \frac{sin\,\theta}{1 - cos\,\theta} \times \frac{1 + cos\,\theta}{1 + cos\,\theta}$

$= \frac{sin\theta\,(1 + cos\theta)}{1 - cos^2\theta}$

$= \frac{sin\,\theta\,(1 + cos\,\theta)}{sin^2\theta}$

$= \frac{1 + cos\,\theta}{sin\,\theta} = \frac{1}{sin\,\theta} + \frac{cos\,\theta}{sin\,\theta} = cosec\theta + cot\theta$ = R.H.S.

Example 23 : Show that the following equalities are not trigonometric identities :

(1) $sin^2\theta - cos^2\theta = 1$ (2) $tan\theta - cot\theta = 0$ (3) $cos^2\theta + sin^2\theta = \frac{2tan\theta}{1 + tan^2\theta}$

Solution : (1) When $\theta = 60$, we have

$sin^2 60 - cos^2 60 = \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2} \neq 1$

So, when $\theta = 60$, $sin^2\theta - cos^2\theta = 1$ is not satisfied.

Thus, $sin^2\theta - cos^2\theta = 1$ is not a trigonometric identity.

(2) When $\theta = 30$, we have

$tan30 - cot30 = \frac{1}{\sqrt{3}} - \sqrt{3} = \frac{1-3}{\sqrt{3}} = \frac{-2}{\sqrt{3}} \neq 0$

Thus, the given equality is not true for all possible values of θ. Hence it is not an identity.

(3) Here left hand side is the fundamental identity $sin^2\theta + cos^2\theta = 1$. So, for $0 \leq \theta \leq 90$, left hand side is always 1.

But, when $\theta = 60$, we have

R.H.S. $= \frac{2tan\,60}{1 + tan^2 60} = \frac{2\sqrt{3}}{1+3} = \frac{\sqrt{3}}{2} \neq 1$

$\therefore$ L.H.S. $\neq$ R.H.S.

Thus, $sin^2\theta + cos^2\theta = \frac{2tan\theta}{1 + tan^2\theta}$ is not true for the measure θ of all acute angles.

Hence, the given equality is not a trigonometric identity.

Example 24 : Prove that, (1) $(cosec\theta - cot\theta)^2 = \frac{1 - cos\,\theta}{1 + cos\,\theta}$

(2) $(1 + cot\theta - cosec\theta)(1 + tan\theta + sec\theta) = 2$

(3) $\frac{sin\,\theta - cos\,\theta + 1}{sin\,\theta + cos\,\theta - 1} = \frac{1}{sec\,\theta - tan\,\theta}$, using the identity $sec^2\theta = 1 + tan^2\theta$.

(4) $\frac{1 + tan^2 A}{1 + cot^2 A} = \left(\frac{1 - tan\,A}{1 - cot\,A}\right)^2$

Solution : (1) L.H.S. $= (cosec\theta - cot\theta)^2$

$= \left(\frac{1}{sin\,\theta} - \frac{cos\,\theta}{sin\,\theta}\right)^2$

$= \frac{(1 - cos\,\theta)^2}{sin^2\theta}$

$$= \frac{(1 - \cos\theta)^2}{1 - \cos^2\theta}$$

$$= \frac{(1 - \cos\theta)^2}{(1 - \cos\theta)(1 + \cos\theta)}$$

$$= \frac{1 - \cos\theta}{1 + \cos\theta} = \text{R.H.S.}$$

(2) L.H.S. = $(1 + \cot\theta - \operatorname{cosec}\theta)(1 + \tan\theta + \sec\theta)$

$$= \left(1 + \frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta}\right)\left(1 + \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}\right)$$

$$= \left(\frac{\sin\theta + \cos\theta - 1}{\sin\theta}\right)\left(\frac{\cos\theta + \sin\theta + 1}{\cos\theta}\right)$$

$$= \frac{[(\sin\theta + \cos\theta) - 1]\,[(\sin\theta + \cos\theta) + 1]}{\sin\theta \cdot \cos\theta}$$

$$= \frac{(\sin\theta + \cos\theta)^2 - (1)^2}{\sin\theta \cdot \cos\theta} \qquad ((x - y)(x + y) = x^2 - y^2)$$

$$= \frac{\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta - 1}{\sin\theta \cdot \cos\theta}$$

$$= \frac{1 + 2\sin\theta\cos\theta - 1}{\sin\theta \cdot \cos\theta}$$

$$= \frac{2\sin\theta\cos\theta}{\sin\theta \cdot \cos\theta} = 2 = \text{R.H.S.}$$

(3) L.H.S. = $\dfrac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1}$

Dividing numerator and denominator by $\cos\theta$.

$$= \frac{\dfrac{\sin\theta - \cos\theta + 1}{\cos\theta}}{\dfrac{\sin\theta + \cos\theta - 1}{\cos\theta}}$$

$$= \frac{\tan\theta - 1 + \sec\theta}{\tan\theta + 1 - \sec\theta}$$

$$= \frac{(\tan\theta + \sec\theta) - 1}{(\tan\theta - \sec\theta) + 1}$$

$$= \frac{(\tan\theta + \sec\theta) - (\sec^2\theta - \tan^2\theta)}{(\tan\theta - \sec\theta) + 1}$$

$$= \frac{(\tan\theta + \sec\theta) - (\sec\theta - \tan\theta)(\sec\theta + \tan\theta)}{(\tan\theta - \sec\theta) + 1}$$

$$= \frac{(\sec\theta + \tan\theta)(1 - (\sec\theta - \tan\theta))}{(\tan\theta - \sec\theta + 1)}$$

$$= \frac{(\sec\theta + \tan\theta)(1 - \sec\theta + \tan\theta)}{(1 - \sec\theta + \tan\theta)}$$

$$= \sec\theta + \tan\theta$$

$$= \frac{(sec\,\theta + tan\,\theta)(sec\theta - tan\theta)}{(sec\,\theta - tan\,\theta)}$$

$$= \frac{sec^2\theta - tan^2\theta}{sec\theta - tan\theta}$$

$$= \frac{1}{sec\,\theta - tan\,\theta} = \text{R.H.S.}$$

(4) R.H.S. $= \left(\frac{1 - tan\,A}{1 - cot\,A}\right)^2$

$$= \frac{(1 - tan\,A)^2}{\left(1 - \frac{1}{tan A}\right)^2}$$

$$= \frac{(1 - tan\,A)^2\, tan^2\,A}{(tan\,A - 1)^2}$$

$$= \frac{sin^2 A}{cos^2 A} \qquad ((a - b)^2 = (b - a)^2)$$

$$= \frac{sec^2 A}{cosec^2 A}$$

$$= \frac{1 + tan^2\,A}{1 + cot^2\,A} = \text{L.H.S.}$$

Example 25 : If $sec\theta + tan\theta = p$, then prove that $\frac{p^2 - 1}{p^2 + 1} = sin\theta$.

Solution : L.H.S. $= \frac{p^2 - 1}{p^2 + 1} = \frac{(sec\,\theta + tan\,\theta)^2 - 1}{(sec\,\theta + tan\,\theta)^2 + 1}$

$$= \frac{sec^2\theta + tan^2\theta + 2sec\,\theta\,tan\,\theta - 1}{sec^2\theta + tan^2\theta + 2sec\,\theta\,tan\,\theta + 1}$$

$$= \frac{(sec^2\theta - 1) + tan^2\theta + 2sec\,\theta\,tan\,\theta}{sec^2\theta + (1 + tan^2\theta) + 2sec\,\theta\,tan\,\theta}$$

$$= \frac{2tan^2\theta + 2sec\,\theta\,tan\,\theta}{2sec^2\theta + 2sec\,\theta\,tan\,\theta}$$

$$= \frac{2tan\,\theta\,(tan\,\theta + sec\,\theta)}{2sec\,\theta\,(sec\,\theta + tan\,\theta)}$$

$$= \frac{tan\,\theta}{sec\,\theta} = \frac{sin\,\theta}{cos\,\theta \cdot sec\,\theta} = sin\theta = \text{R.H.S.}$$

Second Method : $sec\theta + tan\theta = p$ **(i)**

$$\therefore \quad sec\theta - tan\theta = \frac{sec\,\theta - tan\,\theta}{1}$$

$$= \frac{sec\,\theta - tan\,\theta}{sec^2\,\theta - tan^2\,\theta}$$

$$= \frac{1}{sec\,\theta + tan\,\theta} = \frac{1}{p} \qquad \textbf{(ii)}$$

(Since $sec\theta + tan\theta > 1$, $p \neq 0$)

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$\therefore$ Solving (i) and (ii), $sec\theta = \frac{1}{2}\left(p + \frac{1}{p}\right) = \frac{p^2+1}{2p}$

$$tan\theta = \frac{1}{2}\left(p - \frac{1}{p}\right) = \frac{p^2-1}{2p}$$

Now, $sin\theta = \frac{sin\theta}{cos\theta} \cdot cos\theta = \frac{tan\theta}{sec\theta} = \frac{\frac{p^2-1}{2p}}{\frac{p^2+1}{2p}} = \frac{p^2-1}{p^2+1}$

Example 26 : If $sin\theta + cos\theta = \sqrt{2}\,cos\theta$, then prove that $cos\theta - sin\theta = \sqrt{2}\,sin\theta$.

$cos\theta + sin\theta = \sqrt{2}\,cos\theta$

$\therefore$ $sin\theta = \sqrt{2}\,cos\theta - cos\theta$

$= (\sqrt{2} - 1)\,cos\theta$

$\therefore$ $sin\theta = (\sqrt{2} - 1)\,cos\theta \times \frac{\sqrt{2}+1}{\sqrt{2}+1}$

$\therefore$ $(\sqrt{2} + 1)\,sin\theta = ((\sqrt{2})^2 - 1) \cdot cos\theta$

$\therefore$ $\sqrt{2}\,sin\theta + sin\theta = cos\theta$

$\therefore$ $cos\theta - sin\theta = \sqrt{2}\,sin\theta$

Example 27 : Evaluate :

$$\frac{sin^2 20 + sin^2 70}{sec^2 50 - cot^2 40} + 2cosec^2 58 - 2cot58\ tan32 - 4tan13\ tan37\ tan45\ tan53\ tan77$$

Solution :

$$\frac{sin^2 20 + sin^2 70}{sec^2 50 - cot^2 40} + 2cosec^2 58 - 2cot58\ tan32 - 4tan13\ tan37\ tan45\ tan53\ tan77$$

$= \frac{sin^2 20 + cos^2(90-70)}{sec^2 50 - tan^2(90-40)} + 2cosec^2 58 - 2cot58\ cot(90-32) -$

$4tan13\ tan37\ tan45\ cot(90-53)\ cot(90-77)$

$= \frac{sin^2 20 + cos^2 20}{sec^2 50 - tan^2 50} + 2cosec^2 58 - 2cot58\ cot58 - 4tan13\ tan37\ tan45\ cot37\ cot13$

$= \frac{1}{1} + 2cosec^2 58 - 2cot^2 58 - 4(tan13\ cot13)(tan37\ cot37)(tan45)$

$= 1 + 2(cosec^2 58 - 2cot^2 58) - 4(1)(1)(1)$

$= 1 + 2 - 4 = 3 - 4 = -1$

Example 28 : If $cot\theta = \sqrt{7}$, find the value of $\frac{cosec^2\theta - sec^2\theta}{cosec^2\theta + sec^2\theta}$.

Solution : We have, $cot\theta = \sqrt{7}$

$\therefore$ $tan\theta = \frac{1}{cot\,\theta} = \frac{1}{\sqrt{7}}$

Now, $sec^2\theta = 1 + tan^2\theta$

$\therefore$ $sec^2\theta = 1 + \frac{1}{7} = \frac{8}{7}$ and $cosec^2\theta = 1 + cot^2\theta = 1 + 7 = 8$

$\therefore$ $\frac{cosec^2\theta - sec^2\theta}{cosec^2\theta + sec^2\theta} = \frac{8 - \frac{8}{7}}{8 + \frac{8}{7}} = \frac{\frac{48}{7}}{\frac{64}{7}} = \frac{48}{64} = \frac{3}{4}$

Example 29 : ΔABC and ΔPQR are right angled at C and R respectively. If $\angle B$ and $\angle Q$ are acute angles such that $sin B = sin Q$, then prove that $\angle B \cong \angle Q$.

Solution : Consider two right triangles ABC and PQR as shown in figure 9.18.

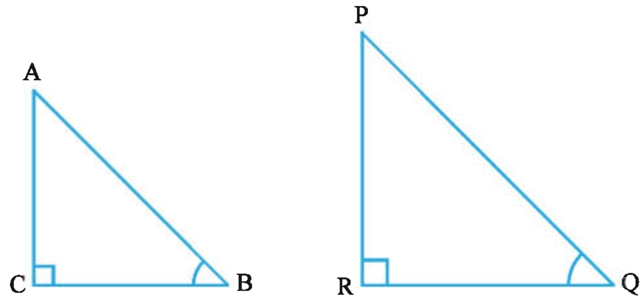

Figure 9.18

We have, $sin B = \frac{AC}{AB}$, $sin Q = \frac{PR}{PQ}$

Now, $sin B = sin Q$

$\therefore \quad \frac{AC}{AB} = \frac{PR}{PQ}$

$\therefore \quad$ Let $\frac{AC}{PR} = \frac{AB}{PQ} = k, (k > 0)$ **(i)**

$\therefore \quad AC = kPR$ and $AB = kPQ$. **(ii)**

Using Pythagoras theorem,

$AB^2 = AC^2 + BC^2$ and $PQ^2 = PR^2 + QR^2$

$\therefore \quad BC = \sqrt{AB^2 - AC^2}$ and $QR = \sqrt{PQ^2 - PR^2}$

Now, $\frac{BC}{QR} = \frac{\sqrt{AB^2 - AC^2}}{\sqrt{PQ^2 - PR^2}} = \frac{\sqrt{k^2PQ^2 - k^2PR^2}}{\sqrt{PQ^2 - PR^2}}$ **(by (ii))**

$= \frac{k\sqrt{PQ^2 - PR^2}}{\sqrt{PQ^2 - PR^2}} = k$ **($k > 0$)**

$\therefore \quad \frac{BC}{QR} = k$ **(iii)**

From (i) and (iii), we get,

$\frac{AC}{PR} = \frac{AB}{PQ} = \frac{BC}{QR}$

Then by using SSS theorem on similarity.

$\therefore \quad$ The correspondence $ABC \leftrightarrow PQR$ is a similarity.

$\therefore \quad \angle B \cong \angle Q$

Exercise 9

Prove the following by using trigonometric identities : (1 to 19)

1. $cos^2\theta + \frac{1}{1 + cot^2\theta} = 1$

2. $2sin^2\theta + 4sec^2\theta + 5cot^2\theta + 2cos^2\theta - 4tan^2\theta - 5cosec^2\theta = 1$

3. $\frac{1}{1+\cos\theta} + \frac{1}{1-\cos\theta} = 2cosec^2\theta$

4. $\frac{\tan\theta+\sin\theta}{\tan\theta-\sin\theta} = \frac{\sec\theta+1}{\sec\theta-1}$

5. $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$

6. $\frac{\sec\theta+\tan\theta}{cosec\,\theta+\cot\theta} = \frac{cosec\,\theta-\cot\theta}{\sec\theta-\tan\theta}$

7. $\frac{\cot\theta+cosec\,\theta-1}{\cot\theta-cosec\,\theta+1} = cosec\theta + \cot\theta$

8. $(\sin\theta + cosec\theta)^2 + (\cos\theta + \sec\theta)^2 = 7 + \tan^2\theta + \cot^2\theta$.

9. $2\sec^2\theta - \sec^4\theta - 2cosec^2\theta + cosec^4\theta = \cot^4\theta - \tan^4\theta$.

10. $(\sin\theta - \sec\theta)^2 + (\cos\theta - cosec\theta)^2 = (1 - \sec\theta \cdot cosec\theta)^2$.

11. $\frac{\sin A+\cos A}{\sin A-\cos A} + \frac{\sin A-\cos A}{\sin A+\cos A} = \frac{2}{\sin^2 A-\cos^2 A} = \frac{2}{1-2\cos^2 A}$

12. $\frac{\tan\theta-\cot\theta}{\sin\theta\cos\theta} = \sec^2\theta - cosec^2\theta = \tan^2\theta - \cot^2\theta$

13. $\frac{\sec\theta-\tan\theta}{\sec\theta+\tan\theta} = 1 - 2\sec\theta\tan\theta + 2\tan^2\theta$

14. $\sqrt{\sec^2\theta+cosec^2\theta} = \tan\theta + \cot\theta$

15. $\frac{1}{cosec\,A-\cot A} - \frac{1}{\sin A} = \frac{1}{\sin A} - \frac{1}{cosec\,A+\cot A}$

16. $\frac{\tan\theta}{1-\cot\theta} + \frac{\cot\theta}{1-\tan\theta} = 1 + \tan\theta + \cot\theta = 1 + \sec\theta \cdot cosec\theta$

17. $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta = 2\sin^2\theta - 1 = 1 - 2\cos^2\theta$.

18. $\tan^2 A - \tan^2 B = \frac{\cos^2 B-\cos^2 A}{\cos^2 B\cos^2 A} = \frac{\sin^2 A-\sin^2 B}{\cos^2 B\cos^2 A}$

19. $2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1 = 0$

20. If $\sin\theta + \cos\theta = p$ and $\sec\theta + cosec\theta = q$, show that $q(p^2 - 1) = 2p$.

21. If $\tan\theta + \sin\theta = a$ and $\tan\theta - \sin\theta = b$, then prove that $a^2 - b^2 = 4\sqrt{ab}$

22. $a\cos\theta + b\sin\theta = p$ and $a\sin\theta - b\cos\theta = q$, then prove that $a^2 + b^2 = p^2 + q^2$.

23. $\sec\theta + \tan\theta = p$, then obtain the values of $\sec\theta$, $\tan\theta$ and $\sin\theta$ in terms of p.

24. **Evaluate the following :**

(1) $\frac{\sec 38}{cosec\,52} + \frac{2}{\sqrt{3}} \cdot \tan 17\tan 38\tan 60\tan 52\tan 73 - 3(\sin^2 32 + \sin^2 58)$

(2) $\frac{-\cot\theta\tan(90-\theta)+cosec\theta\sec(90-\theta)+\sin^2 37+\sin^2 53}{\tan 10\tan 20\tan 30\tan 70\tan 80}$

25. If $\sin A + \cos A = \sqrt{2}\sin(90 - A)$, then obtain the value of $\cot A$.

26. If $cosec\theta = \sqrt{2}$, then find the value of $\frac{2\sin^2\theta+3\cot^2\theta}{4\tan^2\theta-\cos^2\theta}$.

27. If $tan\theta = \frac{8}{15}$, then evaluate $\frac{(1+sin\theta)(2-2sin\theta)}{(2+2cos\theta)(1-cos\theta)}$.

28. If $cos\theta = \frac{b}{\sqrt{a^2+b^2}}$, $0 < \theta < 90$, find the value of $sin\theta$ and $tan\theta$.

29. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) If θ is the measure of an acute angle such that $bsin\theta = acos\theta$, then $\frac{asin\theta - bcos\theta}{asin\theta + bcos\theta}$ is ...... ☐

(a) $\frac{a^2+b^2}{a^2-b^2}$ (b) $\frac{a^2-b^2}{a^2+b^2}$ (c) $\frac{a+b}{a-b}$ (d) $\frac{a-b}{a+b}$

(2) Which of the following is correct for some θ such that $0 \le \theta < 90$? ☐

(a) $\frac{1}{sec\theta} > 1$ (b) $\frac{1}{sec\theta} = 1$ (c) $sec\theta = 0$ (d) $\frac{1}{cos\theta} < 1$

(3) If $tan\theta = \frac{1}{\sqrt{5}}$, then $\frac{cosec^2\theta - sec^2\theta}{cosec^2\theta + sec^2\theta}$ is .... ☐

(a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) $\frac{1}{3}$ (d) 3

(4) If $tan^2\theta = \frac{8}{7}$, then the value of $\frac{(1+sin\theta)(1-sin\theta)}{(1-cos\theta)(1+cos\theta)}$ is .... ☐

(a) $\frac{7}{8}$ (b) $\frac{8}{7}$ (c) $\frac{49}{64}$ (d) $\frac{64}{49}$

(5) If $cot\theta = \frac{4}{3}$, then the value of $\frac{cos\theta - sin\theta}{cos\theta + sin\theta}$ is .... ☐

(a) 7 (b) $\frac{1}{7}$ (c) $\frac{4}{3}$ (d) $\frac{-4}{3}$

(6) If $cosecA = \frac{4}{3}$ and $A + B = 90$, then $secB$ is .... ☐

(a) $\frac{3}{4}$ (b) $\frac{1}{3}$ (c) $\frac{4}{3}$ (d) $\frac{7}{3}$

(7) If θ is the measure of an acute angle and $\sqrt{3}sin\theta = cos\theta$, then θ is .... ☐

(a) 30 (b) 45 (c) 60 (d) 90

(8) If $tanA = \frac{5}{12}$, then the value of $(sinA + cosA)\, secA$ is .... ☐

(a) $\frac{12}{5}$ (b) $\frac{7}{12}$ (c) $\frac{17}{12}$ (d) $\frac{-7}{12}$

(9) If $tan\theta = \frac{4}{3}$, then the value of $\sqrt{\frac{1-sin\theta}{1+sin\theta}}$ is .... ☐

(a) $\frac{1}{3}$ (b) 3 (c) $\frac{3}{4}$ (d) $\frac{9}{16}$

(10) In ΔABC, if $m\angle ABC = 90$, $m\angle ACB = 45$ and $AC = 6$, then area of ΔABC is .... ☐

(a) 18 (b) 36 (c) 9 (d) $\frac{9}{2}$

(11) If $cos^2 45 - cos^2 30 = x \cdot cos45 \cdot sin45$, then x is .... ☐

(a) 2 (b) $\frac{3}{2}$ (c) $-\frac{1}{2}$ (d) $\frac{3}{4}$

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(12) If A and B are complementary angles, then $sin A \cdot sec B$ is ....

(a) 1 (b) 0 (c) -1 (d) 2

(13) The value of $tan 20\ tan 25\ tan 45\ tan 65\ tan 70$ is ....

(a) -1 (b) 1 (c) 0 (d) $\sqrt{3}$

(14) If 7θ and 2θ are measure of acute angles such that $sin 7\theta = cos 2\theta$, then $2 sin 3\theta - \sqrt{3} tan 3\theta$ is ....

(a) 1 (b) 0 (c) -1 (d) $1 - \sqrt{3}$

(15) If A + B = 90, then $\dfrac{cot\ A\ cot\ B + cot\ A\ tan\ B}{sin\ A \cdot sec\ B} - \dfrac{sin^2 B}{cos^2 A}$ is ....

(a) $cot^2 B$ (b) $tan^2 A$ (c) $cot^2 A$ (d) $-cot^2 A$

(16) For ΔABC, $sin\left(\dfrac{B+C}{2}\right)$ = ...

(a) $sin\frac{A}{2}$ (b) $sin A$ (c) $cos\frac{A}{2}$ (d) $cos A$

(17) $\dfrac{sin^4\theta - cos^4\theta}{sin^2\theta - cos^2\theta}$ = ...

(a) 1 (b) 2 (c) 3 (d) 0

(18) If $7 cos^2\theta + 3 sin^2\theta = 4$, then $cot\theta$ is ....

(a) 7 (b) $\frac{7}{3}$ (c) $\sqrt{3}$ (d) $\frac{1}{\sqrt{3}}$

(19) If $tan 5\theta \cdot tan 4\theta = 1$, θ is ....

(a) 7 (b) 3 (c) 10 (d) 9

(20) If A and B are measures of acute angles and $tan A = \frac{1}{\sqrt{3}}$ and $sin B = \frac{1}{2}$, then $cos(A + B)$ is ....

(a) 0 (b) $\frac{1}{2}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{\sqrt{2}}$

*

Summary

In this chapter we have studied following points :

1. In a right ΔABC, right angled at B,

$$sin A = \frac{\text{Side opposite to angle A}}{\text{Hypotenuse}}$$

$$cos A = \frac{\text{Side adjacent to angle A}}{\text{Hypotenuse}}$$

$$tan A = \frac{\text{Side opposite to angle A}}{\text{Side adjacent to angle A}}$$

2. $cosec A = \frac{1}{sin\ A}$, $sec A = \frac{1}{cos\ A}$ and $cot A = \frac{1}{tan\ A}$

Also, $tan A = \frac{sin\ A}{cos\ A}$, $cot A = \frac{cos\ A}{sin\ A}$.

3. If one of the trigonometric ratios of an acute angle is known, the remaining trigonometric ratios of the angle can be obtained.

4. The value of trigonometric ratios for angles 0, 30, 45, 60 and 90.

Measure of an angle A	0	30	45	60	90
*sin*A	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
*cos*A	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
*tan*A	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
*cosec*A	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
*sec*A	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
*cot*A	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

5. If θ is the measure of an acute angle, then

$\sin(90 - \theta) = \cos\theta$, $\cos(90 - \theta) = \sin\theta$

$\tan(90 - \theta) = \cot\theta$, $\cot(90 - \theta) = \tan\theta$

$\sec(90 - \theta) = \operatorname{cosec}\theta$, $\operatorname{cosec}(90 - \theta) = \sec\theta$

6. $\sin^2\theta + \cos^2\theta = 1$, $0 \le \theta \le 90$

$\sec^2\theta - \tan^2\theta = 1$, $0 \le \theta < 90$

$\operatorname{cosec}^2\theta - \cot^2\theta = 1$, $0 < \theta \le 90$

Kaprekar number :

Another class of numbers Kaprekar described are the Kaprekar numbers. A Kaprekar number is a positive integer with the property that if it is squared, then its representation can be partitioned into two positive integer parts whose sum is equal to the original number (e.g. 45, since $45^2 = 2025$, and $20 + 25 = 45$, also 9, 55, 99 etc.) However, note the restriction that the two numbers are positive; for example, 100 is not a Kaprekar number even though $100^2 = 10000$, and $100 + 00 = 100$. This operation, of taking the rightmost digits of a square, and adding it to the integer formed by the leftmost digits, is known as the Kaprekar operation.

Some examples of Kaprekar numbers in base 10, besides the numbers 9, 99, 999, …, are :

Number	Square	Decomposition
703	$703^2 = 494209$	494 + 209 = 703
2728	$2728^2 = 7441984$	744 + 1984 = 2728
5292	$5292^2 = 28005264$	28 + 005264 = 5292
857143	$857143^2 = 734694122449$	734694 + 122449 = 857143

Height and Distance 10

The mathematician does not study mathematics because it is useful.
He studies it because he delights in it and he delights in it because it is beautiful.

- Henri Poincare

10.1 Introduction

In previous chapter, we have studied about trigonometric ratios and techniques of solving right angled triangles. We shall now see how these techniques are used to solve problems regarding heights and distances in life around us. Trigonometry is one of the most ancient subjects studied by scholars all over the world. Note that in practice only some distances can be measured but not all. For instance, height of a hill (distance between its foot and summit), width of a river, distance between two celestial objects can not be measured by a measure tape. So, method of trigonometric ratios is very useful in measuring such distances. When dealing with heights or depths, we have to measure two kinds of angles (upward and downward from our eye-level). We describe these two kinds of angles more precisely as follows.

10.2 Angle of Elevation and Angle of Depression

Horizontal Ray : A ray parallel to the surface of the earth emerging from the eye of an observer is called a horizontal ray.

Ray of Vision : The ray from the eye of an observer towards the object is called the ray of vision or ray of sight.

Angle of Elevation : If the object under observation is above an observer, but not directly above the observer, then the angle formed by the horizontal ray and the ray of sight in a vertical plane is called the angle of elevation. Here horizontal ray, observer and object are in the same vertical plane.

In figure 10.1, the object P under observation is at a higher level than the observer O but not directly above O. Let $\overrightarrow{OM}$ be the horizontal ray in the vertical plane containing O and P. Then the union of the ray of vision $\overrightarrow{OP}$ and horizontal ray $\overrightarrow{OX}$ is $\angle POM$. If $m\angle POM = e$, then e is called the measure of the angle of elevation $\angle POM$, of the object P at the point of observation O.

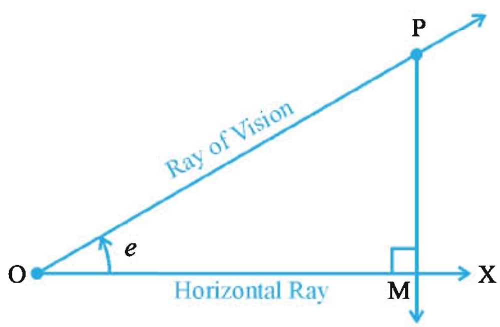

Figure 10.1

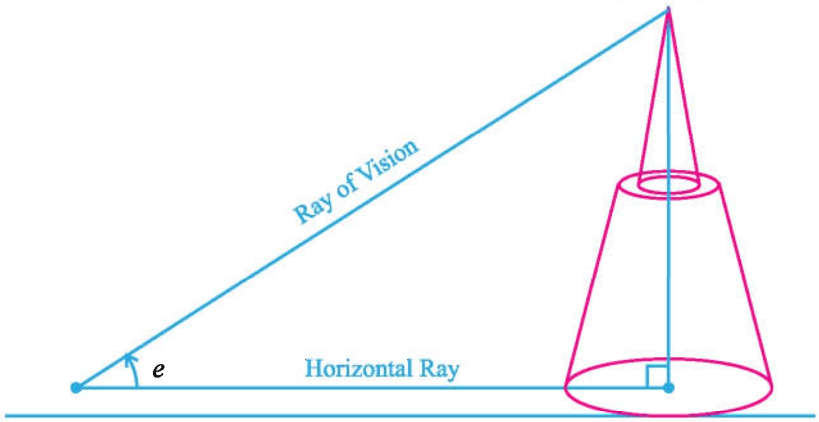

Figure 10.2

Angle of Depression : If the object under observation is at a lower level than an observer but not directly under the observer, then the angle formed by the horizontal ray and the ray of sight is called the angle of depression. Here horizontal ray, observer and the object are in the same vertical plane.

In figure 10.3, the object under observation is at a lower level than the observer O but not directly under O. Let $\overrightarrow{ON}$ be the horizontal ray in the vertical plane containing O and Q. Then the union of the ray of vision $\overrightarrow{OQ}$ and horizontal ray $\overrightarrow{ON}$ is $\angle NOQ$.

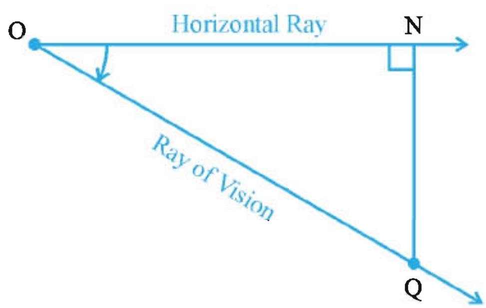

Figure 10.3

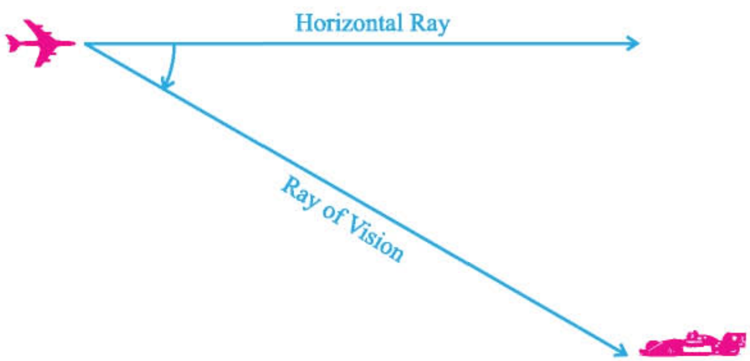

Figure 10.4

10.3 Solution of a Right Triangle

If the measure of any one side and any other element of a right angled triangle are given, then the solution of the right angled triangle can be obtained.

Suppose In ΔABC, $m\angle ABC = 90$, $m\angle ACB = 30$ and $AC = 20\ m$.

Here, $m\angle ACB + m\angle BAC = 90$

$\therefore \quad 30 + m\angle BAC = 90$

$\therefore \quad m\angle BAC = 60$

Now, $sin30 = \frac{AB}{AC}$

$\therefore \quad sin30 = \frac{AB}{20}$

$\therefore \quad \frac{1}{2} \times 20 = AB$

$\therefore \quad AB = 10\ m$

$tan30 = \frac{AB}{BC}$

$\therefore \quad tan30 = \frac{10}{BC}$

$\therefore \quad \frac{1}{\sqrt{3}} = \frac{10}{BC}$

$\therefore \quad BC = 10 \times \sqrt{3}$

$= 10 \times 1.73$

$BC = 17.3\ m$

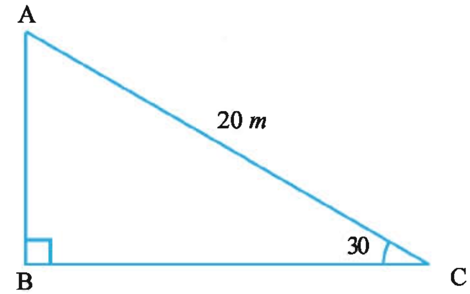

Figure 10.5

Note : In solving these examples, we shall take the values of $\sqrt{3}$ as 1.73, and $\sqrt{2}$ as 1.41.

Example 1 : A tower stands vertically on the ground. From a point on the ground which is 100 *m* away from the foot of the tower, the angle of elevation of the top of the tower is found to have measure 60. Find the height of the tower.

Solution : Suppose $\overline{AB}$ respresents the tower. O is the point 100 *m* away from the tower, OB is the distance of the point from the tower and $\angle AOB$ is the angle of elevation.

Then, OB = 100 *m* and $m\angle BOA = 60$.

In ΔAOB, $tan60 = \frac{AB}{OB}$

$\therefore \quad \sqrt{3} = \frac{AB}{100}$

$\therefore \quad AB = 100 \times \sqrt{3}$

$\therefore \quad AB = 100 \times 1.73$

$= 173\ m$

$\therefore$ The height of the tower is 173 *m*.

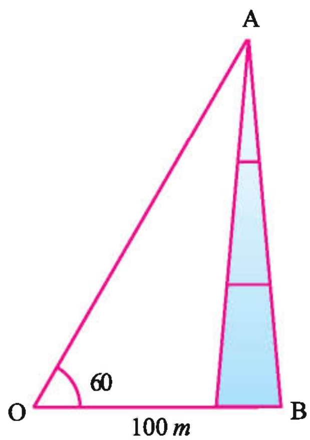

Figure 10.6

Example 2 : As observed from a fixed point on the bank of a river, the angle of elevation of a temple on the opposite bank has measure 30. If the height of the temple is 20 *m*, find the width of the river.

Solution : Here $\overline{AB}$ is the temple on the opposite bank of the river and C is the point of observation on the other bank of the river. So $\overline{BC}$ is the width of the river.

Then, AB = 20 *m* and $m\angle ACB = 30$.

In ΔABC, $tan30 = \frac{AB}{BC}$

$\therefore \quad \frac{1}{\sqrt{3}} = \frac{20}{BC}$

$\therefore \quad BC = 20 \times \sqrt{3}$

$\therefore \quad BC = 20 \times 1.73$

$= 34.6\ m$

$\therefore$ Thus, the width of the river is 34.6 *m*.

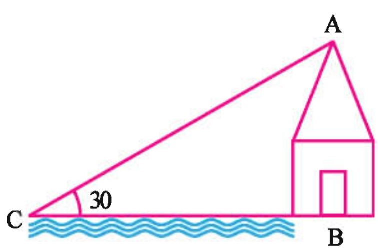

Figure 10.7

Example 3 : An observer 1.5 *m* tall is 28.5 *m* away from a tower. The angle of elevation of the top of the tower from her eyes has measure 45. What is the height of the tower ?

Solution : Here, $\overline{AD}$ is the tower having height h and $\overline{EC}$ be the observer of height 1.5 *m* at a distance of 28.5 *m* from the tower $\overline{AD}$.

Then, BC = DE = 28.5 *m* and

BD = CE = 1.5 *m*, $m\angle ACB = 45$.

In ΔABC, $tan45 = \frac{AB}{BC}$

$\therefore \quad 1 = \frac{AB}{28.5}$

$\therefore \quad AB = 28.5$

Now, $h = AB + BD = 28.5 + 1.5 = 30$

$\therefore \quad h = 30\ m$

Hence, height of the tower is 30 *m*.

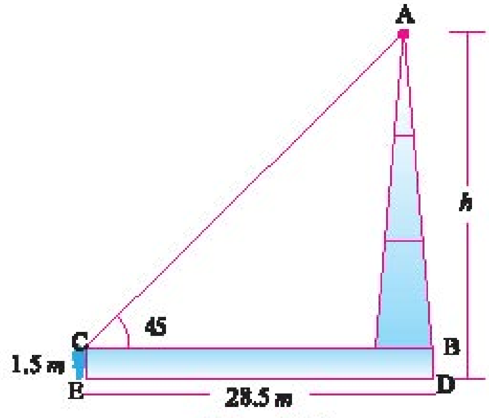

Figure 10.8

Example 4 : A Palm tree breaks due to storm and its upper end touches the ground and makes an angle of measure 30 with the ground. If the top of the tree touches the ground 15 *m* away from the bottom, find the height of the tree.

Solution : Here, $\overline{AC}$ is the tree broken at point B such that broken part $\overline{CB}$ takes the position $\overline{BD}$ and touches the ground at D.

Then, AD = 15 *m* and $m\angle ADB = 30$.

In ΔDAB, we have $tan30 = \frac{AB}{AD}$

$\therefore \quad \frac{1}{\sqrt{3}} = \frac{AB}{15}$

$\therefore \quad AB = \frac{15}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{15\sqrt{3}}{3} = 5\sqrt{3}$

Now, $cos30 = \frac{AD}{BD}$

$\therefore \quad \frac{\sqrt{3}}{2} = \frac{15}{BD}$

$\therefore \quad BD = \frac{30}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = 10\sqrt{3}$

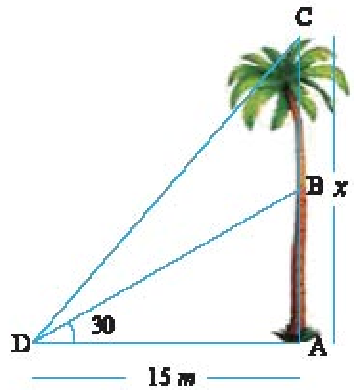

Figure 10.9

So, the height of the tree AC = AB + BC

$= AB + BD$

$= 5\sqrt{3} + 10\sqrt{3}$

$= 15\sqrt{3}$

$= 15(1.73) = 25.95\ m$

Hence the height of the tree is 25.95 *m*.

Example 5 : The angle of elevation of the top of a tower as observed from the foot of a temple has measure 60. The angle of elevation of the top of the temple as observed from the foot of the tower has measure 30. If the temple is 50 *m* high, find the height of the tower.

Solution : Here $\overline{CD}$ is the tower and $\overline{AB}$ is the temple Their feet are the points B and C respectively. $m\angle ACB = 30$ and $m\angle CBD = 60$, Also AB = 50. Let BC = y and CD = x.

In ΔABC, $tan30 = \frac{AB}{BC}$

$\therefore \quad \frac{1}{\sqrt{3}} = \frac{50}{y}$

$\therefore \quad y = 50\sqrt{3}$

Now in ΔDBC,

$tan60 = \frac{DC}{BC}$

$\sqrt{3} = \frac{x}{y}$

$\therefore \quad x = \sqrt{3} \times y = \sqrt{3} \times 50\sqrt{3}$

$\therefore \quad x = 50 \times 3 = 150\ m$

$\therefore$ The height of the tower is 150 *m*.

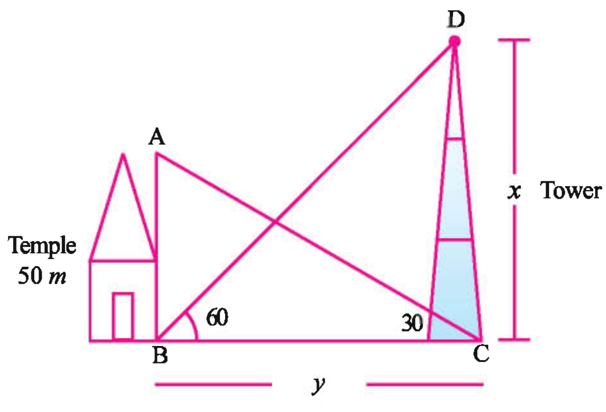

Figure 10.10

Example 6 : As the angle of elevation of the sun increases from 30 to 60, the length of the shadow of a building gets reduced by 10 *m*. Find the height of the building.

Solution : Here $\overline{AB}$ is the building and $\overline{BD}$ is its shadow when angle of elevation of the sun has measure 30 and $\overline{BC}$ is its shadow when angle of elevation of the sun has measure 60.

Then, $m\angle ADB = 30$, $m\angle ACB = 60$, DC = 10 *m*

Let AB = *h*, BC = *x*, then BD = BC + CD

$\therefore$ BD = $x + 10$

In ΔABC, $tan60 = \frac{AB}{BC}$

$\therefore \quad \sqrt{3} = \frac{h}{x}$

$\therefore \quad h = \sqrt{3}x$ **(i)**

In ΔABD, $tan30 = \frac{AB}{BD}$

$\therefore \quad \frac{1}{\sqrt{3}} = \frac{h}{x+10}$

$\therefore \quad x + 10 = \sqrt{3}h$

$\therefore \quad x + 10 = \sqrt{3}(\sqrt{3}x)$ **by (i)**

$\therefore \quad x + 10 = 3x$

$\therefore \quad 2x = 10$

$\therefore \quad x = 5$

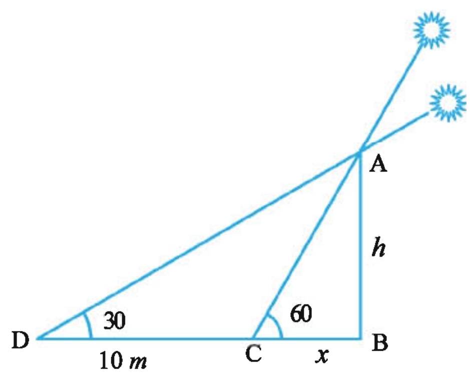

Figure 10.11

Now, by (i) $h = \sqrt{3}x$

$\therefore \quad h = \sqrt{3} \times 5$

$\therefore \quad h = 5(1.73)$

$\therefore \quad h = 8.65\ m$

Hence, the height of the building is 8.65 *m*.

Another Method : $m\angle ACB = 60 = m\angle ADC + m\angle DAC = 30 + m\angle DAC$

(Interior Opposite Angles)

$\therefore \quad m\angle DAC = 30$

$\therefore \quad AC = CD = 10$

Now, $sin60 = \frac{AB}{AC}$

$\therefore \quad AB = AC\ sin60$

$= (10)\frac{\sqrt{3}}{2} = 5\sqrt{3} = 5(1.73) = 8.65\ m$

Example 7 : A man is standing on the top of a building 60 *m* high. He observes that the angle of depression of the top and the bottom of a tower has measure 30 and 60 respectively. Find the height of the tower.

Solution : Let $\overline{AB}$ be the building and $\overline{CD}$ be the tower.

Let CD = h

Let $\overline{CE}$ be the perpendicular from C to $\overline{AB}$. The angles of depression of the top C and the bottom D of the tower $\overline{CD}$ have measures 30 and 60 respectively from A.

Then, $m\angle ACE = 30$ and $m\angle ADB = 60$

Let BD = CE = x

Here, AB = 60, CD = EB = h

AB = AE + EB

$\therefore$ 60 = AE + h

$\therefore$ AE = 60 − h

In ΔAEC, $tan30 = \frac{AE}{CE}$

$\therefore \frac{1}{\sqrt{3}} = \frac{60-h}{x}$

$\therefore x = (60 - h)\sqrt{3}$ **(i)**

In ΔABD, $tan60 = \frac{AB}{BD}$

$\therefore \sqrt{3} = \frac{60}{x}$

$\therefore x = \frac{60}{\sqrt{3}}$ **(ii)**

From (i) and (ii) we have,

$(60 - h)\sqrt{3} = \frac{60}{\sqrt{3}}$

$\therefore 3(60 - h) = 60$

$\therefore 60 - h = 20$

$\therefore h = 40\ m$

Hence, the height of the tower is 40 *m*.

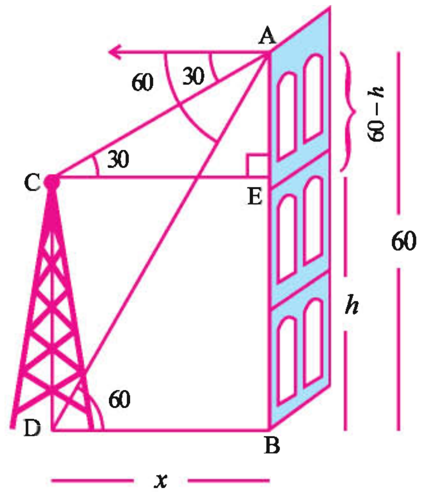

Figure 10.12

Example 8 : The angle of elevation of a jet plane from a point on the ground has measure 60. After a flight of 30 seconds. The angle of elevation has measure 30. If the jet plane is flying at a constant height of $4500\sqrt{3}$ *m*, find the speed of the jet plane.

Solution : Let O be the point of observation, C and D be the two positions of the jet plane. The angles of elevation of the jet plane in two positions C and D from the point O have measures 60 and 30 respectively. A and B are feet of perpendiculars from C and D to the ground.

$m\angle COB = 60$, $m\angle DOB = 30$ and

BD = AC = $4500\sqrt{3}\ m$

In ΔOAC, $tan60 = \frac{AC}{OA}$

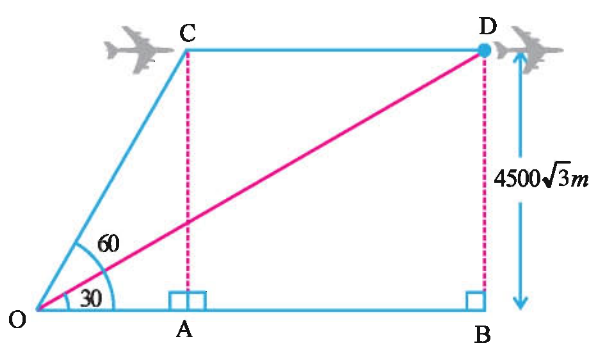

Figure 10.13

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$\therefore \quad \sqrt{3} = \frac{4500\sqrt{3}}{OA}$

$\therefore \quad OA = 4500 \ m$

In ΔOBD, $tan30 = \frac{BD}{OB}$

$\therefore \quad \frac{1}{\sqrt{3}} = \frac{4500\sqrt{3}}{OB}$

$\therefore \quad OB = 4500 \times 3 = 13500 \ m$

Now, $CD = AB = OB - OA$

$\therefore \quad CD = 13500 - 4500$

$\therefore \quad CD = 9000 \ m$

Thus, the jet plane travels 9000 *m* in 30 sec.

Hence speed $= \frac{9000}{30} = 300 \ m/sec = \frac{300 \times 60 \times 60}{1000} \ km/hr$

$\therefore$ Speed of the jet plane $= 1080 \ km/hr$

Example 9 : A straight highway leads to the foot of a tower. A man standing on the top of the tower observes a car at an angle of depression with measure 30. The car is approaching the foot of the tower with a uniform speed. Six seconds later, the angle of depression of the car has measure 60. Find the further time taken by the car to reach the foot of the tower.

Solution : Let $\overline{AB}$ be the tower and height of the tower AB = *h m*. At C the angle of depression of the car has measure 30 and six seconds later it reaches D where the angle of depression is 60.

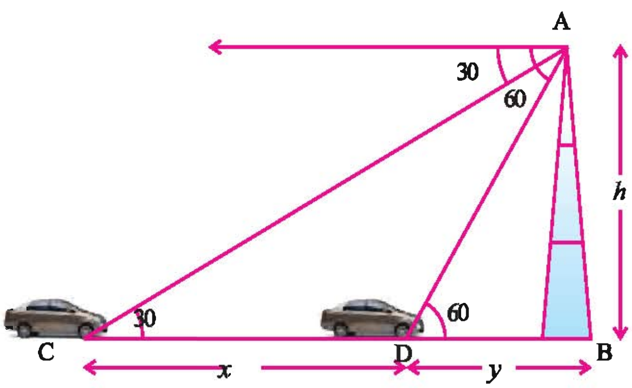

Figure 10.14

Let CD = *x*, DB = *y*

Here, AB = *h m*, $m\angle ACB = 30$ and $m\angle ADB = 60$.

In ΔACB, $tan30 = \frac{AB}{CB}$

$\therefore \quad \frac{1}{\sqrt{3}} = \frac{h}{x+y}$

$\therefore \quad x + y = \sqrt{3}h$ **(i)**

In ΔABD, $tan60 = \frac{AB}{BD}$

$\therefore \quad \sqrt{3} = \frac{h}{y}$

$\therefore \quad h = \sqrt{3}y$ **(ii)**

From (i) and (ii), we have

$x + y = \sqrt{3}(\sqrt{3}y)$

$\therefore \quad x + y = 3y$

$\therefore \quad x = 2y$

The car has uniform speed. Suppose the car travels distance at *v* metere / sec

It travels $x = 2y$ in six seconds

$\therefore$ It travels distance $y = BD$ in 3 seconds.

Hence, further time taken by the car to reach the foot of the tower is 3 seconds.

Example 10 : A 1.3 *m* tall girl spots a ballon moving with the wind in horizontal line at a constant height of 91.3 *m* from the ground. The angle of elevation of the balloon from the eyes of the girl at an instant has measure 60. After some time, the angle of elevation is reduced in measure to 30. Find the distance travelled by the balloon during the interval.

Solution : Let A and P be the positions of the balloon when its angles of elevation from the eyes of the girl at O have measures 60 and 30 respectively.

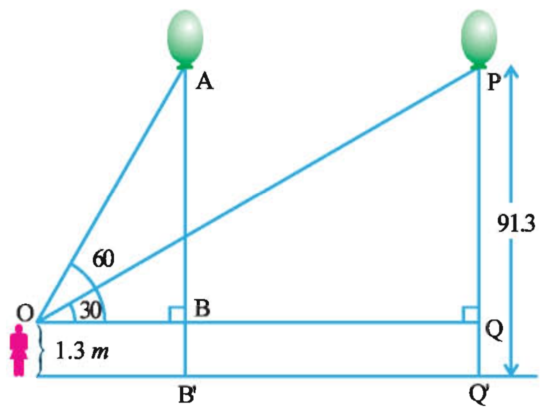

Figure 10.15

Here, AB' = PQ' = 91.3 and

BB' = QQ' = 1.3

$\therefore$ PQ = PQ' $-$ QQ'

= 91.3 $-$ 1.3 = 90 *m*

$\therefore$ PQ = AB = 90 *m*

In ΔABO, $tan 60 = \frac{AB}{OB}$

$\therefore \quad \sqrt{3} = \frac{90}{OB}$

$\therefore \quad OB = \frac{90}{\sqrt{3}} = \frac{90}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = 30 \times \sqrt{3}$ (i)

In ΔPQO, $tan 30 = \frac{PQ}{OQ}$

$\therefore \quad \frac{1}{\sqrt{3}} = \frac{90}{OQ}$

$\therefore \quad OQ = 90\sqrt{3}$ (ii)

$\therefore$ The distance travelled by the balloon = BQ = OQ $-$ OB

$\therefore \quad BQ = 90\sqrt{3} - 30\sqrt{3}$

$= 60\sqrt{3}$

$= 60 \times 1.73$

= 103.8 *m*

Hence, the distance travelled by the balloon is 103.8 *m*.

Example 11 : As observed from the top of a building 60 *m* above the surface of a lake, the angle of elevation of a kite flying in the sky has measure 30 and the angle of depression of the image of the kite in the lake has measure 60. Find the height of the kite above the surface of the lake.

Solution : Let $\overline{BE}$ be the surface of the lake and $\overline{AB}$ be the building. Let F be the reflection of kite C. Horizontal line $\overline{AD}$ intersect $\overline{CE}$ in D.

$m\angle CAD = 30$, $m\angle FAD = 60$

AB = 60 *m*. Let CE = *h*, BE = *l*.

Then CD = *h* $-$ 60 and DF = *h* + 60

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In ΔADC, $tan30 = \frac{CD}{AD}$

$\therefore \quad \frac{1}{\sqrt{3}} = \frac{h-60}{l}$

$\therefore \quad l = \sqrt{3}(h - 60)$ **(i)**

In ΔADF, $tan60 = \frac{DF}{AD}$

$\therefore \quad \sqrt{3} = \frac{h+60}{l}$

$\therefore \quad \sqrt{3}l = h + 60$ **(ii)**

From (i) and (ii)

$\sqrt{3}[\sqrt{3}(h - 60)] = h + 60$

$3(h - 60) = h + 60$

$3h - 180 = h + 60$

$2h = 240$

$h = 120\ m$

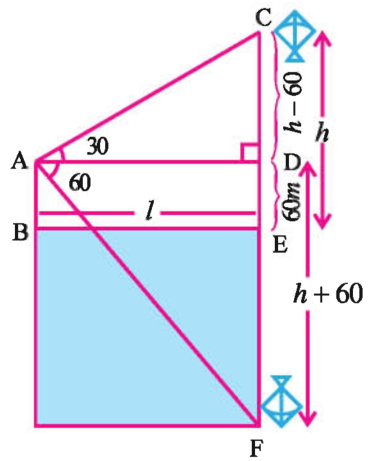


Figure 10.16

Hence, the height of the kite above the surface of the lake is 120 *m*.

Example 12 : A flag-staff of height *h* stands on the top of a school building. If the angles of elevation of the top and bottom of the flag-staff have measures α and β are respectively from a point on the ground, prove that the height of the building is $\frac{h tan\beta}{tan\alpha - tan\beta}$.

Solution : Let $\overline{AB}$ be the flag-staff, $\overline{BC}$ be the school building and D be the point of observation.

Now, AB = *h*. Let BC = H and CD = *d*. $m\angle ADC = \alpha$ and $m\angle BDC = \beta$.

In ΔADC, $tan\alpha = \frac{h + H}{d}$

$d = \frac{h + H}{tan\,\alpha}$ **(i)**

In ΔBDC, $tan\beta = \frac{H}{d}$

$d = \frac{H}{tan\,\beta}$ **(ii)**

From (i) and (ii)

$\frac{h + H}{tan\,\alpha} = \frac{H}{tan\,\beta}$

$\therefore \quad h tan\beta + H tan\beta = H tan\alpha$

$\therefore \quad h tan\beta = H tan\alpha - H tan\beta$

$\therefore \quad H(tan\alpha - tan\beta) = h tan\beta$

$\therefore \quad H = \frac{h tan\beta}{tan\alpha - tan\beta}$

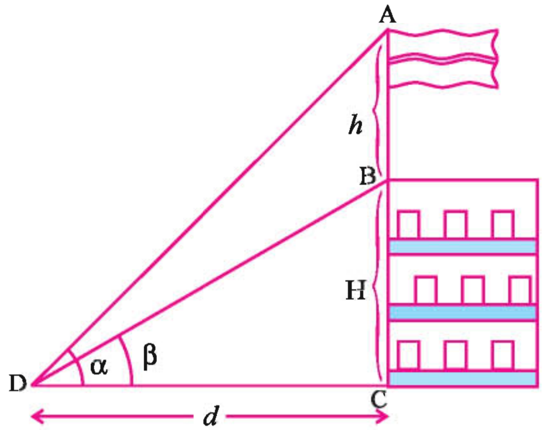


Figure 10.17

Example 13 : A ladder rests against a wall at an angle having measure α with the ground. Its foot is pulled away from the wall by a *m* keeping ladder on the ground. By doing this, its upper end on the wall slides down by b *m*. Now the ladder makes an angle of measure β with the ground. Then prove that $\frac{a}{b} = \frac{\cos\beta - \cos\alpha}{\sin\alpha - \sin\beta}$.

Solution : Let $\overline{AB}$ be the ladder when its top A is on the wall and bottom B is on the ground such that $m\angle ABC = \alpha$. Now the ladder is pulled away from the wall through a distance *a*, so that its top A slides and takes position A' and bottom B slides and takes position B', such that $m\angle A'B'C = \beta$. Let BC = *x*, A'C = *y*.

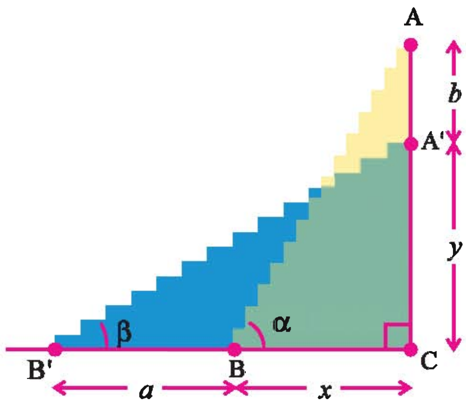


Figure 10.18

Now, AA' = *b* and B'B = *a*.

In ΔABC, $\sin\alpha = \frac{AC}{AB}$, $\cos\alpha = \frac{BC}{AB}$

$\therefore \quad \sin\alpha = \frac{b+y}{AB}$, $\cos\alpha = \frac{x}{AB}$ (i)

In $\Delta A'B'C$, $\sin\beta = \frac{A'C}{A'B'}$, $\cos\beta = \frac{B'C}{A'B'}$

$\therefore \quad \sin\beta = \frac{y}{A'B'}$, $\cos\beta = \frac{a+x}{A'B'}$

Now, AB = A'B'

$\therefore \quad \sin\beta = \frac{y}{AB}$, $\cos\beta = \frac{a+x}{AB}$ (ii)

From (i) and (ii)

$$\sin\alpha - \sin\beta = \frac{b+y}{AB} - \frac{y}{AB} \text{ and } \cos\beta - \cos\alpha = \frac{a+x}{AB} - \frac{x}{AB}$$

$\therefore \quad \sin\alpha - \sin\beta = \frac{b}{AB}$ and $\cos\beta - \cos\alpha = \frac{a}{AB}$

$$\therefore \quad \frac{\cos\beta - \cos\alpha}{\sin\alpha - \sin\beta} = \frac{\frac{a}{AB}}{\frac{b}{AB}}$$

$$\therefore \quad \frac{\cos\beta - \cos\alpha}{\sin\alpha - \sin\beta} = \frac{a}{b}$$

Example 14 : A jet plane is at a vertical height of *h*. The angles of depression of two tanks on the horizontal ground are found to have measures α and β ($\alpha > \beta$) Prove that the distance between the tanks is $\frac{h(\tan\alpha - \tan\beta)}{\tan\alpha \cdot \tan\beta}$.

Solution : Let A be the jet plane, C and D are two tanks.

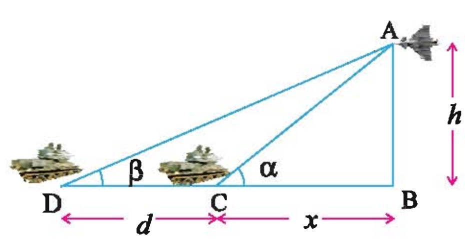


Figure 10.19

Here AB = *h*, BC = *x* and CD = *d*.

$m\angle ACB = \alpha$ and $m\angle ADB = \beta$

In ΔABC, $tan\alpha = \frac{AB}{BC}$

$\therefore \quad tan\alpha = \frac{h}{x}$

$\therefore \quad x = \frac{h}{\tan\alpha}$ **(i)**

In ΔABD, $tan\beta = \frac{AB}{BD}$

$\therefore \quad tan\beta = \frac{h}{x+d}$

$\therefore \quad x + d = \frac{h}{\tan\beta}$ **(ii)**

From (i) and (ii)

$$\frac{h}{\tan\alpha} + d = \frac{h}{\tan\beta}$$

$$\therefore \quad d = \frac{h}{\tan\beta} - \frac{h}{\tan\alpha}$$

$$\therefore \quad d = \frac{h(tan\alpha - tan\beta)}{tan\alpha \cdot tan\beta}$$

Hence, the distance between the tanks is $\frac{h(tan\alpha - tan\beta)}{tan\alpha \cdot tan\beta}$.

EXERCISE 10

1. A pole stands vertically on the ground. If the angle of elevation of the top of the pole from a point 90 *m* away from the pole has measure 30, find the height of the pole.
2. A string of a kite is 100 *m* long and it makes an angle of measure 60 with the horizontal. Find the height of the kite, assuming that there is no slack in the string.
3. A circus artist is climbing from the ground along a rope stretched from the top of a vertical pole and tied at the ground. The height of pole is 10 *m* and the angle made by the rope with ground level has measure 30. Calculate the distance covered by the artist in climbing to the top of the pole.
4. A tree breaks due to a storm and the broken part bends such that the top of the tree touches the ground making an angle having measure 30 with the ground. The distance from the foot of the tree to the point where the top touches the ground is 30 *m*. Find the height of the tree.
5. An electrician has to repair an electric fault on the pole of height 5 *m*. He needs to reach a point 2 *m* below the top of the pole to undertake the repair work. What should be the length of the ladder that he should use, which when inclined at an angle of measure 60 to the horizontal would enable him to reach the required position.
6. As observed from a fixed point on the bank of a river, the angle of elevation of a temple on the opposite bank has measure 30. If the height of the temple is 20 *m*, find the width of the river.
7. As observed from the top of a hill 200 *m* high, the angles of depression of two vehicles situated on the same side of the hill are found to have measure 30 and 60 respectively. Find the distance between the two vehicles.
8. A person standing on the bank of a river, observes that the angle subtended by a tree on the opposite bank has measure 60. When he retreats 20 *m* from the bank, he finds the angle to have measure 30. Find the height of the tree and the breadth of the river.

9. The shadow of a tower is 27 *m*, when the angle of elevation of the sun has measure 30. When the angle of elevation of the sun has measure 60, find the length of the shadow of the tower.

10. From a point at the height 100 *m* above the sea level, the angles of depression of a ship in the sea is found to have measure 30. After some time the angle of depression of the ship has measure 45. Find the distance travelled by the ship during that time interval.

11. From the top of a 300 *m* high light-house, the angles of depression of the top and foot of a tower have measure 30 and 60. Find the height of the tower.

12. As observed from a point 60 *m* above a lake, the angle of elevation of an advertising ballon has measure 30 and from the same point the angle of depression of the image of the ballon in the lake has measure 60. Calculate the height of the balloon above the lake.

13. Watching from a window 40 *m* high of a multistoreyed building, the angle of elevation of the top of a tower is found to have measure 45. The angle of elevation of the top of the same tower from the bottom of the building is found to have measure 60. Find the height of the tower.

14. Two pillars of equal height stand on either side of a road, which is 100 *m* wide. The angles of elevation of the top of the pillars have measure 60 and 30 at a point on the road between the pillars. Find the position of the point from the nearest end of a pillars and the height of pillars.

15. The angles of elevation of the top of a tower from two points at distance *a* and *b* metres from the base and in the same straight line with it are complementary. Prove that the height of the tower is $\sqrt{ab}$ metres.

16. A man on the top of a vertical tower observes a car moving at a uniform speed coming directly towards it. If it takes 12 minutes for the angle of depression to change its measure from 30 to 45, how soon after this, will the car reach the tower ?

17. If the angle of elevation of a cloud from a point *h* metres above a lake has measure α and the angle of depression of its reflection in the lake has measure β, prove that the height of the cloud is $\frac{h(tan\beta + tan\alpha)}{tan\beta - tan\alpha} m$.

18. From the top of a building $\overline{AB}$, 60 *m* high, the angles of depression of the top and bottom at a vertical lamp post $\overline{CD}$ are observed to have measure 30 and 60 respectively. Find,

    (1) the horizontal distance between building and lamp post.

    (2) the height of the lamp post.

    (3) the difference between the heights of the building and the lamp post.

19. A bridge across a valley is *h* metres long. There is a temple in the valley directly below the bridge. The angles of depression of the top of the temple from the two ends of the bridge have measures α and β. Prove that the height of the bridge above the top of the temple is $\frac{h(tan\alpha \cdot tan\beta)}{tan\alpha + tan\beta} m$.

20. At a point on level ground, the angle of elevation of a vertical tower is found to be such that its tangent is $\frac{5}{12}$. On walking 192 metres towards the tower, the tangent of the angle is found to be $\frac{3}{4}$. Find the height of the tower.

21. A statue 1.46 *m* tall, stands on the top of a pedestal. From the point on the ground the angle of elevation of the top of the statue has measure 60 and from the same point, the angle of elevation of the top of the pedestal has measure 45. Find the height of the pedestal.

22. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) On walking ...... metres on a hill making an angle of measure 30 with the ground, one can reach the height of 'a' metres from the ground. ☐

(a) $\frac{\sqrt{3}}{2}a$ (b) $\frac{2a}{\sqrt{3}}$ (c) $2a$ (d) $\frac{a}{2}$

(2) The angle of elevation of the top of the tower from a point P on the ground has measure 45. The distance of the tower from the point P is a and height of the tower is b. Then, ... ☐

(a) $a > b$ (b) $a < b$ (c) $a = b$ (d) $a = 2b$

(3) A 3 m long ladder leans on the wall such that its lower end remains 1.5 m away from the base of the wall. Then, the ladder makes an angle of measure ...... with the ground. ☐

(a) 30 (b) 45 (c) 60 (d) 20

(4) A tower is $50\sqrt{3}$ m high. The angle of elevation of its top from a point 50 m away from its foot has measure ...... ☐

(a) 45 (b) 60 (c) 30 (d) 15

(5) If the ratio of the height of a tower and the length of its shadow is $1 : \sqrt{3}$, then the angle of elevation of the sun has measure ...... ☐

(a) 30 (b) 45 (c) 60 (d) 75

(6) If the angles of elevation of a tower from two points distance a and b ($a > b$) from its foot on the same side of the tower have measure 30 and 60, then the height of the tower is ...... ☐

(a) $\sqrt{a+b}$ (b) $\sqrt{ab}$ (c) $\sqrt{a-b}$ (d) $\sqrt{\frac{a}{b}}$

(7) The tops of two poles of height 18 m and 12 m are connected by a wire. If the wire makes an angle of measure 30 with horizontal, then the length of the wire is ...... ☐

(a) 12 m (b) 10 m (c) 8 m (d) 4 m

(8) The angle of elevation of the top of the building A from the base of building B has measure 50. The angle of elevation of the top of the building B from the base of building A has measure 70. Then, ... ☐

(a) building A is taller than building B.

(b) Building B is taller than building A.

(c) Building A and building B are equally tall.

(d) The relation about the heights of A and B cannot be determined.

(9) If the angle of elevation of the top of a tower of a distance 400 m from its foot has measure 30, then the height of the tower is ...... ☐

(a) $200\sqrt{2}$ (b) $\frac{400}{\sqrt{3}}$ (c) $200\sqrt{3}$ (d) $\frac{400}{\sqrt{2}}$

(10) The angle of depression of a ship from the top of a tower 30 m height has measure 60. Then, the distance of the ship from the base of the tower is ...... ☐

(a) 10 (b) 30 (c) $10\sqrt{3}$ (d) $30\sqrt{3}$

(11) When the length of the shadow of the pole is equal to the height of the pole, then the angle of elevation of the source of light has measure ...... ☐

(a) 45 (b) 30 (c) 60 (d) 75

(12) From the top of a building h metre high, the angle of depression of an object on the ground has measure θ. The distance (in metres) of the object from the foot of the building is ...... ☐

(a) $h \sin\theta$ (b) $h \tan\theta$ (c) $h \cot\theta$ (d) $h \cos\theta$

(13) As observed from the top of the light house the angle of depression of the two ships P and Q anchored in the sea to the same side are found to have measure 35 and 50 respectively. Then from the light house.... ☐

(a) P and Q are at equal distance.

(b) The distance of Q is more then P.

(c) The distance of P is more than Q.

(d) The relation about the distance of P and Q cannot be determined.

(14) Two poles are x metres apart and the height of one is double than that of the other. If from the mid-point of the line joining their feet, an observer finds the angle of elevation of their tops to be complementary, then the height of the shorter pole is ...... ☐

(a) $\frac{x}{4}$ (b) $\frac{x}{\sqrt{2}}$ (c) $\sqrt{2}x$ (d) $\frac{x}{2\sqrt{2}}$

*

Summary

In this chapter we have studied following points :

1. **Horizontal Ray :** A ray parallel to the surface of the earth emerging from the eye of the observer is called a horizontal ray.
2. **Ray of Vision :** The ray from the eye of an observer towards the object is called the ray of vision or ray of sight.
3. **Angle of Elevation :** If the object under observation is above an observer, but not directly above the observer, then the measure of the angle formed by the horizontal ray and the ray of sight is called the angle of elevation. Here horizontal ray, observer and object are in the same vertical plane.
4. **Angle of Depression :** If the object under observation is at a lower level than an observer but not directly under the observer, then the measure of the angle formed by the horizontal ray and the ray of sight is called the angle of depression.
5. The height of length of an object on the distance between two distant objects can be determined with the help of trigonometric ratios.

◆

CIRCLE 11

A mathematician like a painter or a poet is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas.

- G. H. Hardy

11.1 Introduction

We have learnt about circle in standard IX. We defined a circle and also defined terms related to a circle like a radius, a chord, an arc, a segment, a sector of a circle etc.. We also studied some properties of circle. Let us recollect them in brief.

(i) A **circle** is the set of points in a plane which are at the same distance from a fixed point in the plane. The fixed point is called the **centre** of the circle. The line-segment joining the centre and a point on circle is called **radius**. We are using the word radius for a line-segment as well as the length of the line-segment.

(ii) The congruent chords of a circle (or congruent circles) subtend congruent angles at the centre of the circle.

(iii) If the angles subtended by two chords of a circle (or congruent circles) at the centre (or the centres of the respective circles) are congruent, then the chords are congruent.

(iv) A line passing through the centre of a circle and perpendicular to a chord of the circle bisects the chord.

(v) Three non-collinear points always determine a circle uniquely.

(vi) Congruent chords of a circle (or of congruent circles) are equidistant from the centre (or from the respective centres) of the circle (or circles).

The converse of the above statement is also true.

(vii) If two arcs of a circle (or congruent circles) are congruent, then their corresponding chords are also congruent.

(viii) Congruent arcs of a circle subtend congruent angles at the centre of the circle.

(ix) The angle subtended by a minor arc at the centre of a circle has double the measure than the measure of the angle subtended by the same arc at a point on corresponding major arc.

(x) Angles in the same segment of a circle are congruent.

(xi) Angle inscribed in a semicircle is a right angle.

(xii) If a line-segment joining two points subtends congruent angles at two distinct points lying in the same halfplane of the line containing the segment, then there is a circle passing through the four points. We say that these points are concyclic or they are the vertices of a cyclic quadrilateral.

(xiii) The sum of measures of the opposite angles of a cyclic quadrilateral is 180 that is to say that opposite angles of a cyclic quadrilateral are supplementary.

(xiv) If sum of measures of angles of a pair of opposite angles of a quadrilateral is 180, the quadrilateral is cyclic.

Intersection of a line and circle in the same plane :

Now let us consider the intersection of a line and a circle both lying in the same plane. Let us denote the set of points on the circle by S and the line by l. We observe the following three possibilities :

(1) $l \cap S = \emptyset$

In this case we say that the line does not intersect the circle. See figure 11.1.

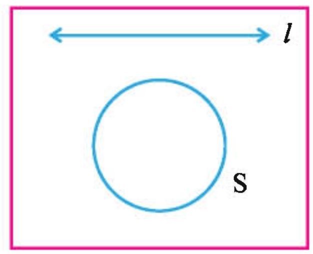


Figure 11.1

(2) $l \cap S$ is singleton.

In this case there is exactly one point common to the line and the circle.

In figure 11.2 the line intersects the circle at point P and only at P.

$\therefore$ $S \cap l = \{P\}$

In this case we say that line l touches the circle S at point P. We also say that l is a tangent to the circle S and point P is the point of contact.

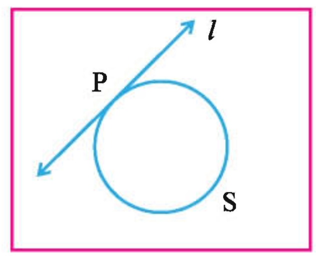


Figure 11.2

(3) $S \cap l$ = consists set of two points.

In figure 11.3 line l intersects the circle in two distinct points P and Q. So $S \cap l = \{P, Q\}$. When a line intersects a circle in two distinct points, the line is called a secant of the circle.

In this chapter, we are interested particularly in case (2). Let us examine case (2) in detail.

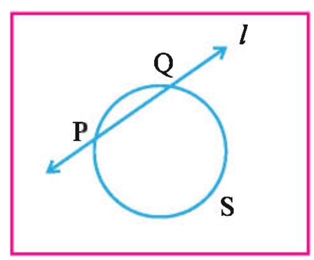


Figure 11.3

11.2 Tangent to a Circle

If a line drawn in the plane of a circle intersects the circle in one and only one point, then the line is called a tangent to the circle and the point at which the line intersects the circle is called the point of contact of the line with the circle.

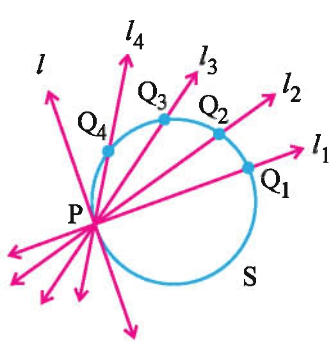


Figure 11.4

Let us see another view point by which we can understand the tangent.

In figure 11.4, line l_1 intersects the circle S in P and Q_1, line l_1 is a secant of the circle S. Now lines l_2, l_3, l_4,... are drawn in such a way that they all pass through P but they intersect the circle in another point which are respectively Q_2, Q_3, Q_4,.... . In a way we can say that line l, passing through point P rotates around point P and the sequence of points Q_1, Q_2, Q_3, Q_4,....approaches nearer and nearer to P. When P and Q coincide with each other, line l no longer remains a secant of the circle S. In this case line l becomes a tangent to the circle S.

So, we can say that,

The tangent to a circle is the limiting case of a secant, when the two end points of its corresponding chord coincide.

This approach of defining a tangent to a circle or in general to a curve was given by **René Descartes**, a great Geometrician and later on it was adopted by Newton, Leibnitz and other mathematicians. Can we draw a tangent at each point of a given circle ? How many tangents can be drawn from a given point on a circle ? Let us have an activity.

Activity : Let us draw a circle and denote its centre by O. Draw a line l passing through O and intersecting the circle at A and B. (You already know that $\overline{AB}$ is a diameter of the circle). We can draw a line perpendicular to l from any point P_1, P_2, P_3,... on $\overrightarrow{OA}$ such that A is between O and P_i (i = 1, 2, 3,...).

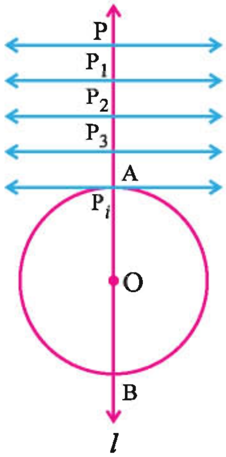


Figure 11.5

From P_1, P_2, P_3,... draw lines perpendicular to l. All points P_1, P_2, P_3,... are outside the circle and as shown in figure 11.5 distances OP_1, OP_2, OP_3 are greater than the radius r (OA) and the sequence of P_1, P_2, P_3,... is approaching A, so that distances OP_1, OP_2, OP_3,... become smaller and smaller as P_i approaches A. The line perpendicular to $\overrightarrow{OA}$ ultimately becomes a tangent to the circle.

So, for each point A on a circle, there is a line $l = \overleftrightarrow{OA}$ and there is one and only one line passing through A which is perpendicular to $\overleftrightarrow{OA}$. Hence we can say that **there is one and only one tangent passing through each point of a circle.**

The activity described above not only shows the existence of a tangent and its uniqueness but it also suggests an important property of the tangent that tangent at a point on the circle is perpendicular to the radius of the circle passing through that point. We are going to prove this statement as theorem 11.1. Let us recall one more property of circle.

If a point P in the plane of ⊙(O, r) is in the exterior of the circle then $OP > r$ and if P is in the plane of ⊙(O, r) such that $OP > r$, then P is in the exterior of the circle. In fact this is the definition of the exterior of a circle.

Now let us prove Theorem 11.1.

Theorem 11.1 : A tangent to a circle is perpendicular to the radius drawn from the point of contact.

Given : Line l is tangent to the ⊙(O, r) at point A.

To prove : $\overline{OA} \perp l$.

Proof : Let $P \in l$, $P \neq A$.

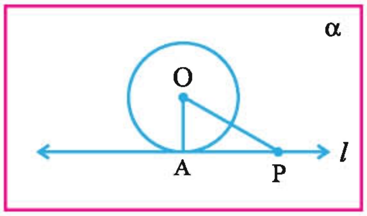


Figure 11.6

If P is in the interior of ⊙(O, r), then the line l will be a secant of the circle and not a tangent. But l is a tangent of the circle, so P is not in the interior of the circle. Also $P \neq A$.

∴ P is the point in the exterior of the circle.

∴ OP > OA. **($\overline{OA}$ is the radius of the circle)**

Therefore each point P ∈ l except A satisfies the inequality OP > OA..

Therefore OA is the shortest distance of line l from O.

∴ $\overline{OA} \perp l$.

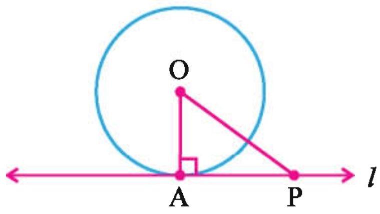


Figure 11.7

Now what about the converse of the theorem 11.1 ? The converse of the statement of theorem 11.1 can be writen as

A line drawn perpendicular to a radius at its end point on the circle is a tangent to the circle.

Is this statement true ? Yes, if the line drawn is in the plane of the circle, the statement is true. We will accept this theorem without proof.

Theorem 11.2 : If a line is in the plane of a circle such that it is perpendicular to the radius of the circle at its end point on the circle, then the line is a tangent to the circle.

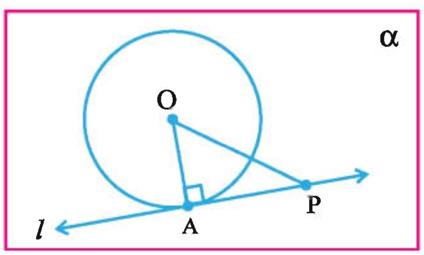


Figure 11.8

In figure 11.8 line l and ⊙(O, r) are in plane α and the line l is perpendicular to radius $\overline{OA}$ at the end point A which is on the circle. If P is any point on l, then

OA < OP because $\overline{OA} \perp l$

∴ OP > OA. Therefor OP > r

Therefore all points like P on l are in the exterior of ⊙(O, r).

∴ Line l intersects the ⊙(O, r) at only one point A. l is also in the plane of ⊙(O, r). Hence l is a tangent to the circle at O.

Notes : (1) This discussion also shows that from every point on a circle one and only one tangent can be drawn.

(2) If a line is tangent to a circle it intersects the circle at one and only one point. This property of tangent is true for a circle, but it is not necessarily true for all curves. In chapter 2 you have studied some curves in different context. A curve known as a cubic curve is drawn in figure 11.9. We can see that the tangent at point P, again intersects the curve at Q.

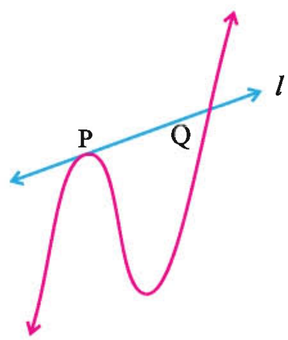


Figure 11.9

(3) The line perpendicular to the tangent to a curve, drawn at the point of contact, in the plane of the curve is called a normal to the curve. Particularly the normal to a circle drawn at each point of the circle passes through the centre. Using this fact, we can define a circle as a plane curve whose normals at all points are concurrent. The point of concurrence is known as the centre of the circle.

Example 1 : A line passing through the centre O of the circle intersects a tangent of the circle in Q. P is the point of contact of the tangent. If radius of the circle is 5 and OQ = 13, find PQ.

Solution : P is the point of contact of tangent and O is the centre of the circle.

$\therefore$ OP = radius of the circle.

$\therefore$ OP = 5, OQ = 13

and $\angle OPQ$ is a right angle because the radius of the circle is perpendicular to the tangent at the point of contact.

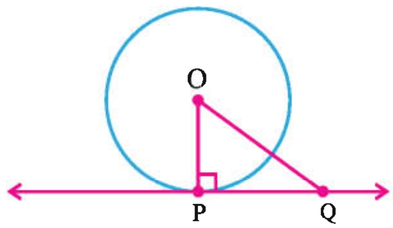


Figure 11.10

$\therefore$ In ΔOPQ, $OP^2 + PQ^2 = OQ^2$

$\therefore$ $5^2 + PQ^2 = 13^2$

$\therefore$ $PQ^2 = 13^2 - 5^2 = 169 - 25 = 144$

$\therefore$ PQ = 12

Example 2 : $\overline{AB}$ is a diameter of the circle. Show that the tangents at A and B are parallel.

Solution : $\overline{AB}$ is the diameter of the circle having centre O. l_1 and l_2 are tangents to the circle at A and B respectively. We have to prove $l_1 \parallel l_2$.

l_1 and l_2 are the lines in the plane of the circle and $\overleftrightarrow{AB}$ is the transversal.

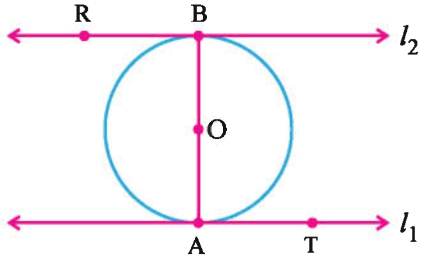


Figure 11.11

Let T be a point on l_1 such that $T \neq A$.

Let R be a point on l_2, other than B such that T and R are in different half planes of $\overleftrightarrow{AB}$. l_1 and l_2 are tangents to the circle with centre O.

$\therefore$ $l_1 \perp \overline{OA}$ and $l_2 \perp \overline{OB}$

But $\overline{AB}$ is a diameter.

$\therefore$ A–O–B

$\therefore$ $l_1 \perp \overline{AB}$ and $l_2 \perp \overline{AB}$ **(both right angles)**

$\therefore$ $\angle TAB \cong \angle RBA$

But these are alternate angles made by transversal $\overleftrightarrow{AB}$ of l_1 and l_2.

$\therefore$ $l_1 \parallel l_2$

Example 3 : Two concentric circles $\odot(O, r_1)$ and $\odot(O, r_2)$ are such that $r_1 > r_2$. Chord $\overline{AB}$ of $\odot(O, r_1)$ touches $\odot(O, r_2)$ at P. Prove that P is the mid-point of $\overline{AB}$.

Solution : $\overline{AB}$ is the chord of $\odot(O, r_1)$.

$\overline{AB}$ touces $\odot(O, r_2)$ at P.

$\therefore$ $\overline{OP} \perp \overline{AB}$

The foot of the perpendicular from O (the centre of the circle) to chord $\overline{AB}$ of $\odot(O, r_1)$ is P.

$\therefore$ P is the mid-point of the chord $\overline{AB}$.

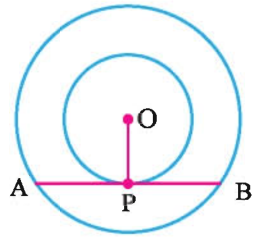


Figure 11.12

Example 4 : Radii of two concentric circles are 26 and 24. A chord of the circle with larger radius touches the circle with smaller radius. Find the length of the chord..

Solution : Let O be the centre of concentric circles. Let the chord $\overline{AB}$ of the circle with larger radius touch the circle with smaller radius at P.

∴ OP = radius of the circle with smaller radius = 24

∴ OA = radius of the circle with larger radius = 26

Since $\overline{AB}$ touches ⊙(O, 24) at P, $\overline{AB} \perp \overline{OP}$.

ΔOPA is right angled triangle with $m\angle OPA = 90$

∴ $OP^2 + AP^2 = OA^2$

∴ $24^2 + AP^2 = 26^2$

∴ $AP^2 = 26^2 - 24^2$

∴ $AP^2 = (26 + 24)(26 - 24) = 100$

∴ AP = 10

∴ $\overline{OP} \perp \overline{AB}$ and $\overline{AB}$ is the chord of ⊙(O, 26).

∴ P is the mid-point of the chord $\overline{AB}$.

∴ AB = 2AP = 20

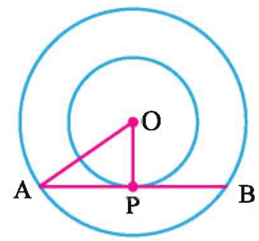


Figure 11.13

Example 5 : A and B are two distinct points on a circle with centre O. $\overline{AB}$ is not a diameter of the circle. The tangents at A and B intersect in point P. Show that ∠AOB and ∠APB are supplementary angles. Also show that PA = PB.

Solution : $\overline{AB}$ is not a diameter of the circle.

∴ The tangents at A and B are not parallel.

∴ They intersect at point P.

Also, $\overline{OA} \perp \overline{AP}$, $\overline{OB} \perp \overline{BP}$

∴ In □ OAPB, $m\angle A + m\angle B = 90 + 90 = 180$

∴ $m\angle AOB + m\angle APB = 180$

(The sum of measures of angles of a quadrilateral is 360)

∴ ∠AOB and ∠APB are supplementary angles.

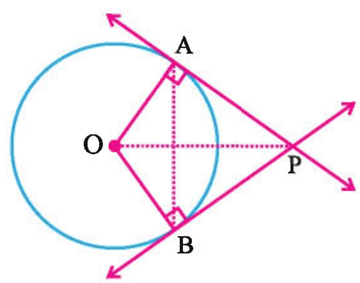


Figure 11.14

In ΔOAP and ΔOBP consider the correspondence OAP ↔ OBP.

∴ $\overline{OP} \cong \overline{OP}$

$\overline{OA} \cong \overline{OB}$ **(radii)**

∴ ∠OAP ≅ ∠OBP **(both being right angles)**

∴ ΔOAP ≅ ΔOBP **(R.H.S. criterian)**

∴ $\overline{PA} \cong \overline{PB}$

∴ PA = PB

Important remarks : (1) □ OAPB is a cyclic quadrilateral. **(Why ?)**

(2) Circle with $\overline{OP}$ as a diameter passes through A and B. **(Why ?)**

EXERCISE 11.1

1. A and B are the points on ⊙(O, r). $\overline{AB}$ is not a diameter of the circle. Prove that the tangents to the circle at A and B are not parallel.
2. A, B are the points on ⊙(O, r) such that tangents at A and B intersect in P. Prove that $\overrightarrow{OP}$ is the bisector of ∠AOB and $\overrightarrow{PO}$ is the bisector of ∠APB.
3. A, B are the points on ⊙(O, r) such that tangents at A and B to the circle intersect in P. Show that the circle with $\overline{OP}$ as a diameter passes through A and B.
4. ⊙(O, r_1) and ⊙(O, r_2) are such that $r_1 > r_2$. Chord $\overline{AB}$ of ⊙(O, r_1) touches ⊙(O, r_2). Find AB in terms of r_1 and r_2.
5. In example 4, if $r_1 = 41$ and $r_2 = 9$, find AB.

*

11.3 Number of Tangents from a Point in the Plane of a Circle

Let P be a point in the plane of a circle. There are three possibilities : (1) P may be in the interior of a circle (2) P may be on the circle (3) P may be in the exterior of a circle.

(1) If P is in the interior of a circle. Can we draw a tangent to the circle passing through P ? The answer is no. Any line passing through this point will intersect the circle in two distinct points. Such lines are secants of the circles, not tangents.

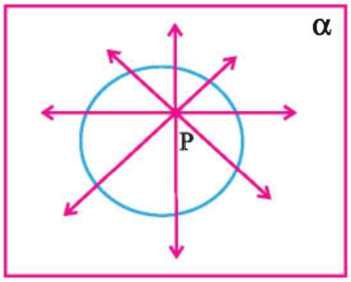


Figure 11.15

(2) We have discussed the case when P is on the circle in detail. There is one and only one line passing through such a point which is be a tangent to the circle.

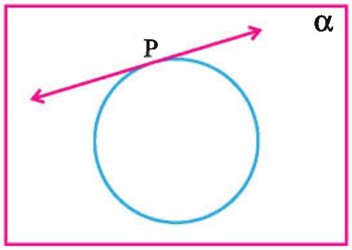


Figure 11.16

(3) **Activity :** Let us draw two radii of the circle $\overline{OA}$ and $\overline{OB}$ with centre O such that $\overline{AB}$ is not a diameter of the circle.

Let us draw tangent l_1 to the circle passing through A. We have already seen that such a construction is possible. In the plane of the circle we have to draw a line passing through A and perpendicular to $\overline{OA}$. Let us draw tangent l_2 to the circle at point B.

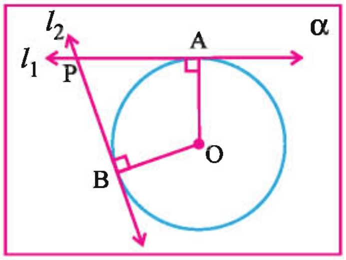


Figure 11.17

l_1 and l_2 are coplanar lines and $\overline{AB}$ is not a diameter. So l_1 and l_2 will intersect. Let the point of intersection be P. P will be in the exterior of the circle. Because all points on a tangent except the point of contact are in the exterior of the circle. So there is a point P in the exterior of the circle from which two tangents can be drawn. Will this be true for all points in the exterior of a circle ? The answer is yes. In the next chapter we will learn a construction to draw tangents to a given circle from a point in the exterior of the circle.

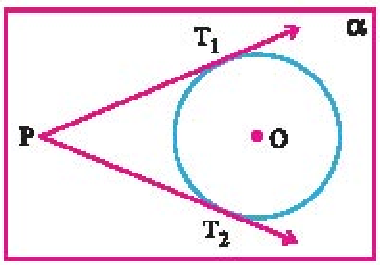


Figure 11.18

If P is a point in the exterior of the circle, then there are two tangents, $\overleftrightarrow{PT_1}$ and $\overleftrightarrow{PT_2}$, to the circle. T_1 and T_2 are the points of contact of those tangents as shown in the figure 11.18.

Lengths PT_1 and PT_1 are called the lengths of those tangents.

If a tangent is drawn from an exterior point, then the distance between this exterior point and the point of contact of the tangent is called the length of the tangent.

We will accept the following property of the circle as a theorem without giving a formal proof.

Theorem 11.3 : The tangents drawn to a circle from a point in the exterior of the circle are congruent.

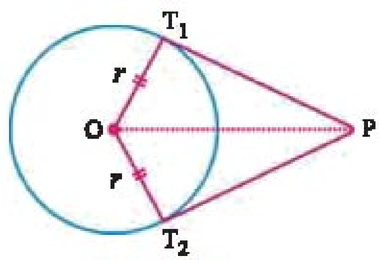


Figure 11.19

In figure 11.19, P is a point in the exterior of $\odot(O, r)$. Tangents to the circle from P touch the circle at T_1 and T_2. Then according to the theorem $PT_1 = PT_2$.

Join O to P.

In ΔOPT_1 and ΔOPT_2, consider the correspondence $OPT_1 \leftrightarrow OPT_2$.

$\angle OT_1P \cong \angle OT_2P$, **(both are right angles)**

$\overline{OT_1} \cong \overline{OT_2}$ **(both are radii)**

$\overline{OP} \cong \overline{OP}$

$\therefore \quad \Delta OPT_1 \cong \Delta OPT_2$. **(R.H.S. theorem)**

Hence $\overline{PT_1} \cong \overline{PT_2}$

$\therefore \quad PT_1 = PT_2$

Example 6 : A circle touches all the four sides of $\square$ ABCD.

Prove that AB + CD = AD + BC.

Proof : Let the circle touch the sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ of $\square$ ABCD at points P, Q, R, S respectively.

$\therefore \quad AP = AS, DS = DR, CR = CQ, BQ = BP$ **(i)**

and A–P–B, B–Q–C, C–R–D, A–S–D **(ii)**

Now, AB + CD = AP + PB + CR + RD

(A–P–B and C–R–D)

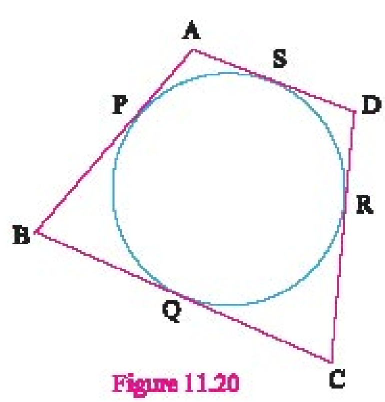


Figure 11.20

= AS + BQ + CQ + DS

= AS + DS + BQ + CQ

= AD + BC **(A–S–D and B–Q–C)**

Thus, AB + CD = AD + BC.

Note : (1) The circle which touches all the sides of a quadrilateral is called the circle inscribed in the quadrilateral.

(2) Whenever we can inscribe a circle in a quadrilateral the sums of the opposite sides of the quadrilateral in both the pairs are equal.

The converse of this property is also true. If in □ ABCD, AB + CD = AD + BC, then there is a circle touching all the sides of □ ABCD. From this result, we can observe that not every quadrilateral has an incircle.

(3) For a triangle there is always a circle which touches all the sides of the triangle. This circle is known as incircle of the triangle and the radius of this circle is called the inradius of the triangle.

Example 7 : If a circle touches all the four sides of a parallelogram, the parallelogram is a rhombus.

Solution : In example 6, we have proved that if there is a circle touching all the four sides of □ ABCD, then AB + CD = AD + BC.

If □ ABCD in the question is a parallelogram.

Then, AB + CD = AD + BC and AB = CD, AD = BC.

∴ 2CD = 2BC

∴ BC = CD. But BC = AD and CD = AB

∴ We have AB = BC = CD = AD

∴ $\square^m$ABCD is a rhombus.

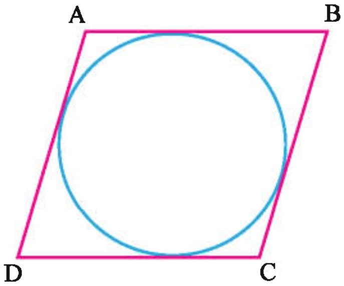


Figure 11.21

Example 8 : In ΔABC, $m\angle B$ = 90. A circle touches all the sides of ΔABC. If AB = 5, BC = 12, find the radius of the circle.

Solution : We have remarked that there exists a circle touching all the sides of a triangle.

If the radius of such a circle is r, then as shown in the figure 11.22.

ID = IE = IF = r

Now, the given triangle is a right triangle and ∠B is right angle.

Moreover $\overline{ID} \perp \overline{BC}$, and $\overline{AB} \perp \overline{BC}$

∴ $\overline{ID} \parallel \overline{FB}$ and similarly $\overline{IF} \parallel \overline{BD}$

∴ □IFBD is a parallelogram.

∴ ID = FB = r and BD = IF = r

∴ $\square^m$IFBD is a rhombus.

Also ∠B is right angle.

∴ □IFBD is a square.

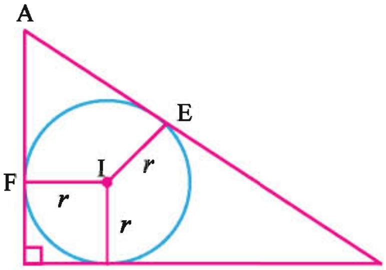


Figure 11.22

Downloaded from https:// www.studiestoday.com

Now, $AB^2 + BC^2 = AC^2$ **($\angle B$ is right angle)**

$\therefore$ $AC^2 = 5^2 + 12^2 = 13^2$

$\therefore$ $AC = 13$

$\therefore$ $AB + BC + AC = 5 + 12 + 13$

$\therefore$ $AF + FB + BD + DC + AC = 30$

$\therefore$ $AE + r + r + CE + AC = 30$ **(AF = AE, DC = CE)**

$\therefore$ $2r + (AE + CE) + AC = 30$

$\therefore$ $2r + 2AC = 30$

$\therefore$ $2r + 2(13) = 30$

$\therefore$ $r + 13 = 15$

$\therefore$ $r = 2$

$\therefore$ The radius of the circle is 2.

Note : In ΔABC if $\angle B$ is right angle, then the radius of the circle touching all the three sides of the triangle is $\frac{AB + BC - AC}{2}$.

Example 9 : A circle touches the sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ of ΔABC at the points D, E, F respectively. The radius of the circle is 4. If BD = 8, DC = 6, find AB and AC.

Solution : Let I be the centre of the circle which touches $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F respectively.

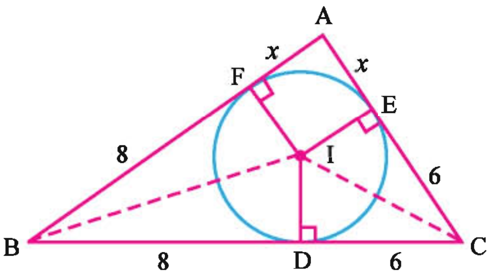


Figure 11.23

$\therefore$ $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AC}$, $\overline{IF} \perp \overline{AB}$ and

$ID = IE = IF =$ radius of the circle $= 4$ **(Given)**

Moreover it is given that BD = 8 and DC = 6

$\therefore$ $BF = BD = 8$, $CE = CD = 6$

Let $AF = AE = x$. Also $BC = a$, $CA = b$, $AB = c$

$\therefore$ $AB = x + 8$, $AC = x + 6$, $BC = 14$

$\therefore$ Perimeter of $\Delta ABC = AB + BC + AC = 2x + 28$

$\therefore$ Semi-perimeter of $\Delta ABC = s = x + 14$

$\therefore$ $s - a = x + 14 - 14 = x$, $\quad s - b = x + 14 - (x + 6) = 8$

$s - c = x + 14 - (x + 8) = 6$

Area of $\Delta ABC = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{(x+14)\cdot x\cdot 8\cdot 6}$

$= \sqrt{48x\,(x+14)}$

Area of $\Delta AIB = \frac{1}{2}AB \cdot IF = \frac{1}{2}(x + 8) \cdot 4 = 2(x + 8)$

Area of $\Delta BIC = \frac{1}{2}BC \cdot ID = \frac{1}{2}(14) \cdot 4 = 28$

Area of $\Delta CIA = \frac{1}{2}AC \cdot IE = \frac{1}{2}(x + 6) \cdot 4 = 2(x + 6)$

$ID = IF$

$\therefore$ $\overrightarrow{BI}$ is the bisector of $\angle B$

Similarly $\overrightarrow{AI}$ is the bisector of $\angle A$

and $\overrightarrow{CI}$ is the bisector of $\angle C$

$\therefore$ I is a point in the interior of ΔABC.

$\therefore$ Area of ΔAIB + Area of ΔBIC + Area of ΔCIA = Area of ΔABC

$\therefore$ $2(x + 8) + 28 + 2(x + 6) = \sqrt{48x(x+14)}$

$\therefore$ $x + 14 = 3x$ **($x + 14 \neq 0$, $x > 0$)**

$\therefore$ $x = 7$

$\therefore$ $AB = x + 8 = 15$, $AC = x + 6 = 13$

EXERCISE 11.2

1. P is the point in the exterior of $\odot(O, r)$ and the tangents from P to the circle touch the circle at X and Y.

(1) Find OP, if $r = 12$, $XP = 5$

(2) Find $m\angle XPO$, if $m\angle XOY = 110$

(3) Find r, if $OP = 25$ and $PY = 24$

(4) Find $m\angle XOP$, if $m\angle XPO = 80$

2. Two concentric circles having radii 73 and 55 are given. The chord of the circle with larger radius touches the circle with smaller radius. Find the length of the chord.

3. $\overline{AB}$ is a diameter of $\odot(O, 10)$. A tangent is drawn from B to $\odot(O, 8)$ which touches $\odot(O, 8)$ at D. $\overrightarrow{BD}$ intersects $\odot(O, 10)$ in C. Find AC.

4. P is in the exterior of a circle at distance 34 from the centre O. A line through P touches the circle at Q. PQ = 16, find the diameter of the circle.

5. In figure 11.24, two tangents are drawn to a circle from a point A which is in the exterior of the circle. The points of contact of the tangents are P and Q as shown in the figure. A line l touches the circle at R and intersects $\overline{AP}$ and $\overline{AQ}$ in B and C respectively. If $AB = c$, $BC = a$, $CA = b$, then prove that

(1) $AP + AQ = a + b + c$

(2) $AB + BR = AC + CR = AP = AQ = \dfrac{a+b+c}{2}$

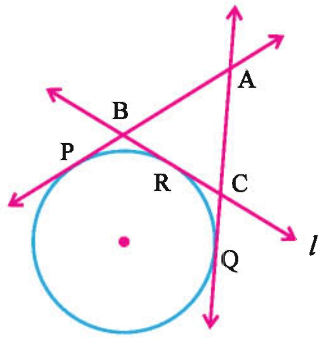


Figure 11.24

6. Prove that the perpendicular drawn to a tangent to the circle at the point of contact of the tangent passes through the centre of the circle.

7. Tangents from P, a point in the exterior of $\odot(O, r)$ touch the circle at A and B. Prove that $\overline{OP} \perp \overline{AB}$ and $\overline{OP}$ bisects $\overline{AB}$.

8. $\overleftrightarrow{PT}$ and $\overleftrightarrow{PR}$ are the tangents drawn to $\odot(O, r)$ from point P lying in the exterior of the circle and T and R are their points of contact respectively. Prove that $m\angle TPR = 2m\angle OTR$.

9. $\overline{AB}$ is a chord of $\odot(O, 5)$ such that AB = 8. Tangents at A and B to the circle intersect in P. Find PA.

10. P lies in the exterior of $\odot(O, 5)$ such that OP = 13. Two tangents are drawn to the circle which touch the circle in A and B. Find AB.

EXERCISE 11

1. A circle touches the sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ of ΔABC at points D, E, F respectively. BD = x, CE = y, AF = z. Prove that the area of $\Delta ABC = \sqrt{xyz\,(x + y + z)}$.

2. ΔABC is an isosceles triangle in which $\overline{AB} \cong \overline{AC}$. A circle touching all the three sides of ΔABC touches $\overline{BC}$ at D. Prove that D is the mid-point of $\overline{BC}$.

3. $\angle B$ is a right angle in ΔABC. If AB = 24, BC = 7, then find the radius of the circle which touches all the three sides of ΔABC.

4. A circle touches all the three sides of a right angled ΔABC in which $\angle B$ is right angle. Prove that the radius of the circle is $\frac{AB + BC - AC}{2}$.

5. In $\square$ ABCD, $m\angle D = 90$. A circle with centre O and radius r touches its sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ in P, Q, R and S respectively. If BC = 40, CD = 30 and BP = 25, then find the radius of the circle.

6. Two concentric circles are given. Prove that all chords of the circle with larger radius which touch the circle with smaller radius are congruent.

7. A circle touches all the sides of $\square$ ABCD. If AB = 5, BC = 8, CD = 6. Find AD.

8. A circle touches all the sides of $\square$ ABCD. If $\overline{AB}$ is the largest side then prove that $\overline{CD}$ is the smallest side.

9. P is a point in the exterior of a circle having centre O and radius 24. OP = 25. A tangent from P touches the circle at Q. Find PQ.

10. **Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :**

(1) P is in exterior of $\odot(O, 15)$. A tangent from P touches the circle at T. If PT = 8, then OP = ... ☐

(a) 17 (b) 13 (c) 23 (d) 7

(2) $\overleftrightarrow{PA}$, $\overleftrightarrow{PB}$ touch $\odot(O, r)$ at A and B. If $m\angle AOB = 80$, then $m\angle OPB$ = ... ☐

(a) 80 (b) 50 (c) 10 (d) 100

(3) A tangent from P, a point in the exterior of a circle, touches the circle at Q. If OP = 13, PQ = 5, then the diameter of the circle is ...... ☐

(a) 576 (b) 15 (c) 8 (d) 24

(4) In ΔABC, AB = 3, BC = 4, AC = 5, then the radius of the circle touching all the three sides is ...... ☐

(a) 2 (b) 1 (c) 4 (d) 3

(5) $\overleftrightarrow{PQ}$ and $\overleftrightarrow{PR}$ touch the circle with centre O at A and B respectively. If $m\angle OPB = 30$ and OP = 10, then radius of the circle = ... ☐

(a) 5 (b) 20 (c) 60 (d) 10

(6) The points of contact of the tangents from an exterior point P to the circle with centre O are A and B. If $m\angle OPB = 30$, then $m\angle AOB$ = ... ☐

(a) 30 (b) 60 (c) 90 (d) 120

(7) A chord of ⊙(O, 5) touches ⊙(O, 3). Therefore the length of the chord = ... ☐

(a) 8 (b) 10 (c) 7 (d) 6

*

Summary

In this chapter we have studied the following :

1. Meaning of the phrase "tangent to a circle".
2. Number of tangents from a point in the plane of the circle.
3. A tangent to a circle is perpendicular to the radius which passes through the point of contact.
4. A line drawn perpendicular to a radius at its end-point on the circle is a tangent to the circle.
5. A unique tangent can be drawn to a circle at each point of the circle.
6. Meaning of the phrase "length of a tangent to a circle from a point in the exterior of a circle".
7. The lengths of tangents from a point in the exterior of a circle are equal.

Constructions 12

Some mathematicians, I believe, has said that true pleasure lies not in the discovery but in the search for it.

- Tolstoy

12.1 Introduction

Why we learn about constructions :

Euclid

The ancient Greek mathematician Euclid is the acknowledged inventor of geometry. He did this over 2300 years ago and his book 'Elements' is still regarded as the ultimate geometry reference. In that work, he uses these construction techniques extensively and so they have become a part of the field of study in geometry. They also provide a greater insight into geometric concepts and give us tools to draw figures when direct measurement is not appropriate.

Why did Euclid do it this way ?

Why did Euclid not just measure things with a ruler and calculate lengths ? For example, one of the basic constructions is bisecting a line-segment (dividing it into two congruent parts). Why not just measure it with a ruler and divide it by two ?

The answer is surprising. The Greeks could not do arithmetic. They had only whole numbers, no zero and no negative numbers. This means they could not, for example divide 5 by 2 and get 2.5, because 2.5 is not a whole number - the only kind they had. Also, their numbers did not use a positional system like ours, with units, tenths, hundredths, etc, but more like the Roman numerals. In short, they could perform very little useful arithmetic.

So, faced with the problem of finding the mid-point of a line, they could not do the obvious - measure it and divide by two. They had to have other ways and this lead to the constructions using compass and straight-edge or unmarked ruler. It is also why the straight-edge has no markings. It is definitely not a graduated ruler, but simply a pencil guide for making straight lines. Euclid and the Greeks solved problems graphically, by drawing shapes, as a *substitute* for using arithmetic.

We have studied some constructions in standard IX using straight-edge (ruler) and compass only. We have learnt how to draw the bisector of a given angle, the perpendicular bisector of a given line-segment and angles whose measures are multiple of 15. Some constructions of triangles were also done. We have also given their justifications.

In this chapter we shall study some more constructions by using the knowledge of the constructions studied in standard IX and also using some principles of geometry. We shall also give the mathematical reasoning (justification) for the constructions, i.e. the principles behind the method of construction done.

12.2 Division of a Line-segment

If we are asked to divide a given line-segment in the ratio 3 : 4, we need to measure the length of line-segment and we mark a point on it, which divides the given line-segment in the given ratio, i.e. 3 : 4. (This is possible if the length of the line-segment is a multiple of 7.)

If we want to divide the line-segment in the ratio $m : n$ ($m, n \in$ N) without measuring the actual length, then there is one interesting method to do it. Let us see how to do it. First of all, we will do the following construction. We will take one example, where $m = 3$ and $n = 5$.

Construction 1 : To divide a given line-segment in the ratio 3 : 5.

Note : (1) As the ratio is 3 : 5, so we would like to divide the line-segment into eight (= 3 + 5) congruent parts and we select first three parts together as one part and the next five together as the other part.

(2) To divide $\overline{AB}$ in the ratio 3 : 5 means to find a point P on $\overline{AB}$ such that $\frac{AP}{PB} = \frac{3}{5}$.

Data : $\overline{AB}$ is given.

To construct : To divide $\overline{AB}$ into two segments, such that the ratio of their lengths is 3 : 5.

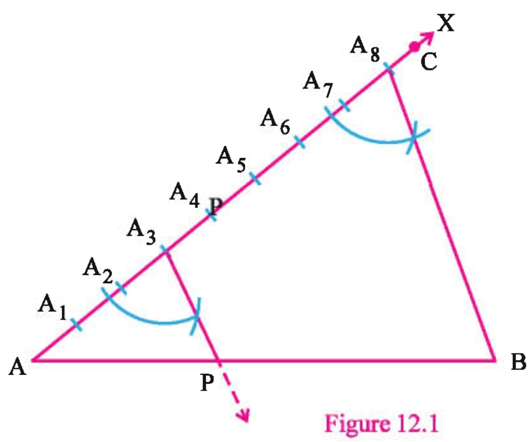


Figure 12.1

Steps of construction : (1) Draw $\overrightarrow{AX}$ making an acute angle with $\overrightarrow{AB}$. Select arbitrary point C on $\overrightarrow{AX}$.

(2) Select an arbitrary radius less than $\frac{1}{8}$AC. Draw an arc with centre A and this radius intersecting $\overline{AC}$ in A_1. Similarly with centre A_1 and the same radius draw an arc intersecting $\overline{AC}$ in A_2 such that $A{-}A_1{-}A_2$. Similarly continue the procedure with centres respectively A_k and arc having same radius intersecting $\overline{AC}$ in A_{k+1} such that $A_{k-1}{-}A_k{-}A_{k+1}$, where $k = 2, 3, 4, ..., 7$. Thus we get 8 points $A_1, A_2, ..., A_8$ on $\overline{AC}$ such that $AA_1 = A_1A_2 = A_2A_3 = ... = A_7A_8$.

(3) Join A_8 with B.

(4) We draw a ray parallel to $\overline{A_8B}$, through A_3, so that it intersects $\overline{AB}$ at P.

(To draw a ray parallel to $\overline{A_8B}$, we shall construct $\angle AA_3P$ congruent to $\angle AA_8B$ as we have learnt earlier.)

Thus we have obtained P $\in \overline{AB}$ such that AP : PB = 3 : 5.

Justification : As $\overleftrightarrow{A_3P} \parallel \overleftrightarrow{A_8B}$ and $\overleftrightarrow{AC}$, $\overleftrightarrow{AB}$ are the two transversals,

$$\frac{AA_3}{A_3A_8} = \frac{AP}{PB}$$ **(Theorem on proportionality)**

Here, $\frac{AA_3}{A_3A_8} = \frac{3}{5}$. So $\frac{AP}{PB} = \frac{3}{5}$

Hence point P divides $\overline{AB}$ in the ratio 3 : 5.

Alternative Method :

Step of Construction : (1) Construct acute congruent angles $\angle XAB$ and $\angle YBA$ in such a way that X and Y are in the opposite half planes of $\overleftrightarrow{AB}$.

(2) Select an arbitrary radius less than $\frac{1}{5}$BY and $\frac{1}{3}$AX. Draw an arc with centre A and this radius intersecting $\overline{AX}$ in A_1. Similarly with centre A_1 and the same radius, draw an arc intersecting $\overline{AX}$ in A_2 such that $A-A_1-A_2$. Similarly with centres A_2 and same radius draw an arc intersecting $\overline{AX}$ in A_3 such that $A_1-A_2-A_3$. Thus we get 3 points A_1, A_2, A_3 on $\overline{AX}$ such that $AA_1 = A_1A_2 = A_2A_3$.

Now draw an arc with centre B and the same radius intersecting $\overline{BY}$ in B_1. Again with centre B_1 and the same radius, draw an arc intersecting $\overline{BY}$ in B_2, such that $B-B_1-B_2$. Similarly continue the procedure with centres respectively B_k and arc having the same radius intersecting $\overline{BY}$ in B_{k+1} such that $B_{k-1}-B_k-B_{k+1}$, where $k = 2, 3, 4$. Thus we get 5 points on $\overline{BY}$ such that $BB_1 = B_1B_2 = B_2B_3 = ... = B_4B_5$.

(3) Join A_3, B_5 intersecting $\overline{AB}$ at the point P.

Thus we have obtained a point $P \in \overline{AB}$ such that AP : PB = 3 : 5.

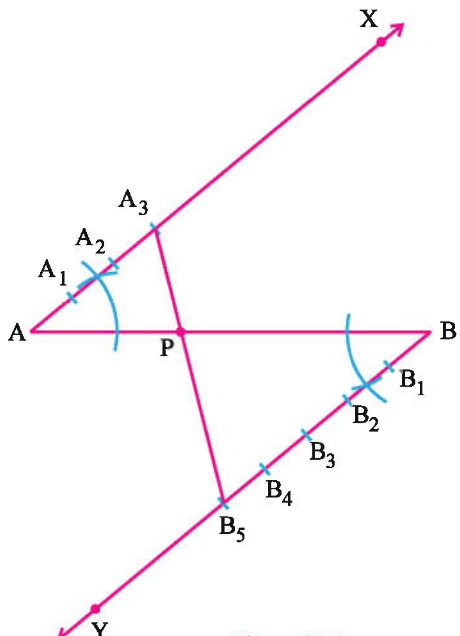


Figure 12.2

Justification : $\Delta AA_3P \sim \Delta BB_5P$ **(AA)**

$\therefore \quad \frac{AA_3}{BB_5} = \frac{AP}{PB}$

But $\frac{AA_3}{BB_5} = \frac{3}{5}$. So $\frac{AP}{PB} = \frac{3}{5}$

$\therefore \quad AP : PB = 3 : 5$

Example 1 : Divide a line-segment into three congruent parts.

Solution : Data : $\overline{AB}$ is given.

To construct : To divide $\overline{AB}$ into three congruent parts.

Steps of construction : (1) Construct $\overrightarrow{AX}$ and $\overrightarrow{BY}$ at A and B respectively in different half planes of $\overleftrightarrow{AB}$ such that $\angle XAB \cong \angle YBA$ and also they are acute.

(2) Select an arbitrary radius less then $\frac{1}{3}$AX ($\frac{1}{3}$BY also). Draw an arc with centre A and this radius intersecting $\overline{AX}$ in A_1. Similarly with centre A_1 and the same radius, draw an arc intersecting $\overline{AX}$ in A_2, such that $A-A_1-A_2$. Here $AA_1 = A_1A_2$.

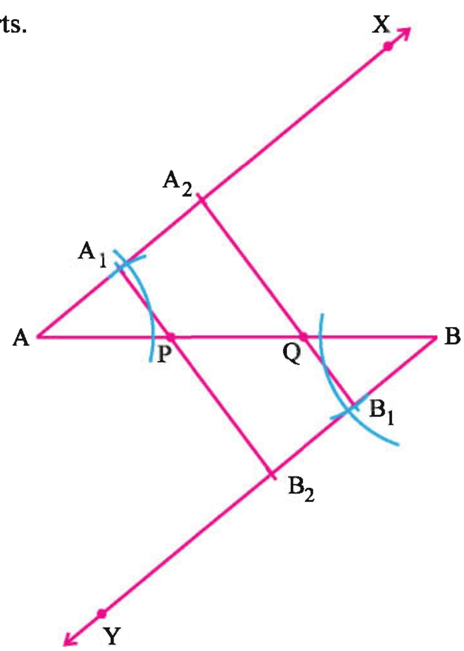


Figure 12.3

(3) Similarly draw an arc with centre B and the same radius (as in (2)) intersecting $\overline{BY}$ in B_1. Again with centre B_1 and the same radius, draw an arc intersecting $\overline{BY}$ in B_2, such that $B-B_1-B_2$. Here also $BB_1 = B_1B_2$.

(4) Join A_1 with B_2 and A_2 with B_1 intersecting $\overline{AB}$ at P and Q respectively.

Thus, points P and Q divide $\overline{AB}$ into three congruent parts, i.e. $AP = PQ = QB\left(=\frac{1}{3}AB\right)$.

Now we shall use this idea of dividing a line-segment in a given ratio with the corresponding sides of similar triangles. Such a ratio is called a **Scale factor**.

Scale Factor : The ratio of the measures of the corresponding sides of two similar triangles is called a scale factor.

Construction 2 : To construct a triangle similar to a given triangle as per given scale factor.

Case (1) : Let the scale factor less than 1, i.e. the triangle to be constructed has sides of measure less than the measures of the corresponding sides of given triangle.

Construct ΔABC whose sides have lengths equal to $\frac{2}{5}$ of the lengths of the corresponding sides of ΔAPQ.

Data : ΔAPQ is given.

To construct : To construct ΔABC such that the ratio of the measures of sides of ΔABC to measures of sides of ΔAPQ is 2 : 5.

Steps of construction : (1) Draw $\overrightarrow{AX}$ making an acute angle with $\overline{AP}$ and Q and X lie in different half planes of $\overleftrightarrow{AP}$.

(2) Select an arbitrary radius less than $\frac{1}{5}AX$. Draw an arc with centre A and this radius intersecting $\overline{AX}$ in A_1. Similarly with centre A_1 and the same radius draw an arc intersecting $\overline{AX}$ in A_2, such that $A-A_1-A_2$. Similarly continue the procedure with centres respectively A_k and arcs having same radius intersecting $\overline{AX}$ in A_{k+1} such that $A_{k-1}-A_k-A_{k+1}$, where $k = 2, 3, 4$. Thus we get 5 points on $\overline{AX}$ such that $AA_1 = A_1A_2 = A_2A_3 = A_3A_4 = A_4A_5$.

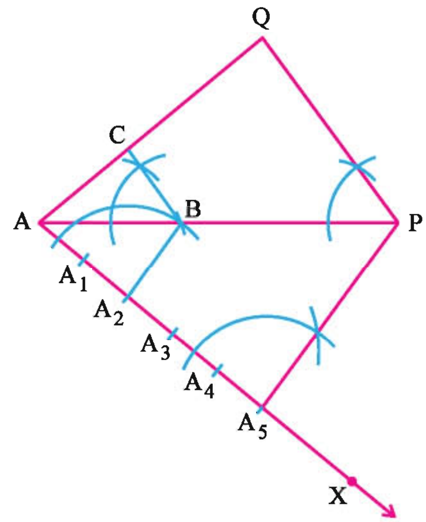


Figure 12.4

(3) Draw $\overline{A_5P}$.

(4) Draw $\overleftrightarrow{A_2B}$ parallel to $\overleftrightarrow{A_5P}$ intersecting $\overline{AP}$ in B.

(5) Draw $\overleftrightarrow{BC}$ parallel to $\overline{PQ}$ intersecting $\overline{AQ}$ in C.

Thus ΔABC is the required triangle of desired measure.

Justification : According to the construction 1, since $\frac{AA_2}{A_2A_5} = \frac{2}{3}$

$$\frac{AB}{BP} = \frac{2}{3}$$

$$\therefore \quad \frac{BP}{AB} = \frac{3}{2}$$

$$\therefore \quad \frac{BP+AB}{AB} = \frac{3+2}{2} \qquad \textbf{(Componendo)}$$

$$\therefore \quad \frac{AP}{AB} = \frac{5}{2} \qquad \textbf{(A–B–P)}$$

$$\therefore \quad \frac{AB}{AP} = \frac{2}{5}$$

Also, $\overline{BC} \parallel \overline{PQ}$. So $\Delta ABC \sim \Delta APQ$

$\therefore \quad \frac{AC}{AQ} = \frac{BC}{PQ} = \frac{AB}{AP} = \frac{2}{5}$.

Case (2) : The scale factor is greater than 1, i.e. construction of a triangle which has sides larger than the sides of the given triangle.

To construct ΔABC similar to ΔAPQ with its sides having lengths equal to $\frac{4}{3}$ times the lengths of the corresponding sides of ΔAPQ.

Data : ΔAPQ is given.

To construct : To construct ΔABC such that the ratio of the lengths of the sides of ΔABC to the lengths of the sides of ΔAPQ is 4 : 3.

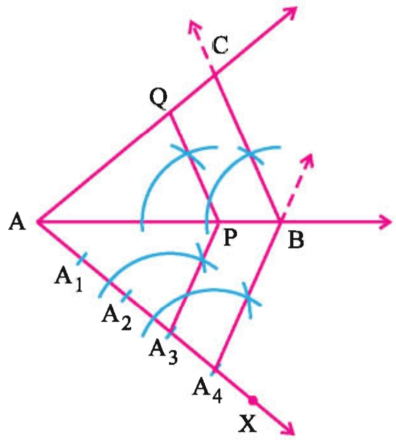


Figure 12.5

Steps of construction : (1) Draw $\overrightarrow{AX}$ making an acute angle with $\overline{AP}$ lying in the half plane of $\overleftrightarrow{AP}$ not containing vertex Q.

(2) Select an arbitrary radius less than $\frac{1}{4}AX$. Draw an arc with centre A and this radius intersecting $\overline{AX}$ in A_1. Similarly with centre A_1 and the same radius draw an arc intersecting $\overline{AX}$ in A_2, such that $A-A_1-A_2$. Similarly continue the procedure with centres respectively A_k and arcs having same radius intersecting $\overline{AX}$ in A_{k+1} such that $A_{k-1}-A_k-A_{k+1}$, where $k = 2, 3$. Thus we get 4 points on $\overline{AX}$ such that $AA_1 = A_1A_2 = A_2A_3 = A_3A_4$.

(3) Draw $\overline{A_3P}$.

(4) Draw a ray parallel to $\overline{A_3P}$ originating from A_4 and intersecting $\overrightarrow{AP}$ at B. **(AB > AP)**

(5) Draw a ray parallel to $\overline{PQ}$ originating from B and intersecting $\overrightarrow{AQ}$ at C.

Thus ΔABC is the required triangle with desired measure.

(**Explanation :** $\frac{AP}{AB} = \frac{AA_3}{AA_4} = \frac{3}{4}$ So, $\frac{AB}{AP} = \frac{4}{3}$ etc.)

EXERCISE 12.1

Construct the following with the help of straight-edge and compass only :

1. Draw $\overline{AB}$ of length 7.4 *cm* and divide it in the ratio 5 : 7.
2. Divide a line-segment into three parts in the ratio 2 : 3 : 4 in the same order.
3. Construct a triangle with sides 4 *cm*, 5 *cm*, 7 *cm* and then construct a triangle similar to it whose sides have lengths in the ratio 2 : 3 to the lengths of the corresponding sides of the first triangle.
4. Draw ΔPQR with $m\angle P = 60$, $m\angle Q = 45$ and $PQ = 6$ *cm*. Then construct ΔPBC whose sides have lengths $\frac{5}{3}$ times the lengths of the corresponding sides of ΔPQR.
5. Draw ΔABC having $m\angle ABC = 90$, $BC = 4$ *cm* and $AC = 5$ *cm*. Then construct ΔBXY, where scale factor is $\frac{4}{3}$.
6. Draw $\overline{PQ}$ of length 6.5 *cm* and divide it in the ratio 4 : 7. Measure the two parts.

*

12.3 Construction of Tangents to a Circle

In this section we shall study a construction based on 'circle and its tangents'. So we shall revise some necessary points.

(1) If a point is in the interior of a circle, then we can not draw a tangent to the circle through this point.

(2) If a point is on the circle, then we have only one tangent to the circle at this point.

As a tangent to the circle is perpendicular to the radius drawn through this point (i.e. point of contact or point of tangency), to construct such type of tangent is very simple. We draw a radius through this point and a line perpendicular to this radius at this point of contact is the required tangent. (See figure 12.6)

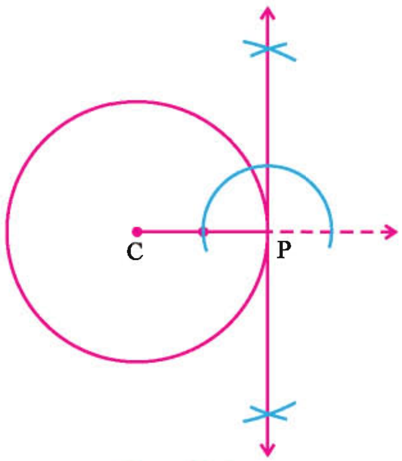


Figure 12.6

(3) If a point is in the exterior of a circle, then there will be two tangents to the circle from such point.

Construction 3 : To construct tangents to a circle through a point in the exterior of a circle.

A circle with centre O and radius 3 *cm* is given. Point P is such that OP = 7 *cm*. Draw the tangents to the circle through P.

Data : ⊙(O, 3) is given. Point P is an exterior point of the circle.

To construct : To draw tangents to ⊙(O, 3) through P.

Steps of construction : (1) ⊙(O, 3) is constructed and P is chosen such that OP = 7 *cm*.

(2) $\overline{OP}$ is drawn.

(3) Mid-point M of $\overline{OP}$ is obtained by constructing perpendicular bisector of $\overline{OP}$.

(4) ⊙(M, OM) is constructed intersecting ⊙(O, 3) at Q and R.

(5) Draw $\overleftrightarrow{PQ}$ and $\overleftrightarrow{PR}$.

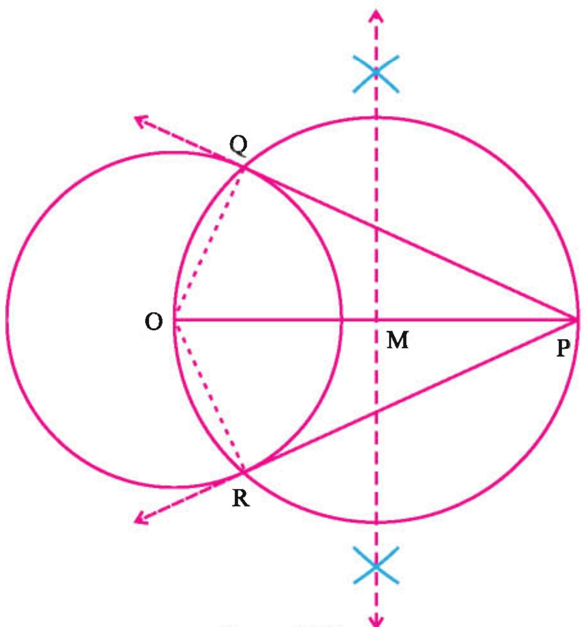


Figure 12.7

Then $\overleftrightarrow{PQ}$ and $\overleftrightarrow{PR}$ are the required tangents. (What can you say about the measures of $\overline{PQ}$ and $\overline{PR}$?)

Justification : ∠PQO and ∠PRO are the angles in the semi circle of ⊙(M, OM).

∴ $m\angle PQO = 90$ and $m\angle PRO = 90$

∴ $\overline{PQ} \perp \overline{OQ}$ and $\overline{PR} \perp \overline{OR}$.

∴ $\overleftrightarrow{PQ}$ and $\overleftrightarrow{PR}$ are the tangents to the ⊙(O, 3) at P and R respectively.

Example 2 : Construct the pair of tangents from the point in the exterior of a circle whose centre is not given.

Solution : Data : A circle of arbitrary radius is given and a point A exterior to this circle is given.

To construct : To draw two tangents from the exterior point A to the given circle.

Steps of construction : (1) Two non-parallel chords $\overline{PQ}$ and $\overline{RS}$ are drawn in given circle.

(2) Perpendicular bisectors of $\overline{PQ}$ and $\overline{RS}$ intersect at O. O is the centre of the given circle.

(3) Draw $\overline{OA}$.

(4) Perpendicular bisector of $\overline{OA}$ is drawn intersecting $\overline{OA}$ at M.

(5) ⊙(M, OM) is drawn which intersects ⊙(O, OP) at B and C.

(6) Draw $\overline{AB}$ and $\overline{AC}$.

Thus, $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ is the required pair of tangents.

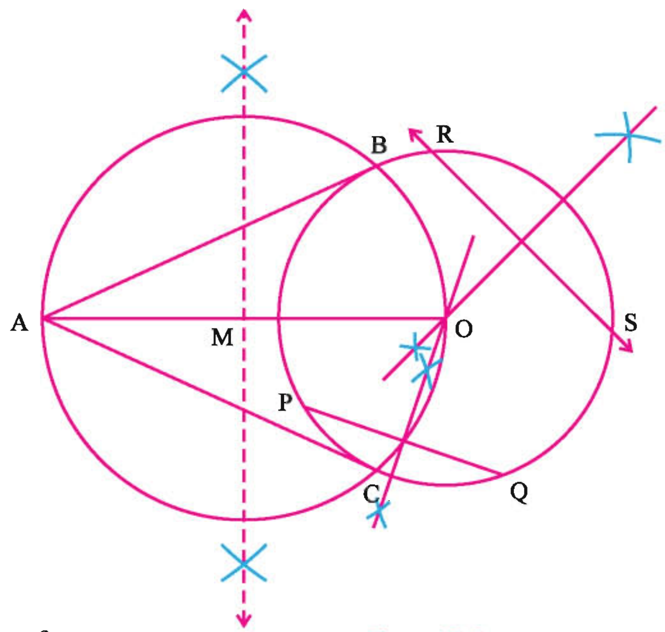


Figure 12.8

EXERCISE 12

1. Draw a circle of radius 5 *cm*. From a point 8 *cm* away from the centre, construct two tangents to the circle from this point. Measure them.
2. Draw ⊙(O, 4). Construct a pair of tangents from A where OA = 10 units.
3. Draw a circle with the help of a circular bangle. Construct two tangets to this circle through a point in the exterior of the circle.
4. Draw ⊙(O, r). $\overline{PQ}$ is a diameter of ⊙(O, r). Points A and B are on the $\overleftrightarrow{PQ}$ such that A–P–Q and P–Q–B. Construct tangents through A and B to ⊙(O, r).
5. Draw $\overline{AB}$ such that AB = 10 *cm*. Draw ⊙(A, 3) and ⊙(B, 4). Construct tangents to each circle through the centre of the other circle.
6. ⊙(P, 4) is given. Draw a pair of tangents through A which is in the exterior of ⊙(P, 4) such that measure of an angle between the tangents is 60.

*

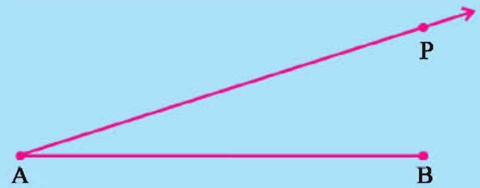
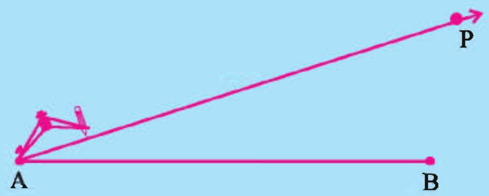
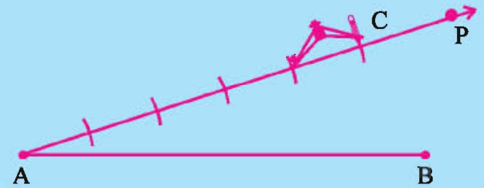
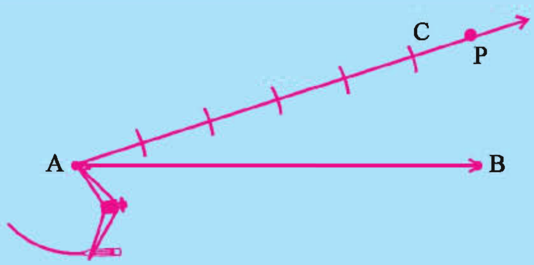
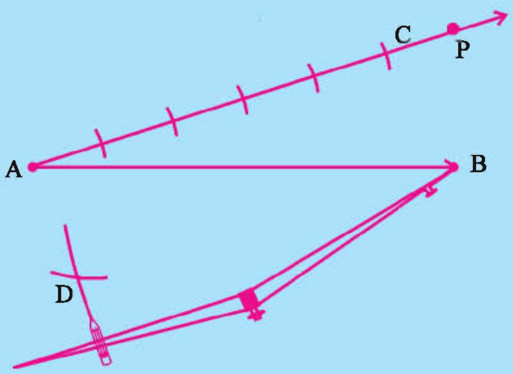
Summary

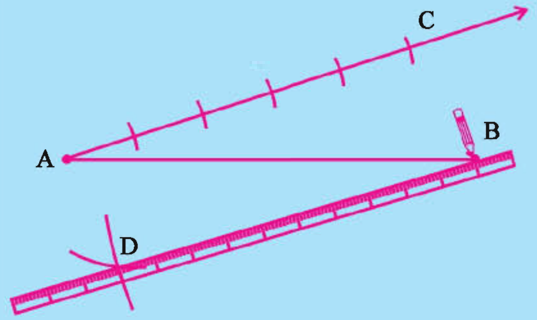
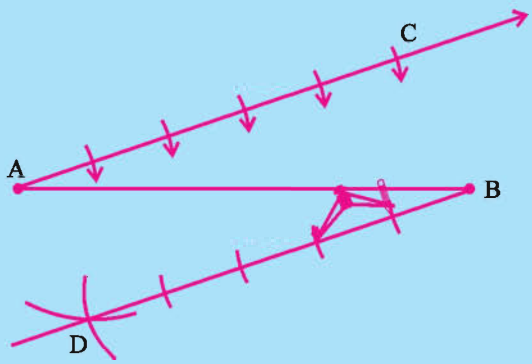
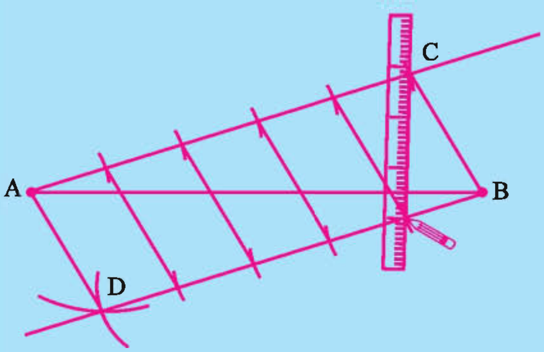
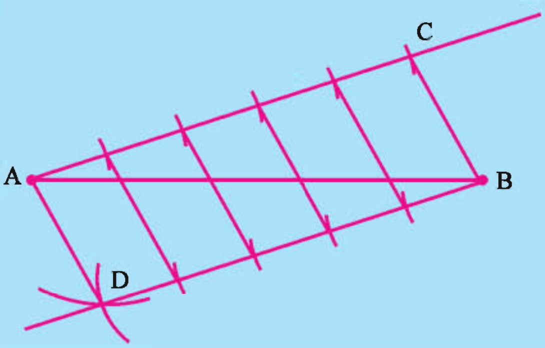
In this chapter we have learnt how to do the following constructions with the help of straight edge and compass only.

1. To divide a given line-segment in the given ratio.
2. To construct a triangle similar to the given triangle whose corresponding sides are in some ratio, which may be less than 1 or greater than 1.
3. To construct a pair of tangents to a circle through a point in the exterior of the circle.

◆

We can summarise the process of construction 1 as follows :

	After doing this	Your work should look like this
	Start with a line-segment AB that we will divide into 5 (in this case) congruent parts.	
Step 1	Draw $\overrightarrow{AP}$ at an acute angle to the given $\overrightarrow{AB}$.	
Step 2	Set the compass on A, and set its width to a bit less than one fifth of AP.	
Step 3	Step the compass along $\overrightarrow{AP}$, marking off 5 congruent arcs. Label the last one C.	
Step 4	With the compass width set to CB, draw an arc from A just below it.	
Step 5	With the compass width set to AC, draw an arc from B intersecting the one drawn in step 4. D is the point of intersection.	

	After doing this	Your work should look like this
Step 6	Join D and B.	
Step 7	Using the same compass width as used to step along $\overline{AC}$, step the compass from B along $\overline{DB}$ making 5 new congruent arcs	
Step 8	Draw line-segments joining corresponding points along $\overline{AC}$ and $\overline{DB}$.	
Step 9	Construction is over. The line-segments divide $\overline{AB}$ into 5 congruent parts.	

Area Related to a Circle 13

No mathematician can be a complete mathematician unless he is also something of a poet.

- Karl Weierstrass

13.1 Introduction

We have already learnt about the methods of finding area and perimeter of simple plane figures having regular shapes like a triangle, a square, a rectangle, a circle, a parallelogram etc. If we see the objects around us, then we can see many objects or their parts in the shape of a circle or parts of a circle. Some shapes are combinations of a circle and other plane figures or some of the parts of the circle along with a triangle, a rectangle, a square etc, for example a circular table cloth, a rectangular table cloth with some circular shaped figures patched on it, washers, bangles, wall clock, flower beds etc. So here we shall study the methods of finding areas and perimeters of the plane figures which we come across in day-to-day life. Here we shall also focus our study specially to the area of a sector and the area of a segment of a circle.

Perimeter and Area of a circle : Perimeter of a circle is known as circumference. We know that the ratio of circumference to the diameter of a circle is a constant, denoted by π.

π is an irrational real number. Its approximate value is taken as $\frac{22}{7}$ or 3.14.

Unless otherwise specified, we will take $\pi = \frac{22}{7}$.

So, circumference = πd, d is the diameter

= $2\pi r$, r is the radius.

Example 1 : A bicycle wheel makes 6250 revolutions in traveling 11 *km*. Find the diameter of the wheel.

Solution : Circumference of the wheel = $\frac{\text{Distance travelled}}{\text{Total number of revolution}}$

$$= \frac{11}{6250} = \frac{11 \times 100000}{6250}$$

$$= 176 \ cm$$

Circumference of a circle = $2\pi r$

$\therefore \quad 176 = 2 \times \frac{22}{7} \times r$

$\therefore \quad r = \frac{176 \times 7}{2 \times 22}$

$\therefore \quad r = 28$

$\therefore$ The radius of the wheel is 28 *cm*.

$\therefore$ The diameter of the wheel 56 *cm*.

Example 2 : The cost of ploughing a circular field at the rate of ₹ 0.75 per m^2 is ₹ 4158. Find the cost of fencing the field at the rate of ₹ 30 per m.

Solution : The area of the field $= \frac{\text{Total cost of ploughing}}{\text{Rate of ploughing}} = \frac{4158}{0.75} = 5544\ m^2$

Now, area of a circular field $= \pi r^2$

$\therefore \quad 5544 = \frac{22}{7} \times r^2$

$\therefore \quad r^2 = \frac{5544 \times 7}{22}$

$\therefore \quad r^2 = 1764$

$\therefore \quad r = 42\ m$

The circumference of the circular field $= 2\pi r = 2 \times \frac{22}{7} \times 42 = 264\ m$

$\therefore$ The cost of fencing the field = 264 × 30 = ₹ 7920

$\therefore$ The cost of fencing is ₹ 7920.

EXERCISE 13.1

1. Find the circumference and the area of the circle whose radius is 8.4 cm.
2. Find the circumference of the circle whose area is 38.5 m^2.
3. The inner circumference of a circular race track is 44 m less than the outer circumference. If the outer circumference is 396 m, then find the width of the track.
4. The radius of the wheel of a truck is 70 cm. It takes 250 revolution per minute. Find the speed of the truck in *km/hr*.

*

13.2 Area of a Sector and a Segment of a Circle

We have already learnt the terms **Sector** and **Segment** of a circle in our earlier classes. Let us recall them.

Sector : The region enclosed by an arc and the radii from the end points of the arc is called a sector. Here, $\overset{\frown}{ABC} \cup \overline{OA} \cup \overline{OC}$ is called a minor sector and denoted by OABC, while $\overset{\frown}{ADC} \cup \overline{OA} \cup \overline{OC}$ is called a major sector and denoted by OADC. Area enclosed by minor sector OABC $= \frac{\pi r^2\theta}{360}$, where θ is the measure of the angle subtended by the minor arc $\overset{\frown}{ABC}$ at the centre.

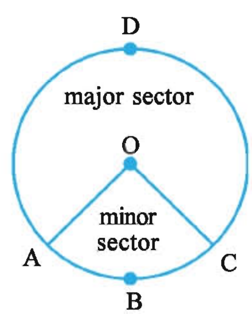


Figure 13.1

Minor sector OABC and major sector OADC are called corresponding sectors of each other.

Area enclosed by major sector OADC

= area of the circle − area enclosed by the minor sector OABC

$= \pi r^2 - \frac{\pi r^2\theta}{360}$

Length of the minor $\overset{\frown}{ABC} = \frac{\pi r\theta}{180}$

The area enclosed by a sector is also called the area of the sector.

A positive real number is associated with every segment called the area of the segment.

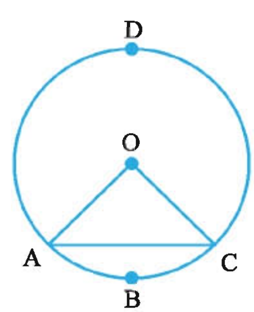


Figure 13.2

Minor Segment : $\overline{AC} \cup \overset{\frown}{ABC}$ is a minor segment.

Area of the region enclosed by minor segment $(\overline{AC} \cup \overset{\frown}{ABC})$

= Area of minor sector OABC − area of ΔOAC

Major Segment : $\overline{AC} \cup \overset{\frown}{ADC}$ is a major segment

Area of the region enclosed by a major segment ($\overline{AC} \cup \overset{\frown}{ADC}$)

= Area of a circle − Area of a minor segment ($\overline{AC} \cup \overset{\frown}{ABC}$).

The area enclosed by the segment is called the area of the segment.

Area of major segment ($\overline{AC} \cup \overset{\frown}{ADC}$) = area of major sector OADC + area of ΔOAC

Example 3 : The length of a minute hand of a clock is 12 *cm*. How much area will it cover on the circular dial in an interval of 5 minutes ? How much area remains to complete the revolution ? (π = 3.14)

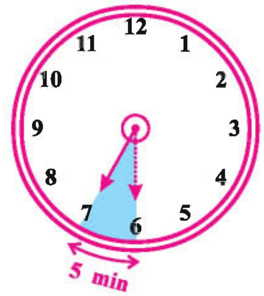


Figure 13.3

Solution : In 5 minutes, the minute hand revolves through an angle of measure $\frac{360}{60} \times 5 = 30$

Now, the area of a minor sector = $\frac{\pi r^2 \theta}{360}$

$\therefore$ The area covered by the minute hand in an interval of 5 minutes $= \frac{3.14 \times 12 \times 12 \times 30}{360}$

$= 37.68\ cm^2$

The area remaining to complete the revolution = The area of the circle − The area of the minor sector

$= \pi r^2 - 37.68$

$= 3.14 \times 12 \times 12 - 37.68$

$= 452.16 - 37.68$

$= 414.48\ cm^2$

Example 4 : Find the area of the minor segment whose chord subtends an angle of measure 120 and radius of the circle is 42 *cm*.

Solution : Area of the sector OACB $= \frac{\pi r^2 \theta}{360} = \frac{22}{7} \times \frac{42 \times 42 \times 120}{360} = 1848\ cm^2$

Draw $\overline{OM}$ perpendicular to $\overline{AB}$. M $\in$ $\overline{AB}$

$m\angle AOM = \frac{1}{2} m\angle AOB = \frac{1}{2} \times 120 = 60$

In the right ΔAMO, $sin60 = \frac{AM}{OA}$

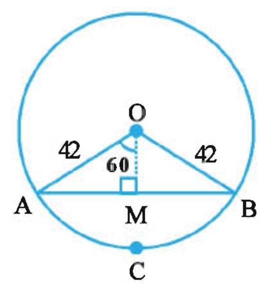


Figure 13.4

$\therefore$ $AM = OA \cdot sin60 = 42 \cdot \frac{\sqrt{3}}{2} = 21\sqrt{3}$

$\therefore$ $AB = 2 \cdot AM = 42\sqrt{3}$

$cos60 = \frac{OM}{OA}$

$\therefore$ $OM = OA \cdot cos60 = 42 \cdot \frac{1}{2} = 21$

$\therefore$ The area of ΔAOB $= \frac{1}{2} \times AB \times OM$

$= \frac{1}{2} \times 42\sqrt{3} \times 21 = 441\sqrt{3}\ cm^2$

$\therefore$ The area of minor segment ($\overline{AB} \cup \overset{\frown}{ACB}$)

= The area of minor sector OACB − The area of ΔAOB = $(1848 - 441\sqrt{3})\ cm^2$

Example 5 : $\overline{OA}$ and $\overline{OB}$ are two mutually perpendicular radii of a circle. Find the area of the minor segment, if the perimeter of the corresponding minor sector is 20 *cm*.

Solution : Let l be the length of the minor arc $\widehat{ACB}$.

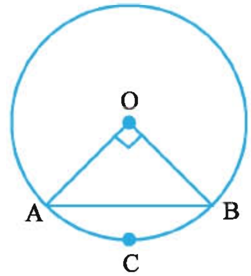


Figure 13.5

The perimeter of the minor sector $= l + 2r = \frac{\pi r\theta}{180} + 2r$

$\therefore \quad 20 = \frac{22}{7} \times r \times \frac{90}{180} + 2r$

$\therefore \quad r\left(\frac{11}{7} + 2\right) = 20$

$\therefore \quad \frac{25}{7} r = 20$

$\therefore \quad r = \frac{20 \times 7}{25} = 5.6\ cm$

The area of the minor segment $(\widehat{ACB} \cup \overline{AB})$

= The area of the minor sector OACB $-$ The area of ΔOAB

$= \frac{\pi r^2\theta}{360} - \frac{1}{2} \times OA \times OB$

$= \frac{22}{7} \times \frac{5.6 \times 5.6 \times 90}{360} - \frac{1}{2} \times 5.6 \times 5.6$

$= 4.4 \times 5.6 - 2.8 \times 5.6$

$= 24.64 - 15.68 = 8.96\ cm^2$

Example 6 : The radius of a circular ground is 70 *m*. There is 7 *m* wide track inside the ground near the boundary. The blue coloured portion of the road shown in figure 13.6 has to be repaired. Find the cost of repair at the rate of ₹ 40 per m^2. Measure of the angle subtended by the arc at the centre is 72.

Solution : The area of the shaded portion

= The area of minor sector OABC $-$ The area of the minor sector ODEF

($m\angle AOC = 72 = m\angle DOF$)

$= \frac{\pi r_1^2\theta}{360} - \frac{\pi r_2^2\theta}{360}$, where $r_1 = 70\ m$, $r_2 = 63\ m$

$= \frac{22}{7} \times 70 \times 70 \times \frac{72}{360} - \frac{22}{7} \times 63 \times 63 \times \frac{72}{360}$

$= 3080 - \frac{12474}{5}$

$= 3080 - 2494.8 = 585.2\ m^2$

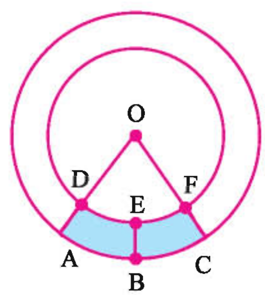


Figure 13.6

The cost of repair of 1 m^2 of road = ₹ 40

$\therefore$ The cost of repair of 585.2 m^2 of road = 40 × 585.2 = 23,408 ₹

Example 7 : An umbrella has 8 ribs which are equally spaced. Assuming the umbrella to be a flat circle of radius 56 *cm*. Find the area between the two consecutive ribs.

Figure 13.7

Figure 13.8

Solution : The measure of the angle made by two consecutive ribs at the centre = $\frac{360}{8}$ = 45

The area between two consecutive ribs = The area of the sector of a circle having radius 56 *cm* and making an angle having measure 45 at the centre.

$$= \frac{\pi r^2 \theta}{360}$$

$$= \frac{22}{7} \times \frac{56 \times 56 \times 45}{360} = 1232 \ cm^2$$

Example 8 : The length of a diagonal of a square garden is 50 *m*. There are two circular flower beds as shown in figure 13.9 on the opposite walls of the garden having centre as the point of intersection of diagonals of the square. Find the area of flower beds. (π = 3.14)

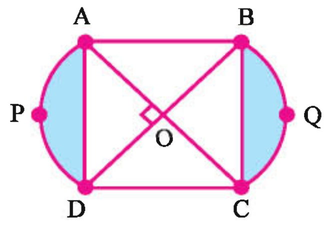


Figure 13.9

Solution : □ABCD is a square and $m\angle AOD = 90$

The radius of the sector OAPD and OBQC is $\frac{50}{2}$ = 25 *m*.

The area of the segment $\overparen{APD} \cup \overline{AD}$ = The area of the sector OAPD − Area of ΔOAD

$$= \frac{\pi r^2 \theta}{360} - \frac{1}{2} \times OA \times OD$$

$$= 3.14 \times 25 \times 25 \times \frac{90}{360} - \frac{1}{2} \times 25 \times 25$$

$$= 625(0.785 - 0.5)$$

$$= 625 \times 0.285$$

$$= 178.125 \ m^2$$

∴ The area of flower beds = 2 × 178.125

= 356.25 m^2

EXERCISE 13.2

1. An arc of a circle whose radius is 21 *cm* subtends an angle of measure 120 at the centre. Find the length of the arc and area of the sector.

2. The radius of a circular ground is 63 *m*. There is 7 *m* wide road inside the ground as shown in figure 13.10. The blue coloured portion of the road, shown in figure 13.10 is to be repaired. If the rate of repair work of the road costs ₹ 25 per m^2, find the total cost of repair.

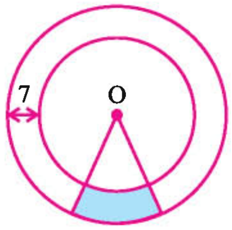


Figure 13.10

3. A regular hexagon of side 10 *cm* is cut from a plane circular sheet of radius 10 *cm* as shown in the figure 13.11. Find the area of the remaining part of the sheet. ($\sqrt{3}$ = 1.73) (π = 3.14)

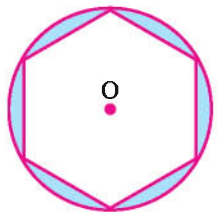


Figure 13.11

4. The length of a minute hand of a circular dial is 10 *cm*. Find the area of the sector formed by the present position and the position after five minute of the minute hand. ($\pi = 3.14$)
5. The radius of a field in the form of a sector is 21 *m*. The cost of constructing a wall around the field is ₹ 1875 at the rate of ₹ 25 per meter. If it costs ₹ 10 per m^2 to till the field, what will be the cost of tilling the whole field ?
6. The length of a side of a square field is 20 *m*. A cow is tied at the corner by means of a 6 *m* long rope. Find the area of the field which the cow can graze. Also find the increase in the grazing area, if length of the rope is increased by 2 *m*. ($\pi = 3.14$)


Figure 13.12

7. A chord of a circle of radius 42 *cm* subtends an angle of measure 60 at the centre. Find the area of the minor segment of the circle. ($\sqrt{3} = 1.73$)
8. A chord of a circle, of length 10 *cm*, subtends a right angle at the centre. Find the areas of the minor segment and the major segment formed by the chord. ($\pi = 3.14$)

*

13.3 Areas of Combinations of Plane Figures

We have learnt about finding areas of a circle, a sector and a segment. Now let us see how we apply this knowledge to find the area of some plane figures involving a circle, a sector or a segment with some other figures like a triangle, a square, a rectangle, etc. There are no additional formulae to find these areas for them but we will learn by some examples how to calculate the area of the figures formed by such combinations.

Example 9 : In a given square of side 14 *cm*, a design is constructed by semicircles, as shown in figure 13.13. Find the area of the region covered by the design.

Solution : We mark four regions of the square which are not coloured blue by I, II, III, IV.

The area of the region I and III together

= The area of square ABCD − The area of the semi-circle AOD and the semi-circle BOC.

$= (\text{side of the square})^2 - 2\left(\frac{1}{2}\pi r^2\right)$. Here $r = \frac{14}{2} = 7$ *cm*

$= 14 \times 14 - \frac{22}{7} \times 7 \times 7$

$= 196 - 154$

$= 42$ cm^2

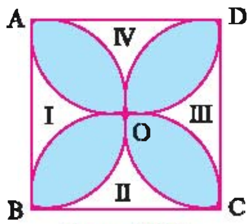


Figure 13.13

Similarly the area of the region II and IV = 42 cm^2

$\therefore$ The area of the design = The area of the square − The sum of the areas of the region I to IV

$= 196 - 84 = 112$ cm^2

Example 10 : A square is inscribed in a circular table cloth of radius 35 *cm*. If the cloth is to be coloured blue leaving the square, then find the area to be coloured.

Solution : Diagonal $\overline{BD}$ is the diameter of the circle having length 70 *cm*.

Let the length of the side of a square be *x cm*.

Now, $AB^2 + AD^2 = BD^2$

$\therefore$ $x^2 + x^2 = (70)^2$

$\therefore$ $2x^2 = 70 \times 70$

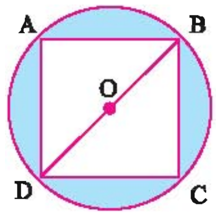


Figure 13.14

$\therefore \quad x^2 = \frac{70 \times 70}{2}$

$= 35 \times 35 \times 2$

$\therefore \quad x^2 = 2450\ cm^2$

The area of the square ABCD $= x^2$

$= 2450\ cm^2$

and the area of the circle $= \pi r^2$

$= \frac{22}{7} \times 35 \times 35$

$= 3850\ cm^2$

$\therefore$ The area of the design = Area of circle − Area of square

$= 3850 - 2450$

$= 1400\ cm^2$

Example 11 : An athletic track whose two ends are semi-circles as shown in the figure 13.15 is 7 *m* wide. The linear section joining these ends is of 110 *m* long. The distance between two inner parallel section is 70 *m*. Find the area of the track.

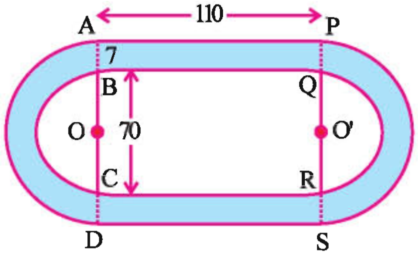


Figure 13.15

Solution :

The area of the track = The area of the rectangle ABQP + The area of the rectangle CDSR + 2(Area of semi-circle with radius OA (42 *m*) − 2(Area of semi-circle with radius OB (35 *m*).

$= AP \times PQ + DS \times CD + 2\left(\frac{1}{2}\pi(OA)^2\right) - 2\left(\frac{1}{2}\pi(OB)^2\right)$

$= 110 \times 7 + 110 \times 7 + \frac{22}{7} \times 42 \times 42 - \frac{22}{7} \times 35 \times 35$

$= 770 + 770 + 5544 - 3850$

$= 3234\ m^2$

Example 12 : What will be the cost of making design in the blue coloured region in figure 13.16 at the rate of ₹ 25 per cm^2.

Solution : The area of the sector ABCP $= \frac{\pi r^2 \theta}{360}$

$= \frac{22}{7} \times 14 \times 14 \times \frac{90}{360}$

$= 154\ cm^2$

= The area of the sector ADCQ.

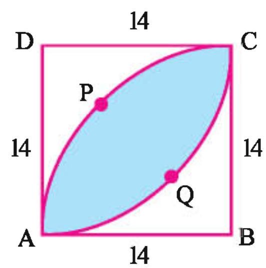


Figure 13.16

The area of the square ABCD $= 196\ cm^2$

$\therefore$ The area of the coloured region = The area of the sector ABCP + The area of the sector ADCQ − The area of the square ABCD.

(Remember, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$)

$= 154 + 154 - 196$

$= 112\ cm^2$

∴ The cost of making design in the coloured region at the rate of ₹ 25 per cm^2 is (112 × 25) = ₹ 2800

∴ The cost of making design is ₹ 2800.

Example 13 : On a square handkerchief, 16 circular designs each of radius 3.5 cm are made (see fig. 13.17). Find the area of the remaining portion of the handkerchief.

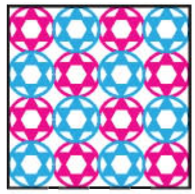


Figure 13.17

Solution : The area of each circular design $= \pi r^2$

$= \frac{22}{7} \times 3.5 \times 3.5$

$= 38.5\ cm^2$

The area of 16 circular designs $= 38.5 \times 16$

$= 616\ cm^2$

The length of the square handkerchief = 4 × diameter of a circle

$= 4 \times (2 \times 3.5)$

$= 28\ cm$

∴ The area of square handkerchief $= 28 \times 28$

$= 784\ cm^2$

∴ The area of the remaining portion of the handkerchief = Total area of the handkerchief − The area occupied by the design

$= 784 - 616$

$= 168\ cm^2$

Example 14 : In figure 13.17, $\overline{PQ}$ and $\overline{RS}$ are two diameters of a circle with centre O. They are perpendicular to each other, $\overline{OS}$ is a diameter of a smaller circle. If OP = 14 *cm*, find the area of the blue coloured region.

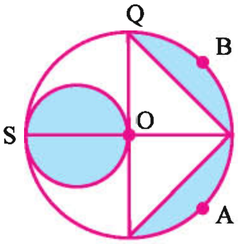


Figure 13.18

Solution :

The diameter of the smaller circle = radius of the big circle = r = 14 *cm*

∴ The radius of the smaller circle, $r_1 = \frac{1}{2} \times 14 = 7\ cm$

∴ The area of the smaller circle $= \pi {r_1}^2$

$= \frac{22}{7} \times 7 \times 7$

$= 154\ cm^2$ **(i)**

∴ The area of the segment PAR + The area of the segment RBQ

= The area of semi-circle PRQ − The area of ΔPQR (i.e 2 · The area of ΔQOR)

$= \frac{1}{2}\pi r^2 - 2 \times \left(\frac{1}{2} \times OQ \times OR\right)$ where OQ = OR = r = 14 *cm*

$= \frac{1}{2} \times \frac{22}{7} \times 14 \times 14 - 14 \times 14$

$= 308 - 196 = 112\ cm^2$

∴ The area of the coloured region = The sum of the result (i) and (ii)

$= 154 + 112$

$= 266\ cm^2$

EXERCISE 13.3

1. A rectangle whose length and breadth are 12 *cm* and 5 *cm* respectively is inscribed in a circle. Find the area of the blue coloured region, as shown in the figure 13.19.

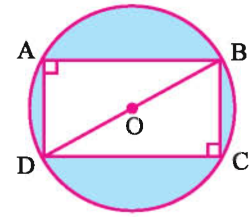


Figure 13.19

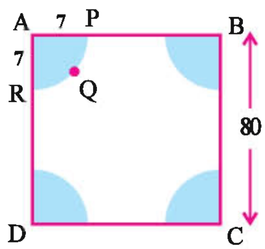


Figure 13.20

2. ABCD, square park, has each side of length 80 *m*. There is a flower bed at each corner in the form of a sector of radius 7 *m*, as shown in figure 13.20. Find the area of the remaining part of the park.

3. What will be the cost of covering the white portion in figure 13.21 with a silver foil if the rate is ₹ 100 per m^2 ? (π = 3.14)

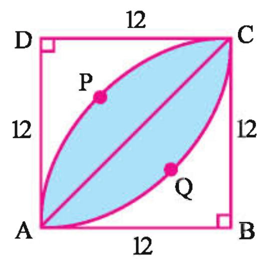


Figure 13.21

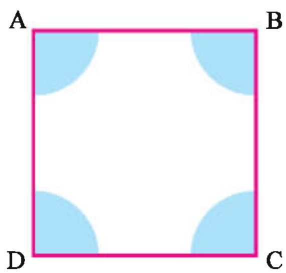


Figure 13.22

4. ABCD is a square plate of 1 *m* length. As shown in figure circles are drawn with their center at A, B, C, D respectively, each with radius equal to 42 *cm*. The blue coloured part at each corner, as shown in the figure 13.22 is cut. What is the area of the remaining portion of the plate ?

5. $\overline{OA}$ and $\overline{OB}$ are two mutually perpendicular radii of a circle of radius 10.5 *cm*. D ∈ $\overline{OB}$ and OD = 6 *cm*. Find the area of blue coloured region shown in figure 13.23.

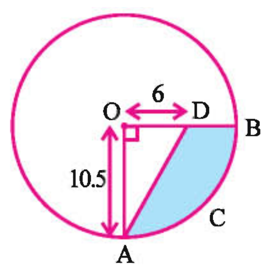


Figure 13.23

EXERCISE 13

1. The area of a circular park is 616 m^2. There is a 3.5 *m* wide track around the park running parallel to the boundary. Calculate the cost of fencing on the outer circle at the rate of ₹ 5 per meter.

2. A man is cycling in such a way that the wheels of the cycle are making 140 revolutions per minute. If the diameter of the wheel is 60 *cm*, then how much distance will he cover in 2 *hours* ?

3. If a chord $\overline{AB}$ of ⊙(O, 20) subtends right angle at O, find the area of the minor segment.

4. There are two arcs $\overparen{APB}$ of ⊙(O, OA) and $\overparen{AQB}$ of ⊙(M, MA) as shown in figure 13.24. Find the area enclosed by two arcs.

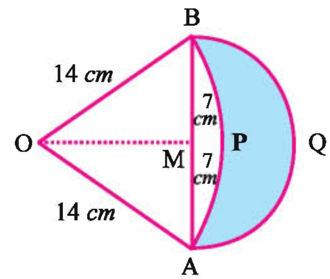


Figure 13.24

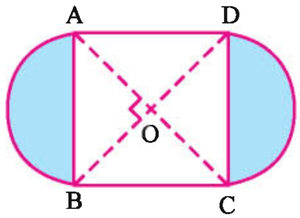


Figure 13.25

5. The length of a side of a square garden ABCD is 70 *m*. A minor segment of ⊙(O, OA) is drawn on each of two opposite sides for developing lawn, as shown in figure 13.24. Find the area of the lawn.

6. ABCD is a square of side 20 *cm*. Find the area of blue coloured region formed by the semi-circles drawn on each side as shown in figure 13.26. (π = 3.14)

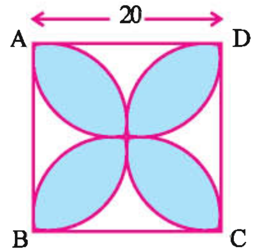


Figure 13.26

7. On a circular table top of radius 30 *cm* a design is formed leaving an equilateral triangle inscribed in a circle. Find the area of the design. (π = 3.14)

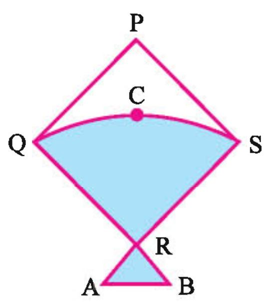


Figure 13.27

8. Figure 13.27 shows a kite formed by a square PQRS and an iscosceles right triangle ARB whose congruent sides are 5 *cm* long. $\overparen{QCS}$ is an arc of a ⊙(R, 42). Find the area of the blue coloured region.

9. In figure 13.28, ABCD is a square with sides having length 8 cm. Find the area of the blue coloured region. (π = 3.14)

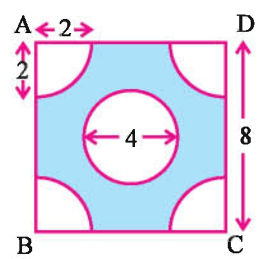


Figure 13.28

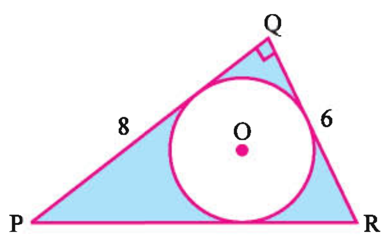


Figure 13.29

10. A circle is inscribed in ΔPQR where $m\angle Q$ = 90, PQ = 8 *cm* and QR = 6 *cm*. Find the area of the blue coloured region shown in figure 13.29. (π = 3.14)

11. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) If an arc of a circle subtends an angle of measure θ at the centre, then the area of the minor sector is ....

(a) $\frac{\pi r\theta}{180}$ (b) $\frac{\pi r^2\theta}{180}$ (c) $\frac{\pi r\theta}{360}$ (d) $\frac{\pi r^2\theta}{360}$

(2) The area of a sector is given by the formula ....
(r is the radius and l is the length of an arc.)

(a) $\frac{1}{2}rl$ (b) $\frac{3}{2}r^2l$ (c) $\frac{4}{3}rl$ (d) $\frac{3}{2}rl$

(3) $\overline{OA}$ and $\overline{OB}$ are the two mutually perpendicular radii of a circle having radius 9 *cm*. The area of the minor sector corresponding to $\angle AOB$ is ...... cm^2. (π = 3.14)

(a) 63.575 (b) 63.585 (c) 63.595 (d) 63.60

(4) A sector subtends an angle of measure 120 at the centre of a circle having radius of 21 *cm*. The area of the sector is ...... cm^2.

(a) 462 (b) 460 (c) 465 (d) 470

(5) If the area and the circumference of a circle are numerically equal, then r = ...

(a) π (b) $\frac{\pi}{2}$ (c) 1 (d) 2

(6) The length of an arc subtending an angle of measure 60 at the centre of a circle whose area is 616 is ....

(a) $\frac{22}{3}$ (b) 66 (c) $\frac{44}{3}$ (d) 33

(7) The area of a minor sector of $\odot$(O, 15) is 150. The length of the corresponding arc is .... (π = 3.14)

(a) 30 (b) 20 (c) 90 (d) 15

(8) If the radius of a circle is increased by 10 %, then corresponding increase in the area of the circle is .... (π = 3.14)

(a) 19 % (b) 10 % (c) 21 % (d) 20 %

(9) If the ratio of the area of two circles is 1 : 4, then the ratio of their circumference ....

(a) 1 : 4 (b) 1 : 2 (c) 4 : 1 (d) 2 : 1

(10) The area of the largest triangle inscribed in a semi-circle of radius 8 is ....

(a) 8 (b) 16 (c) 64 (d) 256

(11) If the circumference of a circle is 44 then the length of a side of a square inscribed in the circle is ....

(a) $\frac{44}{\pi}$ (b) $7\sqrt{2}$ (c) $14\sqrt{2}$ (d) $\frac{7\sqrt{2}}{\pi}$

(12) The length of minute hand of a clock is 14 *cm*. If the minute hand moves from 2 to 11 on the circular dial, then area covered by it is ...... cm^2.

(a) 154 (b) 308 (c) 462 (d) 616

(13) The length of minute hand of a clock is 15 *cm*. If the minute hand moves for 20 minutes on a circular dial of a clock, the area covered by it is ...... cm^2. (π = 3.14)

(a) 235.5 (b) 471 (c) 141.3 (d) 706.5

*

Summary

In this chapter we have studied the following points :

1. Circumference of a circle = $2\pi r$
2. Area of a circle = πr^2
3. Length of an arc of a circle having radius 'r' and angle subtended by the arc at the centre of measure θ is $l = \frac{\pi r\theta}{180}$
4. Area of a minor sector = $\frac{\pi r^2\theta}{360}$, where θ is measure of the angle subtended by the corresponding arc at the centre.
5. Area of the major sector = $\pi r^2 - \frac{\pi r^2\theta}{360}$
6. Area of minor segment = The area of minor sector − The area of the triangle formed by the chord and radii of the circle drawn at the end-points of the chord.
7. Area of major segment = The area of major sector + The area of the triangle formed by the chord and radii of the circle drawn at the end-points of the chord.
8. Areas of combination of plane figures like, square and semicircle, rectangle and semicircle, circle and triangle etc.

Devlali or Self number

In 1963, Kaprekar defined the property which has come to be known as self numbers, which are integers that cannot be generated by taking some other number and adding its own digits to it. For example, 21 is not a self number, since it can be generated from 15: 15 + 1 + 5 = 21. But 20 is a self number, since it cannot be generated from any other integer. He also gave a test for verifying this property in any number. These are sometimes referred to as Devlali numbers (after the town where he lived); though this appears to have been his preferred designation, the term self number is more widespread. Sometimes these are also designated Colombian numbers after a later designation.

Harshad number

Kaprekar also described the Harshad numbers which he named harshad, meaning "giving joy" (Sanskrit harsha, joy +da taddhita pratyaya, causative); these are defined by the property that they are divisible by the sum of their digits. Thus 12, which is divisible by 1 + 2 = 3, is a Harshad number. These were later also called Niven numbers after a 1997 lecture on these by the Canadian mathematician Ivan M. Niven. Numbers which are Harshad in all bases (only 1, 2, 4, and 6) are called all-Harshad numbers. Much work has been done on Harshad numbers, and their distribution, frequency, etc. are a matter of considerable interest in number theory today.

Demlo number

Kaprekar also studied the Demlo numbers, named after a train station where he had the idea of studying them.] These are the numbers 1, 121, 12321, …, which are the squares of the repunits 1, 11, 111, ….

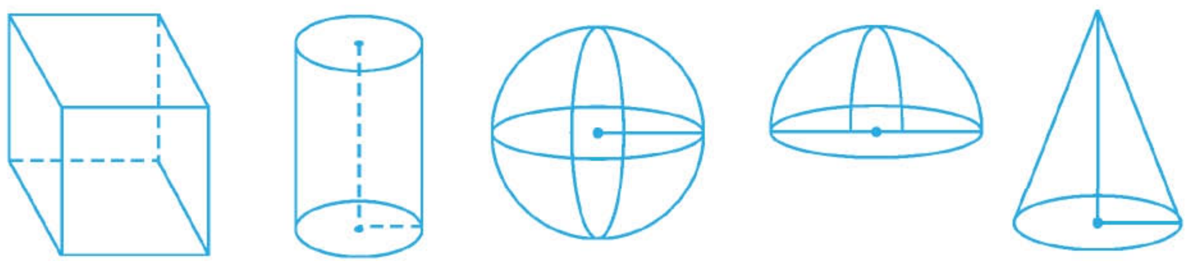
SURFACE AREA AND VOLUME 14

If there is a problem you can't solve, then there is an easier problem you can solve. Find it.

- George Polya

14.1 Introduction

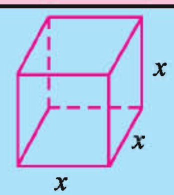
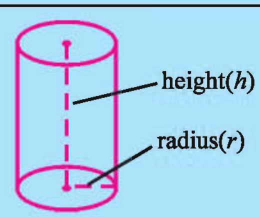
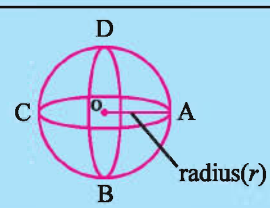
We are already familiar with the surface area and volume of some regular solids like cuboid, cylinder, sphere, hemisphere and right circular cone (see figure 14.1)

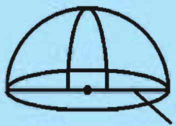
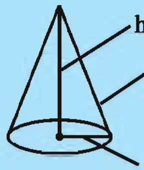


(i) (ii) (iii) (iv) (v)

Figure 14.1

Surface area of some familiar solids :

Sr. No.	Solid	Figure	State of solid	Surface area
1.	Cube		Open cube Closed cube	$5x^2$ $6x^2$
2.	Cylinder		Curved surface area Total surface area	$2\pi rh$ $2\pi r(r + h)$
3.	Sphere		Surface area	$4\pi r^2$

Sr. No.	Solid	Figure	State of solid	Surface area
4.	Hemisphere		Open hemisphere Closed hemisphere	$2\pi r^2$ $3\pi r^2$
5.	Right circular cone		Lateral surface area Total surface area	πrl $\pi r(r + l)$

Note : For simple calculations take $\pi = \frac{22}{7}$ unless otherwise stated.

In our daily life, we come across some solids made up of combinations of two or more of the basic solids as shown in figure 14.1.

We have seen the container on the back of truck or on the train which contains either water or oil or milk. The shape of the container is made of a cylinder with two hemispheres at its ends.

In our science laboratory we have seen a test-tube. This tube is also a combination of a cylinder and a hemisphere at one end.

We have seen a toy top also, it is a combination of a cone and hemisphere at the base of cone.

14.2 Surface Area of a Combination of Solids

Let us consider the cylindrical vessel (see figure 14.2). How do we find the surface area of such a solid ? Whenever we come across a new problem, we first divide (or break it down) into smaller problems which we have solved earlier. We can see that this solid is made up of a cylinder with a conical lid surmounted on it. It looks like what we have in figure 14.3 after we put both the pieces together. To find the surface area of cylindrical vessel, we have to find the surface area of a cone and the surface area of a cylinder individually.

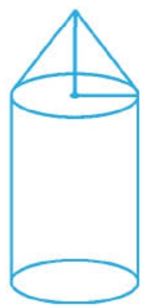


Figure 14.2

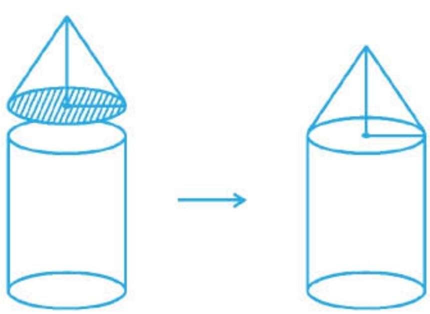


Figure 14.3

Total surface area of a cylindrical vessel (TSA) = Curved surface area of cylinder (CSA) + Curved surface area of cone (CSA).

Let us consider another solid. Suppose we are making a toy by putting cone and hemisphere together. Now let us see how to find the total surface area of this toy. (See figure 14.4)

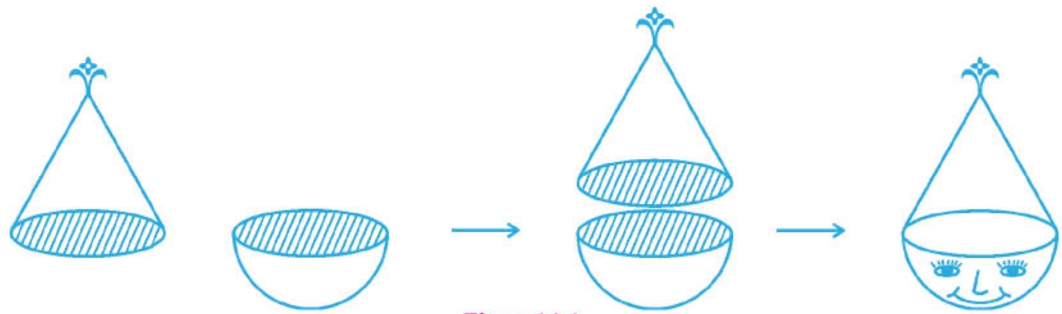


Figure 14.4

First we take a cone and a hemisphere and bring their flat faces together, of course we take the radius of the base of the cone and radius of the hemisphere same. So the steps are as shown in figure 14.4. At the end we get a nice round-bottomed toy. Now if we want to find the surface area of this toy, what should we do ? We need to find total surface area of the toy. We need the curved surface area of a cone and curved surface area of a hemisphere. So we get,

Total surface area of the toy = Curved surface area of cone + CSA of hemisphere.

Now let us learn some examples.

Example 1 : How many square meters of cloth is required to prepare four conical tents of diameter 8 *m* and height 3 *m*. ($\pi = 3.14$)

Solution : Here diameter of the tent is 8 *m*, so the radius is 4 *m* and height of the tent is 3 *m*.

Now the slant height $l = \sqrt{h^2 + r^2}$

$$\therefore \quad l = \sqrt{9 + 16} = 5\ m$$

Now, the curved surface area of cone $= \pi r l$

$= 3.14 \times 4 \times 5$

$= 62.8\ m^2$

So, the cloth required for one tent 62.8 m^2.

Therefore the cloth required for four tents is 251.2 m^2.

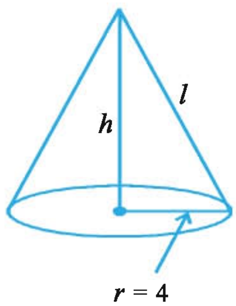


Figure 14.5

Example 2 : A cylinder has hemispherical ends having radius 14 *cm* and height 50 *cm*. Find the total surface area.

Solution : Here the radius of cylinder and hemisphere is 14 *cm* and the height is 50 *cm* as in figure 14.6.

Total surface area of the solid composed of a cylinder and two hemispherical ends,

= CSA of cylinder + 2 × CSA of hemispheres.

$= 2\pi rh + 2(2\pi r^2)$

$= 2\pi r\,(h + 2r)$

$= 2 \times \frac{22}{7} \times 14\ (50 + 28)$

$= 6864\ cm^2$

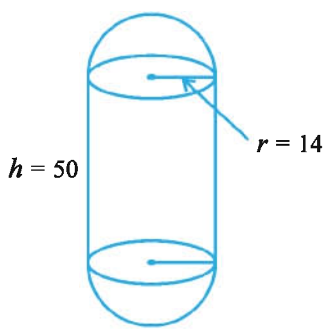


Figure 14.6

Example 3 : A box is made up of a cylinder surmounted by a cone. The radius of the cylinder and cone is 12 *cm* and slant height of the cone is 13 *cm*. The height of the cylinder is 11 *cm*. Find the curved surface area of the box.

Solution : Here radius of the cylinder = radius of the cone = 12 *cm* = r

Height of the cylinder = 11 *cm*. Slant height of the cone (l) = 13 *cm*.

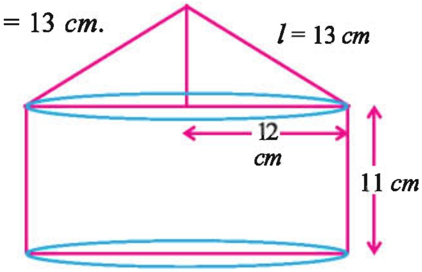


Figure 14.7

∴ Total curved surface area of the given solid

$= \text{CSA of the cylinder} + \text{CSA of the cone}$

$= 2\pi rh + \pi rl$

$= \pi r(2h + l)$

$= \frac{22}{7} \times 12(2 \times 11 + 13) = 1320\ cm^2$

Thus, the curved surface area of the given solid is 1320 cm^2.

Example 4 : A metallic cylinder has diameter 1 *m* and height 3.2 *m*. Find the cost of painting its outer surface at the rate of ₹ 35 per square meter. ($\pi = 3.14$)

Solution : The diameter of the cylinder is 1 *m*.

∴ The radius (r) of the cylinder = 0.5 *m* and the height (h) of the cylinder is 3.2 *m*.

∴ The total surface area of the cylinder (including top and bottom)

$= 2\pi r(r + h)$

$= 2 \times 3.14 \times 0.5\ (0.5 + 3.2) = 11.618\ m^2$

Now, the cost of painting is ₹ 35 per square meter, so the total cost of painting this cylinder

= ₹ 35 × 11.618 = ₹ 406.63

∴ The total cost of painting this cylinder is ₹ 407 (to nearest rupee).

Example 5 : The total surface area of a hemisphere is 763.72 cm^2. Find its diameter.

Solution : Let r be the radius of the hemisphere.

∴ Total surface area of the hemisphere = $3\pi r^2$

∴ $r^2 = \frac{763.72}{3\pi} = \frac{763.72}{3} \times \frac{7}{22} = 81.033 = 81$ (approx.)

∴ $r = 9$ *cm*

∴ The diameter = $2r$ = 18 *cm*

∴ The diameter of the hemisphere is 18 *cm*.

Example 6 : The radius of a conical shaped dome of a temple is 7 *m* and its height is 24 *m*. Find the cost of painting both the sides (inside and outside) of the dome of the temple at the rate of ₹ 15 per square meter. (neglect thickness)

Solution : Here the height of the dome is 24 *m* and the radius of the dome is 7 *m*.

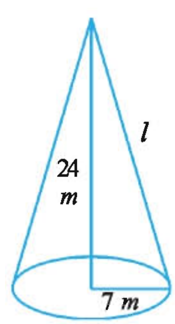


Figure 14.8

∴ The slant height is $l = \sqrt{24^2 + 7^2} = 25$ *m*

The curved surface area of cone = πrl

$= \frac{22}{7} \times 7 \times 25$

$= 550\ m^2$

Now, the cost of painting is ₹ 15 per square meter.

∴ The cost of painting the outer side of the dome is 15 × 550 = ₹ 8250

The total cost of painting both the sides is 2 × 8250 = ₹ 16500

Example 7 : The total surface area of a solid composed of a cone with hemispherical base is 361.1 cm^2. (π = 3.14) The dimension are shown in figure 14.9. Find the total height of the solid.

Solution : Suppose the radius of the hemisphere and the base of cone is r.

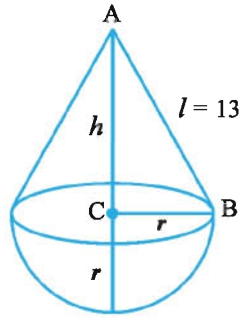


Figure 14.9

$\therefore$ The total surface area of the given solid = $\pi rl + 2\pi r^2$

$\therefore$ $361.1 = 3.14(r \times 13 + 2 \cdot r^2)$

$\therefore$ $\frac{361.1}{3.14} = 13r + 2r^2$

$\therefore$ $115 = 2r^2 + 13r$

$\therefore$ $2r^2 + 13r - 115 = 0$

$\therefore$ $(r - 5)(2r + 23) = 0$

$\therefore$ $r = 5$ or $r = \frac{-23}{2}$

But radius is positive. So, $r = 5$ cm

Now, from the figure 14.9 the height of the cone = $h = \sqrt{l^2 - r^2}$

$= \sqrt{169 - 25}$

$= 12$ cm

$\therefore$ The total height of the solid = $h + r = 17$ cm

EXERCISE 14.1

1. A toy is made by mounting a cone onto a hemisphere. The radius of the cone and a hemisphere is 5 cm. The total height of the toy is 17 cm. Find the total surface area of the toy.

2. A show-piece shown in figure 14.10 is made of two solids - a cube and a hemisphere. The base of the block is a cube with edge 7 cm and the hemisphere fixed on the top has diameter 5.2 cm. Find the total surface area of the piece.

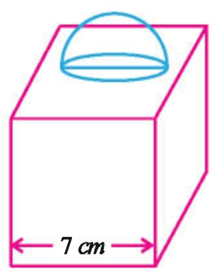


Figure 14.10

3. A vessel is in the form of a hemisphere mounted on a hollow cylinder. The diameter of the hemisphere is 21 cm and the height of vessel is 25 cm. If the vessel is to be painted at the rate of ₹ 3.5 per cm^2, then find the total cost to paint the vessel from outside.

4. Chirag made a bird-bath for his garden in the shape of a cylinder with a hemispherical depression at one end, (see the figure 14.11). The height of the cylinder is 1.5 m and its radius is 50 cm. Find the total area of the bird-bath. (π = 3.14)

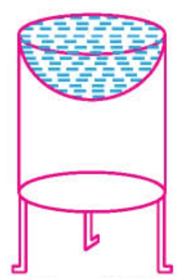


Figure 14.11

5. A solid is composed of a cylinder with hemisperical ends on both the sides. The radius and the height of the cylinder are 20 cm and 35 cm respectively. Find the total surface area of the solid.

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6. The radius of a conical tent is 4 *m* and slant height is 5 *m*. How many meters of canvas of width 125 *cm* will be used to prepare 12 tents ? If the cost of canvas is ₹ 20 per meter, then what is total cost of 12 tents ? (π = 3.14)
7. If the radius of a cone is 60 *cm* and its curved surface area is 23.55 m^2, then find its slant height. (π = 3.14)
8. The cost of painting the surface of sphere is ₹ 1526 at the rate of ₹ 6 per m^2. Find the radius of sphere.

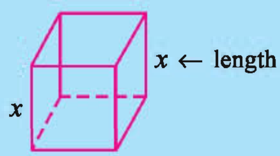
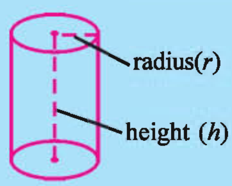
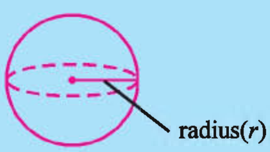
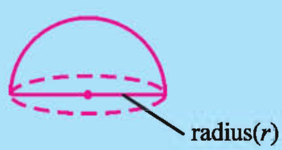
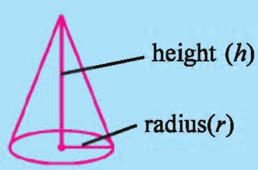
*

14.3 Volume of a combination of Solids

We have seen how to find the surface area of given solids made up of a combination of two basic solids in the previous section. Here we will study how to find the volume of such solids. It is noted that in the calculation of surface area, we cannot add the surface areas of the two constituents, because some part of the surface area have disappeared in the process of joining them. But this will not happen in the calculation of the volume. The volume of the solid formed by joining two basic solids will actually be the sum of the volumes of the constituents.

Note that 1 litre = 1000 cm^3. 1 m^3 = 1000 litre

Volume of some familiar solids

Sr. No.	Solid	Figure	Volume
1.	Cube		x^3
2.	Cylinder		$\pi r^2 h$
3.	Sphere		$\frac{4}{3}\pi r^3$
4.	Hemisphere		$\frac{2}{3}\pi r^3$
5.	Cone		$\frac{1}{3}\pi r^2 h$

Let us see some examples to understand the concept given above.

Example 8 : What will be the volume of the cone whose height is 21 *cm* and radius of the base is 6 *cm* ?

Solution : Volume of cone $= \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3} \times \frac{22}{7} \times 6 \times 6 \times 21$$

$$= 792 \ cm^3$$

$\therefore$ The volume of the cone is 792 cm^3.

Example 9 : Find the capacity of a cylindrical water tank whose radius is 2.1 *m* and height 5 *m*.

Solution : Here radius of the cylinder is $r = 2.1$ *m* and height $h = 5$ *m*.

Volume of the cylindrical tank $= \pi r^2 h$

$$= \frac{22}{7} \times 2.1 \times 2.1 \times 5 = 69.3 \ m^3$$

Now, 1 m^3 = 1000 litres

$\therefore$ 69.3 m^3 = 69.3 × 1000 = 69300 litres

Example 10 : How many maximum litres of petrol can be contained in a cylindrical tank with hemispherical ends having radius 0.42 *m* and total height 3.84 *m* ?

Solution : Here radius of the hemisphere $r = 0.42$ *m* = radius of the cylinder.

Now, the height of cylinder = Total height – 2(radius of hemisphere)

$= 3.84 - 2(0.42) = 3$ *m*.

The volume of the cylindrical tank with hemispherical ends

$$= \pi r^2 h + \frac{4}{3}\pi r^3 \qquad \left(2 \times \frac{2}{3}\pi r^3\right)$$

$$= \frac{22}{7} \times (0.42)^2 \times 3 + \frac{4}{3} \times \frac{22}{7} \times (0.42)^3$$

$$= 22 \times 0.06 \times 0.42 \times 3 + 4 \times 22 \times 0.02 \times 0.42 \times 0.42$$

$$= 1.6632 + 0.310464 = 1.973664 \ m^3.$$

Now, 1 m^3 = 1000 litres

$\therefore$ 1.973664 m^3 = 1.973664 × 1000

= 1973.66 litres

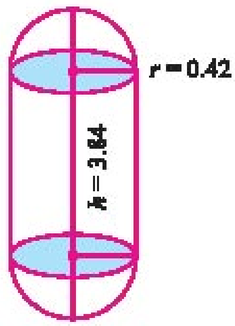


Figure 14.12

Example 11 : A common plot of a society is in the form of circle having diameter 5.6 *m*. Find how many cubic meters of soil is required to raise the level of ground by 25 *cm* ?

Solution : Here the diameter of the circle is 5.6 *m*.

$\therefore$ The radius r is 2.8 *m*

The area of a circle is πr^2, and we want to raise this region by 25 *cm* i.e. 0.25 *m* (say)

$\therefore$ Volume of the soil required to raise the ground $= \pi r^2 h$

$$= \frac{22}{7} \times 2.8 \times 2.8 \times 0.25$$

$$= 6.16 \ m^3$$

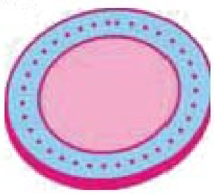


Figure 14.13

$\therefore$ The volume of the soil required is 6.16 m^3.

Example 12 : The volume of a cone is 9504 cm^3 and the radius of the base is 18 *cm*. Find the height of cone.

Solution : Here the radius of base of cone is 18 *cm* and the volume of the cone is 9504 cm^3.

The volume of the cone $= \frac{1}{3}\pi r^2 h$

$\therefore \quad 9504 = \frac{1}{3} \times \frac{22}{7} \times 18 \times 18 \times h$

$\therefore \quad h = \frac{9504 \times 3 \times 7}{22 \times 18 \times 18}$

$\therefore \quad h = 28\ cm$

$\therefore$ The height of the cone is 28 *cm*.

Example 13 : Mayank, an engineering student, was asked to make a model shaped like a cylinder with two cones attached at both ends with thin film-sheet. The radius of the model is 4 *cm* and the total height is 13 *cm*. If each cone has height 3 *cm*, find the volume of the air contained in the model.

Solution : Here radius of the cone and cylinder is $r = 4\ cm$ and the height of cone is $h = 3\ cm$. The height of the cylinder H = 13 – 2(3) = 7 *cm*

As shown in the figure 14.14, the total volume is divided into three parts, namely two conical parts and one cylindrical part.

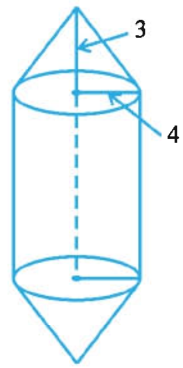


Figure 14.14

The volume of the cone $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3} \times \frac{22}{7} \times 4 \times 4 \times 3$

$= 50.29\ cm^3$

The volume of the cylinder $= \pi r^2 H$

$= \frac{22}{7} \times 4 \times 4 \times 7$

$= 352\ cm^3$

$\therefore$ The total volume of the model = 2 × volume of cone + volume of cylinder

$= 2 \times 50.29 + 352$

$= 452.58\ cm^3$.

The volume of air in the model is $452.58\ cm^3$.

EXERCISE 14.2

1. The curved surface area of a cone is $550\ cm^2$. If its diameter is 14 *cm*, find its volume.
2. A solid is in the form of cone with hemispherical base. The radius of the cone is 15 *cm* and the total height of the solid is 55 *cm*. Find the volume of the solid. ($\pi = 3.14$)
3. How many litres of milk can be stored in a cylindrical tank with radius 1.4 *m* and height 3 *m* ?
4. The spherical balloon with radius 21 *cm* is filled with air. Find the volume of air contained in it.
5. A solid has hemi-spherical base with diameter 8.5 *cm* and it is surmounted by a cylinder with height 8 *cm* and diameter of cylinder is 2 *cm*. Find the volume of this solid. ($\pi = 3.14$)
6. A playing top is made up of steel. The top is shaped like a cone surmounted by a hemisphere. The total height of top is 5 *cm* and the diameter of the top is 3.5 *cm*. Find the volume of the top.
7. How many litres of petrol will be contained in a closed cylindrical tank with hemisphere at one end having radius 4.2 *cm* and total height 27.5 *cm* ?

8. The capacity of a cylindrical tank at a petrol pump is 57750 litres. If its diameter is 3.5 *m*, find the height of cylinder.
9. A hemispherical pond is filled with 523.908 m^3 of water. Find the maximum depth of pond.
10. A gulab-jamun contain 40 % sugar syrup in it. Find how much syrup would be there in 50 gulab-jamuns, each shaped like a cylinder with two hemispherical ends with total length 5 *cm* and diameter 2.8 *cm*.
11. The height and the slant height of a cone are 12 *cm* and 20 *cm* respectively. Find its volume. ($\pi = 3.14$)
12. Find the total volume of a cone having a hemispherical base. If the radius of the base is 21 *cm* and height 60 *cm*.
13. If the slant height of a cone is 18.7 *cm* and the curved surface area is 602.8 cm^2, find the volume of cone. ($\pi = 3.14$)
14. If the surface area of a spherical ball is 1256 cm^2, then find the volume of sphere. (Take $\pi = 3.14$)

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14.4 Conversion of a Solid from one Shape to Another

We know that some of the solids can be melted and can be converted into another shapes, for example wax candle, iron piece, copper etc.

Let us understand the concept of conversion of a solid form into another solid form by examples.

Example 14 : How many balls of radius 0.5 *cm* can be prepared by melting a metal cylinder of radius 5 *cm* and height 7 *cm* ?

Solution : Radius (r) of the cylinder is 5 *cm* and the height (h) is 7 *cm*.

$\therefore$ The volume of the cylinder $= \pi r^2 h$

$$= \frac{22}{7} \times 5 \times 5 \times 7 = 550\ cm^3$$

Let the radius of a ball be R.

Now, the volume of a ball $= \frac{4}{3}\pi R^3$

$$= \frac{4}{3} \times \frac{22}{7} \times (0.5)^3 = 0.5238\ cm^3$$

Now, the volume of 1 ball $= 0.5238\ cm^3$

$\therefore$ The number of balls $= \dfrac{\text{Volume of the cylinder}}{\text{Volume of a ball}}$

$$= \frac{550}{0.5238} = 1050 \text{ (approximately)}$$

$\therefore$ Total number of balls formed is 1050.

Example 15 : A metallic sphere of radius 3.6 *cm* is melted and a wire of diameter 0.4 *cm* of uniform cross-section is drawn from it. Find the length of the wire.

Solution : Let the length (or height) of the wire be h and the radius be r. Also the radius of the sphere is R.

$\therefore$ $r = 0.2\ cm$, $R = 3.6\ cm$

$\therefore$ The volume of the wire = The volume of the sphere

$\therefore$ $\pi r^2 h = \frac{4}{3}\pi R^3$

$\therefore$ $h = \frac{4}{3} \times \frac{3.6 \times 3.6 \times 3.6}{0.2 \times 0.2}$

$= 1555.2\ cm$

$= 15.552\ m$

$\therefore$ The length of the wire is 15.552 *m*.

Example 16 : A hemispherical tank full of water is emptied by a pipe at the rate of $14\frac{2}{7}$ litres per second. How much time will it take to empty three fourth of the tank, if it is 4 *m* in diameter ?

Solution : Radius of hemisphere = 2 *m*

The volume of the tank $= \frac{2}{3}\pi r^3$

$= \frac{2}{3} \times \frac{22}{7} \times (2)^3$

$= \frac{352}{21}\ m^3$

So, the volume of the water to be emptied $= \frac{3}{4} \times \frac{352}{21} \times 1000 = \frac{88}{7} \times 1000$

$= \frac{88000}{7}$ litres

Since $\frac{100}{7}$ litres of water is emptied in 1 second.

$\therefore$ $\frac{88000}{7}$ litres of water will be emptied in $\frac{88000}{7} \times \frac{7}{100} = 880$ seconds

Example 17 : A 30 *m* deep cylindrical well with diameter 7 *m* is dug and the soil obtained by digging is evenly spread out to form a platform 30 *m* × 10 *m*. Find the height of the platform.

Solution : The radius of the well is $r = \frac{7}{2} = 3.5\ m$

The volume of the soil digged out from the well $= \pi r^2 h$ **(h = height of the well)**

$= \frac{22}{7} \times 3.5 \times 3.5 \times 30$

$= 1155\ m^3$

The volume of the soil = The volume of the platform

$\therefore$ $1155 = l \times b \times H = 30 \times 10 \times H$ **(H = height of the platform)**

$\therefore$ $H = \frac{1155}{30 \times 10} = 3.85\ m$

$\therefore$ The height of the platform is 3.85 *m*

Example 18 : How many spherical balls of diameter 0.5 *cm* can be cast by melting a metal cone with radius 6 *cm* and height 14 *cm* ?

Solution : Radius of cone = 6 *cm* = R

The height of the cone = h = 14 *cm*, the radius of the sphere = r = 0.5 *cm* = $\frac{1}{2}$ *cm*

Now, Number of balls $= \frac{\text{The volume of the cone}}{\text{The volume of the sphere}}$

$= \frac{\frac{1}{3}\pi R^2 h}{\frac{4}{3}\pi r^3} = \frac{(6)^2 \times 14}{4 \times \left(\frac{1}{2}\right)^3}$

$= 1008$

$\therefore$ The number of spherical balls is 1008.

EXERCISE 14.3

1. A hemispherical bowl of internal radius 12 *cm* contains some liquid. This liquid is to be filled into cylindrical bottles of diameter 4 *cm* and height 6 *cm*. How many bottles can be filled with this liquid ?
2. A cylindrical container having diameter 16 *cm* and height 40 *cm* is full of ice-cream. The ice-cream is to be filled into cones of height 12 *cm* and diameter 4 *cm*, having a hemispherical shape on the top. Find the number of such cones which can be filled with the ice-cream.
3. A cylindrical tank of diameter 3 *m* and height 7 *m* is completely filled with groundnut oil. It is to be emptied in 15 tins each of capacity 15 litres. Find the number of such tins required.
4. A cylinder of radius 2 *cm* and height 10 *cm* is melted into small spherical balls of diameter 1 *cm*. Find the number of such balls.
5. A metallic sphere of radius 15 *cm* is melted and a wire of diameter 1 *cm* is drawn from it. Find the length of the wire.
6. There are 45 conical heaps of wheat, each of them having diameter 80 *cm* and height 30 *cm*. To store the wheat in a cylindrical container of the same radius, what will be the height of cylinder ?
7. A cylindrical bucket, 44 *cm* high and having radius of base 21 *cm*, is filled with sand. This bucket is emptied on the ground and a conical heap of sand is formed. If the height of the conical heap is 33 *cm*, find the radius and the slant height of the heap.

*

14.5 Frustum of a Cone

In section 14.2, we had seen the objects that are formed when two basic solids were joined together. Here we will do something different. We will take a right circular cone and remove a portion of it. We can do this in many ways. But we will take one particular case that we will remove a smaller right circular cone by cutting the given cone by a plane parallel to its base. We have seen the glasses or tumblers, in general, used for drinking water, are of this shape. (See figure 14.13)

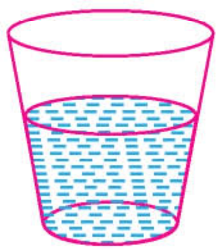
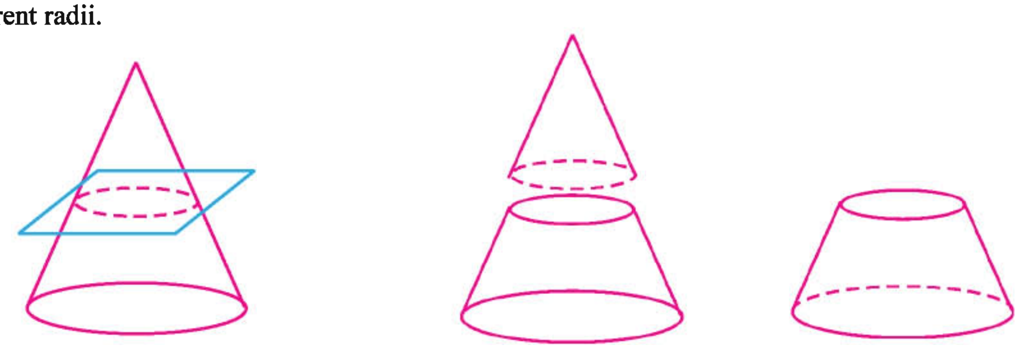


Figure 14.13

Activity : Take some clay or paper or any other such material (like plastic) and form a cone. Cut it with knife parallel to its base. Remove the smaller cone. What will be the remaining portion left ? The portion left with solid is called a **frustum** of the cone. It has two circular ends with different radii.



A cone sliced by a plane parallel to base   The two part separated   Frustum of a cone

Figure 14.14

So, given a cone, when we slice (or cut) through it with a plane parallel to its base and remove the cone that is formed on one side of the plane, the part which is left on the other side of the plane is called a **frustum of the cone.** (Frustum is a latin word which means "piece cut off" and its plural is 'frusta'.) (See figure 14.14)

Now let us see how to find the surface area and volume of a frustum of a cone by an example given below :

Example 19 : The radii of the ends of a frustum are 32 *cm* and 8 *cm* and the height of the frustum of the cone is 54 *cm*. Find its volume, the curved surface area and the total surface area.

(See figure 14.15)

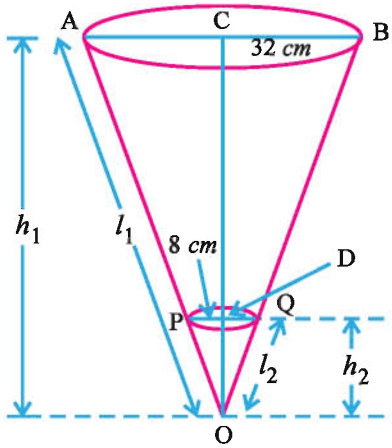


Figure 14.15

Solution : We can see that the volume of the frustum is the difference of volumes of two right circular cones OAB and OPQ. Let the heights and the slant heights of cones OAB and OPQ be respectively h_1, h_2 and l_1, l_2 and radii r_1 and r_2.

Here we have $r_1 = 32\ cm$, $r_2 = 8\ cm$

The height of the frustum is 54 *cm*.

Also, $h_1 = 54 + h_2$ **(i)**

Now, ΔOCB and ΔODQ are similar right angle triangles.

$\therefore \quad \frac{h_1}{h_2} = \frac{r_1}{r_2} = \frac{32}{8} = 4$

$\therefore \quad h_1 = 4h_2$

$\therefore \quad 4h_2 = 54 + h_2$. So $h_2 = 18$ **(by (i))**

Now, the volume of the frustum = Volume of the cone OAB − Volume of the cone OPQ

$$= \frac{1}{3}\pi r_1^2 h_1 - \frac{1}{3}\pi r_2^2 h_2$$

$$= \frac{1}{3}\pi \times (4r_2)^2 \times 4h_2 - \frac{1}{3}\pi \times r_2^2 \times h_2$$

$$= \frac{1}{3} \times \pi \times 63r_2^2 \times h_2$$

$$= \frac{1}{3} \times \frac{22}{7} \times 63 \times 8 \times 8 \times 18 = 76032\ cm^3$$

Now, slant heights l_1 and l_2 of cones OAB and OPQ respectively are given by

$$l_2 = \sqrt{8^2 + 18^2} = \sqrt{388} = 19.698\ cm \text{ (approximately)}$$

$$l_1 = \sqrt{32^2 + 72^2} = \sqrt{6208} = 78.79\ cm \text{ (approximately)}$$

Therefore the curved surface area of the frustum $= \pi r_1 l_1 - \pi r_2 l_2$

$$= \pi \times 4r_2 \times 4l_2 - \pi \times r_2 \times l_2$$

$$= \frac{22}{7} \times 15 \times 8 \times 19.698$$

$$= 7428.96\ cm^2$$

The total surface area of the frustum = the curved surface area $+ \pi r_1^2 + \pi r_2^2$

$$= 7428.96 + \frac{22}{7} \times (32)^2 + \frac{22}{7} \times (8)^2$$

$$= 7428.96 + 3218.29 + 201.14$$

$$= 10848.39\ cm^2$$

Note : Let h be the height, l be the slant height, r_1 and r_2 be radii of the ends ($r_1 > r_2$) of a frustum of a cone. Then we will accept the following formulae for the volume of frustum, the curved surface area and total surface area of the frustum as given below :

(i) Volume of the frustum of a cone = $\frac{1}{3}\pi h\,[r_1^2 + r_2^2 + r_1 r_2]$

(ii) The curved surface area of a frustum of a cone = $\pi(r_1 + r_2) \cdot l$ where $l = \sqrt{h^2 + (r_1 - r_2)^2}$

(iii) Total surface area of a frustum of a cone = $\pi l(r_1 + r_2) + \pi r_1^2 + \pi r_2^2$, where $l = \sqrt{h^2 + (r_1 - r_2)^2}$

Let us solve example 19 by the above formula :

$$\text{Volume of the frustum} = \frac{1}{3}\pi h\,[r_1^2 + r_2^2 + r_1 r_2]$$
$$= \frac{1}{3} \times \frac{22}{7} \times 54\,[(32)^2 + (8)^2 + 32 \times 8]$$
$$= \frac{1}{3} \times \frac{22}{7} \times 54 \times 1344 = 76032\ cm^3$$

The curved surface area of frustum $= \pi(r_1 + r_2) \cdot l$ where

$$l = \sqrt{h^2 + (r_1 - r_2)^2} = \sqrt{54^2 + (32 - 8)^2} = 59.09\ cm \text{ (approximately)}$$

$\therefore$ Curved surface area of the frustum $= \frac{22}{7}(32 + 8) \times (59.09) = 7428.46\ cm^2$

The total surface area of the frustum = curved surface area + $\pi r_1^2 + \pi r_2^2$

$$= 7428.46 + \frac{22}{7} \times (32)^2 + \frac{22}{7} \times (8)^2$$
$$= 7428.46 + 3218.29 + 201.14 = 10847.89\ cm^2$$

Example 20 : A drinking glass is in the shape of frustum of a cone of height 21 cm. The radii of its two circular ends are 3 cm and 2 cm. Find the capacity of the glass.

Solution : The height of the glass is 21 cm and the radii of two circular ends are 3 cm and 2 cm.

$\therefore$ $h = 21\ cm$, $r_1 = 3\ cm$ and $r_2 = 2\ cm$

The capacity of the glass = the volume of the glass $= \frac{1}{3}\pi h\,[r_1^2 + r_2^2 + r_1 r_2]$

$$= \frac{1}{3} \times \frac{22}{7} \times 21 \times [3^2 + 2^2 + 3 \times 2]$$
$$= \frac{1}{3} \times \frac{22}{7} \times [19] \times 21$$
$$= 418\ cm^3$$

Example 21 : An oil funnel made of tin sheet consists of a 20 cm long cylindrical portion attached to a frustum of a cone. If the total height is 40 cm, diameter of the cylindrical portion is 14 cm and the diameter of the top of the funnel is 24 cm, find the area of the tin sheet required to make the funnel. (See fig. 14.16)

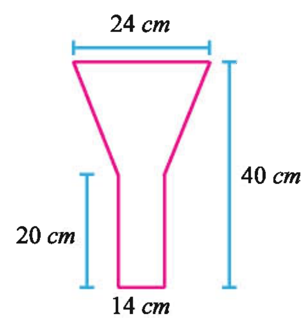


Figure 14.16

Solution : The curved surface area of the cylinder $= 2\pi rh$

$$= 2 \times \frac{22}{7} \times 7 \times 20$$
$$= 880\ cm^2$$

The curved surface area of the frustum $= \pi(r_1 + r_2) \cdot l$

Here $l = \sqrt{h^2 + (r_1 - r_2)^2}$, $r_1 = 12\ cm$, $r_2 = 7\ cm$ and $h = 20\ cm$

$\therefore \quad l = \sqrt{400 + (12-7)^2} = \sqrt{425} = 5\sqrt{17}$

$\therefore$ The curved surface area of the frustum $= \pi(r_1 + r_2) \cdot l$

$= \frac{22}{7} \times (12 + 7) \times 5\sqrt{17}$

$= 1231.04\ cm^2$

$\therefore$ The total area of the tin sheet $= 880 + 1231.04 = 2111.04\ cm^2$

EXERCISE 14.4

1. A metal bucket is in the shape of a frustum of a cone, mounted on a hollow cylindrical base made of the same metalic sheet. The total vertical height of the bucket is 40 *cm* and that of cylindrical base is 10 *cm*, radii of two circular ends are 60 *cm* and 20 *cm*. Find the area of the metalic sheet used. Also find the volume of water the bucket can hold. ($\pi = 3.14$)
2. A container, open from the top and made up of a metal sheet is the form of frustum of a cone of height 30 cm with radii 30 *cm* and 10 *cm*. Find the cost of the milk which can completely fill container at the rate of ₹ 30 per litre. Also find the cost of metal sheet used to make the container, if it costs ₹ 50 per 100 cm^2. ($\pi = 3.14$)

*

EXERCISE 14

1. A tent is in the shape of cylinder surmounted by a conical top. If the height and the radius of the cylindrical part are 3.5 *m* and 2 *m* respectively and the slant height of the top is 3.5 *m*, find the area of the canvas used for making the tent. Also find the cost of canvas of the tent at the rate of ₹ 1000 per m^2.
2. A metallic sphere of radius 5.6 *cm* is melted and recast into the shape of a cylinder of radius 6 *cm*. Find the height of the cylinder.
3. How many spherical balls of radius 2 *cm* can be made out of a solid cube of lead whose side measures 44 *cm* ?
4. A hemispherical bowl of internal radius 18 *cm* contains an edible oil to be filled in cylindrical bottles of radius 3 *cm* and height 9 *cm*. How many bottles are required to empty the bowl ?
5. A hemispherical tank of radius 2.4 *m* is full of water. It is connected with a pipe which empties it at the rate of 7 litres per second. How much time will it take to empty the tank compeletely ?
6. A shuttle cock used for playing badminton has the shape of a frustum of a cone mounted on a hemisphere. The external diameter of the frustum are 5 *cm* and 2 *cm*. The height of the entire shuttle cock is 7 *cm*. Find its external surface area.
7. A fez, the headgear cap used by the trucks is shaped like the frustum of a cone. If its radius on the open side is 12 *cm* and radius at the upper base is 5 *cm* and its slant height is 15 *cm*, find the area of material used for making it. ($\pi = 3.14$)
8. A bucket is in the form of a frustum of a cone with capacity of 12308.8 cm^3 of water. The radii of the top and bottom circular ends are 20 *cm* and 12 *cm* respectively. Find the height of bucket and the cost of making it at the rate of ₹ 10 per cm^2.

9. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) The volume of sphere with diameter 1 *cm* is ...... cm^3. ☐

(a) $\frac{2}{3}\pi$ (b) $\frac{1}{6}\pi$ (c) $\frac{1}{24}\pi$ (d) $\frac{4}{3}\pi$

(2) The volume of hemisphere with radius 1.2 *cm* is ...... cm^3. ☐

(a) 1.152π (b) 0.96π (c) 2.152π (d) 3.456π

(3) The volume of sphere is $\frac{4}{3}\pi$ cm^3. Then its diameter is ...... *cm*. ☐

(a) 0.5 (b) 1 (c) 2 (d) 2.5

(4) The volume of cone with radius 2 *cm* and height 6 *cm* is ...... cm^3. ☐

(a) 8π (b) 12π (c) 14π (d) 16π

(5) The diameter of the base of cone is 10 *cm* and its slant height is 17 *cm*. Then the curved surface area of the cone is ...... cm^2. ☐

(a) 85π (b) 170π (c) 95π (d) 88π

(6) The diameter and the height of the cylinder are 14 *cm* and 10 *cm* respectively. Then the total surface area is ...... cm^2. ☐

(a) 44 (b) 308 (c) 748 (d) 1010

(7) The ratio of the radii of two cones having equal height is 2 : 3. Then, the ratio of their volumes is .... ☐

(a) 4 : 6 (b) 8 : 27 (c) 3 : 2 (d) 4 : 9

(8) If the radii of a frustum of a cone are 7 *cm* and 3 *cm* and the height is 3 *cm*, then the curved surface area is ...... cm^2. ☐

(a) 50π (b) 25π (c) 35π (d) 63π

(9) The radii of a frustum of a cone are 5 *cm* and 9 *cm* and height is 6 *cm*, then the volume is ...... cm^3. ☐

(a) 320π (b) 151π (c) 302π (d) 98π

*

Summary

In this chapter we have studied the following points :

1. The surface area of some solids as under :

(1) Open cube with edge x : $5x^2$ and closed cube : $6x^2$

(2) Cylinder with radius r and height h

(a) Curve surface of a cylinder : $2\pi rh$

(b) Total surface of a cylinder : $2\pi r(r + h)$

(3) Sphere with radius r : $4\pi r^2$

(4) Hemisphere with radius r is

(a) Open hemisphere : $2\pi r^2$

(b) Closed hemisphere : $3\pi r^2$

(5) (a) Lateral surface area of cone : πrl

(b) Total surface area of cone : $\pi r(r + l)$

2. The volume :

(1) of cube with edge x is x^3

(2) of cylinder with radius r and height h is $\pi r^2 h$

(3) of sphere with radius r is $\frac{4}{3}\pi r^3$

(4) of hemisphere with radius r is $\frac{2}{3}\pi r^3$

(5) of cone is $\frac{1}{3}\pi r^2 h$

3. Conversion of solid from one shape to another.

4. For the given cone when we slice through it with a plane parallel to its base and remove the cone that is formed on one side of the plane, the portion which is left on the other side of the plane is called a frustum of a cone.

Let h be the height, l be the slant height, r_1 and r_2 are radii of ends ($r_1 > r_2$) of the frustum of a cone, then

(i) Volume of the frustum of a cone = $\frac{1}{3}\pi h\,[r_1^2 + r_2^2 + r_1 r_2]$

(ii) The curved surface area of a frustum of a cone = $\pi(r_1 + r_2) \cdot l$ where $l = \sqrt{h^2 + (r_1 - r_2)^2}$

(iii) Total surface area of a frustum of a cone = $\pi l(r_1 + r_2) + \pi r_1^2 + \pi r_2^2$

◆

Sharadchandra Shankar Shrikhande (born on October 19, 1917) is an Indian mathematician with distinguished and well-recognized achievements in combinatorial mathematics. He is notable for his breakthrough work along with R. C. Bose and E. T. Parker in their disproof of the famous conjecture made by Leonhard Euler dated 1782 that there do not exist two mutually orthogonal latin squares of order $4n + 2$ for every n. Shrikhande's specialty was combinatorics, and statistical designs. Shrikhande graph is used in statistical designs.

Shrikhande received a Ph.D. in the year 1950 from the University of North Carolina at Chapel Hill under the direction of R. C. Bose. Shrikhande taught at various universities in the USA and in India. Shrikhande was a professor of mathematics at Banaras Hindu University, Banaras and the founding head of the department of mathematics, University of Mumbai and the founding director of the Center of Advanced Study in Mathematics, Mumbai until he retired in 1978. He is a fellow of the Indian National Science Academy, the Indian Academy of Sciences and the Institute of Mathematical Institute, USA.

Statistics 15

A mathematics teacher is a mid-wife to ideas.
The first rule of discovery is to have brains and goodluck.
The second rule of discovery is to sit tight and wait till you get a braight idea.

- George Polya

15.1 Introduction

We have studied in class IX about the classification of given data into ungrouped data as well as grouped data frequency distributions. We have also learnt how to represent the data pictorially in the form of various graphs such as bar graphs, histogram with equal and unequal lengths of class and frequency polygons. We have studied the measures of central tendency, namely mean, median and mode of ungrouped data and grouped data. We have studied the concept of cumulative frequency curves called ogives.

15.2 Mean of grouped data

We know that the mean (or average) of ungrouped data is the sum of all the observations divided by the total number of observations. Recall that if x_1, x_2, x_3,..., x_k are observations with frequencies $f_1, f_2, f_3,..., f_k$ respectively, then the sum of values of all the observations is $f_1x_1 + f_2x_2 + f_3x_3 +...+ f_kx_k$ and the total number of observations is $n = f_1 + f_2 +...+ f_k$.

So, the mean of the data is given by,

$$\bar{x} = \frac{f_1x_1 + f_2x_2 +...+ f_kx_k}{f_1 + f_2 +...+ f_k}$$

We have also symbolised the sum by the greek letter Σ (capital sigma). So,

$$\bar{x} = \frac{\sum_{i=1}^{k} f_i x_i}{\sum_{i=1}^{k} f_i}$$

When there is no doubt then $\bar{x}$ can be written as $\bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i}$ where i varies from 1 to k.

Example 1 : The marks obtained by 100 students of two classes in mathematics paper consisting of 100 marks are as follows :

Marks obtained (x_i)	15	20	25	32	35	45	50	60	70	77	80
Number of students (f_i)	2	3	7	4	10	12	9	8	6	8	11

Marks obtained (x_i)	85	90	92	95	99
Number of students (f_i)	9	4	2	3	2

Find the mean of the marks obtained by the students.

Solution : To find the mean we need the product of x_i with corresponding f_i. For that we prepare the following Table 15.1

Table 15.1

Marks obtained (x_i)	Number of students (f_i)	$f_i x_i$
15	2	30
20	3	60
25	7	175
32	4	128
35	10	350
45	12	540
50	9	450
60	8	480
70	6	420
77	8	616
80	11	880
85	9	765
90	4	360
92	2	184
95	3	285
99	2	198
	$\Sigma f_i = 100$	$\Sigma f_i x_i = 5921$

Now, $\bar{x} = \dfrac{\sum f_i x_i}{\sum f_i}$

$= \dfrac{5921}{100}$

$= 59.21$

Therefore the mean of the marks obtained by the students is 59.21.

In practice the data are large. So, for a meaningful study we have to convert the data into grouped data.

Now, let us convert the above data into grouped data by forming class-intervals of width 15 (because generally we take 6 to 8 classes and range of our data is 90. So we take class-interval as 15). Remember that, while allocating frequencies to each class-interval, a student achieving value equal to upper class-limit would be considered to be in the next class. For example, 7 students who have obtained 25 marks would be considered in the class 25-40 and not in the class 10-25. The grouped frequency distribution table is as Table 15.2.

Table 15.2

Class-interval	10-25	25-40	40-55	55-70	70-85	85-100
Number of students	5	21	21	8	25	20

Now we will see how to calculate mean in continuous frequency distribution. Here we use the formula $\bar{x} = \dfrac{\sum f_i x_i}{\sum f_i}$ where we take x_i as the mid-point of a class (which would serve as the representative of the whole class) and f_i as frequency of that class. It is assumed that the frequency of each class-interval is centred around its mid-point (or class-mark).

$$\text{Class mid-point} = \frac{\text{Upper class limit of the class + Lower class limit of the class}}{2}$$

For the above table the mid-point of the class 10-25 is $\frac{10+25}{2} = 17.5$

Similarly, we can find the mid-point of each class as shown in Table 15.3.

Table 15.3

Class-interval	Number of students (f_i)	Mid-point (x_i)	f_ix_i
10-25	5	17.5	87.5
25-40	21	32.5	682.5
40-55	21	47.5	997.5
55-70	8	62.5	500.0
70-85	25	77.5	1937.5
85-100	20	92.5	1850.0
	Total Σf_i = 100		**Σf_ix_i = 6055**

So, the mean of given data is given by $\overline{x} = \frac{\sum f_ix_i}{\sum f_i} = \frac{6055}{100} = 60.55$

This method of finding the mean is called **Direct Method.**

We can see that Table 15.1 and 15.3 are the same data and applying the same formula for the calculation of the mean but the results obtained are different. Why is it so, and which one is more accurate ? The difference in two values occurs because of the mid-point assumption in Table 15.3. 59.21 is the exact mean, while 60.55 is an approximate mean. It is assumed that the observation of a class is centred around mid-value.

Every time the values x_i s and f_i s are not small. So when the numerical values of x_i and f_i are large, finding the product of x_i and f_i becomes tedious and time consuming also. So for such situation, let us think of a method of reducing these calculations. For this we cannot do anything for f_i s but we can reduce x_i s to a smaller number so the calculation becomes easy. How can we do this ? Let us understand the method.

The first step is to choose one of the x as **assumed mean** and denote it by A. We may take as A an x_i which is the middle of $x_1, x_2, \ldots, x_n$. Let $d_i = x_i - \text{A}$.

[**Note :** In fact A can be any convenient number. There is no change in the proof given below.]

$$\overline{d} = \frac{\sum f_id_i}{\sum f_i}$$

$$= \frac{\sum f_i(x_i - \text{A})}{\sum f_i}$$

$$= \frac{\sum f_ix_i}{\sum f_i} - \text{A} \cdot \frac{\sum f_i}{\sum f_i}$$

$$= \frac{\sum f_ix_i}{\sum f_i} - \text{A}$$

$$\bar{d} = \bar{x} - A$$

$$\therefore \quad \bar{x} = A + \bar{d}$$

$$\bar{x} = A + \frac{\Sigma f_i d_i}{\Sigma f_i}$$

Find $f_i d_i$ and $\Sigma f_i d_i$ as shown in Table 15.4. Let A = 62.5.

Table 15.4

Class-interval	Number of students (f_i)	Mid-point (x_i)	$d_i = x_i - A$	$f_i d_i$
10-25	05	17.5	–45	–225
25-40	21	32.5	–30	–630
40-55	21	47.5	–15	–315
55-70	08	62.5 = A	0	0
70-85	25	77.5	15	375
85-100	20	92.5	30	600
	$n = 100$			$\Sigma f_i d_i = -195$

Now, $\bar{x} = A + \frac{\Sigma f_i d_i}{\Sigma f_i}$

Now, substituting the values, we get

$$\bar{x} = 62.5 + \frac{(-195)}{100}$$

$$= 62.5 - 1.95$$

$$= 60.55$$

Therefore, the mean of the marks obtained by the students is 60.55

The method discussed above is called the method of **Assumed Mean.**

Activity 1 : From Table 15.3, taking the value of A as 17.5, 32.5 and so on and calculate the mean. The mean determined in each case is the same, i.e. 60.55.

So, we can say that the mean does not depend upon the value of A. Therefore we can take value of A as any non-zero number, not necessary that it is one of the value of x_i s.

See that in Table 15.4, the values in column 4 are multiple of 15 (i.e. class-interval), so if we divide the value of column 4 by 15, we get a smaller number to multiply with f_i.

So, let $u_i = \frac{x_i - A}{c}$, where A is the assumed mean and c is the class-size.

Suppose $\bar{u} = \frac{\Sigma f_i u_i}{\Sigma f_i}$

Now, let us find the relation between $\bar{u}$ and $\bar{x}$.

We have $u_i = \frac{x_i - A}{c}$

So, $\bar{u} = \dfrac{\sum f_i\left(\frac{x_i - A}{c}\right)}{\sum f_i} = \dfrac{1}{c}\left[\dfrac{\sum f_i x_i - A\sum f_i}{\sum f_i}\right]$

$= \dfrac{1}{c}\left[\dfrac{\sum f_i x_i}{\sum f_i} - A\right]$

$= \dfrac{1}{c}[\bar{x} - A]$

$\therefore \quad c\bar{u} = \bar{x} - A$

$\therefore \quad \bar{x} = A + c \cdot \bar{u}$

$= A + c \cdot \dfrac{\sum f_i u_i}{\sum f_i}$

Now let us calculate u_i as shown in Table 15.5. Here c is 15.

Table 15.5

Class-interval	f_i	x_i	$u_i = \frac{x_i - A}{c}$	$f_i u_i$
10-25	05	17.5	–3	–15
25-40	21	32.5	–2	–42
40-55	21	47.5	–1	–21
55-70	08	62.5	0	0
70-85	25	77.5	1	25
85-100	20	92.5	2	40
	$\sum f_i = 100$			$\sum f_i u_i = -13$

$\bar{x} = A + c \cdot \bar{u}$

$= A + c \cdot \dfrac{\sum f_i u_i}{\sum f_i}$

$= 62.5 + 15\left(\dfrac{-13}{100}\right)$

$= 62.5 + 15(-0.13)$

$= 62.5 - 1.95$

$= 60.55$

The method discussed above is called the method of **Step-Deviation.**

We note that :

– the step-deviation method will be convenient to apply if the class length is constant.

– the mean obtained by all the three methods is same.

– the assumed mean method and step-deviation method simplify the calculations involved in the direct method.

Example 2 : Find the mean of the data given below by all the three methods :

Class	0-10	10-20	20-30	30-40	40-50	50-60	60-70
Frequency	4	8	3	20	3	4	8

Solution : Let A = 35 and c = 10

Class	f_i	x_i	$d_i = x_i - A$	$u_i = \frac{x_i - A}{c}$	$f_i x_i$	$f_i d_i$	$f_i u_i$
0-10	4	5	–30	–3	20	–120	–12
10-20	8	15	–20	–2	120	–160	–16
20-30	3	25	–10	–1	75	–30	–3
30-40	20	35 = A	0	0	700	0	0
40-50	3	45	10	1	135	30	3
50-60	4	55	20	2	220	80	8
60-70	8	65	30	3	520	240	24
	$\Sigma f_i = 50$				$\Sigma f_i x_i = 1790$	$\Sigma f_i d_i = 40$	$\Sigma f_i u_i = 4$

Using the direct method, $\bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{1790}{50} = 35.8$

Using the assumed mean method, $\bar{x} = A + \frac{\sum f_i d_i}{\sum f_i}$

$= 35 + \frac{40}{50} = 35 + 0.8 = 35.8$

Using the step-deviation method, $\bar{x} = A + \left(\frac{\sum f_i u_i}{\sum f_i}\right) \times c$

$= 35 + \left(\frac{4}{50}\right) \times 10$

$= 35 + 0.8 = 35.8$

Therefore the mean of the data is 35.8

Example 3 : The mean of the following frequency distribution is 16, find the missing frequency :

Class	0-4	4-8	8-12	12-16	16-20	20-24	24-28	28-32	32-36
Frequency	6	8	17	23	16	15	–	4	3

Solution : Let the missing frequency be x, take A = 26, c = 4

Class	Frequency	x_i	$u_i = \frac{x_i - A}{c}$	$f_i u_i$
0-4	6	2	–6	–36
4-8	8	6	–5	–40
8-12	17	10	–4	–68
12-16	23	14	–3	–69
16-20	16	18	–2	–32
20-24	15	22	–1	–15
24-28	x	26 = A	0	0
28-32	4	30	1	4
32-36	3	34	2	6
	$\Sigma f_i = 92 + x$			$\Sigma f_i u_i = -250$

We take A = 26, so that the product $f_i u_i$ is zero when $f_i = x$.

$$\overline{x} = A + \left(\frac{\Sigma f_i u_i}{\Sigma f_i}\right) \times c$$

$$16 = 26 + \left(\frac{-250}{92 + x}\right) \times 4$$

$$-10 = \frac{-1000}{92 + x}$$

$\therefore$ $92 + x = 100$

$\therefore$ $x = 8$

$\therefore$ The missing frequency is 8.

Example 4 : The distribution below shows the number of wickets taken by a bowler in one-day cricket matches. Find the mean number of wickets.

Number of wickets	20-60	60-100	100-150	150-250	250-350	350-450
Number of bowlers	7	5	16	12	2	3

Solution : Here the class size varies and x_i's are large. So we take A = 200 and c = 10 and apply the step-deviation method.

Number of wickets	Number of bowlers (f_i)	x_i	$d_i = x_i - 200$	$u_i = \frac{x_i - A}{c}$	$f_i u_i$
20-60	7	40	−160	−16	−112
60-100	5	80	−120	−12	−60
100-150	16	125	−75	−7.5	−120
150-250	12	200 = A	0	0	0
250-350	2	300	100	10	20
350-450	3	400	200	20	60
	$\Sigma f_i = 45$				$\Sigma f_i u_i = -212$

$\therefore$
$$\overline{x} = A + \left(\frac{\Sigma f_i u_i}{\Sigma f_i}\right) \times c$$

$$= 200 + \left(\frac{-212}{45}\right) \times 10$$

$$= 200 - 47.11$$

$$= 152.89$$

$\therefore$ The mean wickets taken by the bowler is 152.89.

Example 5 : The mean of the following frequency distribution of 125 observations is 22.12. Find the missing frequencies.

Class	0-4	5-9	10-14	15-19	20-24	25-29	30-34	35-39	40-44
Frequency	3	8	12	–	35	21	–	6	2

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Solution : Let the missing frequencies for the classes 15-19 and 30-34 be respectively f_1 and f_2. Let A = 17 and c = 5

Class	Frequency	x_i	$u_i = \frac{x_i - A}{c}$	$f_i u_i$
0-4	3	2	−3	−9
5-9	8	7	−2	−16
10-14	12	12	−1	−12
15-19	f_1	17	0	0
20-24	35	22	1	35
25-29	21	27	2	42
30-34	f_2	32	3	$3f_2$
35-39	6	37	4	24
40-44	2	42	5	10
	$\Sigma f_i = 87 + f_1 + f_2$			$\Sigma f_i u_i = 74 + 3f_2$

Here the total number of observations is 125 and

$\Sigma f_i = 87 + f_1 + f_2$

$\therefore$ $125 = 87 + f_1 + f_2$

$\therefore$ $f_1 + f_2 = 38$ **(i)**

Now, $\bar{x} = A + \left(\frac{\Sigma f_i u_i}{\Sigma f_i}\right) \times c$

$22.12 = 17 + \left(\frac{74 + 3f_2}{125}\right) \times 5$

$\therefore$ $5.12 = \frac{74 + 3f_2}{25}$

$\therefore$ $5.12 \times 25 = 74 + 3f_2$

$\therefore$ $128 - 74 = 3f_2$

$\therefore$ $54 = 3f_2$

$\therefore$ $f_2 = 18$. Also $f_1 + f_2 = 38$. So, $f_1 = 20$

$\therefore$ The missing frequencies are 20 and 18.

EXERCISE 15.1

1. Find the mean of the following frequency distribution :

Class	0-50	50-100	100-150	150-200	200-250	250-300	300-350
Frequency	10	15	30	20	15	8	2

2. Find the mean wage of 200 workers of a factory where wages are classified as follows :

Class	100-150	150-200	200-250	250-300	300-350	350-400	400-450	450-500	500-550
Frequency	4	8	14	42	50	40	32	6	4

3. Marks obtained by 140 students of class X out of 50 in mathematics are given in the following distribution. Find the mean by method of assumed mean method :

Class	0-10	10-20	20-30	30-40	40-50
Frequency	20	24	40	36	20

4. Find the mean of the following frequency distribution by step-deviation method :

Class	40-50	50-60	60-70	70-80	80-90	90-100
Frequency	5	10	20	9	6	2

5. Find the mean for the following frequency distribution :

Class	1-5	6-10	11-15	16-20	21-25	26-30	31-35
Frequency	18	32	30	40	25	15	40

6. A survey conducted by a student of B.B.A. for daily income of 600 families is as follows, find the mean income of a family :

Income	200-299	300-399	400-499	500-599	600-699	700-799	800-899
Number of families	3	61	118	139	126	151	2

7. The number of shares held by a person of various companies are as follows. Find the mean :

Number of shares	100-200	200-300	300-400	400-500	500-600	600-700
Number of companies	5	3	3	6	2	1

8. The mean of the following frequency distribution of 100 observations is 148. Find the missing frequencies f_1 and f_2 :

Class	0-49	50-99	100-149	150-199	200-249	250-299	300-349
Frequency	10	15	f_1	20	15	f_2	2

9. The table below gives the percentage of girls in higher secondary science stream of rural areas of various states of India. Find the mean percentage of girls by step-deviation method :

Percentage of girls	15-25	25-35	35-45	45-55	55-65	65-75	75-85
Number of states	6	10	5	6	4	2	2

10. The following distribution shows the number of out door patients in 64 hospitals as follows. If the mean is 18, find the missing frequencies f_1 and f_2 :

Number of patients	11-13	13-15	15-17	17-19	19-21	21-23	23-25
Number of hospitals	7	6	f_1	13	f_2	5	4

*

15.3 Mode of Grouped Data

Let us recall that the observation which is repeated most often in an ungrouped data is called the mode of the data. In this chapter we shall discuss the way of obtaining the mode of grouped data, denoted by Z.

Let us recall how to find mode of ungrouped data through the following example.

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Example 6 : The wickets taken by a bowler in 10 one-day matches are as follows : 4, 5, 6, 3, 4, 0, 3, 2, 3, 5. Find the mode of the data.

Solution : Here 3 is the number of wickets taken by a bowler in maximum number of matches. i.e. 3 times. So the mode of this data is 3.

In grouped frequency distribution, it is not possible to determine the mode of the data by looking at the frequencies. Here, we can only locate a class with the largest frequency, called the **modal class**. The mode is a value inside the modal class and it is given by the formula :

$$Z = l + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right) \times c$$

where l = lower boundary point of the modal class

c = size of class interval (assuming all class sizes to be equal)

f_1 = frequency of the modal class

f_0 = frequency of the class preceding the modal class.

f_2 = frequency of the class succeeding the modal class.

Let us use this formula in the following examples.

Example 7 : A survey conducted on 20 hostel students for their reading hours per day resulted in the following frequency table :

Number of reading hours	1–3	3–5	5–7	7–9	9–11
Number of hostel students	7	2	8	2	1

Find the mode of this data.

Solution : Here the maximum class freqency is 8 and the class corresponding to this frequency is 5–7. So, the modal class is 5–7.

∴ The lower limit of the modal class 5–7 is $l = 5$.

Class size $c = 2$ and frequency of the modal class $f_1 = 8$. Frequency of the class preceding the modal class $= f_0 = 2$ and frequency of the class suceeding the modal class $= f_2 = 2$.

Now let us substitute these values in the formula :

$$Z = l + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right) \times c$$

$$= 5 + \left(\frac{8-2}{2\times 8 - 2 - 2}\right) \times 2$$

$$= 5 + \frac{6}{12} \times 2$$

$$= 5 + 1 = 6$$

So, the mode of above data is 6.

Example 8 : The mark distribution of 30 students at mathematics examination in a class is as below :

Marks	10-25	25-40	40-55	55-70	70-85	85-100
Number of students	05	21	21	08	25	20

Find the mode of this data.

Solution : Here the maximum class frequency is 25 and the class corresponding to this frequency is 70-85. So, the modal class is 70-85.

The lower limit l of modal class 70-85 = 70 and class size c = 15

Frequency of the modal class f_1 = 25

Frequency of the class preceding the modal class = f_0 = 08,

Frequency of the class succeeding the modal class = f_2 = 20.

Now, let us substitute there values in the formula :

$$\text{Mode} = l + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right) \times c$$

$$= 70 + \left(\frac{25 - 8}{2 \times 25 - 8 - 20}\right) \times 15$$

$$= 70 + \frac{17 \times 15}{22}$$

$$= 70 + 11.59 = 81.59$$

So, the mode of the above data is 81.59.

EXERCISE 15.2

1. Find the mode for the following frequency distribution :

Class	4-8	8-12	12-16	16-20	20-24	24-28
Frequency	9	6	12	7	15	1

2. The data obtained for 100 shops for their daily profit per shop are as follows :

Daily profit per shop (in ₹)	0-100	100-200	200-300	300-400	400-500	500-600
Number of shops	12	18	27	20	17	6

Find the modal profit per shop.

3. Daily wages of 90 employees of a factory are as follows :

Daily wages (in ₹)	150-250	250-350	350-450	450-550	550-650	650-750	750-850	850-950
Number of employees	4	6	8	12	33	17	8	2

Find the modal wage of an employee.

4. Find the mode for the following data : (4 and 5)

Class	0-7	7-14	14-21	21-28	28-35	35-42	42-49	49-56
Frequency	26	31	35	42	82	71	54	19

5.

Class	0-20	20-40	40-60	60-80	80-100	100-120	120-140	140-160	160-180
Frequency	11	14	18	21	31	27	12	11	10

6. The following data gives the information of life of 200 electric bulbs (in hours) as follows :

Life in hours	0-20	20-40	40-60	60-80	80-100	100-120
Number of electric bulbs	26	31	35	42	82	71

Find the modal life of the electric bulbs.

*

15.4 Median of Grouped Data (M)

We have seen the definition of median for ungrouped data in standard IX as : "After arranging the observations in ascending or descending order, the number which is obtained in the middle is called the median." **Also, if the number of observations n is odd, then $\left(\frac{n+1}{2}\right)^{th}$ observation is the median and if the number of observations n is even, then median $M = \dfrac{\left(\frac{n}{2}\right)^{th} \text{observation} + \left(\frac{n}{2}+1\right)^{th} \text{observation}}{2}$.**

Suppose we have to find the median of the following data, which shows the marks of 50 students in mathematics out of 50 marks :

Marks obtained	18	22	30	35	39	42	45	47
Number of Students	4	5	8	8	16	4	2	3

Here $n = 50$ which is even. The median will be the average of $\frac{n}{2}^{th}$ and $\left(\frac{n}{2}+1\right)^{th}$ observation i.e. 25th and 26th observations. To find this observation we proceed as follows :

Table 15.6

Marks obtained	Number of students
18	4
less than or equal to 22	4 + 5 = 9
less than or equal to 30	9 + 8 = 17
less than or equal to 35	17 + 8 = 25
less than or equal to 39	25 + 16 = 41
less than or equal to 42	41 + 4 = 45
less than or equal to 45	45 + 2 = 47
less than or equal to 47	47 + 3 = 50

We have formed a column which shows the number of students getting marks less than or equal to a particular number. It is known as cumulative frequency column.

Table 15.7

Marks obtained	Number of students (f)	Cumulative frequency (cf)
18	4	4
22	5	9
30	8	17
35	8	25
39	16	41
42	4	45
45	2	47
47	3	50

From the above table, we see that 25th observation is 35
26th observation is 39

$\therefore$ Median $= \frac{35+39}{2} = 37$

Note : The column 1 and column 3 of table 15.7 form cumulative frequency table. The median 37 shows the information that 50 % students obtained marks less than 37 and remaining 50 % students obtained marks more than 37.

Now let us see how to obtain the median of a grouped data from the following.

Consider a grouped frequency distribution of marks obtained, out of 100, by 55 students, in a certain examination, as follows :

Table 15.8

Marks	Number of students
0-10	2
10-20	3
20-30	3
30-40	4
40-50	3
50-60	4
60-70	7
70-80	11
80-90	8
90-100	10

We can see that 2 students obtained marks between 0 and 10, 3 students obtained marks between 10 and 20 and so on. So number of students who obtained marks less than 30 is $2 + 3 + 3 = 8$. Therefore the cumulative frequency of class 20-30 is 8. Similarly, we can obtaine the cumulative frequency of each class as shown in Table 15.9 as follow :

Table 15.9

Marks obtained	Number of students (cumulative frequency)
Less than 10	2
Less than 20	$2 + 3 = 5$
Less than 30	$5 + 3 = 8$
Less than 40	$8 + 4 = 12$
Less than 50	$12 + 3 = 15$
Less than 60	$15 + 4 = 19$
Less than 70	$19 + 7 = 26$
Less than 80	$26 + 11 = 37$
Less than 90	$37 + 8 = 45$
Less than 100	$45 + 10 = 55$

The distribution given in Table 15.9 is called **cumulative frequency distribution of less than type.** Here 10, 20, 30,..., 100 are the upper limits of the respective class intervals.

Similarly, we can make the table for the number of students with scores, more than or equal to 0, more than or equal to 10, more than or equal to 20 and so on. From Table 15.8, we can see that all

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55 students have obtained marks more than or equal to 0. Since 2 students obtained marks in the interval 0-10, there are 55 − 2 = 53 students getting more than or equal to 10 marks and so on, as shown in Table 15.10.

Table 15.10

Marks obtained	Number of students (cumulative frequency)
More than or equal to 0	55
More than or equal to 10	55 − 2 = 53
More than or equal to 20	53 − 3 = 50
More than or equal to 30	50 − 3 = 47
More than or equal to 40	47 − 4 = 43
More than or equal to 50	43 − 3 = 40
More than or equal to 60	40 − 4 = 36
More than or equal to 70	36 − 7 = 29
More than or equal to 80	29 − 11 = 18
More than or equal to 90	18 − 8 = 10

The above table is called **cumulative frequency distribution of more than type.** Here 0, 10, 20,.., 90 are the lower limits of the respective class intervals.

Now, to find the median of this grouped data, we will make a table showing cumulative frequency with class interval and frequency, as shown in Table 15.11.

Table 15.11

Marks	Number of students (f)	Cumulative frequency (cf)
0-10	2	2
10-20	3	5
20-30	3	8
30-40	4	12
40-50	3	15
50-60	4	19
60-70	7	26
70-80	11	37
80-90	8	45
90-100	10	55

Here in a grouped data, we are not able to find the middle observation by looking at the cumulative frequencies as the middle observation will be some value in a class interval. So, it is necessary to find the value inside a class which divides the whole distribution into the halves. Which class is this ?

To find this class, we find the cumulative frequency n of all the classes and find $\frac{n}{2}$. Now we find the class whose cumulative frequency is greater than $\frac{n}{2}$ and nearest to $\frac{n}{2}$ is called the **median class**. In the distribution above, $n = 55$. So $\frac{n}{2} = 27.5$. Now, 70 − 80 is the class whose cumulative frequency is 37 which is just greater than 27.5. Therefore, 70-80 is the **median class.**

[**Note :** The cumulative frequency is just greater than $\frac{n}{2}$ means the smallest cumulative frequency which is cumulative frequency greater than $\frac{n}{2}$.]

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After finding the median class, we use the formula given below for calculation of the median.

Median (M) $= l + \left(\dfrac{\frac{n}{2} - cf}{f}\right) \times c$

where l **= lower boundary point of the median class**

n **= total number of observations (sum of the frequencies)**

cf **= cumulative frequency of the class preceding the median class.**

f **= the frequency of the median class**

c **= class size (assuming class sizes to be equal)**

Substituting the values $\frac{n}{2} = \frac{55}{2} = 27.5$, $l = 70$, $cf = 26$, $f = 11$, $c = 10$ in the above formula, we get

$$\text{Median (M)} = 70 + \left(\frac{27.5 - 26}{11}\right) \times 10$$

$$= 70 + \left(\frac{1.5 \times 10}{11}\right) = 71.36$$

So, the half of the students have obtained marks less than 71.36 and the other half have obtained marks more than 71.36.

Example 9 : A survey regarding the weights (in *kg*) of 45 students of class X of a school was conducted and the following data was obtained :

Weight (in *kg*)	Number of students
20-25	2
25-30	5
30-35	8
35-40	10
40-45	7
45-50	10
50-55	3

Find the median weight.

Solution : Here the number of observations is 45.

$\therefore$ $n = 45$. So, $\frac{n}{2} = 22.5$

Now, we will prepare the table containing the cumulative frequency as below :

Weight (in kg)	Number of students (f)	Cumulative frequency (cf)
20-25	2	2
25-30	5	7
30-35	8	15
35-40	10	25
40-45	7	32
45-50	10	42
50-55	3	45

$\frac{n}{2} = 22.5$. This observation lies in the class 35-40. So the median class is 35-40.

So, $l = 35$, $cf = 15$, $f = 10$, $c = 5$

Using the formula $M = l + \left(\frac{\frac{n}{2} - cf}{f}\right) \times c$

$$= 35 + \left(\frac{22.5 - 15}{10}\right) \times 5$$

$$= 35 + \left(\frac{7.5 \times 5}{10}\right) = 38.75$$

So, the median weight is 38.75 *kg*.

This means that the 50 % students have more weight than 38.75 *kg* and other 50 % students have weight less than 38.75 *kg*.

Example 10 : The median of the following frequency distribution is 38.2. Find the value of x and y, where sum of the frequencies is 165.

Class	5-14	14-23	23-32	32-41	41-50	50-59	59-68
Frequency	5	11	x	53	y	16	10

Solution :

Class	Frequency	Cumulative frequency
5-14	5	5
14-23	11	16
23-32	x	$16 + x$
32-41	53	$69 + x$
41-50	y	$69 + x + y$
50-59	16	$85 + x + y$
59-68	10	$95 + x + y$

Solution : It is given that $n = 165$. So, $95 + x + y = 165$, i.e. $x + y = 70$

Also, the median is 38.2 which lies in the class 32-41.

So, median class is 32-41.

$\frac{n}{2} = \frac{165}{2} = 82.5$, $l = 32$, $cf = 16 + x$, $f = 53$, $c = 9$

Using the formula $M = l + \left(\frac{\frac{n}{2} - cf}{f}\right) \times c$

$$\therefore \quad 38.2 = 32 + \left(\frac{82.5 - 16 - x}{53}\right) \times 9$$

$$6.2 = \frac{66.5 - x}{53} \times 9$$

$$\therefore \quad \frac{6.2 \times 53}{9} = 66.5 - x$$

$$\therefore \quad 36.5 = 66.5 - x$$

$$\therefore \quad x = 30$$

but $x + y = 70$. So, $y = 40$

∴ The value of x and y are respectively 30 and 40.

Note : There is a relation between the measures of central tendency :

Mode (Z) = 3 Median (M) − 2 Mean ($\bar{x}$)

EXERCISE 15.3

1. Find the median for the following :

Value of variable	12	13	14	15	16	17	18	19	20
Frequency	7	10	15	18	20	10	9	8	3

2. Find the median for the following frequency distribution :

Class	4-8	8-12	12-16	16-20	20-24	24-28
Frequency	9	16	12	7	15	1

3. Find the median from following frequency distribution :

Class	0-100	100-200	200-300	300-400	400-500	500-600
Frequency	64	62	84	72	66	52

4. The following frequency distribution represents the deposits (in thousand rupees) and the number of depositors in a bank. Find the median of the data :

Deposit (₹ in thousand)	0-10	10-20	20-30	30-40	40-50	50-60
Number of depositors	1071	1245	150	171	131	8

5. The median of the following frequency distribution is 38. Find the value of a and b if the sum of frequences is 400 :

Class	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Frequency	42	38	a	54	b	36	32

6. The median of 230 observations of the following frequency distribution is 46. Find a and b :

Class	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Frequency	12	30	a	65	b	25	18

7. The following table gives the frequency distribution of marks scored by 50 students of class X in mathematics examination of 80 marks. Find the median of the data :

Class	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Frequency	2	5	8	16	9	5	3	2

*

15.5 Graphical Representation of Cumulative Frequency Distribution

We know that 'one picture is better than thousand words.' In class IX, we have represented the data through bar graphs, histogram, frequency polygons. Let us now represent a cumulative frequency distribution graphically.

For example, let us consider the cumulative frequency distribution given in table 15.9.

Remember that 10, 20, 30,..., 100 are the upper limits of the class intervals. To represent the data of table 15.9 graphically, we represent the upper limits of the class intervals on X-axis and their corresponding cumulative frequencies on Y-axis, choosing a convenient scale. The scale may not be same on both the axes. Now, let us plot the points corresponding to the ordered pairs given by (upper limit, corresponding cumulative frequency). i.e. (10, 2), (20, 5), (30, 8), (40, 12), (50, 15), (60, 19), (70, 26), (80, 37), (90, 45), (100, 55) on a graph paper. By joining them by a free hand smooth curve (See figure 15.1), we get a curve.

The curve we obtain is called a **cumulative frequency curve** or an **Ogive** (of the less than type). (See figure 15.1)

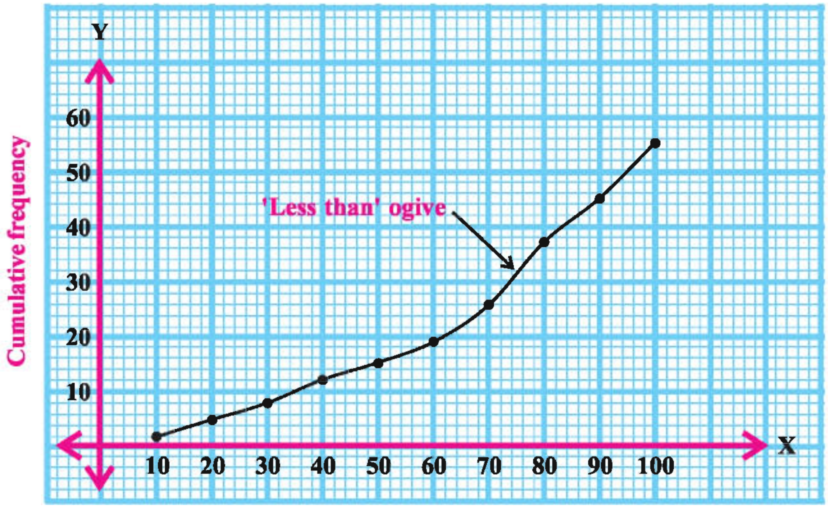


Figure 15.1

Now we draw the ogive (of more than type) of the cumulative frequency distribution in table 15.10.

Here, 0, 10, 20,...., 90 are the lower limits of the class intervals. To represent 'more than type' cumulative frequency curve, we plot the lower limits on X-axis and corresponding cumulative

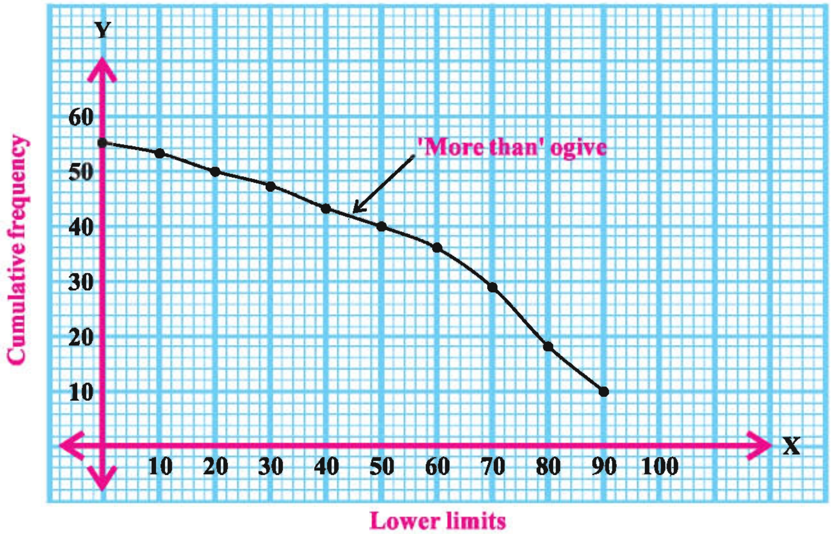


Figure 15.2

frequencies on Y-axis. Then we plot the points (lower limit, corresponding cumulative frequency), i.e. (0, 55), (10, 53), (20, 50), (30, 47), (40, 43), (50, 40), (60, 36), (70, 29), (80, 18), (90, 10) on a graph paper and join them by a free hand smooth curve. The curve we obtain is a **cumulative frequency curve** or **an ogive** (of more than type) (See figure 15.2)

In any way, are the ogives related to the median ?

One obvious way is to locate $\frac{n}{2} = \frac{55}{2} = 27.5$ on the Y-axis. From this point, draw a line parallel to X-axis intersecting the curve at a point. (See figure 15.3) From this point, draw a perpendicular to the X-axis. The point of intersection of this perpendicular with the X-axis determines the median of the data. (See figure 15.3)

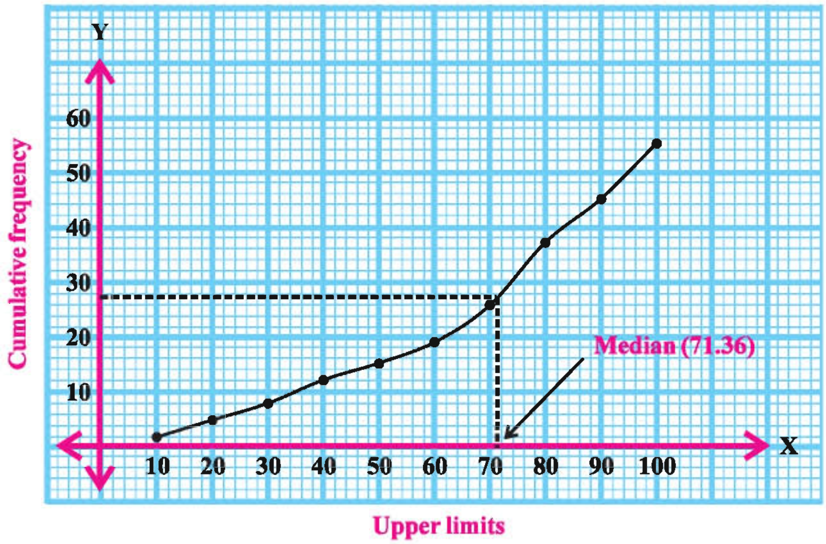


Figure 15.3

Another way of obtaining the median is as follows :

Draw both ogives (i.e. of less than type and of more than type) on the same graph-paper. The two ogives intersect each other at a point. From this point, if we draw perpendicular on the X-axis, the point at which it intersects the X-axis gives us the median (See figure 15.4)

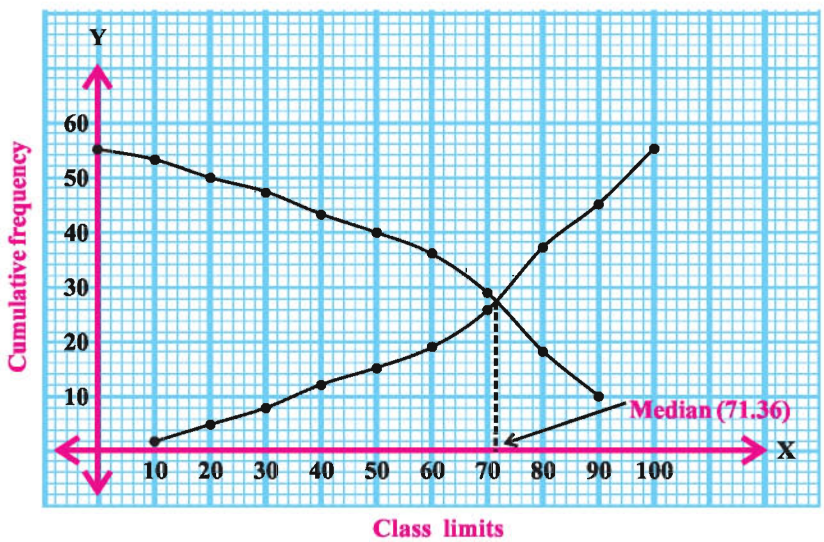


Figure 15.4

Example 11 : The annual income (in lakhs) of 30 officers in a factory gives rise to the following distribution :

Annual income (in lakh)	5-10	10-15	15-20	20-25	25-30	30-35	35-40
Number of officers	2	9	3	6	4	4	2

Draw both ogives for the data above. Hence obtain the median annual income.

Solution :

Annual Income	Number of officers (f)	Cumulative frequency (cf)
5–10	2	2
10–15	9	11
15–20	3	14
20–25	6	20
25–30	4	24
30–35	4	28
35–40	2	30

First we draw the coordinate axes, with lower limits along the X-axis and cumulative frequency along the Y-axis. Then we plot the points (10, 2), (15, 11), (20, 14), (25, 20), (30, 24), (35, 28), (40, 30) for 'less than' ogive and (5, 30), (10, 28), (15, 19), (20, 16), (25, 10), (30, 6), (35, 2), for 'more than' ogive as shown in figure 15.5.

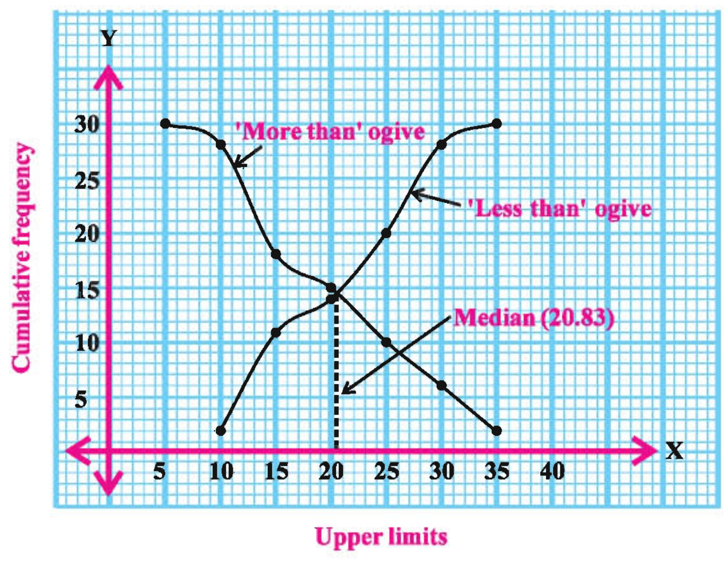


Figure 15.5

The x-coordinate of point of intersection is nearly 20.83, which is the median. It can also be verified by using the formula. Hence, the median annual income (in lakhs) is ₹ 20.83. (See figure 15.5)

EXERCISE 15

1. In a retail market, a fruit vendor was selling apples kept in packed boxes. These boxes contained varying number of apples. The following was the distribution of apples according to the number of boxes. Find the mean by the assumed mean number of apples kept in the box.

Number of apples	50-53	53-56	56-59	59-62	62-65
Number of boxes	20	150	115	95	20

2. **The daily expenditure of 50 hostel students are as follows :**

Daily expenditure (in ₹)	100-120	120-140	140-160	160-180	180-200
Number of students	12	14	8	6	10

Find the mean daily expenditure of the students of hostel using appropriate method.

3. **The mean of the following frequency distribution of 200 observations is 332. Find the value of x and y.**

Class	100-150	150-200	200-250	250-300	300-350	350-400	400-450	450-500	500-550
Frequency	4	8	x	42	50	y	32	6	4

4. **Find the mode of the following frequency distribution :**

Class	0-15	15-30	30-45	45-60	60-75	75-90	90-105
Frequency	8	16	23	57	33	23	13

5. **Find the mode of the following data :**

Class	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Frequency	12	17	28	23	7	8	5

6. **The mode of the following frequency distribution of 165 observations is 34.5. Find the value of a and b.**

Class	5-14	14-23	23-32	32-41	41-50	50-59	59-68
Frequency	5	11	a	53	b	16	10

7. **Find the mode of the following frequency distribution :**

Class	1500-2000	2000-2500	2500-3000	3000-3500	3500-4000	4000-4500	4500-5000
Frequency	14	56	60	86	74	62	48

8. **Find the median of the following frequency distribution :**

Class	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90
Frequency	9	11	15	24	19	9	8	5

9. **The median of the following data is 525. Find the value of x and y, if the sum of frequency is 100 :**

Class	0-100	100-200	200-300	300-400	400-500	500-600	600-700	700-800	800-900	900-1000
Frequency	3	4	x	12	17	20	9	y	8	3

10. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) For some data, if Z = 25 and $\overline{x}$ = 25, then M = ...

(a) 25 (b) 75 (c) 50 (d) 0

(2) For some data Z − M = 2.5. If the mean of the data is 20, then Z = ...

(a) 21.25 (b) 22.75 (c) 23.75 (d) 22.25

(3) If $\overline{x}$ − Z = 3 and $\overline{x}$ + Z = 45, then M = ...

(a) 24 (b) 22 (c) 26 (d) 23

(4) If Z = 24, $\overline{x}$ = 18, then M = ...

(a) 10 (b) 20 (c) 30 (d) 40

(5) If M = 15, $\overline{x}$ = 10, then Z = ...

(a) 15 (b) 20 (c) 25 (d) 30

(6) If M = 22, Z = 16, then $\overline{x}$ = ...

(a) 22 (b) 25 (c) 32 (d) 66

(7) If $\overline{x}$ = 21.44 and Z = 19.13, then M = ...

(a) 21.10 (b) 19.67 (c) 20.10 (d) 20.67

(8) If M = 26, $\overline{x}$ = 36, then Z = ...

(a) 6 (b) 5 (c) 4 (d) 3

(9) The modal class of the frequency distribution given below is ......

Class	0-10	10-20	20-30	30-40	40-50
Frequency	7	15	13	17	10

(a) 10-20 (b) 20-30 (c) 30-40 (d) 40-50

(10) The cumulative frequency of class 20-30 of the frequency distribution given in (9) is ......

(a) 25 (b) 35 (c) 15 (d) 40

(11) The median class of the frequency distribution given in (9) is ......

(a) 40-50 (b) 30-40 (c) 20-30 (d) 10-20

*

Summary

In this chapter we have studied the following points :

1. The mean of the grouped data can be obtained by

(i) the direct method : $\overline{x} = \dfrac{\Sigma f_i x_i}{\Sigma f_i}$

(ii) the assumed mean method : $\overline{x} = A + \dfrac{\Sigma f_i d_i}{\Sigma f_i}$

(iii) the step deviation method : $\overline{x} = A + \left(\dfrac{\Sigma f_i u_i}{\Sigma f_i}\right) \times c$

assuming that the frequency of a class is centered at its mid-point.

2. The mode for the grouped data can be obtained by using the formula :

$$Z = l + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right) \times c$$

where all symbols are in usual notations.

3. The cumulative frequency (cf) of a class is the frequency obtained by adding the frequencies of all the classes preceding the given class. The median for grouped data can be obtained by using the formula :

$$\text{Median (M)} = l + \left(\frac{\frac{n}{2} - cf}{f}\right) \times c$$

where all the symbols have their usual meaning.

Also $Z = 3M - 2\overline{x}$

4. Representing a cumulative frequency distribution graphically as a cumulative frequency curve or an ogive of 'less than type' and 'more than type' the median of the grouped data can be obtained graphically as the x-coordinate of the point of intersection of the two ogives.

Baudhayana, (fl. c. 800 BCE) was an Indian mathematician, who was most likely also a priest. He is noted as the author of the earliest Sulba Sutra—appendices to the Vedas giving rules for the construction of altars—called the Baudhayana Sulbasûtra, which contained several important mathematical results. He is older than the other famous mathematician Apastambha. He belongs to the Yajurveda school.

He is accredited with calculating the value of pi to some degree of precision, and with discovering what is now known as the Pythagorean theorem.

The sutras of Baudhayana :

The Shrautasutra

His shrauta sutras related to performing Vedic sacrifices has followers in some Smarta brahmanas (Iyers) and some Iyengars of Tamil Nadu, Kongu of Tamil nadu, Yajurvedis or Namboothiris of Kerala, Gurukkal brahmins, among others. The followers of this sutra follow a different method and do 24 Tila-tarpana, as Lord Krishna had done tarpana on the day before Amavasya; they call themselves Baudhayana Amavasya.

PROBABILITY 16

Pure mathematics is, in its way, the poetry of logical ideas.

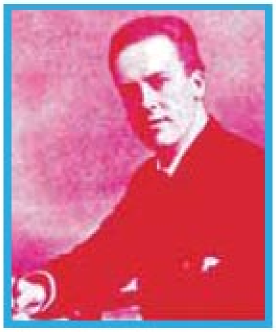
- Albert Einstein

*

The last thing one knows when writing a book is what to put first.

- Blaise Pascal

16.1 Introduction



Karl Pearson
(1857-1936)

In standard IX, we have studied about experimental (or empirical) probability of events which were based on the results of actual experiments. Let us discuss an experiment of tossing a coin 100 times in which the frequencies of the outcomes were 47 times Head and 53 times Tail. Based on this experiment, the empirical probability of getting a head is $\frac{47}{100} = 0.47$ and getting a tail is $\frac{53}{100} = 0.53$. Note that these probabilities are based on the result of an actual experiment of tossing a coin 100 times. For this reason, **these probability are called an experimental or empirical probability.** In fact, the experimental probability is based on the result of actual experiments and adequate recordings of the happening of the results. Moreover, these probabilties are only 'estimates'. If we perform the same experiment of tossing a coin 100 times again, then we may get different results which gives different probability estimates.

In standard IX, we had done activities of tossing a coin many times and noted the results of getting heads (or tails). We had noted that as the number of tosses of the coin increased, the experimental probability of getting a head (or tail) came closer and closer to $\frac{1}{2}$. Many persons from different parts of the world have done this kind of experiment and recorded the number of heads (or tails) turned up.

Statistician **Karl Pearson** had tossed the coin 24000 times and he got 12012 heads and thus, the experimental probability of getting head was $\frac{12012}{24000} = 0.5005$. In the eighteenth century French **de Buffon** tossed a coin 4040 times and he got 2048 heads. The experimental probability of getting head in this case was found to be $\frac{2048}{4040} = 0.507$.

Now, let us think, 'What is the empirical probability of a tail, if the experiment is carried out upto one lac times ? or 10 lacs time ? and so on ?' We will feel that as the number of tossing a coin increases, the experimental probability of getting tail (or head) seems to be around $\frac{1}{2} = 0.5$; which is what we call the **theoretical probability** of getting tail (or head). In this chapter we provide an introduction to the theoretical **(also called classical)** probability of an event and discuss simple problems based on this concept.

16.2 Probability - A Theoretical Approach

Let us start with an example.

Suppose a fair coin is tossed at random.

Note : When we say a fair coin or a balance die we mean, it is 'symmetrical' so that there is no reason for it to come down more often on one side than the other. We call this property of a coin or die as being 'unbaised'. Random toss means that the coin or die is allowed to fall freely without any bias or interference.

We know that when we toss a coin, then there are only two possible outcomes namely head or tail. We can reasonably assume that each outcome, head or tail, **is as likely to occur as the other.** We refer to this by saying that the outcomes, head and tail, are **equally likely**.

For another example of equally likely outcomes, suppose we toss two coins once. What are the possible outcomes ? They are HH, HT, TH, TT. Each outcome has the same probability of showing up. So, the equally likely outcomes of tossing two coins are HH, HT, TH and TT.

Now, the question arise in our mind that for every experiment are the outcomes equally likely ? Let us see.

Suppose that a bag contains 5 blue and 3 red marbles and we draw a marble without looking into a bag. What are the outcomes ? Are the outcomes a red marble and a blue marble equally likely ? Since there are 5 blue and 3 red marbles, we would agree that we are more likely to get a blue marble than a red marble. So the outcomes (i.e. blue or red marble) are not equally likely. However, the outcomes of drawing a marble of any colour from the bag is equally likely. So, all experiments do not necessarily have equally likely outcomes.

In this chapter, from now onwards, **we will assume that all the experiments have equally likely outcomes.**

We had defined in standard IX, the experimental or empirical probability P(E) of an event E as

$$P(E) = \frac{\text{Number of trials in which the event happened}}{\text{Total number of trials}}$$

We can extend this probability to every event associated with an experiment which can be repeated a large number of times. The repetition of an experiment has some limitations, as it may be very expensive or time consuming or unfeasible in many situations. Of course, it worked well in tossing a coin or throwing a die. But how about the repetition of the phenomenon of an earthquake, Tsunami or flood to compute the empirical probability of multistoreyed building getting destroyed in these phenomenons ?

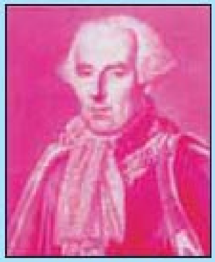
The assumption of equally likely outcomes is one such assumption that leads us to the following definition of probability of an event.

The theoretical probability (or classical probability) of an event E denoted by P(E), is defined as

$$P(E) = \frac{\text{Number of outcomes favourable to E}}{\text{Number of all possible outcomes of the experiment}}$$

where we assume that the outcomes (or results) of the experiment are **equally likely**.

"Probability theory had its origin in the 16th century when an Italian physician and mathematician J.Cardan wrote the first book on the subject, The Book on Games of Chance. Since its inception, the study of probability has attracted the attention of great mathematicians. James Bernoulli (1654 – 1705), A. de Moivre (1667 – 1754) and Pierre Simon Laplace are among those who made significant contributions to this field. Laplace's Theorie Analytique des Probabilités, 1812, is considered to be the greatest contribution by a single person to the theory of probability. In recent years, probability has been used extensively in many areas such as biology, economics, genetics, physics, sociology etc



Pierre Simon Laplace (1749 – 1827)

Let us find the probability of some of the events associated with experiments where the equally likely assumption holds.

Example 1 : Find the probability of getting number 1, 4 or 5 on a die when a fair die is thrown.

Solution : In this experiment of throwing a fair die, the possible outcomes are 1, 2, 3, 4, 5 and 6. Let E be the event that 'getting number 1' on the die. The number of outcomes favourable to E is only one. Therefore

$$P(E) = P(\text{getting } 1) = \frac{\text{Number of outcomes favourable to E}}{\text{Number of all possible outcomes}} = \frac{1}{6}$$

Similarly, let F be the event 'getting number 4', then $P(F) = \frac{1}{6}$ and G be the event 'getting number 5', then $P(G) = \frac{1}{6}$.

Example 2 : A fair coin is tossed twice. Find the probability of getting different the outcomes of this experiment.

Solution : If A fair coin is tossed twice, then the possible outcomes of this experiment are HH, HT, TH and TT.

Let A be the event 'getting two heads', then

$$P(A) = \frac{\text{Number of outcomes favourable to A}}{\text{Number of all possible outcomes}} = \frac{1}{4}$$

Let B be the event 'getting first head and then tail'.

Then $P(B) = P(\{HT\}) = \frac{1}{4}$

Let C be the event 'getting T first and then H'.

So, $P(C) = P(\{TH\}) = \frac{1}{4}$

Let D be the event 'getting both tails', then

$P(D) = P(\{TT\}) = \frac{1}{4}$

Note : (1) An event having only one outcome of an experiment is called **an elementary** or **primary event**. In example 2, all four events A, B, C and D are elementary events.

(2) Note that in example 2 : $P(A) + P(B) + P(C) + P(D) = 1$

Observe that **the sum of the probabilities of all the elementary events of an experiment is 1.** This is true in general also.

Example 3 : Suppose we throw a dice once : **(1)** What is the probability of getting a number greater than 3 ? **(2)** What is the probability of getting a number less than or equal to 3 ?

Solution : (1) Let A be the event "getting a number greater than 3". The number of possible outcomes of this experiment is six : 1, 2, 3, 4, 5 and 6. Therefore, the number of outcomes favourable to A is 3. (namely 4, 5 and 6)

So, P(A) = P(Number greater than 3) = $\frac{3}{6} = \frac{1}{2}$.

(2) Let B be the event "getting a number less than or equal to 3". Outcomes favourable to B are 1, 2, 3. So, the number of outcomes favourable to B is 3.

So, P(B) = P(Number getting less than or equal to 3) = $\frac{3}{6} = \frac{1}{2}$.

Note : Event A is the "getting the number greater than 3" and B is the event "getting the number less than or equal to 3". Remember that not getting the number greater than 3 is the same as getting number less than or equal to 3, and vice versa.

So, "event B" is 'not event A'. We denote the event 'not event A' as A' or $\overline{A}$.

In general, it is true that for an event A, **$P(A') = P(\overline{A}) = 1 - P(A)$.**

$\overline{A}$ is called the complement of the event A. We can say that A and $\overline{A}$ are **complementary** events.

Before proceeding further let us see the following :

Tossing a balance dice once, we have six outcomes namely 1, 2, 3, 4, 5 and 6. Now the question is, 'what is the probability of getting number 7 on the dice ?' Since no face of the die is marked with number 7, so there is no outcome favourable to 7, i.e. the number of outcome 7 is zero. In other words getting a number 7 on a dice is **impossible**.

So, P(getting number 7) = $\frac{0}{6} = 0$.

So, the probability of an **impossible event** is zero.

Again the another question arise in our mind that what is the probability of getting a natural number less than 7 on a dice which is thrown once ?

Here all the faces of a dice are marked with natural numbers less than 7. So, the number of favourable outcomes is the same as the number of all possible outcomes of the experiment, which is 6.

Therefore, P(E) = P(getting number less than 7) = $\frac{6}{6} = 1$

So, the probability of an event which is certain (or sure) to occur is 1. Such an event is called **certain event** or **sure event**.

Note : From the definition of probability we can say that **$0 \leq P(E) \leq 1$** for any event E.

Example 4 : A card is selected at random from well-shuffled pack of 52 cards. Find the probability that the selected card is

(i) a face card

(ii) of diamond

(iii) not an ace

(iv) is an ace of black colour

Solution : Here selection of one card from well-shuffled pack of 52 cards is equally likely outcome.

(i) There are 12 face cards (4 kings, 4 queens, 4 jackals). Let A be the event that 'the selected card is a face card'.

So, the number of outcomes favourable to the event A is 12.

$\therefore$ $P(A) = \frac{12}{52} = \frac{3}{13}$

(ii) There are 13 diamond cards. Let B be the event that 'the selected card is of diamond'.

Therefore, number of outcomes favourable to B is 13.

$\therefore$ $P(B) = \frac{13}{52} = \frac{1}{4}$

(iii) Let C be the event that 'the selected card is not an ace'. Then the event C' is 'the selected card is an ace', then the event C' has 4 elements.

$\therefore$ $P(C') = \frac{4}{52} = \frac{1}{13}$

But $P(C) = 1 - P(C')$

$= 1 - \frac{1}{13} = \frac{12}{13}$

(iv) Let D be the event that 'the selected card is an ace of black colour'. So the number of outcomes favourable to D is 2 (i.e. an ace of spade and club).

$\therefore$ $P(D) = \frac{2}{52} = \frac{1}{26}$

Example 5 : In a pack of 400 screws there are 120 defective screws. Find the probability that the selected screw is non-defective.

Solution : Here in a pack, there are 400 screws. Let A be the event that "the selected screw is non-defective". The number of outcomes favourable to A is 400 − 120 = 280.

$\therefore$ $P(A) = \frac{280}{400} = 0.7$

Example 6 : There are 5 red, 2 yellow and 3 white roses in a flowerpot. Select one rose from it at random. What is the probability that the selected rose is of **(i)** red, **(ii)** yellow **(iii)** not white colour.

Solution : Here there are total 5 + 2 + 3 = 10 roses in a flowerpot.

(i) Let A be the event that 'the selected rose is red'. The number of outcomes favourable to A is 5.

$\therefore$ $P(A) = \frac{5}{10} = \frac{1}{2}$

(ii) Let B be the event that 'the selected rose is yellow'. The number of outcomes favourable to B is 2.

$\therefore$ $P(B) = \frac{2}{10} = \frac{1}{5} = 0.2$

(iii) Let C be the event that 'the selected rose is not white'. The complement of event C (i.e. $\overline{C}$) is 'the selected rose is white'.

The number of outcomes favourable to $\overline{C}$ is 3.

$\therefore$ $P(\overline{C}) = \frac{3}{10} = 0.3$

But $P(C) = 1 - P(\overline{C}) = 1 - 0.3 = 0.7$

Example 7 : Two balance dice are thrown once. Write down all the possible outcomes of this experiment. What is the probability that the sum of numbers on two dice is

(i) 7 (ii) 11 (iii) more than 10

(iv) less than 2 (v) less than 13 (vi) a prime number

Solution : The possible outcomes of the event of throwing two balanced dice are (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6).

So, the total number of outcomes is 36.

(i) Let A be the event that the sum of numbers on two dice is 7. The favourable outcomes are (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1). The number of favourable outcomes of the event A is 6.

$\therefore \quad P(A) = \frac{6}{36} = \frac{1}{6}$

(ii) Let B be the event that "the sum of numbers on two dice is 11". The favourable outcomes are (5, 6) and (6, 5). The number of favourable outcomes is 2.

$\therefore \quad P(B) = \frac{2}{36} = \frac{1}{18}$

(iii) Let C be the event that "the sum of numbers on two dice is more than 10". The favourable outcomes are (5, 6), (6, 5) and (6, 6). The number of favourable outcomes of C is 3.

$\therefore \quad P(C) = \frac{3}{36} = \frac{1}{12}$

(iv) Let D be the event that "the sum of numbers on two dice is less than 2". There is no outcome in which the sum is less than 2. So, the number of outcomes favourable to D is 0.

$\therefore \quad P(D) = \frac{0}{36} = 0$ (i.e. it is an impossible event)

(v) Let F be the event in the sum of number on two dice is less than 13. Here the sum of number on two dice is less than 13 for all outcomes.

$\therefore$ Number of outcomes favourable to F is 36.

$\therefore \quad P(F) = \frac{36}{36} = 1$ (i.e. it is certain event)

(vi) Let E be the event that "the sum of numbers on two dice is a prime". Primes less than 12 are 2, 3, 5, 7 and 11. The favourable outcomes are (1, 1), (1, 2), (2, 1), (1, 4), (2, 3), (3, 2), (4, 1), (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1), (5, 6), (6, 5). The number of favourable outcomes of D is 15.

$\therefore \quad P(E) = \frac{15}{36} = \frac{5}{12}$.

Example 8 : A dice is thrown once. Find the probability of getting (i) a prime number (ii) a number lying between 2 and 5 and (iii) an even number.

Solution : When a dice is thrown once, we get the outcomes as 1, 2, 3, 4, 5 and 6.

(i) Let A be the event that getting the number on the dice is prime.

2, 3 and 5 are prime numbers. Therefore the number favourable to this event is 3.

$\therefore \quad P(A) = \frac{3}{6} = \frac{1}{2}$

(ii) Let B be the event of getting the number on the dice lying between 2 and 5. The numbers lying between 2 and 5 are 3 and 4. So, the number of favourable outcomes is 2.

$\therefore \quad P(B) = \frac{2}{6} = \frac{1}{3}$

(iii) Let C be the event that getting the number on the dice is even. Here 2, 4 and 6 are even numbers. So, number of favourable outcomes is 3.

$\therefore \quad P(C) = \frac{3}{6} = \frac{1}{2}$

Example 10 : Gopi buys a toy for his son, if it is non-defective. Shopkeeper takes out one toy at random from a box of 10 toys containing 3 defective toys and other good ones. Find the probability that (i) Gopi buy the toy, (ii) Gopi does not buy the toy.

Solution : Here there are 10 toys in the box, out of which 3 are defective, so 7 of them are non-defective toys.

(i) Let A be the event that Gopi buys the toy. This means that the toy is not defective. So, the number of favourable outcomes of this event is 7. So, $P(A) = \frac{7}{10} = 0.7$

(ii) Let B be the event that Gopi does not buy the toy. This means that the toy is defective. So, number of favourable outcomes is 3.

So, $P(B) = \frac{3}{10} = 0.3$.

Note that the event C is also taken as 'not A'.

$\therefore \quad P(B) = P(\overline{A}) = 1 - P(A) = 1 - 0.7 = 0.3$

EXERCISE 16

1. 15 defective ballpens are accidentally mixed with 135 good ones. It is not possible to just look at a ballpen and say whether it is defective or not. One ballpen is picked up at random from it. Find the probability that the ballpen selected is a good one.
2. A box contains 5 green, 8 yellow and 7 brown balls. One ball is taken out from a box at random. What is the probability that the ball taken out is (i) yellow ? (ii) brown ? (ii) neither green nor brown ? (iv) not brown ?
3. A bag contains orange flavoured candies only. Rahi takes out one candy without looking into the bag. What is the probability that she takes out (i) the orange flavoured candy ? (ii) a lemon flavoured candy ?
4. A box contain 100 cards marked with numbers 1 to 100. If one card is drawn from the box, find the probability that it bears (i) single digit number, (ii) two-digit numbers (iii) three-digit number (iv) a number divisible by 8 (v) a multiple of 9, (vi) a multiple of 5.
5. A carton consist of 100 trousers of which 73 are good, 12 have minor defects and 15 have major defects. Kanu, a trader, will only accept the trousers which are good, but Radha, another trader, will only reject the trousers which have major defects. One trouser is drawn at random from the carton. What is the probability that,

   (i) it is acceptable to Kanu ? (ii) it is acceptable to Radha ?

6. Marks obtained by 50 students from 100 are as follows

Marks	0-34	35-50	51-70	71-90	91-100
Number of student	8	9	14	11	8

Find the probability that a student get marks :

(i) below 34, (ii) between 71-90

(iii) more than 70 (iv) less than or equal to 50 (v) above 90.

7. Two fair dice are rolled simultaneously. Find the probability of the following events :

(1) A : getting the same number on both dice.

(2) B : the sum of the integers on two dice is more than 4 but less than 8.

(3) C : the product of numbers on two dice is divisible by 2.

(4) D : the sum of numbers on two dice is greater than 12.

8. A coin is tossed three times. Find the probability of the following events :

(1) A : getting at least two heads,

(2) B : getting exactly two heads,

(3) C : getting at most one head,

(4) D : getting more heads than tails.

9. A game of chance consists of spinning an arrow which comes to rest pointing at one of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 (see figure 16.1) and there are equally likely outcomes. What is the probability that it will point at

(1) 7 ?

(2) a number greater than 9 ?

(3) an odd number ?

(4) an even number ?

(5) a number less than 5 ?

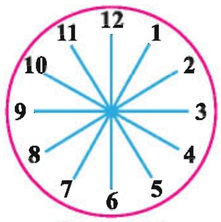


Figure 16.1

10. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) The sum of probability of event A and the probability of an event $\overline{A}$ (not A) is .... ☐

(a) 0 (b) 1 (c) 0.5 (d) 0.4

(2) The probability of the certain event is .... ☐

(a) 0 (b) 0.5 (c) 0.7 (d) 1

(3) The probability of the impossible event is .... ☐

(a) 0 (b) 0.5 (c) 0.6 (d) 1

(4) The probability of an event is greater than or equal to .... ☐

(a) 1 (b) 1.2 (c) 0.2 (d) 0

(5) The probability of an event is less than or equal to .... ☐

(a) −1 (b) −0.1 (c) −0.5 (d) 1

(6) If P(A) = 0.35, then P($\overline{A}$) = ... ☐

(a) 0 (b) 0.35 (c) 0.65 (d) 1

(7) If P($\overline{E}$) = 0.47, then P(E) = ... ☐

(a) 0 (b) 0.20 (c) 0.50 (d) 0.53

(8) The probability that you will get 101 marks in the paper which is in your hand is .... ☐

(a) 1 (b) 0.5 (c) 0 (d) −0.5

(9) The probability of the event "The Sun rises in West" is .... ☐

(a) 1 (b) 0.5 (c) 0 (d) −0.5

(10) The sum of the probabilities of all the elementary events of an experiment is .... ☐

(a) 0 (b) 0.2 (c) 1 (d) 0.8

*

Summary

In this chapter we have studied the following points :

1. The difference between experimental probability and theoretical probability.

2. The theoretical (or classical) probability of an event E, denoted by P(E), is defined as

$$P(E) = \frac{\text{Number of outcomes favourable to E}}{\text{Number of all possible outcomes of the experiment}}$$

3. The probability of sure (certain) event is 1.

4. The probability of an impossible event is 0.

5. The probability of an event E satisfies $0 \le P(E) \le 1$.

6. An event having only one outcome is called an elementary (or primary) event. The sum of the probabilities of all the elementary events of an experiment is 1.

7. For any event A, $P(A) + P(\overline{A}) = 1$ where $\overline{A}$ stands for 'not A'. A and $\overline{A}$ are called complementary events.

◆

The mathematics in Sulbasutra

Pythagorean theorem :

The most notable of the rules (the Sulbasutra-s do not contain any proofs of the rules which they describe, since they are sutra-s, formulae, concise) in the Baudhayana Sulba Sutra says:

dirghasyaksanaya rajjuh parsvamani, tiryadam mani,
cha yatprthagbhute kurutastadubhayan karoti.

A rope stretched along the length of the diagonal produces an area which the vertical and horizontal sides make together.

This appears to be referring to a rectangle, although some interpretations consider this to refer to a square. In either case, it states that the square of the hypotenuse equals the sum of the squares of the sides. If restricted to right-angled isosceles triangles, however, it would constitute a less general claim, but the text seems to be quite open to unequal sides.

If this refers to a rectangle, it is the earliest recorded statement of the Pythagorean theorem.

Baudhayana also provides a non-axiomatic demonstration using a rope measure of the reduced form of the Pythagorean theorem for an isosceles right triangle:

The cord which is stretched across a square produces an area double the size of the original square.

ANSWERS

(Answers to questions involving some calculations only are given.)

Exercise 1.2

1. (1) 1 (2) 5 (3) 1 **4.** 575

Exercise 1.3

1. (1) $7^2 \times 11 \times 13$ (2) $2^2 \times 3 \times 5^4$ (3) $3 \times 7 \times 13 \times 37$ (4) $2 \times 11 \times 701$

2. (1) 2, 42000 (2) 25, 4000 (3) 5, 6525 (4) 7, 25025

3. (1) 1, 105 (2) 20, 240 (3) 7, 3822 **5.** 1365 **8.** 210 *min*

Exercise 1.4

1. (1) Teminating, 0.0192 (2) Teminating, 0.00544 (3) Teminating, 0.00208
(4) Teminating, 0.000896 (5) Teminating, 0.094 (6) Teminating, 0.005625
(7) Non-terminating (8) Teminating, 0.4 (9) Non-terminating
(10) Teminating, 0.0390625

2. (1) Irrational (2) Rational, $\frac{3456789120}{999999999}$ (3) Rational, $\frac{5123456789}{1000000000}$
(4) Rational, $\frac{1}{11}$ (5) Rational, $\frac{763}{330}$ (6) Rational, $\frac{1}{7}$
(7) Rational, 1 (8) Rational, $\frac{5781}{1000}$ (9) Rational, $\frac{289}{125}$
(10) Rational, $\frac{2469}{20000}$

Exercise 1.5

1. (1) $\sqrt{3} + \sqrt{2}$ (2) $\sqrt{7} + \sqrt{2}$ (3) $\frac{\sqrt{6} - \sqrt{2}}{2}$ (4) $\frac{\sqrt{2a + 2b} + \sqrt{2a - 2b}}{2}$ (5) $2 + \sqrt{3}$
(6) $2 + \sqrt{2}$ (7) $\frac{\sqrt{14} + \sqrt{6}}{2}$ (8) $\frac{3\sqrt{2} - \sqrt{14}}{2}$ **2.** $\sqrt{5}$

Exercise 1

1. 5, 175 **2.** 5, 2625 **3.** 44, 660 **4.** 625, 3125 **5.** 5, 109375 **6.** 5, 525
7. 2, 144 **8.** 4, 144 **9.** 7, 1260 **10.** 16, 8064
21. Denominator is 5^3, so it is terminating
22. Denominator is 5^4, so it is terminating
23. & 24. Denominator is not in the form of $2^m \cdot 5^n$, so it is non-terminating
25. Denominator is 2^4, so it is terminating
26. $\sqrt{7} + \sqrt{5}$ **27.** $\sqrt{7} + 1$ **28.** $\frac{\sqrt{15} + \sqrt{3}}{3}$ **29.** $3 + \sqrt{5}$ **30.** $\frac{\sqrt{2n + 2} + \sqrt{2n - 2}}{2}$ **31.** 2
32. 3 **33.** 45 **34.** 7 **35.** 5 *cm* **36.** 30 *ltr* **37.** 1140 **38.** 100080

41. 720 *cm* **42.** 24240

43. (1) c (2) d (3) c (4) a (5) d (6) a (7) b (8) c (9) a (10) c
(11) b (12) b (13) a (14) a (15) a (16) b (17) c (18) c (19) b (20) b
(21) a (22) a (23) d (24) c (25) c

Exercise 2.1

1. (1) Quadratic polynomial (2) Cubic polynomial (3) Quadratic polynomial (4) Cubic polynomial

2. (1) 4 (2) 3 (3) 1 (4) 2

3. (1) 10 (2) -5 (3) -125 (4) -1

4. (1) 9, 21 (2) 4, 20 (3) 8, 20 (4) 47, 27

5. (1), (3) and (4) statements are valid and (2) is invalid.

6. (1) $(x-1)^2(x+1)$ (2) $(x+1)(5x+6)$ (3) $(x-3)(x^2+9)$ (4) $(x+1)(x^2+x+2)$

Exercise 2.2

1. (1) 2 (2) 0 (3) 1 (4) 3 **2.** No. of zeros : 1, Zero : -1 **3.** Zero do not exist

4. (1) 1 (2) 0 (3) 3 (4) 2 (5) 4 (6) 3 **5.** No. of zeros : 2, Zeros : -2, 2

Exercise 2.3

2. (1) -7, 3 (2) $\frac{5}{6}$, 1 (3) -1, $-\frac{5}{4}$ (4) $-\frac{8}{3}$, 1 (5) -9, 9 (6) 3, -2

3. -1, $\frac{4}{3}$ are the zeros. Sum of zeros : $\frac{1}{3}$, Product of zeros $-\frac{4}{3}$

4. (1) $k[x^2-2x-3]$, $k \neq 0$ (2) $k[x^2+3x-4]$, $k \neq 0$ (3) $k\left[\frac{1}{2}(6x^2-2x+3)\right]$, $r \neq 0$

5. (1) $6x^2+17x+11$ (2) x^2-x^2-x+1 (3) $5x^2+7x+2$
(4) x^3-3x^2-x+3 (5) $3x^3-5x^2-11x-3$

Exercise 2.4

2. (1) quotient : x^2-5x+4, remainder : 0 (2) quotient : $\frac{2}{3}x+1$, remainder : 0
(3) quotient : $5x+7$, remainder : 0 (4) quotient : $x+2$, remainder : 0
(5) quotient polynomial : x^2+4x+6, remainder polynomial : $3-3x$

2. 9 **3.** $a = 9$ **4.** $2x^2+5x+3$ **5.** $2x^4+7x^3+14x^2+9x+2$ **6.** 0

7. Each student gets $x^2-4x+14$ pens and no. of pens left undistributed is $9x-13$

8. $4x^2+2x-3$ **9.** -4, $\frac{1}{2}$

Exercise 2

1. (1) False (2) False (3) True (4) True (5) False

2. No. of zeros : 2, Zeros : -6 -3

3. $-\frac{5}{2}$; $-\frac{1}{2}$ are the zeros of $p(x)$, Sum of the zeros : -3, Product of zeros : $\frac{5}{4}$

4. $k(x^2+4x+9)$, $k \neq 0$ **5.** Quotient polynomial : $x+5$, Remainder polynomial : -26 **6.** $2x+1$

7. -2, 1 **8.** $3x^2+8x+5$ **9.** -5, 7

10. (1) b (2) d (3) c (4) d (5) a (6) c (7) d (8) b (9) d

Exercise 3.1

1. $x - 7y + 30 = 0;\ x - 3y - 10 = 0$
2. $x + y = 150;\ x - 2y = 0$
3. $2x - y = 0;\ x + y = 135$
4. $x - 3y + 5 = 0;\ x + y = 55$
5. $x + y = 85;\ x - 4y = 0$
6. $x - 3y = 0;\ x + y = 200$
7. $x - 2y = 0;\ x + y = 1$

Exercise 3.2

1. (1) (3, 2) (2) $\emptyset$ (3) (1, 1) (4) coincident lines, infinitely many solutions (5) (4, 3)
2. $(1, 1), \left(-\frac{1}{2}, 0\right), \left(\frac{5}{3}, 0\right)$ **3.** Number of boys = 5; Number of girls = 10
4. Infinitely many solutions

Exercise 3.3

1. (1) (2, 5) (2) (3, 9) (3) $\emptyset$ (4) infinitely many solutions (5) (2, 3)
2. $m = \frac{3}{14}$; $(x, y) = (42, 14)$ **3.** $\frac{5}{7}$
4. Age of the father : 40 years; Age of the son : 10 years
5. Cost of a ticket from Ahmedabad to Anand is ₹ 60,
   Cost of a ticket from Ahmedabad to Vadodara is ₹ 80.

Exercise 3.4

1. (1) $\left(\frac{9}{13}, -\frac{5}{13}\right)$ (2) $\left(-\frac{2}{5}, \frac{3}{5}\right)$ (3) (a, b) (4) $\left(\frac{2}{a}, \frac{3}{b}\right)$ **2.** Numbers are 20 and 15
3. Number of coins of 25 paise = 80,
   Number of coins of 50 paise = 60
4. 21 **5.** length = 40 *cm*; breadth = 20 *cm*; area = 800 cm^2
6. ₹ 2500 at 8 % per annum; ₹ 5000 at 6 % per annum

Exercise 3.5

1. (1) (3, 4) (2) (2, 1) (3) (3, 0) (4) (3, −4)
2. 48 **3.** 38, 32 **4.** 180 **5.** $m\angle A = 60$, $m\angle B = 90$, $m\angle C = 30$; Right angled triangle

Exercise 3.6

1. (1) $\left(\frac{1}{2}, \frac{1}{3}\right)$ (2) (0, 0), (3, 2) (3) $\left(\frac{7}{3}, -4\right)$ (4) (1, 1) (5) (9, 16)
2. 18 days taken by a man to finish embroidary work and
   36 days taken by a woman to finish embroidary work
3. Speed of the boat in still water is 6 *km/hr* and the speed of the stream is 3 *km/hr*
4. $\left(\frac{1}{2}, 2\right)$ **5.** Speed of the train is 25 *km/hr* and speed of the bus is 50 *km/hr*

Exercise 3

1. $x - 7y = 0$, $2x + 5y = 570$ **2.** {(3, 1)} **3.** (2, 1) **4.** (a, b) **5.** (3, 5)
6. The numbers are 30 and 24

7. Length of the rectangle is 15 units and breadth of the rectangle is 10 units.

8. Cost of food per day is ₹ 80; fixed charge per day ₹ 200. **9.** $\frac{4}{7}$

10. (1) c (2) c (3) c (4) b (5) c (6) d (7) c (8) b (9) c (10) c
(11) c (12) b

Exercise 4.1

1. (1), (2), (3), (4), (6) and (7) are quadratic equations; (5) is not a quadratic equation.

2. (1), (3), (4) Yes; (2) No.

3. (3) 5 (4) -2 **4.** (1) $-\frac{4}{3}, \frac{4}{3}$ (2) 3, 11 (3) $-\frac{3}{2}, -\frac{2}{3}$ (4) $\frac{1}{15}, 1$ (5) $\sqrt{5}, -\frac{1}{\sqrt{5}}$
(6) $\frac{3}{2}, \frac{2}{3}$

Exercise 4.2

1. (1) 25, real, rational, distinct (2) 1, real, distinct (3) 1 real, rational, distinct
(4) -12, no real roots (5) -3, no real roots (6) 147, real, distinct

3. (1) 2, $\frac{2}{9}$ (2) 0, 3

5. (1) $-5-\sqrt{19}, -5+\sqrt{19}$ (2) $\frac{-5+\sqrt{29}}{2}, \frac{-5-\sqrt{29}}{2}$ (3) $\frac{3+\sqrt{17}}{2}, \frac{3-\sqrt{17}}{2}$ (4) $\sqrt{6}, 2\sqrt{6}$
(5) $\frac{-5\sqrt{2}+\sqrt{26}}{6}, \frac{-5\sqrt{2}-\sqrt{26}}{6}$ (6) -3, 3

Exercise 4.3

1. 13, 14 **2.** 13, 14 **3.** D = $-48 < 0$, So, Statements are incorrect. **4.** 40

5. 100, 80 **6.** 5 *km/hour* **7.** 10,800 m^2 **8.** 6

Exercise 4

1. (1) $-2\sqrt{3}, 2\sqrt{3}$ (2) -5, 12 (3) 7, 8 (4) $-\frac{5}{2}, \frac{5}{2}$ (5) 2, 3

2. (1) $12-4\sqrt{10}, 12+4\sqrt{10}$ (2) $-4, \frac{5}{3}$ (3) 5 (4) -20, 15 (5) -18, 13

3. 13, 7 **4.** 40 *km/hour* **5.** 25 *km/hour* **6.** 15 *km/hour* **7.** $\frac{5}{3}$

8. 80 *km/hour*, 100 *km/hour* **9.** 23 years **10.** 38 years **11.** 23 years **12.** 25 **13.** 30

14. 11, 13 **15.** 14, 16 **16.** 24 **17.** 30 *Rs/kg* **18.** 60 *Rs/ltr* **19.** ₹ 60, 60 % **20.** ₹ 40

21. 54 cm^2 **22.** 36 *cm*

23. (1) b (2) a (3) d (4) c (5) a (6) c (7) d (8) b

Exercise 5.1

1. (1) 3, 5, 7, 9, 11,... $T_n = 2n + 1$ (2) $-3, -5, -7, -9, -11$,... $T_n = -2n - 1$
(3) 100, 93, 86, 79, 72,... $T_n = -7n + 107$ (4) $-100, -93, -86, -79, -72$,..., $T_n = 7n - 107$
(5) 1000, 900, 800, 700, 600,... $T_n = -100n + 1100$

2. (1) Not an A.P. (2) Not an A.P. (3) Not an A.P. (4) A.P., $T_n = 10n - 5$
(5) A.P., $T_n = 5n + 12$ (6) A.P., $T_n = -2n + 103$ (7) A.P., $T_n = -3n + 204$
(8) A.P., $T_n = 5n$ (9) Not an A.P.

3. (1) $T_n = 5n - 3$ (2) $T_n = 205 - 5n$ (3) $T_n = 1100 - 100n$ (4) $T_n = 50n$

(5) $T_n = n - \frac{1}{2}$ (6) $T_n = n + 0.1$ (7) $T_n = 1.1n + 0.1$ (8) $T_n = \frac{2}{3}n + 1$

4. (1) $-60, -48, -36, \ldots, T_n = 12n - 72$ (2) $2, 1, 0, -1, -2, \ldots, T_n = 3 - n$

5. (1) $-2, 3, 8, 13, 18, \ldots, T_n = 5n - 7$ (2) $-3, 2, 7, 12, 17, \ldots, T_n = 5n - 8$

6. No term = 0, $T_n = 0$ does not yeild $n \in N$ (increasing and positive)

7. Yes, $T_{50} = 5$ **8.** 89th term **9.** 273 **10.** 930 **11.** −76

Exercise 5.2

1. (1) 800 (2) 1020 (3) −360 (4) 76 (5) 159

2. (1) 15150 (2) 1050 (3) 1090 (4) −100 (5) −260

3. (1) 100 (2) 1365 (3) $S_n = n^2$, $S_{10} = 100$ (4) $T_n = 2n + 21$, $S_n = n^2 + 22n$

(5) 1 (6) 24

4. 20 **5.** 20 **6.** $n = 19$, $S_n = 950$ **7.** $a = 3$, $d = -4$, $T_n = 7 - 4n$ **8.** 165150

9. 10605 **10.** 32th term, 1888

Exercise 5

1. $S_n = 3n^2 + 8n$ **2.** $T_n = 2n + 1$ **3.** No. of terms = 5 or 16, $T_5 = 18$, $T_6 = -15$ **4.** 35

5. 82350 **6.** $\frac{2m-1}{2n-1}$ **8.** $\frac{105}{94}$, $\frac{16m-7}{14m-4}$ **9.** 0, 6, 12 or 12, 6, 0 **10.** 780 m, 20 potatoes

11. 5.5 m **12.** 19 **13.** 12 **14.** $n = 6$, $d = 5$

15. (1) d (2) b (3) a (4) b (5) c (6) b (7) a (8) b (9) a (10) d

(11) c (12) a (13) b (14) b (15) a

Exercise 6.1

1. $\angle D \cong \angle Z$, $\angle E \cong \angle X$, $\angle F \cong \angle Y$ and $\frac{DE}{ZX} = \frac{EF}{XY} = \frac{FD}{ZY}$

2. 40 **3.** $7\frac{1}{2}$, 12 **4.** 4 **5.** 12.6, 5 **6.** $\frac{3}{4}$, $\frac{3}{4}$

8. $7\frac{1}{2}$, $3\frac{3}{4}$ **9.** (1) False (2) True (3) False (4) True (5) True **10.** (1) d (2) b (3) b (4) a

Exercise 6.2

1. (1) AB = 6, AE = 2.7, AC = 4.5 (2) AD = 1.55, DB = 4.65, AE = 1.05

(3) AD = 9.6, AB = 21.6, AC = 14.4 (4) DB = 11.2, EC = 8.4, AC = 13.8

(5) AD = 3.4, AB = 6.8, AE = 2.55

2. 4.8 **3.** (1) 5.2 (2) 5.7, 9.5

Exercise 6.3

7. (1) True (2) False (3) True (4) False (5) True

Exercise 6.4

1. 8.64 **4.** $48\sqrt{3}$ **8.** 40, 160

Exercise 6

3. Yes, two similar triangles can have the same area. If triangles are congruent then they are similar and they have the same area.

5. (1) True (2) False (3) True (4) False (5) False

6. (1) 30.7 (2) RPQ (3) 12 (4) 5 : 4 (5) 16 (6) 33.6 (7) 30 (8) 200 (9) 9 (10) CDB (11) 13.75

7. (1) c (2) b (3) b (4) b (5) a (6) b (7) d (8) c (9) d (10) b (11) d (12) c (13) b (14) b

Exercise 7.1

2. $4\frac{8}{13}$ **3.** $4\sqrt{10}, 4\sqrt{15}, 4\sqrt{6}$ **4.** 25 **5.** 2.5 **6.** (1) 16 (2) $3\sqrt{34}$ or $5\sqrt{34}$ (3) 26 (4) 1.5

7. $\frac{x^2}{\sqrt{x^2+y^2}}, \frac{xy}{\sqrt{x^2+y^2}}, \frac{y^2}{\sqrt{x^2+y^2}}$ **9.** 12 **14.** $\frac{1}{2}a^2$

Exercise 7.2

1. 120 **2.** 56 **3.** 2.5 m **4.** $4\sqrt{3}$ **6.** 8.5 **7.** 10 **8.** 62 **9.** 25 **10.** $2\sqrt{3}$

Exercise 7

1. 15 **6.** 17 m

9. (1) b (2) d (3) c (4) c (5) a (6) c (7) b (8) d (9) c (10) a (11) b (12) b (13) c (14) d (15) d (16) b (17) a (18) d

Exercise 8.1

1. (1) 13 (2) 5 (3) $2\sqrt{a^2+b^2}$ **2.** (16, 0), (6, 0) **4.** $P\left(0, -\frac{2}{3}\right)$ **5.** $3x + y + 1 = 0$

6. 18 or 2 **7.** 7 **8.** $(0, 3\sqrt{3})$ or $(0, -3\sqrt{3})$ **9.** (1) -5 (2) 5 (3) 0 or 3

Exercise 8.2

1. (2, 4) **2.** 1 : 2, 3 **3.** (1, 2), (2, 4), (3, 6) **4.** (2, −3) **5.** (−3, 2), (5, 0), (1, 4)

7. (−2, −11) **8.** (−3, 3), (1, 1) **9.** 1 : 3 **10.** (5, 4) **11.** (4, −1), (0, 3), (−2, 1)

Exercise 8.3

1. 25 **2.** 7 or −3 **3.** 2 **4.** $\frac{9}{2}$ **5.** 5 or 15

Exercise 8

1. $\frac{15}{2}$, 3 **2.** (1, 6) **3.** A–C–B, 1 : 2

4. If $m\angle A = 90$, $k = 3$. If $m\angle B = 90$, $k = 2$, $m\angle C \neq 90$. **5.** $\left(\frac{9}{7}, \frac{16}{7}\right)$

6. Area of $\Delta ABC = 16$, Area of $\Delta DEF = 4$ **8.** (0, −1)

10. (1) c (2) a (3) d (4) b (5) d (6) a (7) c (8) b

Exercise 9.1

1. $\frac{5}{13}, \frac{12}{13}, \frac{12}{5}, \frac{5}{13}, \frac{12}{13}$ **2.** $sinA = \frac{3}{5}, cosA = \frac{4}{5}, tanA = \frac{3}{4}, cosecA = \frac{5}{3}, secA = \frac{5}{4}, cotA = \frac{4}{3}$

3. $\frac{3}{5}, \frac{3}{4}$ **4.** $\frac{5}{12}, \frac{12}{13}$ **5.** $sinB = \frac{2\sqrt{2}}{3}, tanB = 2\sqrt{2}, cosecB = \frac{3}{2\sqrt{2}}, secB = 3, cotB = \frac{1}{2\sqrt{2}}$

6. $\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{3}}$ **7.** $\frac{26}{9}$ **8.** $\frac{-39}{23}$ **12.** $\frac{12}{13}, \frac{5}{13}, \frac{12}{5}$ **13.** (1) Yes (2) Yes (3) Yes

Exercise 9.2

2. (1) $\frac{43-24\sqrt{3}}{11}$ (2) $\frac{67}{12}$ (3) $\frac{\sqrt{3}-2}{2}$ (4) $\frac{67}{12}$

3. (1) $m\angle A = 45, BC = 5, AC = 5\sqrt{2}$ (2) $m\angle C = 60, BC = 5, AB = 5\sqrt{3}$

(3) $m\angle C = 30, m\angle A = 60, AB = 3\sqrt{2}$ (4) $m\angle A = 45, m\angle C = 45, AC = 4\sqrt{2}$

4. $BC = 20\sqrt{3}, AC = 40, BD = 40$ **5.** $\frac{7}{2}$ **8.** $sin15 = \frac{\sqrt{6}-\sqrt{2}}{4}, cos15 = \frac{\sqrt{6}+\sqrt{2}}{4}$

Exercise 9.3

1. (1) 1 (2) 0 (3) 0 (4) 2 (5) 0 (6) 1 (7) 2 (8) $\frac{1}{\sqrt{3}}$ (9) 2 (10) $\frac{6-\sqrt{3}}{3}$

3. (1) $cos5 + sec5$ (2) $sin1 + sec3$ (3) $cosec9 + sec36$

6. 29 **7.** 5 **9.** 44 **10.** $\frac{1}{2}$

Exercise 9

23. $sec\theta = \frac{p^2+1}{2p}, tan\theta = \frac{p^2-1}{2p}, sin\theta = \frac{p^2-1}{p^2+1}$ **24.** (1) 0 (2) $2\sqrt{3}$

25. $\sqrt{2} + 1$ **26.** $\frac{8}{7}$ **27.** $\frac{225}{64}$ **28.** $\frac{a}{\sqrt{a^2+b^2}}, \frac{a}{b}$

29. (1) b (2) b (3) a (4) a (5) b (6) c (7) a (8) c (9) a (10) c
(11) c (12) a (13) b (14) b (15) c (16) c (17) a (18) d (19) c (20) b

Exercise 10

1. 51.9 *m* **2.** 86.5 *m* **3.** 20 *m* **4.** 51.9 *m* **5.** 3.46 *m* **6.** 34.6 *m* **7.** 230.6 *m*
8. 17.3 *m*, 10 *m* **9.** 9 *m* **10.** 73 *m* **11.** 200 *m* **12.** 120 *m* **13.** 94.6 *m*
14. 25 *m*, 43.25 *m* **16.** 16.38 *min* **18.** (1) 34.6 *m* (2) 40 *m* (3) 20 *m*
20. 180 *m* **21.** 2 *m*

22. (1) c (2) c (3) c (4) b (5) a (6) b (7) a (8) b (9) b (10) c
(11) a (12) c (13) c (14) d

Exercise 11.1

4. $2\sqrt{r_1^2 - r_2^2}$ **5.** 80

Exercise 11.2

1. (1) 13 (2) 35 (3) 7 (4) 10 **2.** 96 **3.** 16 **4.** 60 **9.** $\frac{20}{3}$ **10.** $\frac{120}{13}$

Exercise 11

3. 3 **5.** 15 **7.** 3 **9.** 7

10. (1) a (2) b (3) d (4) b (5) a (6) d (7) a

Exercise 13.1

1. 52.8 *cm*, 221.76 cm^2 **2.** 22 *m* **3.** 7 *m* **4.** 66 *km/hr*

Exercise 13.2

1. 44 *cm*, 462 cm^2 **2.** ₹ 10,908.33 **3.** 54.48 cm^2 **4.** 26.17 cm^2 **5.** ₹ 3465

6. 28.26 m^2, 21.98 m^2 **7.** 161.07 cm^2 **8.** 14.25 cm^2, 142.75 cm^2

Exercise 13.3

1. 72.67 cm^2 **2.** 6246 m^2 **3.** ₹ 6192 **4.** 4456 cm^2 **5.** 55.125 cm^2

Exercise 13

1. ₹ 550 **2.** 31.68 *km* **3.** 114 cm^2 **4.** 59.10 cm^2 **5.** 1400 m^2 **6.** 228 cm^2

7. 1658.25 cm^2 **8.** 1398.5 cm^2 **9.** 38.88 cm^2 **10.** 11.44 cm^2

11. (1) d (2) a (3) b (4) a (5) d (6) c (7) b (8) c (9) b (10) c
(11) b (12) c (13) a

Exercise 14.1

1. 361.42 cm^2 **2.** 315.25 cm^2 **3.** ₹ 8200.50 **4.** 62800 cm^2 **5.** 19800 cm^2

6. 602.88 *m*, ₹ 12057.6 **7.** 12.5 *m* **8.** 4.5 *m*

Exercise 14.2

1. 1232 cm^3 **2.** 16500 cm^3 **3.** 18480 *litres* **4.** 38808 cm^3 **5.** 185.82 cm^3

6. 21.66 cm^3 **7.** 1.45 *litres* **8.** 6 *m* **9.** 6.3 *m* **10.** 0.5 *litre* **11.** 3215.36 cm^3

12. 47124 cm^3 **13.** 1725.47 cm^3 **14.** 4186.67 cm^3

Exercise 14.3

1. 48 **2.** 120 **3.** 3300 **4.** 240 **5.** 180 *m* **6.** 450 *cm* **7.** 42 *cm*, 53.41 *cm*

Exercise 14.4

1. 15072 cm^2, 163.28 *litres* **2.** 97.34 *litres*, ₹ 2920.20, for tin ₹ 4867

Exercise 14

1. 66 m^2, ₹ 66000 **2.** 6.5 *cm* **3.** 2541 **4.** 48 **5.** 68.96 *minutes* **6.** 153.4 cm^2

7. 800.7 cm^2 **8.** 15 *cm*, ₹ 17097

9. (1) b (2) a (3) c (4) a (5) a (6) c (7) d (8) a (9) c

Exercise 15.1

1. 148.5 2. 332 3. 25.857 4. 66.346 5. 18.675 6. 580.33 7. 350
8. $f_1 = 30, f_2 = 8$ 9. 41.71 10. $f_1 = 9, f_2 = 20$

Exercise 15.2

1. 21.45 2. 256.25 3. 606.76 4. 33.49 5. 94.29 6. 95.69

Exercise 15.3

1. 15.5 2. 13.67 3. 288.09 4. 12.55 5. $a = 150,\ b = 48$ 6. $a = 34,\ b = 46$
7. 36.25

Exercise 15

1. 57.0875 2. 145.2 3. $x = 14,\ y = 40$ 4. 53.799 5. 56.875 6. $a = 43,\ b = 27$
7. 3342.1 8. 46.25 9. $x = 9,\ y = 15$
10. (1) a (2) c (3) d (4) b (5) c (6) b (7) d (8) a (9) c (10) b
(11) c

Exercise 16

1. 0.9 2. (i) $\frac{2}{5}$ (ii) $\frac{7}{20}$ (iii) $\frac{2}{5}$ (iv) $\frac{13}{20}$ 3. (i) 1 (ii) 0
4. (i) 0.09 (ii) 0.9 (iii) 0.01 (iv) 0.14 (v) 0.11 (vi) 0.2
5. (i) 0.73 (ii) 0.85 6. (i) $\frac{4}{25}$ (ii) $\frac{11}{50}$ (iii) $\frac{19}{50}$ (iv) $\frac{17}{50}$ (v) $\frac{4}{25}$ 7. (1) $\frac{1}{6}$ (2) $\frac{5}{12}$ (3) $\frac{3}{4}$ (4) 0
8. (1) $\frac{1}{2}$ (2) $\frac{3}{8}$ (3) $\frac{1}{2}$ (4) $\frac{1}{2}$ 9. (1) $\frac{1}{12}$ (2) $\frac{1}{4}$ (3) $\frac{1}{2}$ (4) $\frac{1}{2}$ (5) $\frac{1}{3}$
10. (1) b (2) d (3) a (4) d (5) d (6) c (7) d (8) c (9) c (10) c

Circling the Square

Another problem tackled by Baudhayana is that of finding a circle whose area is the same as that of a square (the reverse of squaring the circle). His sutra i.58 gives this construction:

Draw half its diagonal about the centre towards the East-West line; then describe a circle together with a third part of that which lies outside the square.

Explanation :

- Draw the half-diagonal of the square, which is larger than the half-side by $x = \frac{a}{2}\sqrt{2} - \frac{a}{2}$.
- Then draw a circle with radius $\frac{a}{2} + \frac{x}{3}$ or $\frac{a}{2} + \frac{a}{6}(\sqrt{2} - 1)$, which equals $\frac{a}{6}(2 + \sqrt{2})$.
- Now $(2 + \sqrt{2})^2 \approx 11.66 \approx \frac{36.6}{\pi}$ so the area $\pi r^2 \approx \frac{a^2}{6^2} \times \frac{36.6}{\pi} \approx a^2$.

TERMINOLOGY

(In Gujarati)

Algbraic Irrational Number	બૈજિક અસંમેય સંખ્યા
Algorithm	પ્રવિધિ
Angle of Depression	અવસેધકોણ
Angle of Elevation	ઉત્સેધકોણ
Arithmetic Progression (A.P.)	સમાંતર શ્રેણી
Composite Integer	વિભાજ્ય સંખ્યા
Consistant	સુસંગત
Discriminant	વિવેચક
Distance Formula	અંતરસૂત્ર
Division of a Line-segment	રેખાખંડનું વિભાજન
Equality	સમતા
Finite Sequence	સાન્ત શ્રેણી
Frustum of a Cone	શંકુનો આડછેદ
Geometric Transformation	ભૌમિતિક રૂપાંતર
Greatest Common Divisor (*g.c.d.*)	ગુરુત્તમ સામાન્ય અવયવ
Horizontal Ray	ક્ષૈતિજકિરણ
Identity	નિત્યસમ
Incircle	અંતઃવર્તુળ
Least Common Multiple (*l.c.m.*)	લઘુતમ સામાન્ય અવયવ
Mathematical Model	ગાણિતિક પ્રતિકૃતિ
Method of Cross Multiplication	ચોકડી ગુણાકારની રીત
Method of Elimination	લોપની રીત
Method of Substitution	આદેશની રીત
Mid-point	મધ્યબિંદુ
nth term	nમું પદ
Pair of Linear Equations in Two Variables	દ્વિચલ સુરેખ સમીકરણ યુગ્મ
Pattern	ભાત
Point of Contact	સ્પર્શબિંદુ
Prime Integer	અવિભાજ્ય સંખ્યા
Proportionality	સમપ્રમાણતા
Quadratic Equation	દ્વિઘાત સમીકરણ
Ray of Vision	દૃષ્ટિકિરણ
Root	બીજ
Scale factor	સ્કેલમાપન
Section Formula	વિભાજન સૂત્ર
Sequence	શ્રેણી
Surface Area	પૃષ્ઠફળ
Tangent	સ્પર્શક
Trigonometry	ત્રિકોણમિતિ